Online Appendix

Proof of Proposition 1. This proof is organized in 4 parts. To begin, let $J_i(\cdot)$ denote the expected discounted payoff of each member of an n-person team with parameters $\{r_i, V_i\}$ who undertakes a project with volatility σ_i.

Proof for property (i): First, pick $\alpha < 1$ and V such that $V_1 = \alpha V_2 < V_2 = V$, and let $r = r_1 = r_2$ and $\sigma = \sigma_1 = \sigma_2$. Let $D_V(q) = J_1(q) - J_2(q)$, and note that it is smooth, and $D_V(0) = (\alpha - 1) V < 0 = \lim_{q \to -\infty} D_V(q) = 0$. Suppose that $D_V(\cdot)$ has some interior extreme point, which I denote by z^*. Then $D'_V(z^*) = 0$, and by using (**??**) we have

$$rD_V(z^*) = \frac{\sigma^2}{2} D''_V(z^*) \, .$$

Suppose that z^* is a global minimum. Then $D''_V(z^*) \geq 0 \Longrightarrow D_V(z^*) \geq 0$, which contradicts the fact that $D_V(0) < 0$. So z^* must be a global maximum. Then $D''_V(z^*) \leq 0 \Longrightarrow D_V(z^*) \leq 0$, which contradicts the fact that z^* is interior. Hence $D_V(\cdot)$ cannot have any interior extreme points, and thus it must be decreasing for all q; *i.e.*, $D'_V(q) \leq 0$ for all q and $D'_V(q) < 0$ for at least some q.

The next step involves showing that in fact, $D'_V(q) < 0$ for all q. Suppose that there exists a z such that $D'_V(z) = 0$. Then either $D_V(z) = 0$ or $D_V(z) < 0$. First, suppose that $D_V(z) = 0$. Because $\lim_{q \to -\infty} D_V(q) = 0$, any interior maximum on $(-\infty, z]$ must satisfy $D_V(z) = \frac{\sigma^2}{2r} D''_V(z) \leq 0$, and any interior minimum must satisfy $D_V(z) = \frac{\sigma^2}{2r} D''_V(z) \geq 0$. Therefore, $D_V(q) = D'_V(q) = 0$ for all $q < z$. Next, suppose that $D_V(z) < 0$, and let $\hat{z} = \arg\min_{q \leq z} \{D_V(q)\}$. Clearly, $\hat{z} > -\infty$. To show that $\hat{z} < z$, suppose that the contrary is true; *i.e.*, $\hat{z} = z$. Then $D'_V(z) = 0$, $D_V(z) < 0$, and (**??**) imply that $D''_V(z) < 0$, which contradicts the assumption that $\hat{z}$ is a minimum. Hence $\hat{z}$ is interior, so that $D'_V(z) = 0$ and $D''_V(z) \geq 0$, which together with (**??**) imply that $D_V(z) \geq 0$. However, this contradicts the assumption that $D_V(z) < 0$. Therefore, $D_V(z) = 0$, and it follows that $D_V(q) = D'_V(q) = 0$ for all $q < z$. Next, let $M(q) = [J_1(q) - J_2(q)] + [J'_1(q) - J'_2(q)]$, and note that $M(q) \leq 0$ for all q, $M(0) < 0$, and $M(q) = 0$ for all $q < z$. By applying the differential form of Grönwall's inequality, it follows that $M(q) = 0$ for all q, which contradicts the fact that $M(0) < 0$. Hence, I conclude that there does not exist a z such that $D'_V(z) = 0$. Therefore, $D'_V(q) < 0$ for all q, which implies that $a_1(q) < a_2(q)$ for all q.

Proof for property (ii): First pick $\delta > 1$ and r such that $r_1 = \delta r > r = r_2$. Next, define $D_r(q) = J_1(q) - J_2(q)$. By noting that $\lim_{q \to -\infty} D_r(q) = D_r(0) = 0$, observe that either $D_r(\cdot) \equiv 0$, or $D_r(\cdot)$ has at least one interior extreme point. Suppose $D_r(\cdot) \equiv 0$. Then $D_r'(\cdot) \equiv D_r''(\cdot) \equiv 0$, and using (**??**) we have that $\delta J_1(\cdot) \equiv J_2(\cdot)$. However this is a contradiction, because $J_1(\cdot) \equiv J_2(\cdot)$, ,$J_i(\cdot) > 0$ and $\delta > 1$. Therefore $D_r(\cdot)$ must have at least one interior extreme point, which I denote by z^*. By noting that $D_r'(z^*) = 0$ and using (**??**), we have that

$$r\left[\delta J_1(z^*) - J_2(z^*)\right] = \frac{\sigma^2}{2} D_r''(z^*) \ .$$

Suppose that z^* is a global maximum. Then $D_r''(z^*) \leq 0$, and hence $\delta J_1(z^*) - J_2(z^*) \leq 0$. However because $J_i(\cdot) > 0$ and $\delta > 1$, this implies that $D_r(z^*) < 0 = D_r(0)$, which contradicts the assumption that z^* is a global maximum. Therefore, z^* must be a global minimum, and $D_r(q) \leq 0$ for all q.

I next show that $D_r(\cdot)$ is single-troughed. Suppose it is not. Then I can find an interior local minimum z^* followed by an interior a local maximum $\bar{z} > z^*$. Since $\bar{z}$ is an interior maximum, $D_r'(\bar{z}) = 0$ and $D_r''(\bar{z}) \leq 0$, and from (**??**) it follows that $\delta J_1(\bar{z}) \leq J_2(\bar{z})$. Because z^* is an interior minimum, $D_r''(z^*) \geq 0$ implies that $\delta J_1(z^*) \geq J_2(z^*) \Rightarrow -\delta J_1(z^*) \leq -J_2(z^*)$, and by using $\delta J_1(\bar{z}) \leq J_2(\bar{z})$, we have that $0 < \delta\left[J_1(\bar{z}) - J_1(z^*)\right] \leq J_2(\bar{z}) - J_2(z^*)$, where the first inequality follows from Theorem 1 (iii) and the fact that $\bar{z} > z^*$. By assumption $D_r(\bar{z}) > D_r(z^*)$, which implies that $J_2(\bar{z}) - J_2(z^*) < J_1(\bar{z}) - J_1(z^*)$, so that

$$\delta\left[J_1(\bar{z}) - J_1(z^*)\right] \leq J_2(\bar{z}) - J_2(z^*) < J_1(\bar{z}) - J_1(z^*) \ ,$$

which contradicts the facts that $\delta > 1$ and $J_1(\bar{z}) - J_1(z^*) > 0$. Hence $D_r(\cdot)$ must be single-troughed. Because $\lim_{q \to -\infty} D_r(q) = D_r(0) = 0$, there exists a $\Theta_r < 0$ such that $D_r'(q) \leq 0$, and hence $a_1(q) \leq a_2(q)$, if and only if $q \leq \Theta_r$.

