

# Lecture Notes on the Dynamic Provision of Discrete Public Goods

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November 27, 2018

These lecture notes provide an introduction to dynamic public good provision games. These games can be used to model situations in which a group of agents collaborate to complete a project, or computer scientists developing open source software, or individuals contributing to a fundraising campaign. These notes are targeted to advanced masters' or doctoral students.

We focus on a particular type of public good provision problems, in which (i) progress is gradual (*i.e.*, each contribution only gets you a step closer to completion) and (ii) the public good is discrete and generates a lump-sum payoff only once the cumulative contributions reach a threshold. Early contributions to this literature were made by Admati and Perry (1991), Marx and Matthews (2000) and Yildirim (2006), who characterized equilibria of such games, and showed that some of the static inefficiencies are mitigated when agents can contribute over time. These seminal papers modeled the dynamic public good games in discrete time, which restricted tractability and made it difficult to address other economically interesting questions. In this chapter, we model the problem in continuous-time, which as shown by

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Kessing (2007) and later by Georgiadis (2015), makes the model substantially more tractable and enables one to address various organizational questions.

This literature is related to at least four strands of literature: First, to papers that study the dynamic provision of continuous public goods; see, for example, Fershtman and Nitzan (1991) and the citing articles. Second, to models of common resource extraction problems; see, for example, Reinganum and Stokey (1985) and the citing articles. Third, to papers that consider dynamic contests; see, for example, Harris and Vickers (1987) and the citing articles. And finally, to strategic experimentation problems; see, for example, Bolton and Harris (1999), Keller, Rady and Cripps (2005), Bonatti and Hörner (2011) and the citing articles.

## 1 Model Description

Time  $t \in [0, \infty)$  is continuous. Agents are risk neutral and credit constrained, they discount time at rate  $r > 0$ , and each agent's outside option is normalized to 0. The project starts at state  $q_0 = 0$ , and at every moment  $t$ , each agent privately chooses his action to influence the process

$$dq_t = \left( \sum_{i=1}^n a_{i,t} \right) dt,$$

where  $q_t$  denotes the state of the project and  $a_{i,t} \geq 0$  denotes agent  $i$ 's action at time  $t$ . Each agent's flow cost of choosing action  $a$  is  $a^2/2$ . The project is completed at the first  $\tau$  such that  $q_\tau = Q$ . If  $\tau \leq T$ , where  $T \leq \infty$  is a given deadline, then each agent receives a prespecified reward  $V_n$  upon completion, and receives no reward otherwise.

We shall restrict attention to Markov Perfect equilibria (hereafter MPE) with symmetric and differentiable strategies. The restriction to differentiable strategies rules out the possibility that the agents can exploit that the project progresses deterministically to implement non-Markov equilibria, and is discussed below.

## 2 Preliminaries

For a given set of action profiles  $\{a_{i,t}\}$ , the discounted payoff function of agent  $i$  at time  $t$  satisfies

$$\Pi_{i,t} = \underbrace{e^{-r(\tau-t)}V_n \mathbf{1}_{\{\tau \leq T\}}}_{\text{discounted net payoff}} - \underbrace{\int_t^\tau e^{-r(s-t)} \frac{a_{i,s}^2}{2} ds}_{\text{discounted costs}},$$

where  $\tau$  denotes the completion time of the project. The first term captures agent  $i$ 's discounted net payoff from completing the project, while the second term captures her discounted costs along the evolution path of the project.

### 2.1 First-Best Outcome

We begin by considering the situation in which a central planner dictates each agent's action to maximize total surplus

$$TS = n \left[ e^{-r\tau} V_n \mathbf{1}_{\{\tau \leq T\}} - \int_t^\tau e^{-r(s-t)} \frac{a_{i,s}^2}{2} ds \right], \quad (1)$$

where  $\{a_{i,s}\}_{s \in [0, T]}$  is agent  $i$ 's action profile. Because each agent's costs are convex, it is without loss to restrict attention to symmetric action profiles.

This problem can be rewritten as follows:

$$\max_{\tilde{\tau} \leq T} \left\{ nV_n e^{-r\tilde{\tau}} - \min_{a_t} \left\{ \int_0^{\tilde{\tau}} e^{-rs} \frac{na_s^2}{2} ds \quad \text{s.t.} \quad \int_0^{\tilde{\tau}} na_s ds = Q \right\} \right\}, \quad (2)$$

that is, we solve the problem in two stages:

*Stage 1:* For a given completion time  $\tilde{\tau}$ , we find the action profile that minimizes total costs subject to completing the project at some fixed time  $\tilde{\tau}$ ; and

*Stage 2:* We find the completion time  $\tau^{fb}$  that maximizes the total discounted net payoff minus total costs.

