Optimal Feedback in Contests

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with

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Model (1/4): Players & Timing

- **Players**: A principal and \( n \geq 2 \) agents

- At \( t = 0 \), the principal designs a contest comprising
  - a termination rule \( \tau \),
  - a rule for allocating a $1 prize, and
  - a feedback policy.

- At every \( t > 0 \), each agent
  - receives a message per the feedback policy, and
  - chooses effort \( a_{i,t} \in [0,1] \)

- The contest ends at \( \tau \) and prize is awarded according to allocation rule
Model (2/4): Agents’ Output & “Who observes what”

- **Each agent’s output** takes the form of a Poisson “breakthrough”:
  - During \((t, t + dt)\) agent \(i\) “succeeds” with probability \(a_{i,t} dt\)
  - Each agent can succeed at most once
  - Denote \(x_{i,t} = 1\) if agent \(i\) has succeeded by \(t\), and \(x_{i,t} = 0\) otherwise

- **Who observes what:**
  - Principal observes successes but not efforts
  - Each agent observes his effort but not successes

Denote by \(p_{i,t}\) agent \(i\)’s belief at \(t\) that he has succeeded

- *Note:* Effort is worthwhile for an agent only if he hasn’t yet succeeded
Model (3/4): Principal’s Choice Variables

- **A termination rule** $\tau$ is a stopping time w.r.t $x_t = \{x_{1,t}, \ldots x_{n,t}\}_{s \leq t}$
  - e.g., if $\tau = \inf \{t : x_{i,t} = 1 \text{ some } i\}$, contest ends upon first success

- **A prize allocation rule** $q \in [0,1]^n$ specifies the probability each agent wins the prize as function of $x_\tau$; i.e., each agent’s time of success
  - e.g., if $q_i(x_\tau) = \mathbb{I}\{x_{i,t} \geq x_{j,t} \forall j, t\}$, first agent to succeed wins prize w.p 1

- **A feedback policy** $\mathcal{M}$ specifies the message sent to each agent at every $t$ as a function of $x_t$ and past messages
  - e.g., if $m_{i,t} = x_{i,t} \forall i, t$, ea. agent is told whether he has succeeded
  - Alternatives: Random feedback, feedback about others’ successes, feedback about feedback, etc.
Model (4/4): Payoffs

- Given a contest, each agent’s expected utility is

\[ u_{i,t} = \max_{a_i \in [0,1]} \mathbb{E} \left[ q_i(x_{\tau}) - \int_t^\tau c a_{i,s} \, ds \right] \]

- **First term:** Probability agent \( i \) wins the prize
- **Second term:** Cost of effort where \( c \in (1/n, 1) \)
- **BNE:** Each agent chooses effort optimally anticipating rivals’ efforts

- Principal chooses a contest \( \{\tau, q, M\} \) and effort recommendations to

\[
\max_{\tau, q, M, a} \mathbb{E} \left[ \sum_{i=1}^n \int_0^\tau a_{i,t} \, dt \right] \\
s.t. \ a_{i,t} \text{ is IC for all } i, t.
\]
A Motivating Example

- Consider a manager who uses a promotion, acting as the prize, to motivate a group of employees.

- Each agent must clear some “bar” to be eligible for promotion.
  - This “bar” is represented by a success in the model (hence agents can succeed only once).

- Agents don’t definitively know whether they have cleared said “bar”, but the principal can disclose this (or other) information.

- Manager cares about aggregate effort (not clearing the bar per-se).

**Question:** How to design contest to get the most effort for $1 prize?
Remarks

i. *No discounting.*

- Model is equivalent to one in which players discount time at some rate, and the value of the prize appreciates at the same rate.

ii. *Agents don’t observe their own successes.*

- Goal is to give the principal full control of the agents’ information.
- In optimal contest, each agent is fully appraised of his own success; *i.e.*, main result would be unchanged if agents observed own successes.

iii. *Constant hazard rate of success.*

- Success during \((t, t + dt)\) depends only on effort during this interval.
- *Extension:* Arrival rate of success increases with past efforts.
Outline of Results

**Proposition 1:** Optimal contest *without* feedback

- No messages permitted and contest ends at some deterministic \( T \)
- *Egalitarian allocation rule* is optimal: Each agent who succeeds by \( T \) wins the prize with equal probability
**Outline of Results**

**Proposition 2:** Optimal contest *with* feedback

- *Cyclical structure:*
  - Initially, principal sets provisional deadline $T^*$
  - If one or more agents succeed by $T^*$, contest ends at $T^*$
  - Otherwise, the deadline is extended to $t = 2T^*$
  - If no agent succeeds by $2T^*$, deadline extended until $3T^*$. And so on.

