Optimal Feedback in Contests

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with

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Model (1/4): Players & Timing

- **Players:** A principal and $n \geq 2$ agents

- At $t = 0$, the principal designs a contest comprising
  - i. a rule specifying *when* the contest will end,
  - ii. a rule for allocating a $1$ prize, and
  - iii. a feedback policy

- At every $t > 0$, each agent
  - receives a message per the feedback policy, and
  - chooses effort $a_{i,t} \in [0,1]$

- When the contest ends, prize is awarded according to allocation rule
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Model (2/4): Agents’ Output & “Who observes what”

- **Each agent’s output** takes the form of a Poisson “breakthrough”:
  - During \((t, t + dt)\) agent \(i\) “succeeds” with probability \(a_{i,t} dt\)
  - Each agent can succeed at most once
  - Denote \(x_{i,t} = 1\) if agent \(i\) has succeeded by \(t\), and \(x_{i,t} = 0\) otherwise

- **Who observes what:**
  - Principal observes successes but not efforts
  - Each agent observes his effort but not successes
  - **Denote** by \(p_{i,t}\) agent \(i\)’s belief at \(t\) that he has succeeded

\[\text{Note: Effort is worthwhile for an agent only if he hasn’t yet succeeded}\]
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Model (3/4): Principal’s Choice Variables

- A **termination rule** $\tau$ is a stopping time w.r.t $x_t = \{x_{1,t}, \ldots x_{n,t}\}_{s \leq t}$
  - *e.g.*, contest *may* end at prespecified deadline or upon first success

- A **prize allocation rule** $q \in [0, 1]^n$ specifies the probability each agent wins the prize as function of $x_\tau$; *i.e.*, each agent’s time of success
  - *e.g.*, prize *may* be awarded to the first agent to succeed or randomly

- A **feedback policy** $\mathcal{M}$ specifies the message sent to each agent at every $t$ as a function of $x_t$ and past messages
  - *e.g.*, each agent *may* be kept appraised whether he has succeeded
  - **Alternatives:** Random feedback, feedback about others’ successes, feedback about feedback, etc
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Model (4/4): Payoffs

- Given a contest, each agent’s expected utility is

\[ u_{i,t} = \max_{a_i \in [0,1]} \mathbb{E} \left[ q_i(x_\tau) - \int_t^\tau c a_i,s ds \right] \]

- **First term:** Probability agent \( i \) wins the prize
- **Second term:** Cost of effort where \( c \in (1/n, 1) \)

Principal chooses a contest \( \{\tau, q, M\} \) and effort recommendations to

\[ \max_{\tau,q,M,a} \mathbb{E} \left[ \sum_{i=1}^n \int_0^\tau a_{i,t} dt \right] \]

s.t. \( a_{i,t} \) is IC for all \( i, t \)

i.e., she maximizes aggregate effort subject to incentive compatibility
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A Motivating Example

- Consider a manager who uses a promotion, which acts as the prize, to motivate a group of employees.
- Each agent must clear some “bar” to be eligible for the promotion.
  - This “bar” is represented by a success in the model (hence agents can succeed only once).
- Agents don’t definitively know whether they have cleared said “bar”, but the principal can disclose this (or other) information.
- Manager cares about aggregate effort (not clearing the bar per-se).

**Question:** How to design contest to elicit maximum effort?
Remarks

i. *No discounting.*

- Model is equivalent to one in which players discount time at some rate, and the value of the prize appreciates at the same rate.
  
- *E.g.*, promotion pay raise may appreciate with the interest rate.

ii. *Agents don’t observe their own successes.*

- Goal is to give the principal full control of the agents’ information.

- In optimal contest, each agent is fully appraised of his own success; *i.e.*, main result would be unchanged if agents observed own successes.

iii. *Constant hazard rate of success.*

- Success during \((t, t + dt)\) depends only on effort during this interval.

- We extend model to allow success rate to increase with past efforts.
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Outline of Results

**Proposition 1:** Optimal contest *without* feedback

- No messages permitted and contest ends at some deterministic $T$
- *Egalitarian allocation rule* is optimal: Each agent who succeeds by $T$ wins the prize with equal probability
Outline of Results

Proposition 2: Optimal contest with feedback

- Cyclical structure:
  - Initially, principal sets provisional deadline $T^*$
  - If one or more agents succeed by $T^*$, the contest ends at $T^*$
  - Otherwise, the deadline is extended to $t = 2T^*$
  - If no agent succeeds by $2T^*$, deadline extended until $3T^*$. And so on.

