Supplement to "Optimal contracts with a risk-taking agent"

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These proofs refer to results in the text and the Appendices.

D.4 *Proof of existence for* $\underline{u} = -\infty$

Suppose that $\underline{u} = -\infty$, which corresponds to the agent having no limited liability constraint. This section gives conditions under which a unique solution to (P) exists and satisfies certain properties. Say that $u(\cdot)$ is *regular* if $\underline{w} = -\infty$ and $\frac{u'(w)u'''(w)}{[u''(w)]^2} < 3$ for all $w \in \mathbb{R}$. These conditions are quite mild; in particular, the second condition means that $u'(\cdot)$ is not excessively convex, in the sense that it has local concavity everywhere greater than -2. See Prékopa (1973) and Borell (1975) for details.

Proposition 10. *Suppose $\pi(y) \equiv y$, $u(\cdot)$ is strictly concave and regular, and $\underline{u} = -\infty$. Then for any $a \geq 0$, there exists a unique contract $v(\cdot)$ that implements a at maximum profit. Furthermore, there exists $\overline{u} < \infty$ independent of $\underline{u}$ such that $v(\overline{y}) < \overline{u}$ and $v(\underline{y}) > -\overline{u}$.*

Proof. Given Lemma 6, it is enough to show that for some $\underline{u}$, $v_{\underline{u}}(\underline{y}) > \underline{u}$. Assume not, so that, in particular, for all $\underline{u}$, $v_{\underline{u}}(\underline{y}) = \underline{u}$. We show that this leads to a contradiction. We henceforth restrict attention to $\underline{u} \leq 0$. For $\underline{u}$ sufficiently negative, it cannot be the case that $v_{\underline{u}}$ is linear. In particular, if $v_{\underline{u}}$ is linear, then since $v_{\underline{u}}(\overline{y}) > u_0 + c(a)$, we have that

$$\int v_{\underline{u}}(x) f_a(x|a)\, dx = \int v'_{\underline{u}}(x)\big(-F_a(x|a)\big)\, dx \geq \frac{u_0 + c(a) - \underline{u}}{\overline{y} - \underline{y}},$$

which diverges in $\underline{u}$, contradicting that $v_{\underline{u}}$ must satisfy (IC-FOC) with equality. Hence, for each $\underline{u}$, we can take a point $x_{\underline{u}} \in C_{v_{\underline{u}}}$, and derive $\lambda_{\underline{u}}$ and $\mu_{\underline{u}}$ as in the proof of Proposition 3.

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Let $z_{\underline{u}}(\cdot) = \rho(\lambda_{\underline{u}} + \mu_{\underline{u}} l(\cdot|a))$, where we follow the convention that $\rho(s) = -\infty$ for $s \leq 0$. The contract $v_{\underline{u}}$ will, in general, differ from $z_{\underline{u}}$, since $z_{\underline{u}}$ need be neither concave nor satisfy the limited liability constraint. Note that $n_{\underline{u}}(\cdot) = \rho^{-1}(v_{\underline{u}}(\cdot)) - (\lambda_{\underline{u}} + \mu_{\underline{u}} l(\cdot|a)) \underset{s}{=} v_{\underline{u}}(\cdot) - z_{\underline{u}}(\cdot)$.

Step 1. There is $\overline{\mu} < \infty$ such that $\mu_{\underline{u}} \leq \overline{\mu}$ for all $\underline{u}$.

Proof. Applying a small positive amount of $t_{x_{\underline{u}},\overline{y}}$ adds cost at rate at most $\rho^{-1}(\overline{u}) \times \int_{x_{\underline{u}}}^{\overline{y}} (x - x_{\underline{u}}) f(x|a)\, dx$, adds incentives at rate $\int_{x_{\underline{u}}}^{\overline{y}} (x - x_{\underline{u}}) f_a(x|a)\, dx$, and relaxes (IR). It follows that

$$\mu_{\underline{u}} \leq \rho^{-1}(\overline{u}) \frac{\displaystyle\int_{x_{\underline{u}}}^{\overline{y}} (x - x_{\underline{u}}) f(x|a)\, dx}{\displaystyle\int_{x_{\underline{u}}}^{\overline{y}} (x - x_{\underline{u}}) f_a(x|a)\, dx}.$$