- When contest ends, prize is awarded according to egalitarian rule.
  - *i.e.*, every agent who succeeded wins prize with equal probability

- Agents are fully appraised of their own success. They are informed about their rivals’ successes at $T^*, 2T^*, ..$
  - *i.e.*, if deadline is extended, then no one has succeeded yet

- This contest achieves the *first-best payoff* for the principal
Outline of Results

**Proposition 3:** Optimal contest with increasing hazard rate

- Effort today makes success tomorrow more likely
- Similar structure, except that each provisional deadline has a stochastic duration
No-feedback Contests

First, we restrict attention to contests without feedback

- No message transmission permitted (i.e., no direct feedback)
- Principal chooses a deterministic deadline $T$ (i.e., no indirect feedback)

Fix a contest and for each agent, define the **reward function**

$$R_{i,t} = \mathbb{E} \left[ q_i(x_T) \mid dx_{i,t} = 1 \right]$$

i.e., agent’s expected reward conditional on succeeding at $t$

The agent’s payoff can thus be expressed as

$$u_{i,t} = \max_{a_{i,s} \in [0,1]} \int_t^T (1 - p_{i,s}) a_{i,s} R_{i,s} - c a_{i,s} \, ds$$

- During $(t, t + dt)$, succeeds w.p $(1 - p_{i,t}) a_{i,t} dt$, in which case earns $R_{i,t}$,
- and he incurs cost $c a_{i,t} dt$
Agents’ Problem

- Fix an arbitrary deadline $T$ and reward function $R_{i,t}$. Agent solves

$$u_{i,0} = \max_{a_{i,t} \in [0,1]} \int_0^T [(1 - p_{i,t})R_{i,t} - c] a_{i,t} dt$$

s.t. $\dot{p}_{i,t} = (1 - p_{i,t}) a_{i,t}$ with $p_{i,0} = 0$

- On the constraint:
  - Evolution equation for $p_{i,t}$ follows from Bayes’ rule
  - Captures fact that effort today lowers future probability of success

- Std. optimal control problem: Use Pontryagin’s maximum principle
Agents’ Problem: Incentive Compatibility

- Today’s talk: Restrict attention to contests with $a_{i,t} = 1$ for all $[0, T]$

Lemma 1.
- Consider no-feedback contest w. deadline $T$ and reward function $R_{i,t}$
- Effort $a_{i,t} = 1$ is incentive compatible for all $t \in [0, T]$ if and only if

\[
e^{-t} R_{i,t} \geq \left( \frac{c}{MB \text{ at } t} \right) + \int_t^T e^{-s} R_{i,s} ds \quad \text{for all } t.
\]

- 1st term: Success arrives at rate $e^{-t}$, and reward is $R_{i,t}$
- 2nd term: (Direct) marginal cost of effort
- 3rd term: Success today eliminates possibility of success in the future
No-feedback Contest: Principal’s Problem

- Optimal no-feedback contest solves the following problem:

\[
\max_{T, q} \quad n \int_0^T 1 \, dt \\
\text{s.t.} \quad e^{-t} R_{i,t} \geq c + \int_t^T e^{-s} R_{i,s} \, ds \quad \forall \, i, t \\
\]

\[T \geq 0, \, q \text{ is a feasible prize allocation rule}\]

where \(R_{i,t} = \mathbb{E}\left[q_i(x_{\tau}) \mid dx_{i,t} = 1\right].\)

- The principal chooses
  - a terminal date \(T,\) and
  - a prize allocation rule \(q\)

  to maximize aggregate effort s.t IC constraint.