Proof for property (iii): First pick $\alpha > 1$ and σ such that $\sigma_1^2 = \alpha\sigma_2^2 > \sigma_2^2 = \sigma^2$. Let $J_1(\cdot)$ and $J_2(\cdot)$ denote each agent's expected discounted payoff associated with σ_1 and σ_2, respectively. Moreover let $D_\sigma(q) = J_1(q) - J_2(q)$ and observe that $\lim_{q \to -\infty} D_\sigma(q) = D_\sigma(0) = 0$. So either $D_\sigma(\cdot) \equiv 0$ on $(-\infty, 0]$, or $D_\sigma(\cdot)$ has some interior global extreme point. Suppose that $D_\sigma(\cdot) \equiv 0$ on $(-\infty, 0]$. This implies that $D_\sigma(q) = D_\sigma'(q) = D_\sigma''(q) = 0$ for all q, and using (**??**) it follows that for all q

$$rD_\sigma(q) = \frac{\sigma^2}{2}\left[\alpha D_\sigma''(q) + (\alpha - 1) J_2''(q)\right] \Longrightarrow J_2''(q) = 0 \, .$$

However this contradicts Theorem 1 (iii), which implies that $D_\sigma(\cdot)$ has at least one interior global extreme point, denoted by z^*. Then $D_\sigma'(z^*) = 0$, and using (**??**) yields $rD_\sigma(z^*) =$

$\frac{\sigma^2}{2}\left[\alpha D''_\sigma(z^*)+(\alpha-1)J''_2(z^*)\right]$. Suppose that z^* is a global minimum. Then $D''_\sigma(z^*)\geq 0$, $\alpha>1$, and $J''_2(z^*)>0$ imply that $D_\sigma(z^*)>0$. However, this contradicts the fact that $D_\sigma(0)=0$. Therefore z^* must be a maximum. This implies that there exist interior thresholds $\Theta_{\sigma,1}\leq\Theta_{\sigma,2}$ such that $D_\sigma(\cdot)$ is increasing on $(-\infty,\Theta_{\sigma,1}]$ and decreasing on $[\Theta_{\sigma,2},0]$.[1] Finally, because $a_1(q)\geq a_2(q)$ if and only if $D'_\sigma(q)\geq 0$, the desired result follows.

□

Proof of Proposition 2. This proof is organized in 3 parts. I first show that the desired relationships hold with weak inequality. Then I show that they in fact hold with strict inequality.

Part I: $\hat{a}(q)\geq a(q)$ for all q.

Note that $c(a)=\frac{a^{p+1}}{p+1}$ implies that $f(x)=x^{1/p}$ and $c(f(x))=\frac{x^{\frac{p+1}{p}}}{p+1}$. As a result (??) and the first-best HJB equation can be written as

$$\begin{aligned} rJ(q) &= \left(n-\frac{1}{p+1}\right)\left[J'(q)\right]^{\frac{p+1}{p}}+\frac{\sigma^2}{2}J''(q) \text{ and} \\ r\hat{J}(q) &= \frac{p}{p+1}\left[n\hat{J}'(q)\right]^{\frac{p+1}{p}}+\frac{\sigma^2}{2}\hat{J}''(q)\ , \end{aligned}$$

respectively, where the subscript for the i^{th} agent has been suppressed since the equilibria are symmetric. Note that the equilibrium effort level of each agent is given by $f(J'(q))$, while the first-best effort level of each agent is given by $f\left(n\hat{J}'(q)\right)$. Because $f(\cdot)$ is strictly increasing, it suffices to show that $n\hat{J}'(q)\geq J'(q)$ for all q. Let $\alpha=\left[\frac{np}{np+(n-1)}\right]^p n$, and note that $\alpha|_{n=1}=1$, $\alpha\leq n$ and α is strictly increasing in n for all $p>0$ and $n\geq 2$, which implies that $1<\alpha\leq n$ for all $p>0$ and $n\geq 2$. Because $\hat{J}'(q)>0$ and $J'(q)>0$ for all q, it suffices to show that $\alpha\hat{J}'(q)\geq J'(q)$ for all q. Now define $\Delta_\alpha(q)=\alpha\hat{J}(q)-J(q)$ and note that $\Delta_\alpha(\cdot)$ is smooth, $\lim_{q\to-\infty}\Delta_\alpha(q)=0$, and $\Delta_\alpha(0)=(\alpha-1)V>0$. So either $\Delta_\alpha(\cdot)$ is increasing on $(-\infty,0]$ or it has at least one interior global extreme point. If the former is true, then the desired inequality holds. Now suppose the latter is true and let us denote this extreme point by z^*. Using that $\alpha\hat{J}'(z^*)=J'(z^*)$, (??) and the first-best HJB equation, we have that

$$\begin{aligned} r\Delta_\alpha(z^*) &= \left[\frac{\alpha p}{p+1}\left(\frac{n}{\alpha}\right)^{\frac{p+1}{p}}-n+\frac{1}{p+1}\right]\left[J'(q)\right]^{\frac{p+1}{p}}+\frac{\sigma^2}{2}\Delta''_\alpha(z^*) \\ \Longrightarrow r\Delta_\alpha(z^*) &= \frac{\sigma^2}{2}\Delta''_\alpha(z^*)\ . \end{aligned}$$

Suppose that z^* is a global maximum. Then $\Delta''_\alpha(z^*)\leq 0$ implies that $\Delta_\alpha(z^*)\leq 0$, contra-

[1] Unfortunately, it is not possible to prove that $J'''_i(q)$ is unimodal (or monotone) in q, and consequently that $D_\sigma(\cdot)$ does not have any local extrema so that $\Theta_{\sigma,1}=\Theta_{\sigma,2}$, which would in turn imply that $a_1(q)\geq a_2(q)$ if and only if $q\leq\Theta_{\sigma,i}$.

dicting the fact that $\Delta_\alpha(0) > 0$. Therefore, z^* must be a global minimum. Then $\Delta''_\alpha(z^*) \geq 0$ implies that $\Delta_\alpha(z^*) \geq 0$, contradicting the facts that $\lim_{q \to -\infty} \Delta_\alpha(q) = 0$ and that z^* is interior. Therefore $\Delta_\alpha(\cdot)$ cannot have any interior extreme points, which implies that $\Delta_\alpha(\cdot)$ is increasing on $(-\infty, 0]$.

Part II: $\hat{J}(q) \geq J(q)$ for all q.
Let us define $\Delta_1(q) = \hat{J}(q) - J(q)$ and note that $\Delta_1(\cdot)$ is smooth, and $\lim_{q \to -\infty} \Delta_1(q) = \Delta_1(0) = 0$. Therefore either $\Delta_1(\cdot) \equiv 0$, or $\Delta_1(\cdot)$ has at least one local interior extreme point. If the former is true, then $\Delta'_1(q) = \Delta''_1(q) = 0$ for all q. Then using (**??**) and the first-best HJB equation, it follows that $\frac{1}{p+1}\left[pn^{\frac{p+1}{p}} - n(p+1) + 1\right][J'(q)]^{\frac{p+1}{p}} = 0$, which contradicts the facts that $J'(z^*) > 0$ and $\left[pn^{\frac{p+1}{p}} - n(p+1) + 1\right] > 0$ for all $n \geq 2$ and $p > 0$. Therefore it must be the case that $\Delta_1(\cdot)$ has an interior extreme point, which we denote by z^*. Using that $\hat{J}'(z^*) = J'(z^*)$, (**??**) and the first-best HJB equation, we have that

$$r\Delta_1(z^*) = \frac{pn^{\frac{p+1}{p}} - n(p+1) + 1}{p+1}[J'(z^*)]^{\frac{p+1}{p}} + \frac{\sigma^2}{2}\Delta''_1(z^*) .$$

By noting that $pn^{\frac{p+1}{p}} - n(p+1) + 1 > 0$, and any interior minimum must satisfy $\Delta''_1(z^*) \geq 0$, it follows that $\Delta_1(z^*) > 0$, and hence $\Delta_1(q) \geq 0$, or equivalently $\hat{J}(q) \geq J(q)$ for all q.