- When contest ends, prize is awarded according to egalitarian rule
  - *i.e.*, every agent who has succeeded wins prize with equal probability

- Agents are fully appraised of their own success. They are informed about their rivals’ successes periodically at $T^*, 2T^*, ..$
  - *i.e.*, if deadline is extended, then no one has succeeded yet
Outline of Results

**Proposition 3:** Optimal contest with increasing hazard rate

- Effort today makes success tomorrow more likely
- Optimal contest has similar structure to the one with constant hazard rate, except that each provisional deadline has a stochastic duration
No-feedback Contests

- First, we restrict attention to contests without feedback
  - No message transmission permitted (i.e., no direct feedback)
  - Principal chooses a deterministic deadline $T$ (i.e., no indirect feedback)

- Fix a contest and for each agent define the reward function

$$R_{i,t} = \mathbb{E}\left[ q_i(x_T) \mid dx_{i,t} = 1 \right]$$

i.e., agent’s expected reward conditional on succeeding at $t$

- The agent’s payoff can thus be expressed as

$$u_{i,t} = \max_{a_{i,s} \in [0,1]} \int_t^T (1 - p_{i,s}) a_{i,s} R_{i,s} - c a_{i,s} ds$$

- During $(t, t + dt)$, succeeds w.p $(1 - p_{i,t})a_{i,t} dt$, in which case earns $R_{i,t}$, and he incurs cost $c a_{i,t} dt$
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  R_{i,t} = \mathbb{E} \left[ q_i(x_T) \mid dx_{i,t} = 1 \right]
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  - and he incurs cost $c a_{i,t} dt$
Agents’ Problem

- Fix an arbitrary deadline $T$ and reward function $R_{i,t}$. Agent solves

$$u_{i,0} = \max_{a_{i,t} \in [0,1]} \int_0^T \left[ (1 - p_{i,t}) R_{i,t} - c \right] a_{i,t} dt$$

s.t. $\dot{p}_{i,t} = (1 - p_{i,t}) a_{i,t}$ with $p_{i,0} = 0$

- On the constraint:
  - Evolution equation for $p_{i,t}$ follows from Bayes’ rule
  - Captures fact that effort today lowers future probability of success

- Std. optimal control problem: Use Pontryagin’s maximum principle
Agents’ Problem: Incentive Compatibility

- Wolog can restrict attention to contests with \( a_{i,t} = 1 \) for all \([0, T]\)

Lemma 1.
- Consider no-feedback contest w. deadline \( T \) and reward function \( R_{i,t} \)
- Effort \( a_{i,t} = 1 \) is incentive compatible for all \( t \in [0, T] \) if and only if

\[
\underbrace{e^{-t} R_{i,t}}_{\text{MB at } t} \geq \underbrace{c}_{\text{direct MC}} + \underbrace{\int_t^T e^{-s} R_{i,s} ds}_{\text{strategic cost}} \quad \text{for all } t.
\]

- 1\(^{st}\) term: Success arrives at rate \( e^{-t} \), and reward is \( R_{i,t} \)
- 2\(^{nd}\) term: (Direct) marginal cost of effort
- 3\(^{rd}\) term: Success today eliminates possibility of success in the future
Agents’ Problem: Incentive Compatibility

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**Lemma 1.**

- Consider no-feedback contest w. deadline $T$ and reward function $R_{i,t}$
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- **1st term:** Success arrives at rate $e^{-t}$, and reward is $R_{i,t}$
- **2nd term:** (Direct) marginal cost of effort
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No-feedback Contest: Principal’s Problem

Optimal no-feedback contest solves the following problem:

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\max_{T, q} \quad n \int_0^T 1 \, dt
\]
\[
\text{s.t.} \quad e^{-t} R_{i,t} \geq c + \int_t^T e^{-s} R_{i,s} \, ds \quad \forall i, t
\]
\[
T \geq 0, \quad q \text{ is a feasible prize allocation rule}
\]

where \( R_{i,t} = \mathbb{E}[q_i(x_{i \tau}) | d x_{i,t} = 1] \).

The principal chooses
- a terminal date \( T \), and
- a prize allocation rule \( q \)

to maximize aggregate effort subject to IC constraint.

Restriction to symmetric contests with max. effort shown to be wolog.
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Optimal No-feedback Contest

- **Definition 1:** Egalitarian prize allocation rule (EGA)

  \[ q_{i}^{ega}(x_T) = \frac{x_{i,T}}{\sum_{j} x_{j,T}} \]

  i.e., every agent who succeeds by \( T \) wins prize with equal probability

- **Definition 2:** \( \hat{T} \) is the unique solution of \( 1 - e^{-n\hat{T}} = nc(e^{\hat{T}} - 1) \)

  Given EGA & no feedback, this is longest max. effort is IC

**Proposition 1.**

- The optimal no-feedback contest has deadline \( \hat{T} \) and egalitarian prize allocation rule \( q^{ega} \).