- Restriction to symmetric contests with max. effort shown to be wolog
**Definition 1:** Egalitarian prize allocation rule (EGA)

\[ q_{ega}^{i}(x_T) = \frac{x_{i,T}}{\sum_j x_{j,T}} \]

- *i.e.*, every agent who succeeds wins the prize with equal probability

**Definition 2:** \( \hat{T} \) is the unique solution of \( 1 - e^{-n\hat{T}} = nc(e^{\hat{T}} - 1) \)

- Given EGA & no feedback, this is longest max. effort is IC

**Proposition 1.**

- The optimal no-feedback contest has deadline \( \hat{T} \) and egalitarian prize allocation rule \( q^{ega} \).
- In equilibrium, each agent exerts maximum effort for all \( t \in [0, \hat{T}] \).
Optimal No-feedback Contest: Heuristic Derivation (1/3)

- Observation #1: $R_{i,t} = ce^T$ satisfies IC with equality for all $t$
  - Time-invariant & symmetric $R_{i,t}$ corresponds to EGA allocation rule

- Observation #2:
  - Recall $R_{i,t}$ is prob. agent wins prize conditional on succeeding at $t$
  - Given $R_{i,t}$, agent $i$ wins the prize with probability $\int_0^T e^{-t}R_{i,t}dt$. So
    \[
    \sum_i \int_0^T e^{-t}R_{i,t}dt \leq \Pr\{\text{prize is awarded}\} \quad \frac{1 - e^{-nT}}{\Pr\{\text{at least one agent succeeds}\}}
    \]
  - In other words, increasing $e^{-t}R_{i,t}$ entails an opportunity cost, and so the principal wants to minimize $e^{-t}R_{i,t}$ subject to satisfying IC.
Consider alternative contest with $e^{-t}\widetilde{R}_{i,t} > e^{-t}R_{i,t}$ on some interval.

**Egalitarian contest:** IC at $t'$ requires that $e^{-t'}R_{i,t'} \geq c + 1$

**Alternative contest:** IC at $t'$ requires that $e^{-t'}\widetilde{R}_{i,t'} \geq c + 1 + 2$

Thus $e^{-t}\widetilde{R}_{i,t} > e^{-t}R_{i,t}$ for all $t < t'$; i.e., $\widetilde{R}_{i,t}$ is more expensive.
Thus, any non-egalitarian contest with deadline $T$ can be replaced by EGA contest with same deadline that is *cheaper* for principal

- *Cheaper* $\Rightarrow$ Can extend deadline and still satisfy IC for all $t$

It remains to pin down the optimal deadline $\hat{T}$

- Fix a $T$. Given the egalitarian allocation rule,

$$\Pr\{\text{agent } i \text{ wins prize}\} = \int_0^T e^{-t} R_{i,t}^{ega} dt = \frac{1 - e^{-nT}}{n}$$

- Since $R_{i,t}^{ega}$ is time-invariant, we have $R_{i,t}^{ega} = \left[1 - e^{-nT}\right]/\left[n(1 - e^{-T})\right]$

- By def. $\hat{T}$ is largest deadline for which $R_{i,t}^{ega} \geq e^T c$; i.e., max effort IC
Optimality of Egalitarian Contest: Intuition (1/2)

- As an alternative, take “winner-takes-all” contest with deadline $T$
  i.e., at $T$, the prize is awarded to the first agent who succeeded
- Assuming max. effort is IC on $[0, T]$, we have reward functions
  $$R_{i,t}^{\text{wta}} = e^{-(n-1)t}$$
  i.e., if agent $i$ succeeds at $t$, he is the first to do so w.p. $e^{-(n-1)t}$

- Notice that $e^{-t} R_{i,t}^{\text{wta}} > e^{-t} R_{i,t}^{\text{ega}}$; i.e., WTA is more expensive than EGA
The problem is that the WTA contest frontloads incentives too much.

- IC is slack for all $t < T$; i.e., incentives excessively strong early on.

In contrast, EGA (maximally) backloads incentives s.t IC binds $\forall t$.
Key Lemma: Sufficient Condition for Optimality

- Next, we consider contests with an arbitrary feedback policy

Lemma 2:
A contest is guaranteed to be optimal if in equilibrium:

i. The prize is awarded with probability 1; i.e., \( \sum_i \mathbb{E}[q_i(x_T)] = 1 \)

ii. Each agent earns zero rents; i.e., \( u_{i,0} = 0 \) for all \( i \)

- The principal’s object can be rewritten as

\[
\mathbb{E} \left[ \sum_{i=1}^{n} \int_0^T a_i, t dt \right] = \frac{1}{c} \left[ \sum_i \mathbb{E}[q_i(x_T)] - \sum_i u_{i,0} \right]
\]