Part III: $\hat{a}(q) > a(q)$ and $\hat{J}(q) > J(q)$ for all q.
Recall that in proving existence of a MPE in Theorem 1 (Part I), I obtained a bound $|J''(q)| \leq C[|J(q)| + |J'(q)|]$ for all q, where $C > 0$ is a constants. Using an analogous approach, one can obtain a similar bound for $\left|\hat{J}''(q)\right|$; *i.e.*, $\left|\hat{J}''(q)\right| \leq \hat{C}\left[\left|\hat{J}(q)\right| + \left|\hat{J}'(q)\right|\right]$for all q.

Suppose that there exists a $z \leq 0$ such that $\Delta'_\alpha(z) = 0$. Because $r\Delta_\alpha(z) = \frac{\sigma^2}{2}\Delta''_\alpha(z)$, using the same argument used to establish Proposition 1 (ii), it follows that z must be a minimum such that $\Delta_\alpha(z) = 0$, and $\Delta_\alpha(q) = 0$ for all $q \leq z$. The last equality implies that $\Delta'_\alpha(q) = 0$ for all $q < z$. Now define $M_\alpha(q) = \alpha\left[\hat{J}(q) + \hat{J}'(q)\right] - [J(q) + J'(q)]$, and note by parts I and II that $M_\alpha(q) \geq 0$ for all q. Also $M_\alpha(q) = 0$ for all $q < z$, and there exists a constant $C_\alpha > 0$ such that $M'_\alpha(q) \leq C_\alpha \cdot M_\alpha(q)$ for all q. By applying the differential form of Grönwall's inequality, it follows that $M_\alpha(q) = 0$ for all q. However this contradicts the facts that $\alpha\hat{J}(0) - J(0) > 0$ and $\alpha\hat{J}'(0) \geq J'(0)$. Therefore there does not exist a z such that $\Delta'_\alpha(z) = 0$, so that $\alpha\hat{J}'(q) > J'(q)$ for all q, which implies that $\hat{a}(q) > a(q)$ for all q.

To show that $\hat{J}(q) > J(q)$ for all q, I use the same approach as above. First note that if there exists a $\hat{z} < 0$ such that $\Delta_1(\hat{z}) = 0$, then $\Delta_1(q) = 0$ for all $q \leq \hat{z}$. Then by defining

$M(q) = \left[\hat{J}(q) + \hat{J}'(q)\right] - \left[J(q) + J'(q)\right]$, and by using the fact that $M(q) > 0$ for at least some q, and the differential form of Grönwall's inequality, the desired result follows. The details are omitted.

□

Proof of Theorem 3. This proof is organized in 5 parts. I first show that a solution to (**??**) subject to the boundary conditions (**??**) exists. Then I show that property (i) holds. Finally, I show that the solution to the above boundary value problem is unique. The proofs resemble those in Theorem 1 closely.

Part I: Existence of a solution.

First note that $J_i(\cdot)$ depends only on V_i for all i and not on $F(\cdot)$, so for given V_i I can solve $F(\cdot)$ by taking $J_i(\cdot)$ as given for all i. I shall use a similar approach as that used to prove existence for $J_i(\cdot)$. Let us re-write(**??**) and (**??**) as

$$F_N''(q) = \frac{2r}{\sigma^2} F_N(q) + \frac{2}{\sigma^2}\left[\sum_{i=1}^{n} f\left(J_i'(q)\right)\right] F_N'(q) \quad \text{subject to } F_N(-N) = 0 \text{ and } F_N(0) = F_0\,, \tag{1}$$

where $F_0 = U - \sum_{i=1}^{n} V_i > 0$. Let $h(F_N, F_N')$ denote the RHS of (1), and observe that $h(\cdot,\cdot)$ is continuous. Now fix some arbitrary $K > 0$ and define a new function

$$h_K(F_N, F_N') = \max\left\{\min\left\{h(F_N, F_N'),\, K\right\},\, -K\right\}\,.$$

Note that $h_K(\cdot,\cdot)$ is continuous and bounded, so that by the Scorza-Dragoni Lemma (see Lemma 4 in **?**), there exists a solution to $F_{N,K}'' = h_K\left(F_{N,K}, F_{N,K}'\right)[-N, 0]$ subject to $F_{N,K}(-N) = 0$ and $F_{N,K}(0) = F_0$. The next part of the proof involves showing that there exists some $\bar{K}$ such that $h_K\left(F_{N,K}, F_{N,K}'\right) \in \left[-\bar{K}, \bar{K}\right]$ for all K on $[-N, 0]$, which will imply that the solution $F_{N,\bar{K}}''(\cdot)$ satisfies (1). The final step involves showing that a solution exists when $N \to \infty$, so that a solution to (**??**) subject to (**??**) exists.

By part I of Theorem 1, there exists an $\bar{A}$ such that $|J_i'(q)| \leq \bar{A}$ for all q, and it is straightforward to show that $F_{N,K}(q) \in [0\,, F_0]$ and $F_{N,K}'(q) \geq 0$ for all q. Letting $\Omega = nf\left(\bar{A}\right)$, a bound for $\left|F_{N,K}''(q)\right|$ can be obtained by

$$\left|F_{N,K}''(q)\right| \leq \frac{2r}{\sigma^2} F_0 + \frac{2}{\sigma^2} \Omega F_{N,K}'(q)\,.$$

By noting that $F_N(0) > 0$ and using the mean-value theorem, it follows that there exists a $z^* \in [-N, 0]$ such that $F_N'(z^*) = \frac{F_0}{N}$. Hence, for all $z \in [-N, 0]$

$$F_0 \quad > \quad \left| \int_{z^*}^{z} F_N'(q)\, dq \right| \geq \frac{\sigma^2}{2} \left| \int_{z^*}^{z} F_N'(q) \frac{F_N''(q)}{rF_0 + \Omega F_N'(q)} dq \right| \geq \frac{\sigma^2}{2} \left| \int_0^{F_N'(z)} \frac{s}{rF_0 + \Omega s} ds \right|,$$

where I let $s = F_N'(q)$ and used that $F_N'(q) F_N''(q) = F_N'(q)\, dF_N'(q)$. The fact that $\int_0^\infty \frac{s}{rF_0 + \Omega s} ds = \infty$ implies that there exists a $\bar{B} < \infty$ such that $\frac{\sigma^2}{2} \left| \int_0^{\bar{B}} \frac{s}{rF_0+\Omega s} ds \right| = F_0$. This implies that $F_N'(q) \leq \bar{B}$ for all $q \in [-N, 0]$.