- In equilibrium, each agent exerts maximum effort for all \( t \in [0, \hat{T}] \).
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Optimal No-feedback Contest: Heuristic Derivation (1/3)

- **Observation #1:** $R_{i,t} = ce^T$ satisfies IC with equality for all $t$
  - Time-invariant & symmetric $R_{i,t}$ corresponds to EGA allocation rule

- **Observation #2:**
  - Recall $R_{i,t}$ is prob. agent wins prize conditional on succeeding at $t$
  - Given $R_{i,t}$, agent $i$ wins the prize with probability $\int_0^T e^{-t}R_{i,t}dt$. So
    $$\sum_i \int_0^T e^{-t}R_{i,t}dt \leq \frac{1 - e^{-nT}}{Pr\{at least one agent succeeds\}}$$
    - Pr\{prize is awarded\}
  - In other words, increasing $e^{-t}R_{i,t}$ entails an opportunity cost, and so the principal wants to minimize $e^{-t}R_{i,t}$ subject to satisfying IC.
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Consider alternative contest with $e^{-t\tilde{R}_{i,t}} > e^{-t}R_{i,t}$ on some interval.

- **Egalitarian contest:** IC at $t'$ requires that $e^{-t'}R_{i,t'} \geq c + 1$

- **Alternative contest:** IC at $t'$ requires that $e^{-t'}\tilde{R}_{i,t'} \geq c + 1 + 2$

Thus $e^{-t}\tilde{R}_{i,t} > e^{-t}R_{i,t}$ for all $t < t'$; i.e., $\tilde{R}_{i,t}$ is more expensive.
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Thus, any non-egalitarian contest with deadline $T$ can be replaced by EGA contest with same deadline that is cheaper for principal

- **Cheaper** ⇒ Can extend deadline and still satisfy IC for all $t$

It remains to pin down the optimal deadline $\hat{T}$

- Fix a $T$. Given the egalitarian allocation rule,

$$\Pr \{\text{agent } i \text{ wins prize}\} = \int_0^T e^{-t} R_{i,t}^{ega} dt = \frac{1 - e^{-nT}}{n}$$

- Since $R_{i,t}^{ega}$ is time-invariant, we have $R_{i,t}^{ega} = \left[1 - e^{-nT}\right]/\left[n(1 - e^{-T})\right]$

- By def. $\hat{T}$ is largest deadline for which $R_{i,t}^{ega} \geq e^T c$; i.e., max effort IC
Thus, any non-egalitarian contest with deadline $T$ can be replaced by EGA contest with same deadline that is *cheaper* for principal

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By def. $\hat{T}$ is largest deadline for which $R_{i,t}^{ega} \geq e^T c$; *i.e.*, max effort IC
Optimality of Egalitarian Contest: Intuition

- Under the egalitarian contest, IC binds throughout $[0, \hat{T}]$ interval
- As an alternative, take “winner-takes-all” contest: at the deadline, the prize is awarded to the first agent who succeeded
  - This contest frontloads incentives: if agents exert max. effort until $T$, then IC binds at $T$ but it is slack at all $t < T$.
- Principal has single unit of the prize $\Rightarrow$ Limited “stock of incentives”
- So if IC is slack for $t < T$ under WTA contest, the egalitarian contest must motivate more effort
- Turns out the egalitarian contest maximally backloads incentives
- Hence it motivates the most effort among all no-feedback contests
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  - Principal has single unit of the prize $\Rightarrow$ Limited “stock of incentives”
  - So if IC is slack for $t < T$ under WTA contest, the egalitarian contest must motivate more effort
- Turns out the egalitarian contest *maximally* backloads incentives
  - Hence it motivates the most effort among all no-feedback contests
Key Lemma: Sufficient Condition for Optimality

- Next, we consider contests with arbitrary feedback policy

Lemma 2:
A contest is guaranteed to be optimal if in equilibrium:

1. The prize is awarded with probability 1; i.e., \( \sum_i \mathbb{E} [q_i(x_\tau)] = 1 \)

2. Each agent earns zero rents; i.e., \( u_{i,0} = 0 \) for all \( i \)

- The principal’s objective can be rewritten as

\[
\mathbb{E} \left[ \sum_{i=1}^{n} \int_0^\tau a_{i,t} \, dt \right] = \frac{1}{c} \left[ \sum_i \mathbb{E} [q_i(x_\tau)] - \sum_i u_{i,0} \right] \\
\text{Total Surplus} \leq 1 \quad \text{Rents} \geq 0
\]