Pr\{prize awarded\} \leq 1 \quad \text{rents} \geq 0

- If a contest attains those bounds, it must be optimal (and first-best)
Step 1: Constructing a Zero-Rent Contest (1/2)

- We can write each agent's payoff as
  \[ u_{i,t} = \max_{a_{i,s} \in [0,1]} \int_{t}^{\tau} [(1 - p_{i,s})R_{i,s} - c]a_{i,s} ds \]

- For a contest to concede no rents to the agents,
  \[(1 - p_{i,t})R_{i,t} = c \text{ for all } i, t\]

- Claim: Whenever \(a_{i,t} > 0\), such a contest must have \(p_{i,t} = 0\)
  - Suppose there is an interval on which \(p_{i,t} > 0\) and \((1 - p_{i,t})R_{i,t} = c\)
  - Agent can pause effort during first half of interval so \(p_{i,t}^{\text{private}} < p_{i,t}^{\text{eqm}}\)
  - Then \((1 - p_{i,t}^{\text{private}})R_{i,t} > c\), so agent can earn rents during second half

- Thus feedback policy must keep agents appraised of own success
  - Define the feedback policy \(M^{\text{pronto}} = \{m_{i,t} = x_{i,t} \text{ for all } i, t\}\)
Step 2: Constructing a Zero-Rent Contest (2/2)

- Since $p_{i,t} = 0$ until each agent succeeds, contest must have $R_{i,t} = c$ for all $i, t$

- For $R_{i,t}$ to be time-invariant & symmetric, alloc. rule must be EGA

- Suppose prize is awarded according to EGA rule at some fixed $T$

- My reward conditional on succeeding at $t$, $R_{i,t}$, depends on how many rivals I expect to succeed by $T$
  
  - This number $N_T \sim Binom(n-1, 1-e^{-T})$, and
  
  $$R_{i,t}^{ega} = \mathbb{E} \left[ \frac{1}{1 + N_T} \right]$$

- If $T \approx 0$, no rivals will succeed a.s, so $R_{i,t}^{ega} \approx 1$

- As $T \to \infty$, all $n-1$ of my rivals will succeed a.s, so $R_{i,t}^{ega} \to 1/n$

- There is a unique $T^*$ such that $R_{i,t}^{ega} = c$
Step 3: Towards an Optimal Contest

- Consider the contest with:
  i. Deterministic deadline $T^*$
  ii. Egalitarian allocation rule
  iii. Feedback policy $\mathcal{M}^{\text{pronto}}$

- By construction,
  - $R_{i,t}^{\text{ega}} = c$ so each agent exerts max. effort until he succeeds and $u_{i,t} = 0$
  - But the prize is awarded with probability $\sum_i \mathbb{E}[q_i(x_T)] = 1 - e^{-nT^*} < 1$
    i.e., this contest satisfies part (ii) of Lemma 2, but **not** part (i)

- Next, we amend this contest such that $\sum_i \mathbb{E}[q_i(x_T)] = 1$
  - By Lemma 2, such contest will be optimal.
Step 3: Cyclical Structure

- Consider the (cyclical) termination rule:

\[ \tau^* = \inf \left\{ t : t = kT^* \text{, } k \in \mathbb{N} \text{ and } \sum_i x_{i,t} \geq 1 \right\} \]

- This contest comprises “cycles” of length \( T^* \), and is terminated at the end of the first cycle in which one or more agents have succeeded.

- Within each cycle, \( R_{i,t}^{ega} = c \) by construction, so maximum effort is IC, and each agent’s instantaneous payoff is 0. Thus, \( u_{i,t} = 0 \) for all \( t \).

- Since the contest doesn’t end until at least one agent succeeds, the prize is awarded with probability 1.

  - i.e., the contest satisfies conditions of Lemma 2, and is hence optimal.
Proposition 2.