Because $\bar{B}$ is independent of both N and K, $F_{N,K}'(q) \in \left[0,\, \bar{B}\right]$ for all $q \in [-N, 0]$, $N \in \mathbb{N}$, and $K > 0$. In addition, we know that $F_{N,K}(q) \in [0\,, F_0]$ for all $q \in [-N, 0]$, $N \in \mathbb{N}$, and $K > 0$. Now let $\bar{K} = \frac{2r}{\sigma^2} F_0 + \frac{2}{\sigma^2} \Omega \bar{B}$, and observe that a solution to $F_{N,\bar{K}}'' = h_{\bar{K}}\left(F_{N,\bar{K}}, F_{N,\bar{K}}'\right)$ subject to $F_{N,\bar{K}}(-N) = 0$ and $F_{N,\bar{K}}(0) = F_0$ exists, and $h_{\bar{K}}\left(F_{N,\bar{K}}(q)\,, F_{N,\bar{K}}'(q)\right) = h\left(F_{N,\bar{K}}(q)\,, F_{N,\bar{K}}'(q)\right)$ for all $q \in [-N, 0]$. Therefore, $F_{N,\bar{K}}(\cdot)$ solves (1).

To show that a solution for (1) as $N \to \infty$ exists, recall that there exists a constant $\bar{B}$ such that $|F_N'(q)| \leq \bar{B}$ on $[-N, 0]$ for all $N \in \mathbb{N}$. Hence the sequences $\{F_N(\cdot)\}$ and $\{F_N'(\cdot)\}$ are uniformly bounded and equicontinuous on $[-N, 0]$. By applying the Arzela-Ascoli theorem to a sequence of intervals $[-N, 0]$ and letting $N \to \infty$, it follows that the system of ODE defined by (**??**) subject to (**??**) has at least one solution.

Part II: $F(q) > 0$ for all q.
First note that $\lim_{q \to -\infty} F(q) = 0$ and $F(0) > 0$. Suppose that $F(q) < 0$ for some q. Then $F(\cdot)$ has an interior minimum z such that $F(z) < 0$. Then $F'(z) = 0$ together with (**??**) implies that $rF(z) = \frac{\sigma^2}{2} F''(z) \geq 0$, which is a contradiction. Therefore, $F(q) \geq 0$ for all q. Next, suppose that there exists some z^* such that $F(z^*) = 0$. Either $F(q) = 0$ for all $q < z^*$ or $F(q) \neq 0$ for at least some $q < z^*$. Suppose that the latter is true. Then there exists some interior extreme point $\bar{z} < z^*$, which using $F'(\bar{z}) = 0$ and (**??**) implies that $rF(\bar{z}) = \frac{\sigma^2}{2} F''(\bar{z})$. By noting that any maximum must satisfy $F''(\bar{z}) \leq 0 \implies F(\bar{z}) \leq 0$, while any minimum must satisfy $F''(\bar{z}) \geq 0 \implies F(\bar{z}) \geq 0$, it follows that $F(q) = 0$ and $F'(q) = 0$ for all $q < z^*$. By applying the differential form of Grönwall's inequality to $|F(q)| + |F'(q)|$ and using that $|F''(q)| \leq \frac{2r}{\sigma^2} |F(q)| + \frac{2nf(\bar{A})}{\sigma^2} |F'(q)|$, it follows that $F(q) = 0$ for all q. However this contradicts the fact that $F(0) > 0$. Hence $F(\cdot)$ cannot have an interior minimum, and there cannot exist an interior z^* such that $F(z^*) = 0$. Hence $F(q) > 0$ for all q.

Part III: $F'(q) > 0$ for all q.

Because $F(\cdot)$ is continuous and $\lim_{q\to-\infty} F(q) = 0 < F(0)$, there exists an interior Λ such that $F(\Lambda) < F(0)$, and by the mean-value theorem, there exists a $z^* \in (\Lambda, 0)$ such that $F'(z^*) = \frac{F(0)-F(\Lambda)}{-\Lambda} > 0$. Suppose that there exists a z^{**} such that $F'(z^{**}) \leq 0$. Then by the intermediate value theorem, there exists a $\bar{z}$ between z^* and z^{**} such that $F'(\bar{z}) = 0$. Using (**??**) and the fact that $F(q) > 0$ for all q, it follows that $rF(\bar{z}) = \frac{\sigma^2}{2} F''(\bar{z}) > 0$; *i.e.*, $\bar{z}$ is a minimum. Because $\bar{z}$ is interior, $\lim_{q\to-\infty} F(q) = 0$, and $F(\bar{z}) > 0$, there exists an interior local maximum $\hat{z} < \bar{z}$, so that $F'(\hat{z}) = 0$ and $F''(\hat{z}) \leq 0$. Using (**??**), it follows that $F(\hat{z}) \leq 0$, which contradicts the fact that $F(q) > 0$ for all q. Therefore, $F'(q) > 0$ for all q.

Part IV: Uniqueness of a solution.

Because $F(\cdot)$ is a function of $J_i'(\cdot)$ for all i, and Theorem 1 established that the equilibrium is symmetric and unique if the contract is symmetric (*i.e.*, $V_i = V_j$ for all $i \neq j$), I focus only on this case only. Suppose that there exist two solutions that solve (**??**) subject to the initial conditions (**??**), denoted by $F_1(\cdot)$ and $F_2(\cdot)$, respectively. Let $\Delta F(q) = F_1(q) - F_2(q)$, and note that $\Delta F(0) = \lim_{q\to-\infty} \Delta F(q) = 0$, and $\Delta F(\cdot)$ is smooth. Also observe that either $\Delta F(\cdot) \equiv 0$, or $\Delta F(\cdot)$ has a global extreme point. Suppose the latter is true and letting z^* be such extreme point, we have that $\Delta F'(z^*) = 0$. Using (**??**) and the facts that $\Delta F''(z^*) \geq 0$ if z^* is a minimum and $\Delta F''(z^*) \leq 0$ if z^* is a maximum, it follows that $\Delta F(q) = 0$ for all q. Hence $F_1(\cdot) \equiv F_2(\cdot)$ and the proof is complete.

□

Proof of Proposition 4. In preparation, I establish a Lemma that ensures that the single-crossing property of **?** is satisfied.

Lemma 1. *Suppose the manager employs n identical agents, each of whom receives $\frac{B}{n}$ upon completion. Then for all $\delta \in (0, U - B)$, there exists a threshold T_δ such that she is better off increasing each agent's reward by $\frac{\delta}{n}$ so that each agent receives $\frac{B+\delta}{n}$ if and only if the length of the project $|q_0| \geq T_\delta$.*

Proof of Lemma 2. Consider two teams each comprising of n symmetric agents. Upon completion of the project, each member of the first team receives a reward $\frac{B}{n}$, while each member of the second team receives a reward $\frac{B+\delta}{n}$, where $\delta > 0$. Let us denote each agent's expected discounted payoff and equilibrium effort level of the two teams given q by $\{J_0(q)\ ,\ a_0(q)\}$ and $\{J_\delta(q)\ ,\ a_\delta(q)\}$, respectively. From Proposition 1 (i) we know that $a_\delta(q) > a_0(q)$ for all q; *i.e.*, each agent's effort level is strictly increasing in his compensation. Abusing notation, let us denote the manager's expected discounted profit given q for the two cases by