But, as in the proof that $|Q(\mathbf{0})| > 0$,

$$\begin{aligned}\frac{\partial}{\partial x_{\underline{u}}} \frac{\displaystyle\int_{x_{\underline{u}}}^{\overline{y}} (x - x_{\underline{u}}) f(x|a)\, dx}{\displaystyle\int_{x_{\underline{u}}}^{\overline{y}} (x - x_{\underline{u}}) f_a(x|a)\, dx} &\underset{s}{=} - \frac{\displaystyle\int_{x_{\underline{u}}}^{\overline{y}} (x - x_{\underline{u}}) f_a(x|a)\, dx}{\displaystyle\int_{x_{\underline{u}}}^{\overline{y}} (x - x_{\underline{u}}) f(x|a)\, dx} + \frac{\displaystyle\int_{x_{\underline{u}}}^{\overline{y}} f_a(x|a)\, dx}{\displaystyle\int_{x_{\underline{u}}}^{\overline{y}} f(x|a)\, dx} \\ &\leq 0,\end{aligned}$$

and so we can take

$$\overline{\mu} = \rho^{-1}(\overline{u}) \frac{\displaystyle\int (x - \underline{y}) f(x|a)\, dx}{\displaystyle\int (x - \underline{y}) f_a(x|a)\, dx} < \infty.$$

◁

Step 2. There is $\underline{\mu} > 0$ and $\underline{u}^* > -\infty$ such that $\mu_{\underline{u}} \geq \underline{\mu}$ for all $\underline{u} < \underline{u}^*$.

Proof. Choose $-\infty < \underline{u}^* \leq 0$ such that

$$\rho^{-1}\big(\underline{u}^*\big) < \frac{1}{2} \rho^{-1}\big(u_0 + c(a)\big), \tag{15}$$

$$c'(a) < \frac{u_0 + c(a) - \underline{u}^*}{\overline{y} - \underline{y}}, \tag{16}$$

where such a $\underline{u}^*$ exists since by assumption $\lim_{w \to -\infty} \frac{1}{u'(w)} = 0$. Let

$$r \equiv \sup_{\tau \in [\frac{1}{2}\rho^{-1}(u_0 + c(a)), \infty)} \rho'(\tau).$$

Since $\rho(1/u'(w)) = u(w)$, we have that

$$\rho'\left(\frac{1}{u'(w)}\right) = \frac{(u')^3}{-u''}(w),$$

from which

$$\frac{\rho''\left(\dfrac{1}{u'(w)}\right)}{\rho'\left(\dfrac{1}{u'(w)}\right)} = u'(w)\left(\frac{u'''(w)u'(w)}{\left(u''(w)\right)^2} - 3\right). \tag{17}$$

Since u is regular, it follows that $\rho'' < 0$ and so $r < \infty$. Let $\bar{l}_x = \max_x l_x(x|a)$ and choose $\underline{\mu} > 0$ such that

$$\underline{\mu} < \frac{1}{2}\frac{\rho^{-1}\left(u_0 + c(a)\right)}{l(\bar{y}|a) - l(\underline{y}|a)}, \tag{18}$$

$$\underline{\mu} < \frac{1}{r\bar{l}_x}\frac{u_0 + c(a)}{\bar{y} - \underline{y}}. \tag{19}$$

Assume that for some $\underline{u} < \underline{u}^*$, $\mu_{\underline{u}} < \underline{\mu}$. We show that this leads to a contradiction, establishing the result.

Using Corollary 2 (which depends only on the necessity part of the proof of Proposition 3, which is proved in Appendix B) and the fact that $\bar{y}$ is free, $n(\bar{y}) \leq 0$, and so $\lambda_{\underline{u}} + \mu_{\underline{u}} l(\bar{y}|a) \geq \rho^{-1}(v_{\underline{u}}(\bar{y})) \geq \rho^{-1}(u_0 + c(a))$. Thus,

$$\begin{aligned} \lambda_{\underline{u}} + \mu_{\underline{u}} l(\underline{y}|a) &= \lambda_{\underline{u}} + \mu_{\underline{u}} l(\bar{y}|a) - \mu_{\underline{u}}\left(l(\bar{y}|a) - l(\underline{y}|a)\right) \\ &\geq \rho^{-1}\left(u_0 + c(a)\right) - \frac{1}{2}\frac{\rho^{-1}\left(u_0 + c(a)\right)}{l(\bar{y}|a) - l(\underline{y}|a)}\left(l(\bar{y}|a) - l(\underline{y}|a)\right) \\ &= \frac{1}{2}\rho^{-1}\left(u_0 + c(a)\right), \end{aligned} \tag{20}$$

where the inequality follows from $\mu_{\underline{u}} < \underline{\mu}$ and (18).