- The following contest is optimal:
  
  i’. termination rule $\tau^* = \inf \{ t : t = kT^*, \ k \in \mathbb{N} \text{ and } \sum_i x_{i,t} \geq 1 \}$,
  
  ii. egalitarian prize allocation rule, and
  
  iii. feedback policy $M^{\text{pronto}}$

- In equilibrium, each agent exerts max. effort until he succeeds

Intuition for cyclical structure:

- If rivals exert max. effort during $[0, T]$, my expected reward conditional on succeeding $\downarrow T$ (because I will have to share prize with more rivals)
- By construction, $T^*$ is critical value such that $R_{i,t} = c$
- Cycles inform agents noone has succeeded, “resetting” incentives
Optimal feedback is most valuable when

- Marginal costs $c$ are small or large; i.e., close to $1/n$ or 1, or
- Number of agents $n$ is small
Increasing Hazard Rate

- So far, we have assumed constant (unit) hazard rate of success
  - i.e., agent succeeds during \((t, t + dt)\) with probability \(a_{i,t}dt\)

- Suppose instead that success arrives at rate \(\lambda_{i,t}a_{i,t}\), and
  \[
  \dot{\lambda}_{i,t} = f(\lambda_{i,t}) a_{i,t} dt
  \]
  for some function \(f(\cdot)\) and \(\lambda_{i,0} = \lambda\).

I. Case \(f(\lambda) < 0\): Effort today makes future success less likely
  - e.g., Halac et al. (2017): “good news Poisson experimentation”

II. Case \(f(\lambda) > 0\): Effort today makes future success more likely
  - Optimal contest has similar features & properties as in base model:
    it awards the prize with probability 1 and extracts all rents
Building Blocks

- **Assume:** \( f(\lambda) \geq 0 \) and satisfies \( \lambda_{i,t} \in (c, nc) \)
  - Suffices to assume \( \lambda > c \) and \( f(\lambda) = 0 \) for some \( \lambda \in (\lambda, nc) \)

- Let \( \lambda^*_t \) solve \( \dot{\lambda}_{i,t} = f(\lambda_{i,t}) \) subject to \( \lambda_{i,t} = \lambda \)
  - This is the trajectory of \( \lambda_{i,t} \) if agent exerts max. effort

- By an earlier argument, feedback policy \( M^{pronto} \) to extract all rents

- For max. effort to be IC and rents to be 0, we must have

\[
\lambda^*_t R_{i,t} = c \quad \text{for all } i, t
\]

  - Because \( \lambda^*_t \) increases in \( t \), \( R_{i,t} \) must decrease in \( t \)
    - i.e., incentives should be frontloaded since “earlier” success is “tougher”

- Suffices to find prize allocation and termination rules s.t \( R_{i,t} = c/\lambda^*_t \)
Proposition 3.

- There exists an optimal contest from the following class:
  1. **Cyclical stochastic structure**: Each cycle ends with rate $\gamma(t, \lambda_t)$
  2. At the end of each cycle, if a success has occurred, contest ends and prize is awarded according to EGA; otherwise, a new cycle starts
  3. Feedback policy $\mathcal{M}^{pronto}$; i.e., agents appraised of own success

- In equilibrium, each agent exerts max. effort until he succeeds
- *i.e.*, similar structure to before, except cycles have stochastic length
- If $\gamma = \infty$, contest is “winner-takes-all”
- If $\gamma = 0$ for $t < T$ and $\gamma = \infty$ for $t \geq T$, the contest is egalitarian
- By choosing function $\gamma(t, \lambda_t)$, can fine-tune degree of frontloading
Related Literature

- **Static tournaments / contests:**
  - Lazear & Rosen ('81), Green & Stokey ('83), Nalebuff & Stiglitz ('83)
  - *Optimal prize allocation:* Moldovanu & Sela ('01), Drugov & Ryvkin ('18, '19), Olszewski and Siegel ('20)
  - "*Turning down the heat*": Fang et al. ('18) and Letina et al. ('20)

- **Dynamic contests:**
  - Taylor ('95), Benkert & Letina ('20)
  - *Tugs of war:* Moscarini & Smith ('11), Cao ('14)

- **Feedback in contests:**
  - "*Reveal intermediate progress?*": Yildirim ('05), Lizzeri et al. ('05), Aoyagi ('10), Ederer ('10), Goltsman & Mukherjee ('19)
  - *Contests for experimentation:* Halac et al. ('17)
Discussion

- Contest design with endogenous feedback
  - Cyclical structure
  - Egalitarian prize allocation rule (maximally backloads incentives)
  - Each agent is always appraised of own success, but is informed of rivals’ successes periodically

- Future work
  - Continuous effort
  - Decreasing hazard rate
  - Continuous output / more general production functions
  - Asymmetric agents