$F_B(q)$ and $F_{B+\delta}(q)$, respectively. Now let $\Delta_V(\cdot) = F_B(\cdot) - F_{B+\delta}(\cdot)$, and observe that $\lim_{q\to-\infty}\Delta_V(q) = 0 < \delta = \Delta_V(0)$. Because $\Delta_V(\cdot)$ is smooth, it is either increasing on $(-\infty, 0]$, or it has an interior global extreme point. Suppose the latter is true and denote that extreme point by $\bar{z}$. By using (**??**), it follows that

$$r\Delta_V(\bar{z}) = n\left[a_B(\bar{z}) - a_{B+\delta}(\bar{z})\right] F_B'(\bar{z}) + \frac{\sigma^2}{2}\Delta_V''(\bar{z}) .$$

Because $F_B'(\bar{z}) > 0$, $a_B(\bar{z}) < a_{B+\delta}(\bar{z})$, $\Delta_V(0) > 0$, and $\bar{z}$ is interior, it follows that $\bar{z}$ must be a global minimum. By noting that any local maximum $\hat{z}$ must satisfy $\Delta_V(\hat{z}) \le 0$, it follows that $\Delta_V(\cdot)$ is either increasing on $(-\infty, 0]$, or it crosses 0 exactly once from below. Therefore there exists a T_δ such that $\Delta_V(q_0) \le 0$ if and only if $q_0 \le -T_\delta$, or equivalently, the manager is better off increasing each agent's reward by $\frac{\delta}{n}$ if and only if $|q_0| \ge T_\delta$. By noting that $T_\delta = -\infty$ if $\Delta_V(\cdot)$ is increasing on $(-\infty, 0]$, the proof is complete.

□

All other parameters held constant, the manager chooses her budget $B \in [0, U]$ to maximize her expected discounted profit at q_0; *i.e.*, she chooses $B(|q_0|) = \arg\max_{B\in[0,U]}\{F_n(q_0; B)\}$. By noting that the necessary conditions for the Monotonicity Theorem (*i.e.*, Theorem 4) of **?** to hold are satisfied, it follows that the manager's optimal budget $B(|q_0|)$ is (weakly) increasing in the project length $|q_0|$.

□

Proof of Proposition 6. This proof is organized in 2 parts.

Part I: *Agents' Problem*

***(a)** Formulation of the Agents' Problem*

To begin, fix the manager's budget $B < U$ and the retirement state R. Then denote by $\bar{J}(\cdot)$ each agent's expected discounted payoff when both agents carry out the project to completion together. Let us assume by convention that as soon as the project hits R for the first time, agent 2 will retire, and agent 1 will carry out the remainder of the project on his own. Upon completion of the project, each agent i receives V_i, where $V_1 + V_2 = B$. The V_i's will be chosen such that $J_1(R) = J_2(R)$; *i.e.*, the agents have the same expected discounted payoff when the project hits R for the first time. This will ensure that strategies are symmetric before agent 2 retires (which makes the analysis tractable). Therefore, denote by $J_R(\cdot)$ the expected discounted payoff of each agent before agent 2 retires. Note that $\bar{J}(\cdot)$ and $J_i(\cdot)$ are defined on $(-\infty, 0]$, while $J_R(\cdot)$ is defined on $(-\infty, R]$. Using (**??**), $\bar{J}(\cdot)$ satisfies

$$r\bar{J}(q) = -c\left(f\left(\bar{J}'(q)\right)\right) + 2f\left(\bar{J}'(q)\right)\bar{J}'(q) + \frac{\sigma^2}{2}\bar{J}''(q) \text{ s.t. } \lim_{q\to-\infty}\bar{J}(q) = 0 \text{ and } \bar{J}(0) = \frac{B}{2} .$$

Because the state of the project q can drift back below R after agent 2 has retired, $J_1(\cdot)$ and $J_2(\cdot)$ satisfy

$$
\begin{aligned}
rJ_1(q) &= -c(f(J_1'(q))) + f(J_1'(q))J_1'(q) + \frac{\sigma^2}{2}J_1''(q) \text{ s.t. } \lim_{q\to-\infty} J_1(q) = 0 \text{ and } J_1(0) = V_1 \text{ , and} \\
rJ_2(q) &= f(J_1'(q))J_2'(q) + \frac{\sigma^2}{2}J_2''(q) \text{ s.t. } \lim_{q\to-\infty} J_2(q) = 0 \text{ and } J_2(0) = B - V_1
\end{aligned}
$$

on $(-\infty, 0]$, respectively. Observe that after agent 2 retires, his expected discounted payoff depends on the effort of agent 1 and on his net payoff V_2 upon completion of the project. By using the same approach as used to prove Proposition 1 (i), it follows that $J_1(\cdot)$ $\{J_2(\cdot)\}$ increases $\{$decreases$\}$ in V_1, and $J_1(\cdot)$ and $J_2(\cdot)$ depend continuously on V_1. Moreover, $J_1(R) > J_2(R) = 0$ if $V_1 = B$, and it is straightforward to show that $J_1(R) < J_2(R)$ if $V_1 = \frac{B}{2}$. Therefore, by the intermediate value theorem, there exists a $V_1 > \frac{B}{2}$ such that $J_1(R) = J_2(R)$; *i.e.*, when q_t hits R for the first time, the agents are indifferent with respect to which one will retire.

Next, using (**??**), $J_R(\cdot)$ satisfies

$$
rJ_R(q) = -c(f(J_R'(q))) + 2f(J_R'(q))J_R'(q) + \frac{\sigma^2}{2}J_R''(q) \text{ s.t. } \lim_{q\to-\infty} J_R(q) = 0 \text{ and } J_R(R) = J_1(R) \text{ ,}
$$

where the second condition ensures value matching at $q = R$. Because $J_1(\cdot)$ and $J_2(\cdot)$ are pinned down independently of $J_R(\cdot)$, the above boundary conditions completely characterize $J_R(\cdot)$.

(b) *Show that* $J_R(R) \le \bar{J}(R)$ *and* $J_R'(q) \le \bar{J}'(q)$ for all $q \le R$.
Let $D(q) = J_1(q) + J_2(q) - 2\bar{J}(q)$, note that $\lim_{q\to-\infty} D(q) = D(0) = 0$, and $D(\cdot)$ is smooth. Therefore either $D(\cdot) \equiv 0$ on $(-\infty, 0]$, or $D(\cdot)$ has at least one interior extreme point. Suppose the latter is true, and let us denote this extreme point by $\hat{z}$. Then $D'(\hat{z}) = 0$ so that

$$
\begin{aligned}
rD(\hat{z}) &= -c(f(J_1'(\hat{z}))) + 2c\left(f\left(\bar{J}'(\hat{z})\right)\right) + 2\left[f(J_1'(\hat{z})) - 2f\left(\bar{J}'(\hat{z})\right)\right]\bar{J}'(\hat{z}) + \frac{\sigma^2}{2}D''(\hat{z}) \\
&= -\frac{1}{2}\left\{2\left[\bar{J}'(\hat{z})\right]^2 + \left[J_1'(\hat{z}) - 2\bar{J}'(\hat{z})\right]^2\right\} + \frac{\sigma^2}{2}D''(\hat{z}) \text{ .}
\end{aligned}
$$

Suppose that $\hat{z}$ is a maximum. Then $D''(\hat{z}) \le 0$, and because the first term in the RHS is strictly negative, it follows that $D(\hat{z}) < 0$. This implies that any local maximum $\hat{z}$ must satisfy $D(\hat{z}) \le 0$, and hence $D(q) \le 0$ for all q. Moreover, because the inequality is strict, note that it cannot be case that $D(\cdot) \equiv 0$ on $(-\infty, 0]$. Because $J_R(R) = J_1(R) = J_2(R)$, the result implies that $J_R(R) \le \bar{J}(R)$. Finally, by applying Proposition 1 (i), it follows that $J_R'(q) \le \bar{J}'(q)$ for all $q \le R$.