Since $\underline{u} < \underline{u}^*$, and by (15), $\rho^{-1}(v_{\underline{u}}(\underline{y})) = \rho^{-1}(\underline{u}) < \frac{1}{2}\rho^{-1}(u_0 + c(a))$. Thus, using (20), $n(\underline{y})$ is strictly positive and it follows by Corollary 2 that $v_{\underline{u}}$ begins with a linear segment, the slope of which (by concavity) is at least

$$\frac{u_0 + c(a) - \underline{u}}{\bar{y} - \underline{y}} \geq \frac{u_0 + c(a)}{\bar{y} - \underline{y}}.$$

But using (20) and the definition of r, we have that for all x,

$$\begin{aligned} z_{\underline{u}}'(x) &= \rho'\left(\lambda_{\underline{u}} + \mu_{\underline{u}} l(x|a)\right)\mu_{\underline{u}} l_x(x|a) \\ &\leq r\mu_{\underline{u}}\bar{l}_x \\ &< \frac{u_0 + c(a)}{\bar{y} - \underline{y}}, \end{aligned}$$

where the strict inequality follows from (19). Hence, the initial linear segment of $v_{\underline{u}}$ crosses $z_{\underline{u}}$ at most once (from below). This implies that the entire contract is, in fact, linear with slope at least $(u_0 + c(a) - \underline{u})/(\bar{y} - \underline{y})$. In particular, let x_H be the right end of the linear segment. If x_H is at or before the crossing point, then $v_{\underline{u}}$ violates (2) and so cannot be optimal by part (i) of Proposition 3. If $x_H < \overline{y}$ is after the crossing, then we violate Corollary 2. It follows that $v_{\underline{u}}$ generates incentives at least

$$\frac{u_0 + c(a) - \underline{u}^*}{\overline{y} - \underline{y}} > c'(a)$$

using (16). But we have shown that (IC-FOC) binds at $v_{\underline{u}}$, leading to the desired contradiction. $\triangleleft$

STEP 3. There is $u_0 + c(a) > u_c > -\infty$ such that if $\underline{u} < \underline{u}^*$ and $\rho(\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a))) < u_c$, then $z_{\underline{u}}(\cdot)$ is concave at x.

PROOF. Note first that ρ is trivially concave anywhere that it is equal to $-\infty$ and that, by assumption, $\lim_{s\to 0}\rho(s) = -\infty$. Hence, it is enough to prove concavity where $\rho(\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a)))$ is finite. But it follows from (17) and the fact that u is regular that $\lim_{t\downarrow 0}\rho''(t)/\rho'(t) = -\infty$ and so $\rho''(t)/\rho'(t)$ is negative for t below some t'. Assume $\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a)) < t'$. Then

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\rho\big(\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a))\big) &= \frac{\partial}{\partial x}\big(\rho'\big(\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a))\big)\mu_{\underline{u}} l_x(x|a)\big) \\ &= \rho''\big(\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a))\big)\big(\mu_{\underline{u}} l_x(x|a)\big)^2 \\ &\quad + \rho'\big(\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a))\big)\mu_{\underline{u}} l_{xx}(x|a) \\ &\underset{s}{=} \frac{\rho''}{\rho'}\big(\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a))\big)\mu_{\underline{u}} + \frac{l_{xx}}{l_x^2}(x|a) \\ &\le \frac{\rho''}{\rho'}\big(\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a))\big)\underline{\mu} + \frac{l_{xx}}{l_x^2}(x|a).\end{aligned}$$

The second term is bounded by assumption. The first term diverges to $-\infty$ as $\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a)) \to 0$. Hence, since ρ is monotone and since $\lim_{w\to-\infty} u'(w) = \infty$, the result follows. $\triangleleft$

STEP 4. As in the derivation of r in Step 2, let $\hat{r}$ be such that for all $t \ge \rho^{-1}(u_c)$, $\rho'(t) \le \hat{r}$. Let $-\infty < \underline{\hat{u}} \le \underline{u}^*$ satisfy

$$\hat{s} \equiv \frac{u_0 + c(a) - \underline{\hat{u}}}{\overline{y} - \underline{y}} \ge \max\{c'(a), \overline{\mu}\bar{l}_x\hat{r}\}$$

and assume that $\underline{u} < \underline{\hat{u}}$. Then $z_{\underline{u}}(\underline{y}) \le \underline{u}$.