- If a contest attains those bounds, it must be optimal (and first-best)
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E \left[ \sum_{i=1}^{n} \int_{0}^{\tau} a_{i,t} dt \right] = \frac{1}{c} \left[ \sum_i E[q_i(x_{\tau})] - \sum_i u_{i,0} \right]
\]

  - Total Surplus \( \leq 1 \)
  - Rents \( \geq 0 \)

- If a contest attains those bounds, it must be optimal (and first-best)
Step 1: Constructing a Zero-Rent Contest (1/2)

- We can write each agent's payoff as

\[ u_{i,t} = \max_{a_{i,s} \in [0,1]} \int_t^T [(1 - p_{i,s}) R_{i,s} - c] a_{i,s} ds \]

- For a contest to concede no rents to the agents, it must have

\[ (1 - p_{i,t}) R_{i,t} = c \text{ for all } i, t \]

- **Claim:** Whenever \( a_{i,t} > 0 \), such a contest must have \( p_{i,t} = 0 \)
  
  - Suppose there is an interval on which \( \dot{p}_{i,t} > 0 \) and \( (1 - p_{i,t}) R_{i,t} = c \)
  
  - Agent can pause effort during first half of interval so \( p_{i,t}^{\text{private}} < p_{i,t}^{\text{eqm}} \)
  
  - Then \( (1 - p_{i,t}^{\text{private}}) R_{i,t} > c \), so agent can earn rents during second half

- Thus feedback policy must keep agents appraised of own success

- Define the feedback policy \( M^{\text{pronto}} = \{ m_{i,t} = x_{i,t} \text{ for all } i, t \} \)
Step 1: Constructing a Zero-Rent Contest (1/2)

- We can write each agent's payoff as
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Step 2: Constructing a Zero-Rent Contest (2/2)

- Since $p_{i,t} = 0$ until each agent succeeds, contest must have
  $$R_{i,t} = c \quad \text{for all } i, t$$
- For $R_{i,t}$ to be time-invariant & symmetric, alloc. rule must be EGA
- Suppose prize is awarded according to EGA rule at some fixed $T$
- My reward conditional on succeeding at $t$, $R_{i,t}$, depends on how many rivals I expect to succeed by $T$
  - This number $N_T \sim \text{Binom}(n - 1, 1 - e^{-T})$, and
  $$R_{ega,i,t} = \mathbb{E} \left[ \frac{1}{1 + N_T} \right]$$
  - If $T \approx 0$, no rivals will succeed a.s, so $R_{ega,i,t} \approx 1$
  - As $T \to \infty$, all $n - 1$ of my rivals will succeed a.s, so $R_{ega,i,t} \to 1/n$
  - There is a unique $T^*$ such that $R_{ega,i,t} = c$
Step 3: Towards an Optimal Contest

- Consider the contest with:
  
  i. Deterministic deadline $T^*$
  
  ii. Egalitarian allocation rule

  iii. Feedback policy $\mathcal{M}^{pronto}$

- By construction,
  
  \[ R_{i,t}^{ega} = c \] so ea agent exerts max. effort until he succeeds and $u_{i,t} = 0$

  But the prize is awarded with probability $\sum_i \mathbb{E}[q_i(x_T)] = 1 - e^{-nT^*} < 1$

  i.e., this contest satisfies part (ii) of Lemma 2, but \textbf{not} part (i)

- Next, we amend this contest such that $\sum_i \mathbb{E}[q_i(x_T)] = 1$

  By Lemma 2, that contest will be optimal.
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Step 3: Cyclical Structure

- Consider the (cyclical) termination rule:

\[
\tau^* = \inf \left\{ t : t = kT^* , k \in \mathbb{N} \text{ and } \sum_i x_{i,t} \geq 1 \right\}
\]

- This contest comprises “cycles” of length \( T^* \), and is terminated at the end of the first cycle in which one or more agents have succeeded.

- Within each cycle, \( R_{i,t}^{ega} = c \) by construction, so maximum effort is IC, and each agent’s instantaneous payoff is 0. Thus, \( u_{i,t} = 0 \) for all \( t \).

- Since the contest doesn’t end until at least one agent succeeds, the prize is awarded with probability 1.