Part II: *Manager's Problem*

(a) *Formulation of the Manager's Problem*

To begin, denote by $\bar{F}(\cdot)$ the manager's expected discounted profit when both agents carry out the project to completion together. Denote by $F_1(\cdot)$ the manager's expected discounted profit when one agent carries out the project alone (*i.e.*, after agent 2 has retired). Denote by $F_R(\cdot)$ the manager's expected discounted profit taking into account that agent 2 will retire at the first time that the state of the project hits R. Note that $\bar{F}(\cdot)$ and $F_1(\cdot)$ are defined on $(-\infty, 0]$, while $F_R(\cdot)$ is defined on $(-\infty, R]$. Using (**??**), $\bar{F}(\cdot)$ and $F_1(\cdot)$ satisfy

$$\begin{aligned} r\bar{F}(q) &= 2f\left(\bar{J}'(q)\right)\bar{F}'(q) + \frac{\sigma^2}{2}\bar{F}''(q) \text{ s.t. } \lim_{q\to-\infty}\bar{F}(q) = 0 \text{ and } \bar{F}(0) = U - B \text{ , and} \\ rF_1(q) &= f\left(J_1'(q)\right)F_1'(q) + \frac{\sigma^2}{2}F_1''(q) \text{ s.t. } \lim_{q\to-\infty}F_1(q) = 0 \text{ and } F_1(0) = U - B\,, \end{aligned}$$

respectively. Finally, the manager's expected discounted profit before one agent is retired satisfies

$$rF_R(q) = 2f\left(J_R'(q)\right)F_R'(q) + \frac{\sigma^2}{2}F_R''(q) \text{ s.t. } \lim_{q\to-\infty}F_R(q) = 0 \text{ and } F_R(R) = F_1(R)\ ,$$

where the second condition ensures value matching at $q = R$. Because $F_1(\cdot)$ is determined independently of $F_R(\cdot)$, these boundary conditions completely characterize $F_R(\cdot)$.

(b) *Show that* there exists a $\Theta_R > |R|$ *such that* $F_R(q_0) \ge \bar{F}(q_0)$ *if and only if* $|q_0| < \Theta_R$.

First, let $\Delta_1(q) = F_1(q) - \bar{F}(q)$, and note that $\lim_{q\to-\infty}\Delta_1(q) = \Delta_1(0) = 0$, and that $\Delta_1(\cdot)$ is smooth. As a result, either $\Delta_1(\cdot) \equiv 0$ on $(-\infty, 0]$, or it has at least one interior extreme point. Suppose that the latter is true, and let us denote such extreme point by z^*. Then $\Delta_1'(z^*) = 0$, which implies that

$$r\Delta_1(z^*) = \left[f\left(J_1'(z^*)\right) - 2f\left(\bar{J}'(z^*)\right)\right]\bar{F}'(z^*) + \frac{\sigma^2}{2}\Delta_1''(z^*)\ .$$

It is straightforward to prove a result analogous to Theorem 2 (ii): that there exists a threshold Φ such that $f\left(J_1'(z^*)\right) \le 2f\left(\bar{J}'(z^*)\right)$ if and only if $z^* \le \Phi$. As a result $\Delta_1(z^*) \le 0$ if $z^* \le \Phi$, while $\Delta_1(z^*) \ge 0$ if $z^* \ge \Phi$. It follows that $\Delta_1(\cdot)$ crosses 0 at most once from below.

Next, define $\Delta_R(q) = F_R(q) - \bar{F}(q)$ on $(-\infty, R]$. Note that $\lim_{q\to-\infty}\Delta_R(q) = 0$, $\Delta_R(R) =$

$\Delta_1(R)$, and $\Delta_R(\cdot)$ is smooth, where the second equality follows from the value matching condition $F_R(R) = F_1(R)$. Because $\Delta_1(\cdot)$ crosses 0 at most once from below, depending on the choice of the retirement point R, it may be the case that $\Delta_1(R) \lesseqqgtr 0$.

Suppose $\Delta_1(R) \geq 0$. Then either $\Delta_R(\cdot)$ increases in $(-\infty, R]$, or it has at least one interior extreme point. Suppose the latter is true, and let us denote such extreme point by $\bar{z}$. Then $\Delta_R'(\bar{z}) = 0$ implies that

$$r\Delta_R(\bar{z}) = 2\left[f\left(J_R'(\bar{z})\right) - f\left(\bar{J}'(\bar{z})\right)\right]\bar{F}'(\bar{z}) + \frac{\sigma^2}{2}\Delta_R''(\bar{z}) \ .$$

Recall from part I (c) of this proof that $J_R'(q) \leq \bar{J}'(q)$ for all $q \leq R$, which implies that $f\left(J_R'(\bar{z})\right) \leq f\left(\bar{J}'(\bar{z})\right)$. It follows that $\bar{z}$ must satisfy $\Delta_R(\bar{z}) \leq 0$. Because $\Delta_1(R) \geq 0$, it follows that there exists a threshold $\Theta_R > |R|$ such that $\Delta_1(q_0) \geq 0$ if and only if $|q_0| < \Theta_R$. If $\Delta_1(R) < 0$, the same analysis yields that $\Delta_R(\cdot)$ decreases in $(-\infty, R]$, and hence $\Delta_1(q_0) \leq 0$ for all $q_0 \leq R$.

(c) *Conclusion of the Proof*

I have shown that as long as R is chosen such that $F_1(R) \geq \bar{F}(R)$ (so that $\Delta_1(R) \geq 0$), there exists a threshold $\Theta_R > |R|$ such that $F_R(q_0) \geq \bar{F}(q_0)$ for all $|q_0| < \Theta_R$. The last relationship implies that as long as the the length of the project $|q_0| < \Theta_R$, the manager is better off implementing the proposed retirement scheme relative to allowing both agents to carry out the project to completion together. Finally, the requirement that R is chosen such that $F_1(R) \geq \bar{F}(R)$ is equivalent to the requirement that if the project length were $|q_0| = |R|$ and the manager did not use a dynamic team size management scheme, then she would be better off employing one instead of two agents.

$\square$

Proof of Proposition 7. In preparation, I first establish two Lemmas.

Lemma 2. *Consider a project undertaken by two identical agents who differ only in their final rewards such that $V_1 > V_2$. Also, suppose that effort costs are quadratic. Then $\frac{d}{dq}\left[a_1(q) - a_2(q)\right] \geq 0$ for all q.*

Proof of Lemma 3. Observe that when effort costs are quadratic, then $a_i(q) = J_i'(q)$, so it suffices to show that $D_J'(\cdot) = J_1'(\cdot) - J_2'(\cdot)$ is (weakly) increasing on $(-\infty, 0]$. First note that $\lim_{q\to-\infty} D_J'(q) = 0$, and from Proposition 1 (i), it follows that $D_J'(q) > 0$ for all q. Fix $z \leq 0$, and let $\bar{z} = \arg\max\left\{D_J'(q) : q \leq z\right\}$. Clearly, $\bar{z} > -\infty$. Suppose that $\bar{z}$ is interior.