PROOF. Assume that $z_{\underline{u}}(\underline{y}) > \underline{u}$. Then, since $v_{\underline{u}}(\underline{y}) = \underline{u}$, $v_{\underline{u}}$ begins with a linear segment of positive length of slope at least $\hat{s}$, and so by Proposition 3 and part (i) of Definition 2, crosses $z_{\underline{u}}$ from below and is strictly above $z_{\underline{u}}$ for an interval of positive length as well. Let $x_{\underline{u},c}$ be defined by $z_{\underline{u}}(x_{\underline{u},c}) = u_c$. If $v_{\underline{u}}$ has its initial crossing of $z_{\underline{u}}$ at or before $x_{\underline{u},c}$, then since $z_{\underline{u}}$ is concave until $x_{\underline{u},c}$, $v_{\underline{u}}$ remains above $z_{\underline{u}}$ until $x_{\underline{u},c}$. But then, since for $x > x_{\underline{u},c}$, $\hat{s} \geq z'_{\underline{u}}$, $v_{\underline{u}}$ in fact never re-crosses $z_{\underline{u}}$. Alternatively, if the initial crossing of $z_{\underline{u}}$ by $v_{\underline{u}}$ is after $x_{\underline{u},c}$, then again, since $v_{\underline{u}}$ has slope greater than $z'_{\underline{u}}$ for $x > x_{\underline{u},c}$, $v_{\underline{u}}$ never re-crosses $z_{\underline{u}}$. In either case, by Corollary 2, $v_{\underline{u}}$ is thus linear on all of $[\underline{y}, \overline{y}]$, a contradiction. ◁

STEP 5. Let $u_{y_0} = u_0 + c(a) - c'(a)(\overline{y} - y_0) > -\infty$. Then $v_{\underline{u}}(y_0) \geq u_{y_0}$.

PROOF. Since $v_{\underline{u}}(\overline{y}) \geq u_0 + c(a)$, it follows that everywhere on $[\underline{y}, y_0)$, $v_{\underline{u}}(\cdot)$ is below the line $L(\cdot)$ that goes through $(y_0, v_{\underline{u}}(y_0))$ and $(\overline{y}, u_0 + c(a))$, and everywhere on $(y_0, \overline{y}]$, $v_{\underline{u}}(\cdot)$ is above $L(\cdot)$. Hence, since $f_a < 0$ on $[\underline{y}, y_0)$ and $f_a > 0$ on $(y_0, \overline{y}]$,

$$
\begin{aligned}
c'(a) &= \int v_{\underline{u}}(x) f_a(x|a)\, dx \\
&\geq \int L(x) f_a(x|a)\, dx \\
&= \frac{u_0 + c(a) - v_{\underline{u}}(y_0)}{\overline{y} - y_0}.
\end{aligned}
$$

Rearranging yields the desired result. ◁

STEP 6. Choose $\infty < u_s < \min\{u_{y_0}, u_c, \rho(-\overline{\mu} l(\underline{y}|a)), \hat{\underline{u}}\}$ small enough that for all $t \leq u_s$,

$$
\rho'\left(\frac{1}{u'\big(u^{-1}(t)\big)}\right)\overline{\mu}\underline{l_x} \geq \hat{s}, \tag{21}
$$

where $\underline{l_x} = \min_x l_x(x|a) > 0$. Since $\rho'(\tau)$ diverges to ∞ as $\tau \downarrow 0$, and since $1/u'(u^{-1}(t))$ goes to 0 as $t \downarrow -\infty$, such a u_s is guaranteed to exist.

STEP 7. Choose $\underline{u} < u_s$. Let $\overline{z}_{\underline{u}}(\cdot) = \rho(\overline{\lambda}_{\underline{u}} + \overline{\mu} l(\cdot|a))$, where $\overline{\lambda}_{\underline{u}}$ solves $\rho(\overline{\lambda}_{\underline{u}} + \overline{\mu} l(\underline{y}|a)) = \underline{u}$. By Step 4, $z_{\underline{u}}(\underline{y}) \leq \underline{u}$, and so, since $\mu_{\underline{u}} \leq \overline{\mu}$, $z_{\underline{u}}(\cdot) \leq \overline{z}_{\underline{u}}(\cdot)$. Let $x_{\underline{u},s}$ be defined by $\overline{z}_{\underline{u}}(x_{\underline{u},s}) = u_s$. Since $\overline{\lambda}_{\underline{u}} + \overline{\mu} l(\underline{y}|a) = 1/u'(u^{-1}(\underline{u})) > 0$, it follows that $\overline{\lambda}_{\underline{u}} + \overline{\mu} l(y_0|a) \geq -\overline{\mu} l(\underline{y}|a)$ and, hence,