Let us fix an arbitrary completion time  $\tilde{\tau} \leq T$ , and solve the first-stage problem. The

Lagrangian for this problem is

$$L(\lambda) = \min_{a_s} \left\{ n \int_0^{\bar{\tau}} \left( e^{-rs} \frac{a_s^2}{2} - \lambda a_s \right) ds + \lambda Q \right\},$$

where  $\lambda$  denotes the dual multiplier on the constraint, which specifies that the cumulative actions until  $\bar{\tau}$  equal the size of the project,  $Q$ . Noting that the objective is strictly convex, we have the first-order condition

$$a_s = \lambda e^{rs}.$$

Observe that in the first-best outcome, each agent's discounted marginal cost is constant in  $s$ , which is intuitive – due to the convexity of the agents' costs, they perfectly smooth their actions across time. We can pin down  $\lambda$  by substituting  $a_s = \lambda e^{rs}$  into the constraint to obtain

$$\lambda = \frac{rQ}{n} (e^{r\bar{\tau}} - 1)^{-1}, \text{ and hence, } a_s = \frac{rQ}{n} \frac{e^{rs}}{e^{r\bar{\tau}} - 1}. \quad (3)$$

By substituting the first-best action profile given in (3) into (2), we solve

$$\max_{\bar{\tau} \leq T} \left\{ nV_n e^{-r\bar{\tau}} - \frac{rQ^2}{2n} (e^{r\bar{\tau}} - 1)^{-1} \right\}$$

to obtain the first-best completion time,  $\tau^{fb} = \min \{T, \bar{\tau}^{fb}\}$ , where  $\bar{\tau}^{fb} = -\frac{1}{r} \ln \left( 1 - \sqrt{\frac{rQ^2}{2V_n n^2}} \right)$ . Note that it is efficient to complete the project if and only if  $rQ^2 < 2V_n n^2$ ; otherwise, the agents are collectively better off abandoning the project and collecting their outside option (which has been normalized to zero).

Because the project progresses deterministically, using that  $q_t = \int_0^t n a_s$ , we can equivalently write the first-best action profile as a function of the project state,  $q$ , instead of time,  $t$  as

$$a(q) = \frac{r}{n} \left( \left[ q - Q + \sqrt{\frac{2V_n n^2}{r}} \right]^+ \right),$$

where  $[\cdot]^+ := \max \{\cdot, 0\}$ .

### 3 Markov Perfect Equilibria

We will characterize the MPE for the cases in which  $T = \infty$  and the  $T < \infty$  separately, and in doing so, we will illustrate two common techniques for solving such problems. The restriction to MPE implies that action profiles may depend only on payoff-relevant variables (Maskin and Tirole, 2001), so, for example, agents cannot punish each other

#### 3.1 Case 1: No Deadline ( $T = \infty$ )

Absent a deadline, payoffs depend solely on the state of the project,  $q$ , and not on time  $t$ . Therefore, by the restriction to MPE, action profiles may depend only on  $q$  (see, for example, Maskin and Tirole 2001), and so the subscript  $t$  can be dropped.

Using standard arguments, one can derive the Hamilton-Jacobi-Bellman (hereafter HJB) equation for the discounted payoff function of agent  $i$

$$r\Pi_i(q) = \max_{a_i \geq 0} \left\{ -\frac{a_i^2}{2} + \left( \sum_{j=1}^n a_j \right) \Pi'_i(q) \right\}, \quad (4)$$

where  $a_i : [0, Q] \rightarrow \mathbb{R}_+$  is agent  $i$ 's action profile, subject to the boundary conditions

$$\Pi_i(q) \geq 0 \text{ and } \Pi_i(Q) = V_n. \quad (5)$$

Note that (4) admits the following interpretation: at the optimal action profile, each agent's flow costs,  $r\Pi_i(q)$ , must be equal to his flow benefit of bringing the progress closer to completion,  $\left(\sum_{j=1}^n a_j\right) \Pi'_i(q)$ , minus his flow costs,  $a_i^2/2$ . The first boundary condition asserts that at every project state, each agent's discounted payoff must be nonnegative, for otherwise, he can choose action  $a_i = 0$  forever after and guarantee himself payoff zero. The second condition asserts that upon completing the project, each agent receives his reward and the game ends.