Then $D_J''(\bar{z}) = 0$ and $D_J'''(\bar{z}) \leq 0$, and by using (**??**) we have that $rD_J'(\bar{z}) = \frac{\sigma^2}{2} D_J'''(\bar{z}) \leq 0$. However, this contradicts the fact that $D_J'(\bar{z}) > 0$, which implies that $\bar{z} = z$. Since z was chosen arbitrarily, this implies that $D_J'(\cdot)$ is (weakly) increasing on $(-\infty, 0]$.

$\square$

Lemma 3. *Consider a project undertaken by two identical agents, and suppose that effort costs are quadratic. Consider the following two scenarios for the agents' compensation: (i) $V_1 = V_2 = \frac{B}{2}$, and (ii) $V_1 = \frac{B}{2} + \epsilon > \frac{B}{2} - \epsilon = V_2$. Then for all $\epsilon \in \left(0, \frac{B}{2}\right]$ there exists a $\Theta_\epsilon < 0$ such that the aggregate effort of the team is larger under asymmetric rewards (*i.e.*, under scenario (ii)) if and only if $q \geq \Theta_\epsilon$.*

Proof of Lemma 4. First let us denote the expected discounted payoff function of the agents under asymmetric compensation by $J_1(q)$ and $J_2(q)$, respectively, and let us denote the expected discounted payoff function of the agents under symmetric compensation by $J_S(q)$. Because effort costs are quadratic, $a_i(q) = J_i'(q)$. Observe that we are interested in comparing $2a_S(q)$ and $a_1(q) + a_2(q)$, or equivalently $2J_S'(q)$ and $J_1'(q) + J_2'(q)$ on $(-\infty, 0]$. Let us define $M(q) = 2J_S(q) - J_1(q) - J_2(q)$. By noting that $\lim_{q \to -\infty} M(q) = M(0) = 0$ and $M(\cdot)$ is smooth on $(-\infty, 0]$, it follows that either $M(\cdot) \equiv 0$, or it has at least one interior global extreme point. Suppose the latter is true and let us denote that extreme point by z^*. By using (**??**), and the facts that $f(x) = x$ and $c(f(x)) = \frac{x^2}{2}$, it follows that

$$rM(z^*) = \frac{1}{2}\left[6\left(J_S'(z^*)\right)^2 - 2\left(J_1'(z^*) + J_2'(z^*)\right)^2 + \left(J_1'(z^*)\right)^2 - \left(J_2'(z^*)\right)^2\right] + \frac{\sigma^2}{2} M''(z^*) \ .$$

Because z^* is an extreme point, $M'(z^*) = 0$ implies that $J_S'(z^*) = \frac{J_1(z^*) + J_2(z^*)}{2}$. By substituting into the above equality and simplifying the terms, we have

$$rM(z^*) \quad = \quad \frac{1}{4}\left[J_1'(z^*) - J_2'(z^*)\right]^2 + \frac{\sigma^2}{2} M''(z^*) \ .$$

Suppose that z^* is a global interior minimum. Then the facts that $M''(z^*) \geq 0$ and $J_1'(z^*) > J_2'(z^*)$ (which follows from Proposition 1 (i)), imply that $M(z^*) > 0$. However, this contradicts the fact that $M(0) = 0$, which implies that z^* must be a maximum and $M(q) \geq 0$ for all q. Moreover, because $J_1(z^*) > J_2(z^*)$, note that it cannot be the case that $M(\cdot) \equiv 0$.

Now suppose that $M(\cdot)$ has more than one extreme points. Then there must exist a local maximum z^* followed by a local minimum $\bar{z} > z^*$. This implies that $M''(z^*) \leq 0 \leq M''(\bar{z})$, and by Lemma 3, $0 \leq J_1'(z^*) - J_2'(z^*) \leq J_1'(\bar{z}) - J_2'(\bar{z})$. These equalities imply that $M(z^*) \leq M(\bar{z})$, which contradicts the assumption that z^* is a maximum and $\bar{z}$ is a minimum. Hence

$M(\cdot)$ has a global maximum on $(-\infty, 0]$ and no other local extreme points. Therefore there exists a $\Theta_\epsilon < 0$ such that $M'(q) \geq 0$ if and only if $q \leq \Theta_\epsilon$.

$\square$

To begin, let us denote the manager's expected discounted profit by $F_0(q)$ and $F_\epsilon(q)$ under the symmetric ($i.e.$, $\left(\frac{B}{2}, \frac{B}{2}\right)$) and the asymmetric ($i.e.$, $\left(\frac{B}{2} + \epsilon, \frac{B}{2} - \epsilon\right)$) compensation scheme, respectively. Moreover, let us denote the expected discounted payoff of each agent by $J_S(\cdot)$, $J_1(\cdot)$, and $J_2(\cdot)$, where the subscripts follow the convention from Lemma 4. Next, let $\Delta_\epsilon(q) = F_0(q) - F_\epsilon(q)$, and observe that $\lim_{q \to -\infty} \Delta_\epsilon(q) = \Delta_\epsilon(0) = 0$. Therefore, either $\Delta_\epsilon(\cdot) \equiv 0$, or $\Delta_\epsilon(\cdot)$ has at least one interior global extreme point. Suppose the latter is true, and let us denote that extreme point by $\bar{z}$. By using (**??**) and the fact that $\Delta'_\epsilon(\bar{z}) = 0$, it follows that

$$r\Delta_\epsilon(\bar{z}) = \left[2J'_S(\bar{z}) - J'_1(\bar{z}) - J'_2(\bar{z})\right] F'_0(\bar{z}) + \frac{\sigma^2}{2}\Delta''_\epsilon(\bar{z}) \, .$$

From Lemma 3, we know that there exists a threshold Θ_ϵ such that $2J'_S(q) \geq J'_1(q) + J'_2(q)$ if and only if $q \leq \Theta_\epsilon$, and from Theorem 3 (ii) that $F'_0(q) > 0$ for all q. It follows that $\bar{z}$ is a global maximum if $\bar{z} \leq \Theta_\epsilon$, while it is a global minimum if $\bar{z} \geq \Theta_\epsilon$. Moreover, any local extreme point $\bar{z} \leq \Theta_\epsilon$ must satisfy $\Delta_\epsilon(\bar{z}) \geq 0$, while any local extreme point $\bar{z} \geq \Theta_\epsilon$ must satisfy $\Delta_\epsilon(\bar{z}) \leq 0$. Moreover, because $2J'_S(q) > J'_1(q) + J'_2(q)$ for at least some q, and $F'_0(q) > 0$ for all q, it cannot be the case that $\Delta_\epsilon(\cdot) \equiv 0$. Therefore, either one of the following three cases must be true: (i) $\Delta_\epsilon(\cdot) \geq 0$ on $(-\infty, 0]$, (ii) $\Delta_\epsilon(\cdot) \leq 0$ on $(-\infty, 0]$, or (iii) $\Delta_\epsilon(\cdot)$ crosses 0 exactly once from above. Hence, there exists a T_ϵ such that $F_0(q_0) \geq F_\epsilon(q_0)$ if and only if $q_0 \leq -T_\epsilon$, or equivalently if and only if $|q_0| \geq T_\epsilon$.

$\square$

Proof of Proposition 8.