$$
\rho\big(\overline{\lambda}_{\underline{u}} + \overline{\mu} l(y_0|a)\big) > \rho\big(-\overline{\mu} l(\underline{y}|a)\big) > u_s,
$$

where the last inequality is by definition of u_s in Step 6. Thus, $x_{\underline{u},s} < y_0$.

STEP 8. For all $x < x_{\underline{u},s}$, $v_{\underline{u}}(x) \leq \overline{z}_{\underline{u}}(x)$.

PROOF. Let $x_{\underline{u},c}$ be defined by $z_{\underline{u}}(x_{\underline{u},c}) = u_c$. By construction, $\overline{z}_{\underline{u}}(\cdot)$ is concave where $x \leq x_{\underline{u},c}$. Using (21), $\overline{z}'_{\underline{u}}(\cdot) > \hat{s}$ for $x < x_{\underline{u},s}$ and $\overline{z}'_{\underline{u}}(\cdot) < \hat{s}$ for $x \geq x_{\underline{u},c}$. Assume that for some $\tilde{x} < x_{\underline{u},s}$, $v_{\underline{u}}(\tilde{x}) > \overline{z}_{\underline{u}}(\tilde{x}) \geq z_{\underline{u}}(\tilde{x})$. By Corollary 2, $v_{\underline{u}}$ is linear at $\tilde{x}$. If $v'_{\underline{u}}(\tilde{x}) \leq \overline{z}'_{\underline{u}}(\tilde{x})$, then, since $\overline{z}_{\underline{u}}$ is concave on $[\underline{y}, x_{\underline{u},s}]$ and again using Corollary 2, $v_{\underline{u}}$ is also above $\overline{z}_{\underline{u}}$ and, hence, is linear, for all x in $[\underline{y}, \tilde{x}]$. But then

$$
v_{\underline{u}}(\underline{y}) - \overline{z}_{\underline{u}}(\underline{y}) \geq v_{\underline{u}}(\tilde{x}) - \overline{z}_{\underline{u}}(\tilde{x}) > 0,
$$

contradicting that $v_{\underline{u}}(\underline{y}) = \underline{u}$. Thus, $v_{\underline{u}}'(\tilde{x}) > \overline{z}_{\underline{u}}'(\tilde{x}) > \hat{s}$. But then $v_{\underline{u}}$ remains linear and, hence, strictly above the concave function $z_{\underline{u}}$ at least until $x_{\underline{u},c}$. For $x \geq x_{\underline{u},c}$, $\overline{z}_{\underline{u}}'(\tilde{x}) \leq \hat{s}$, and so as before v can never re-cross $\overline{z}_{\underline{u}}$, and so a fortiori can never re-cross $z_{\underline{u}}$. Hence, $v_{\underline{u}}$ is linear on $[\tilde{x}, \overline{y}]$, with slope at least $\hat{s}$. Let L be the line that agrees with $v_{\underline{u}}$ on $[\tilde{x}, \overline{y}]$. To the left of $\tilde{x}$, $v_{\underline{u}}$, being concave, lies below L. But $\tilde{x} < x_{\underline{u},s} < y_0$ and so, since $f_a(\cdot|a)$ is negative on $[\underline{y}, \tilde{x}]$,

$$\int v_{\underline{u}}(x) f_a(x|a)\, dx \geq \int L(x) f_a(x|a)\, dx \geq \hat{s} > c'(a),$$

again a contradiction. ◁

STEP 9. We show that $\lim_{\underline{u}\to-\infty} \int v_{\underline{u}}(x) f_a(x|a)\, dx = \infty$. For $\underline{u}$ sufficiently negative, this provides the necessary contradiction to the original supposition that $v_{\underline{u}}(\underline{y}) = \underline{u}$ for all $\underline{u}$, proving the result.