In a MPE, at every moment  $t$ , each agent  $i$  observes the state of the project  $q$ , and chooses

his action  $a_i$  to maximize his discounted payoff while accounting for the action profiles of all the other agents. Assuming an interior solution to (4), which we will verify later, the first-order condition for agent  $i$ 's problem requires that  $a_i(q) = \Pi'_i(q)$ : at every moment, he chooses his action such that his marginal cost equals the marginal benefit associated with bringing the project closer to completion. Noting that the second-order condition is satisfied and that the first-order condition is necessary and sufficient, and using the symmetry assumption, it follows that in any MPE, the discounted payoff for each agent satisfies

$$r\Pi(q) = \frac{2n-1}{2} [\Pi'(q)]^2 \quad (6)$$

subject to (5).

This is a first-order nonlinear differential equation, which can be solved analytically. Towards this goal, let us conjecture that its solution is of the form  $\Pi(q) = A + Bq + \Gamma q^2$ . By substituting into (6) and the boundary condition, it follows that

$$\Pi(q) = \frac{r}{2(2n-1)} \left( \left[ q - Q + \sqrt{\frac{2V_n}{r} \frac{2n-1}{n}} \right]^+ \right)^2.$$

Therefore, as long as  $rQ^2 < 2V_n \left( \frac{2n-1}{n} \right)$ , there exists a MPE in which the project is completed in finite time. Note that  $\Pi'(q) \geq 0$  for all  $q$ , and so the solution to each agent's problem is indeed interior as we assumed above. Using the first-order condition  $a(q) = \Pi'(q)$ , it follows that along the equilibrium path, each agent chooses action

$$a(q) = \frac{r}{2n-1} \left[ q - Q + \sqrt{\frac{2V_n}{r} \frac{2n-1}{n}} \right]^+. \quad (7)$$

Notice that if  $rQ^2 \geq 2V_n \left( \frac{2n-1}{n} \right)$ , then  $a(0) = 0$ , and so no agent ever chooses  $a_i > 0$ , and the project is not completed in equilibrium. In general, there need not exist a unique MPE. In particular, if  $rQ^2 \in \left( 2V_n, 2V_n \frac{2n-1}{n} \right)$ , then there exists another MPE in which no agent ever chooses  $a_i > 0$ , and the project is not completed. To see why such a MPE exists, suppose

that all agents  $j \neq i$  choose  $a_j(0) = 0$ . Then agent  $i$  finds it optimal to also choose  $a_i(0) = 0$  because he is not willing to undertake the entire project single-handedly (since  $rQ^2 > 2V_n$ ).

A first observation is that  $a(q)$  is increasing in  $q$ , that is, the agents choose a higher action, the closer the project is to completion. This is due to the facts that they are impatient and costs are incurred at the time each action is chosen, while they are rewarded only when the project is completed. As a result, their incentives are stronger, the closer the project is to completion. An implication of this observation, which was first made by Yildirim (2006) and Kessing (2007), is that actions are strategic complements (across time) in this game. That is because by raising his action, an agent brings the project closer to completion, which induces the other agents to raise their future actions.

We have assumed that the project progresses deterministically for the sake of simplicity. One can extend the analysis to the case in which the project progresses stochastically, for example, according to  $dq_t = (\sum_{i=1}^n a_{i,t}) dt + dB_t$  where  $B_t$  is a standard Brownian motion with  $B_0 = 0$ . In that case, in a MPE, each agent's discounted payoff is characterized by a second-order differential equation, which does not admit a closed-form solution, but one can show that the main results and comparative statics continue to hold. See Georgiadis (2015) for details.

**Exercise.** Verify that in the MPE characterized above, at every project state, each agent chooses a strictly lower action compared to the first-best outcome, and the project is completed at a later time.

**Some Comparative Statics.** By differentiating  $a(q)$  with respect to  $V_n$  and  $r$ , while holding all other parameters constant, we find that each agent's effort level possesses the following properties:

- i.  $a(q)$  is increasing in  $V_n$  ; and
- ii. there exists a threshold  $\Theta_r$  such that  $a(q)$  is increasing in  $r$  if and only if  $q \geq \Theta_r$ .