- i.e., the contest satisfies conditions of Lemma 2, and is hence optimal.
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- The following contest is optimal:
  
  i’. termination rule \( \tau^* = \inf \{ t : t = kT^*, \ k \in \mathbb{N} \text{ and } \sum_i x_{i,t} \geq 1 \} \),
  
  ii. egalitarian prize allocation rule, and
  
  iii. feedback policy \( M^{pronto} \)

- In equilibrium, each agent exerts max. effort until he succeeds.

Intuition for cyclical structure:

- If rivals exert max. effort during \([0, T]\), my expected reward conditional on succeeding \(\downarrow T\) (because I will have to share prize with more rivals).
- By construction, \(T^*\) is critical value such that \(R_{i,t} = c\)
- Cycles inform agents no one has succeeded, “replenishing” incentives.
Optimal Contest (with Feedback)

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Optimal feedback is most valuable when

- Marginal costs $c$ are small or large, or
- Number of agents $n$ is small
Increasing Hazard Rate

- So far, we have assumed constant (unit) hazard rate of success
  - \( i.e., \) each agent succeeds during \((t, t + dt)\) with probability \(a_{i,t}dt\)

- Suppose instead that success arrives at rate \(\lambda_{i,t}a_{i,t}\), and

\[
\dot{\lambda}_{i,t} = f(\lambda_{i,t})a_{i,t}dt
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for some function \(f(\cdot)\) and \(\lambda_{i,0} = \lambda\).

I. Case \(f(\lambda) < 0\): Effort today makes future success less likely
   - \(e.g., \) “Good news Poisson experimentation”: Halac et al. (2017)
   - A “zero-rent contest” does not exist ⇒ value creation/capture trade-off

II. Case \(f(\lambda) > 0\): Effort today makes future success more likely
   - Optimal contest has similar features & properties as in base model:
     - it awards the prize with probability 1 and extracts all rents
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Building Blocks

- **Assume:** $f(\lambda) \geq 0$ and $\lambda_{i,t} \in (c, nc)$
  - Suffices to assume $\lambda > c$ and $f(\lambda) = 0$ for some $\lambda \in (\lambda, nc)$

- Let $\lambda^*_t$ solve $\dot{\lambda}_{i,t} = f(\lambda_{i,t})$ subject to $\lambda_{i,t} = \lambda$
  - This is trajectory of $\lambda_{i,t}$ if agent exerts max. effort throughout $[0, t]$

- By an earlier argument, feedback policy $M^{\text{pronto}}$ to extract all rents

- For max. effort to be IC and rents to be 0, we must have
  \[
  \lambda^*_t R_{i,t} = c \quad \text{for all } i, t
  \]
  - Because $\lambda^*_t$ increases in $t$, $R_{i,t}$ must decrease in $t$
    - i.e., incentives should be frontloaded since “earlier” success is “tougher”

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Optimal Contest

Proposition 3.

- There exists an optimal contest from the following class:
  1. **Cyclical stochastic structure**: Each cycle ends with rate $\gamma(t, \lambda_t)$
  2. At the end of each cycle, if a success has occurred, contest ends and prize is awarded according to EGA; otherwise, a new cycle starts
  3. Feedback policy $M^{pronto}$; i.e., agents appraised of own success

- In equilibrium, each agent exerts max. effort until he succeeds

- *i.e.*, similar structure to before, except cycles have stochastic length
  - If $\gamma = \infty$, contest is “winner-takes-all”
  - If $\gamma = 0$ for $t < T$ and $\gamma = \infty$ for $t \geq T$, the contest is egalitarian
  - By choosing function $\gamma(t, \lambda_t)$, can fine-tune degree of frontloading
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Related Literature

- **Static tournaments / contests:**
  - Lazear & Rosen (’81), Green & Stokey (’83), Nalebuff & Stiglitz (’83)
  - *Optimal prize allocation:* Moldovanu & Sela (’01), Drugov & Ryvkin (’18, ’19), Olszewski and Siegel (’20)
  - *“Turning down the heat”:* Fang et al. (’18) and Letina et al. (’20)

- **Dynamic contests:**
  - Taylor (’95), Benkert & Letina (’20)
  - *Tugs of war:* Moscarini & Smith (’11), Cao (’14)

- **Feedback in contests:**
  - *“Reveal intermediate progress?”:* Yildirim (’05), Lizzeri et al. (’05), Aoyagi (’10), Ederer (’10), Goltsman & Mukherjee (’19)
  - *Contests for experimentation:* Halac et al. (’17)
Discussion

- Contest design with endogenous feedback
  - Cyclical structure
  - Egalitarian prize allocation rule (maximally backloads incentives)
  - Each agent is always appraised of own success, but is informed of rivals’ successes only periodically

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  - Continuous effort
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  - Continuous output / more general production functions
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