PROOF. By Step 8, for $\underline{u}$ sufficiently negative, $v_{\underline{u}}(x) \leq \overline{z}_{\underline{u}}(x)$ for all $x \leq x_{\underline{u},s}$. Let $v_{\underline{u}}^T$ truncate $v_{\underline{u}}$ to never pay more than u_s. Since $\max(0, v_{\underline{u}}(x) - u_s)$ is an increasing function, $\int \max(0, v_{\underline{u}}(x) - u_s) f_a(x|a)\, dx \geq 0$ and, hence, $\int v_{\underline{u}}(x) f_a(x|a)\, dx \geq \int v_{\underline{u}}^T(x) f_a(x|a)\, dx$. Note also that since $v_{\underline{u}}(y_0) > u_s$, $v_{\underline{u}}^T(x) = u_s$ for all $x \geq y_0$. Let $\overline{z}_{\underline{u}}^T$ similarly truncate $\overline{z}_{\underline{u}}$ to pay u_s to the right of $x_{\underline{u},s}$. Then $\overline{z}_{\underline{u}}^T$ is everywhere at least as large as $v_{\underline{u}}^T$, but equal to $v_{\underline{u}}^T$ everywhere to right of y_0. Hence, since f_a is negative to the left of y_0, we have

$$\int v_{\underline{u}}(x) f_a(x|a)\, dx \geq \int v_{\underline{u}}^T(x) f_a(x|a)\, dx \geq \int \overline{z}_{\underline{u}}^T(x) f_a(x|a)\, dx.$$

To arrive at a contradiction, it would thus be enough to show that $\int \overline{z}_{\underline{u}}^T(x) f_a(x|a)\, dx$ diverges as $\underline{u} \to -\infty$. But by Moroni and Swinkels (2014, Lemma 4), under our regularity conditions, $\int \overline{z}_{\underline{u}}(x) f_a(x|a)\, dx$ does diverge as $\underline{u} \to -\infty$.

Let

$$u^{**} = \rho\big(1 + \overline{\mu}\big(l(\overline{y}|a) - l(\underline{y}|a)\big)\big) < \infty.$$

Then, for all $\underline{u}$ sufficiently negative that $\frac{1}{u'(u^{-1}(\underline{u}))} \leq 1$, $\overline{z}_{\underline{u}}(\overline{y}) \leq u^{**}$. Hence,

$$\begin{aligned} \int \overline{z}_{\underline{u}}(x) f_a(x|a)\, dx - \int \overline{z}_{\underline{u}}^T(x) f_a(x|a)\, dx &= \int \big(\overline{z}_{\underline{u}}(x) - \overline{z}_{\underline{u}}^T(x)\big) f_a(x|a)\, dx \\ &\leq \int_{y_0}^{\overline{y}} \big(\overline{z}_{\underline{u}}(x) - \overline{z}_{\underline{u}}^T(x)\big) f_a(x|a)\, dx \\ &\leq \big(u^{**} - u_s\big) \int_{y_0}^{\overline{y}} f_a(x|a)\, dx \\ &< \infty, \end{aligned}$$

where the first inequality follows because $\overline{z}_{\underline{u}}(x) - \overline{z}_{\underline{u}}^T$ is weakly positive, and the second inequality follows because it is bounded above by $u^{**} - u_s$. □

D.5 *Agent reports x*

In this section, we allow the agent to send a contractible message $\tilde{x}$ after he observes x but before y is realized. Payments can therefore depend on both $\tilde{x}$ and y, which allows the principal to discipline the agent from engaging in risk-taking. Restricting attention to the case where both parties are risk-neutral, we show that a linear contract is optimal in this setting.

Since the principal does not benefit from risk-taking, it is without loss to restrict attention to mechanisms that punish the agent as much as possible whenever his report does not match the final output: $s(y)\mathbb{I}_{\{y=\tilde{x}\}} - M\mathbb{I}_{\{y\neq\tilde{x}\}}$ for some upper semicontinuous function $s(\cdot)$. Then the principal's problem is

$$\begin{aligned} \max_{a,s(\cdot)} \quad & \mathbb{E}_{F(\cdot|a),G}\big[y - s(y)\mathbb{I}_{\{y=\tilde{x}\}} + M\mathbb{I}_{\{y\neq\tilde{x}\}}\big] \\ \text{subject to} \quad & a, G, \tilde{x} \in \arg\max_{\tilde{a},\tilde{G}\in\mathcal{G},\tilde{\tilde{x}}} \big\{\mathbb{E}_{F(\cdot|\tilde{a}),\tilde{G}}\big[s(y)\mathbb{I}_{\{y=\tilde{\tilde{x}}\}} - M\mathbb{I}_{\{y\neq\tilde{\tilde{x}}\}}\big] - c(\tilde{a})\big\}, \\ & \mathbb{E}_{F(\cdot|a),G}\big[s(y)\mathbb{I}_{\{y=\tilde{x}\}} - M\mathbb{I}_{\{y\neq\tilde{x}\}}\big] - c(a) \geq u_0, \\ & s(\cdot) \geq -M, \end{aligned}$$

where $\tilde{x}$ maps x to a report made to the principal.