The first statement is intuitive: if the agents receive a larger reward upon completion, then their incentives are stronger. The second statement asserts that as agents become less patient, they choose a larger action when the project is close to completion, and a smaller action when it is far from completion. The intuition behind it is more subtle. Notice that the marginal benefit of bringing the completion time forward (which occurs if the agents raise their action) is equal to  $-\frac{d}{d\tau}(e^{-r\tau}V_n) = rV_n e^{-r\tau}$ , and it increases in  $r$  if (and only if)  $\tau$  is sufficiently small; *i.e.*, the project is sufficiently close to completion.

Next, consider the effect of the team size on the agents' incentives. In particular, consider two teams comprising of  $n$  and  $m > n$  identical agents. For this analysis, it is necessary to consider how each agent's reward depends on the group size. We assume that upon completion of the project, each agent receives reward  $V_n = V/n$ . Other parameters held constant, there exist thresholds  $\Theta_{n,m}$  and  $\Phi_{n,m}$  such that

- i.  $a_m(q) \geq a_n(q)$  if and only if  $q \leq \Theta_{n,m}$ ; and
- ii.  $ma_m(q) \geq na_n(q)$  if and only if  $q \leq \Phi_{n,m}$ .

It is left to the reader to verify these results.

By increasing the size of the group, two opposing forces influence the agents' incentives: First, as is well known in the static moral hazard in teams literature starting with Olson (1965), the agents obtain stronger incentives to free-ride. Moreover, because the agents' incentives to free-ride are proportional to their costs, actions increase with progress, and costs are convex, this free-riding effect becomes more intense as the project progresses. Second, because actions are strategic complements in this game, when agents are part of a larger group, they have incentives to raise their actions, because doing so will induce a greater number of other agents to raise their future actions, which in turn makes them better off. This encouragement effect is strong at the outset of the game where a lot of progress remains to complete the project, and it becomes weaker as the project nears completion. In summary, the encouragement effect dominates the free-riding effect if and only if the project is sufficiently



far from completion. This implies that by increasing the group size, the agents obtain stronger incentives when the project is far from completion, while their incentives become weaker near completion.

### 3.2 Case 2: Finite Deadline ( $T < \infty$ )

Suppose now that the agents must complete the project by some prespecified deadline  $T < \infty$ . In this case, time  $t$  is also payoff-relevant, and so now each agent's action profile must be a function of both  $t$ , and the project state  $q$ , so  $a_i : [0, T] \times [0, Q] \rightarrow \mathbb{R}_+$ . In most cases, it is difficult to solve problems with more than one state variables using the technique illustrated in *Case 1*, because the HJB equations take the form of partial differential equations, which are less tractable than ordinary differential equations. To solve the problem with a finite deadline we will use optimal control techniques, and exploit the fact that the project progresses deterministically, which implies that there is a one-to-one mapping between  $t$  and  $q$  along the equilibrium path.

It is convenient to consider an auxiliary game in which the project *must* be completed by the deadline  $T$ . Conditional on a completion time before the deadline,  $\tau \leq T$ , each agent minimizes his discounted costs, while anticipating that each of the other agents behaves in the same cost-minimizing manner. The MPE of this auxiliary game will be a MPE for the original game if each agent's discounted payoff at time 0 is weakly larger than his outside option (which has been normalized to 0), and otherwise, there is no project-completing MPE.

We use the maximum principle of optimal control and write strategies and payoffs as a function of time  $t$  (see, for example, Kamien and Schwartz, 2012).<sup>1</sup> The Hamiltonian corresponding to each agent  $i$ 's objective function is

$$H_{i,t} = -e^{-rt} \frac{a_{i,t}^2}{2} + \lambda_{i,t} \left( \sum_{j=1}^n a_{j,t} \right), \quad (8)$$

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<sup>1</sup>Note that the optimal control approach is equivalent to the HJB approach. However, it is often the case that one is more tractable than the other.

where  $\lambda_{i,t} \geq 0$  is the co-state variable associated with agent  $i$ 's payoff function. Conceptually,  $\lambda_{i,t}$  is analogous to a dual multiplier in Lagrangian optimization, except that in optimal control, it is a function rather than a scalar. Moreover, it can be interpreted as agent  $i$ 's marginal benefit from bringing the project closer to completion at  $t$ .<sup>2</sup> His terminal value function is  $\phi_{i,\tau} = e^{-r\tau}V_n$ , and the requirement that the project be completed by the deadline imposes the constraint

$$\int_0^\tau \sum_{i=1}^n a_{i,t} dt = Q \quad , \text{ where } \tau \leq T. \quad (9)$$