Proof for Statement (i): I shall use a similar approach to that used to prove Theorem 1. By substituting agent i's first order condition into his HJB equation, it follows that his expected discounted payoff satisfies

$$rJ_i(q) = h(q) - c\left(f\left(J'_i(q)\right)\right) + \left[\sum_{j=1}^{n} f\left(J'_j(q)\right)\right] J'_i(q) + \frac{\sigma^2}{2} J''_i(q)$$

subject to the boundary conditions (**??**). By part VI of the proof of Theorem 1, it follows that any solution to the above ODE must be symmetric, so that the above ODE can be re-written as

$$rJ_n(q) = h(q) - c\left(f\left(J'_n(q)\right)\right) + nf\left(J'_n(q)\right) J'_n(q) + \frac{\sigma^2}{2} J''_n(q) \, . \tag{2}$$

By part VII of the proof of Theorem 1, it follows that there may exist at most one solution to (2). Next, I show that any solution to the above ODE must satisfy $0 \leq J_n(q) \leq V$ and $J_n'(q) \geq 0$ for all i and q.

To begin, let $D(q) = J_n(q) - \frac{h(q)}{r}$, and observe that $D(\cdot)$ is smooth and $\lim_{q\to-\infty} D(q) = 0 \leq D(0)$. To obtain a contradiction, suppose that $D(q) < 0$ for some q. Then $D(\cdot)$ must have an interior local minimum z such that $D(z) < 0$, $D'(z) = 0$, and $D''(z) \geq 0$. By substituting this into (2) one obtains

$$rD(z) = \underbrace{-c\left(f\left(\frac{h'(z)}{r}\right)\right) + nf\left(\frac{h'(z)}{r}\right)\left(\frac{h'(z)}{r}\right)}_{\left(n-\frac{1}{p+1}\right)\left(\frac{h'(z)}{r}\right)^{\frac{p+1}{p}}\geq 0} + \frac{\sigma^2}{2}\underbrace{\left(D''(z) + \frac{h''(z)}{r}\right)}_{\geq 0} \geq 0\,,$$

which is a contradiction. Therefore, $D(q) \geq 0$ and hence $J_n(q) \geq 0$ for all q.

Because $\lim_{q\to-\infty} J_n(q) = 0 \leq J_n(0)$ and $J_n(q) \geq 0$ for all q, observe that either $J_n(\cdot)$ is non-decreasing, or it has an interior strict maximum y (in addition to possibly more extreme points). Suppose that the latter is true. Then $J_n'(y) = 0$ and $J_n''(y) < 0$, and by substituting these into (2) one obtains $rD(y) = \frac{\sigma^2}{2} J_n''(y) < 0$, which is a contradiction because $D(y) \geq 0$. Therefore, it must be the case that $J_n'(q) \geq 0$ for all q. This result also implies that the first order condition indeed always binds, and $0 \leq J_n(q) \leq V$ for all q. Insofar, I have established that if a solution to (2) subject to (**??**) exists, then it satisfies statement (i). It is now straightforward to apply the approach used in part I of the proof of Theorem 1 to establish that a MPE exists, and to verify that the verification theorem (p. 123 in **?**) is satisfied, thus ensuring that the solution to (2) subject to (**??**) is optimal for the original problem.

Proof for Statement (ii): Because $\lim_{q\to-\infty} D(q) = 0 \leq D(0)$ and $D(q) \geq 0$ for all q, $D(\cdot)$ either has at most one interior extreme point that is a maximum, or it has an interior local maximum y followed by a local minimum $z > y$ satisfying $D(y) > D(z)$ (in addition to possibly other interior extreme points). Aiming for a contradiction, suppose that the latter is true. By noting that $D'(y) = D'(z) = 0$, $D''(y) \leq 0 \leq D''(z)$, $h'(y) \leq h'(z)$, and $h''(y) \leq h''(z)$, it follows that

$$\begin{aligned} rD(y) &= \left(n - \frac{1}{p+1}\right)\left(\frac{h'(y)}{r}\right)^{\frac{p+1}{p}} + \frac{\sigma^2}{2}\left(D''(y) + \frac{h''(y)}{r}\right) \\ &\leq \left(n - \frac{1}{p+1}\right)\left(\frac{h'(z)}{r}\right)^{\frac{p+1}{p}} + \frac{\sigma^2}{2}\left(D''(z) + \frac{h''(z)}{r}\right) = rD(z)\,, \end{aligned}$$

which is a contradiction. Therefore, there exists some threshold θ (not necessarily interior) such that $D'(q) \geq 0$ if and only if $q \leq \theta$. By applying the envelope theorem to (2) we have that

$$rJ_n'(q) = h'(q) + nf(J_n'(q))\, J_n''(q) + \frac{\sigma^2}{2} J_n'''(q) \; .$$

Suppose that $J_n'(\cdot)$ has an interior extreme point, denoted by $\bar{z}$. Then $J_n''(\bar{z}) = 0$, so that $rD'(\bar{z}) = \frac{\sigma^2}{2} J_n'''(\bar{z})$, and recall that $D'(q) \geq 0$ if and only if $q \leq \theta$. Therefore, $J_n'''(\bar{z}) \geq 0$ and hence $\bar{z}$ is a minimum if and only if $\bar{z} \leq \theta$. Because $\lim_{q \to -\infty} J_n'(q) = 0$ and $J_n(q) \geq 0$ for all q, if $J_n'(\cdot)$ has an interior strict minimum (say $\bar{z}$), then it must also have an interior strict maximum $y < \bar{z}$. However, this is a contradiction, because $\bar{z} \leq \theta$ and $J_n'(\cdot)$ cannot have an interior strict maximum $\bar{y} < \theta$. Therefore, $\bar{z} \geq \theta$ and $\bar{z}$ must be a maximum. Using a similar argument, it follows that $J_n'(\cdot)$ cannot have any other interior extreme points, which implies that there exists some threshold ω (not necessarily interior) such that $J_n''(q) \geq 0$, and hence $a_n'(q) \geq 0$, if and only if $q \leq \omega$.

Proof for Statement (iii): By noting that $J_n'(q)$ being unimodal in q is sufficient for the proof of Theorem 2, it follows that the comparative statics of Theorem 2 continue to hold, which proves statement (iii).

□

Proof of Proposition 9.
Statement (i) follows by noting that the only difference compared to the model analyzed in Section 3, (*i.e.*, without cancellation states) is that it need not be the case that $J_i'(Q_C) = 0$, and that the condition $\lim_{q \to -\infty} J_i(q) = 0$ is only used in the proof of Theorem 1 (iii).

To prove statement (ii), suppose that $J_n'(\cdot)$ has an interior strict maximum y. Then $J_n''(y) = 0$ and $J_n'''(y) < 0$, and by substituting these into (**??**), it follows that $rJ_n'(y) = \frac{\sigma^2}{2} J_n'''(y) < 0$, which is a contradiction, because $J_n'(q) \geq 0$ for all q. Therefore, $J_n'(\cdot)$ cannot have any interior maxima, and hence it can have at most one interior minimum. Therefore, there exists a threshold ω (not necessarily interior) such that $J_n''(q) \geq 0$ and hence $a_n'(q) \geq 0$ if and only if $q \geq \omega$.

Finally, statement (iii) follows by noting that $J_n'(q)$ being unimodal in q is sufficient for the proof of Theorem 2.

□