Fix $s(\cdot)$, and consider the agent's choice of G_x and $\tilde{x}$ following any intermediate output $x > \underline{y}$. Define

$$\lambda_s(x) = \max\big\{\lambda : \lambda(y - \underline{y}) - M = s(y) \text{ for some } y \geq x\big\}.$$

Intuitively, $\lambda_s(x)$ is the smallest slope such that $\lambda_s(x)(y - \underline{y}) - M \geq s(y)$ for all $y \geq x$. We show that following intermediate output $x > \underline{y}$, the agent optimally chooses G_x and $\tilde{x}$ so that his expected payoff is $\lambda_s(x)(x - \underline{y}) - M$.[25]

Lemma 7. *For any $s(\cdot)$ and $x \in \mathcal{Y}$, the principal's expected payment to the agent equals*

$$\sigma_s(x) \equiv \max_{G_x,\tilde{x}}\big\{\mathbb{E}_{G_x}\big[s(y)\mathbb{I}_{\{y=\tilde{x}\}} - M\mathbb{I}_{\{y\neq\tilde{x}\}}\big]\big\} = \begin{cases} s(\underline{y}) & \text{if } x = \underline{y}, \\ \lambda_s(x)(x - \underline{y}) - M & \text{if } x > \underline{y}. \end{cases} \tag{22}$$

Proof. Fix $s(\cdot)$ and $x > \underline{y}$. First, we show that there exists some G_x and $\tilde{x}$ such that $\mathbb{E}_{G_x}[s(y)\mathbb{I}_{\{y=\tilde{x}\}} - M\mathbb{I}_{\{y\neq\tilde{x}\}}] = \lambda_s(x)(x - \underline{y}) - M$. By definition of $\lambda_s(\cdot)$, there exists a $\hat{y} \geq x$ such that $\lambda_s(x)(\hat{y} - \underline{y}) - M = s(\hat{y})$. Let $\tilde{x} = \hat{y}$ and $G_x(y) = (1 - p_{\hat{y}}) + p_{\hat{y}}\mathbb{I}_{\{y\geq\hat{y}\}}$, where $p_{\hat{y}} = \frac{x-\underline{y}}{\hat{y}-\underline{y}}$; i.e., $y = \underline{y}$ with probability $1 - p_{\hat{y}}$ and $y = \hat{y}$ with probability $p_{\hat{y}}$. Then the agent's expected payoff is

$$\begin{aligned} p_{\hat{y}}s(\hat{y}) - (1 - p_{\hat{y}})M &= \frac{x - \underline{y}}{\hat{y} - \underline{y}}s(\hat{y}) - \frac{\hat{y} - x}{\hat{y} - \underline{y}}M \\ &= \frac{x - \underline{y}}{\hat{y} - \underline{y}}\big[\lambda_s(x)(\hat{y} - \underline{y}) - M\big] - \frac{\hat{y} - x}{\hat{y} - \underline{y}}M \\ &= \lambda_s(x)(x - \underline{y}) - M. \end{aligned}$$

[25] If $x = \underline{y}$, then the agent is compelled to choose $G_{\underline{y}}(y) = 1$, so his expected payoff is equal to $s(\underline{y})$.

Next we show that the agent cannot earn more than $\lambda_s(x)(x-\underline{y})-M$ following intermediate output x. For any report $\tilde{x}$, the agent earns more than $-M$ only if $y=\tilde{x}$, so his optimal distribution G_x maximizes the probability that $y=\tilde{x}$ subject to the constraint that $\mathbb{E}_{G_x}[y]=x$. This is accomplished by choosing $G_x(\cdot)$ such that $y=\tilde{x}$ with some probability $p_{\tilde{x}}$ and $y=\underline{y}$ with probability $1-p_{\tilde{x}}$, where $p_{\tilde{x}}\tilde{x}+(1-p_{\tilde{x}})\underline{y}=x$. It suffices to show that the agent's expected payoff under this distribution is maximized if $\tilde{x}=\hat{y}$.