From Pontryagin's maximum principle, we have that each agent's action profile must maximize (8), and so we have the optimality (first-order) condition

$$\frac{dH_{i,t}}{da_{i,t}} = 0 \quad , \text{ or equivalently, } a_{i,t} = \lambda_{i,t}e^{rt}, \quad (10)$$

and the *adjoint equation*

$$\dot{\lambda}_{i,t} = -\frac{dH_{i,t}}{dq_t}, \quad (11)$$

respectively. Intuitively, the adjoint equation specifies the evolution path of  $\lambda_{i,t}$ . Finally, the transversality condition for each agent is

$$H_{i,\tau} + \frac{d\phi_\tau}{d\tau} \geq 0 \quad (= 0 \quad \text{if } \tau < T). \quad (12)$$

The conditions (9)-(12) are necessary conditions for a MPE for the auxiliary game. Therefore, we will proceed by characterizing a solution to this system of equations, and argue that the corresponding action profile  $\{a_{i,t}\}$  constitutes a MPE for the original game if and only if each agent obtains a nonnegative ex-ante discounted payoff.

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<sup>2</sup>Note that each agent  $i$ 's Hamiltonian is a function of  $t, q, \{a_{j,t}\}_{j=1}^n$ , and  $\lambda_{i,t}$ . For notational simplicity, we suppress the latter three arguments and simply write  $H_{i,t}$ .

First, by totally differentiating the Hamiltonian with respect to  $q_t$ , we can rewrite (11) as

$$\dot{\lambda}_{i,t} = - \sum_{j=1}^n \frac{dH_{i,t}}{da_{j,t}} \frac{da_{j,t}}{dt} \frac{dt}{dq_t} = - \sum_{j \neq i} \frac{dH_{i,t}}{da_{j,t}} \frac{da_{j,t}}{dt} \frac{dt}{dq_t} = - \sum_{j \neq i} \frac{\lambda_{i,t} (r\lambda_{j,t} + \dot{\lambda}_{j,t})}{\sum_{l=1}^n \lambda_{l,t}},$$

where the first equality uses that  $dH_{i,t}/da_{i,t} = 0$  by the optimality condition, and the second equality uses that  $da_{j,t}/dt = (r\lambda_{j,t} + \dot{\lambda}_{j,t})e^{rt}$  and  $dq_t/dt = \sum_{l=1}^n \lambda_{l,t}e^{rt}$ . After rearranging terms and imposing symmetry, we have

$$\dot{\lambda}_t = - \frac{n-1}{2n-1} r \lambda_t.$$

This differential equation has a unique non-zero solution,  $\lambda_t = c e^{-rt \frac{n-1}{2n-1}}$ , where  $c$  is a constant to be determined. By substituting  $\lambda_t$  into (9), we can express  $c$  as a function of the completion time  $\tau$  as follows:

$$cn \int_0^\tau e^{rt(1-\frac{n-1}{2n-1})} dt = Q \Rightarrow \frac{(2n-1)c}{r} (e^{\frac{rn\tau}{2n-1}} - 1) = Q \Rightarrow c = \frac{rQ}{2n-1} (e^{\frac{rn\tau}{2n-1}} - 1)^{-1}.$$

The completion time  $\tau$  can be pinned down by the transversality condition, (12), which can be rewritten as

$$1 - \sqrt{\frac{rQ^2}{2V} \frac{1}{2n-1}} \leq e^{-\frac{rn\tau}{2n-1}} \quad (= 0 \text{ if } \tau < T).$$

By noting that the right-hand-side in the last inequality decreases in  $\tau$  and is  $< 1$ , it follows that the project is completed at  $\tau^{mpe} = \min \{T, \bar{\tau}^{mpe}\}$ , where  $\bar{\tau}^{mpe} = -\frac{2n-1}{rn} \ln \left( 1 - \sqrt{\frac{rQ^2}{2V} \frac{1}{2n-1}} \right)$ . Sufficiency follows by noting that  $H_{i,t}$  is strictly concave in  $a_{i,t}$ , and applying the Mangasarian theorem (see Seierstad and Sydsaeter, 1987).

Collecting terms, it follows that there exists a unique candidate for a symmetric, project-

completing MPE, wherein each agent's effort and ex-ante discounted payoff satisfies

$$a_t^{mpe} = \frac{rQ}{2n-1} \frac{e^{\frac{rnt}{2n-1}}}{e^{\frac{rn\tau^{mpe}}{2n-1}} - 1} \quad \text{and} \quad \Pi_0^{mpe} = e^{-r\tau^{mpe}} V_n - \frac{rQ^2}{2(2n-1)} \frac{e^{\frac{r\tau^{mpe}}{2n-1}} - 1}{\left(e^{\frac{rn\tau^{mpe}}{2n-1}} - 1\right)^2}, \quad (13)$$

respectively,  $\tau^{mpe} = \min\{T, \bar{\tau}^{mpe}\}$  and  $\bar{\tau}^{mpe} = -\frac{2n-1}{rn} \ln\left(1 - \sqrt{\frac{rQ^2}{2V_n} \frac{1}{2n-1}}\right)$ . This candidate is an MPE if and only if  $\Pi_0 \geq 0$ .

Notice that in the MPE characterized above, each agent's discounted marginal cost (*i.e.*,  $e^{-rt}a_t^{mpe}$ ) decreases over time, which implies that equilibrium actions are frontloaded. Notice that actions increase with progress and this is a game with positive externalities. Therefore, each agent has an incentive to distort her action profile to induce the other agents to raise their future actions, which is accomplished by raising her action at the early stages of the project; *i.e.*, by frontloading. This is a consequence of actions being strategic complements across time, as discussed earlier.

We conclude by discussing the restriction to MPE with *differentiable strategies*. Heuristically, consider an action profile in which at every moment  $t$ , each agent chooses the first-best action  $a_t^{fb}$  as long as every agent has chosen the first-best action in the past (*i.e.*, if  $q_t = \int_0^t na_s^{fb} ds$ ), and chooses the action that corresponds to the MPE with differentiable strategies otherwise. One can then construct a MPE which implements the first-best outcome using such an action profile. For details, see p. 12 in Georgiadis (2017). This result is an artifact of the assumption that the project progresses deterministically. In this sense, it is economically not very interesting, and so we rule it out.

## Exercise

- i. Verify that if the project is completed strictly before the deadline in equilibrium, then the action profile given in (13) coincides with (7).
- ii. Compare the equilibrium action profile for different deadlines to the first-best outcome. Does there exist a deadline that implements the first-best outcome in a MPE? Explain.

## 4 Wrapping Up

### 4.1 Further Reading

The framework considered in these notes has been used to address various organizational questions. For example, Georgiadis et al. (2014) ask how a principal should set the size of a project that is undertaken by a group of agents to maximize her profits, and they show that the principal faces a commitment problem: as the project progresses, she has an incentive to increase the size of the project (despite the agents being rational and anticipating this behavior). Georgiadis (2015) examines how a profit-maximizing principal would choose the team size and the agents' incentive contracts. See also Georgiadis and Tang (2017) for a summary. Bowen et al. (2019) ask how heterogenous agents will agree to a particular project size under different collective choice institutions (*e.g.*, dictatorship, unanimity). Cvitanic and Georgiadis (2016) construct a mechanism that induces efficient contributions for dynamic public good provision and scarce resource extraction games.

### 4.2 Open Questions

The type of games analyzed in these notes fall under the class of dynamic games with externalities. In particular, the dynamic provision of a discrete public good (*i.e.*, one that generates a payoff *only* upon provision) is one with positive externalities and actions are strategic complements. In contrast, models of dynamic provision of continuous public goods exhibit positive externalities and actions are strategies substitutes, and as shown by (Fershtman and Nitzan (1991)), exhibit rather different dynamics, as do models of dynamic resource extraction, as analyzed by Reinganum and Stokey (1985). It is an open question how to develop a unified framework to analyze such dynamic games with externalities. This framework may be useful to address questions such as how to organize public good provision mechanisms (*e.g.*, Kickstarter campaigns) or tournaments to maximize welfare.

A standard assumption in such models is that at every moment, each player observes the

relevant state variable. As shown by Georgiadis (2017), under certain assumptions, efficiency can be restored in a dynamic public good provision game if the players never observe progress until the project is completed. It is an open question how a designer should control what information each player observes when choosing an action in public good games. Towards this direction, the information design tools developed by Rayo and Segal (2010) and Kamenica and Gentzkow (2011), and extended to a dynamic setting by Ely (2017) and others may be helpful.

Finally, a technical question is to develop techniques for analyzing such games with stochastic progress, a deadline, and non-binary actions. These are features of many real-world settings, but mathematically, they are quite intractable. In particular, the trick employed in these notes is no longer applicable, because the mapping between the project state and time is no longer one-to-one, so different tools are necessary.

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