Suppose that there exists some $\tilde{x}\neq\hat{y}$ such that $p_{\tilde{x}}s(\tilde{x})-(1-p_{\tilde{x}})M>p_{\hat{y}}s(\hat{y})-(1-p_{\hat{y}})M=\lambda_s(x)(x-\underline{y})-M$. Then there must exist some $\tilde{\lambda}>\lambda_s(x)$ such that $\tilde{\lambda}(\tilde{x}-\underline{y})-M=s(\tilde{x})$, which contradicts the definition of $\lambda_s(x)$. Therefore, for all x, the agent's expected payoff equals $\lambda_s(x)(x-\underline{y})-M$. □

To see this result, recall that the agent earns $-M$ whenever his report does not equal the realized output. Therefore, if he misreports $\tilde{x}\neq x$, then he chooses G_x to maximize the probability that $y=\tilde{x}$. In particular, it is optimal for G_x to put weight on only two points, $\tilde{x}$ and $\underline{y}$. Given this $\tilde{x}$, the agent's payoff can be written as $p_{\tilde{x}}s(\tilde{x})-(1-p_{\tilde{x}})M$, where $p_{\tilde{x}}\tilde{x}+(1-p_{\tilde{x}})\underline{y}=x$. It can be shown that the agent's payoff can be rewritten as $\lambda(x-\underline{y})-M$, where $\lambda\leq\lambda_s(x)$. There exists some report $\tilde{x}$ that sets $\lambda=\lambda_s(x)$, proving the result.

Using Lemma 7, we can rewrite the principal's problem as

$$\begin{aligned}\max_{a,s(\cdot)}\quad & \mathbb{E}_{F(\cdot|a)}\big[x-\sigma_s(x)\big]\\ \text{subject to}\quad & a\in\arg\max_{\tilde{a}}\big\{\mathbb{E}_{F(\cdot|\tilde{a})}\big[\sigma_s(x)\big]-c(\tilde{a})\big\},\\ & \mathbb{E}_{F(\cdot|a)}\big[\sigma_s(x)\big]-c(a)\geq u_0,\\ & s(\cdot)\geq -M,\end{aligned}$$

where, for any contract $s(\cdot)$, $\sigma_s(\cdot)$ is given by (22).

Recall the definition of $s_a^{\mathrm{L}}(\cdot)$ from Section 4. We show that if $a\geq 0$ is such that (LL) holds with equality after $\underline{y}$ under $s_a^{\mathrm{L}}(\cdot)$, then $s_a^{\mathrm{L}}(\cdot)$ implements a at maximum profit in this setting. Consequently, if (LL) binds for the optimal $a\geq 0$, then a linear contract is optimal as in Proposition 2.

Proposition 11. *Fix any effort $a\geq 0$. If $s_a^{\mathrm{L}}(\underline{y})=-M$, then $s_a^{\mathrm{L}}(\cdot)$ implements a at maximum profit.*

Proof. Note that $\lambda_s(\cdot)$ is decreasing for any $s(\cdot)$ and, moreover, is constant for all $x\in\mathcal{Y}$ if $s(\cdot)$ is affine. Let $\hat{s}(\cdot)$ implement a at maximum profit and suppose there exists $x_L<x_H$ such that $\lambda_{\hat{s}}(x_L)>\lambda_{\hat{s}}(x_H)$.

Define $s_L(y)=\beta(y-\underline{y})-M$, where β is chosen such that $\mathbb{E}_{F(\cdot|a)}[s_L(y)-\lambda_{\hat{s}}(y)\times(y-\underline{y})+M]=0$. Such a β exists by the intermediate value theorem because $\lambda_{\hat{s}}(y)\geq 0$ is finite. Since $\lambda_{\hat{s}}(\cdot)$ is strictly decreasing over some interval, there exists some $y^*\in(\underline{y},\overline{y})$ such that $\lambda_{\hat{s}}(y)\geq\beta$ if and only if $y\leq y^*$. Then $\beta-\lambda_{\hat{s}}(y)$ is first negative and then positive, $\int [\beta - \lambda_{\hat{s}}(y)](y - \underline{y}) f(y|a)\, dy = 0$ by construction, and $\frac{f_a(\cdot|a)}{f(\cdot|a)}$ is strictly increasing, so Beesack's inequality implies that

$$\int \big[\beta - \lambda_{\hat{s}}(y)\big](y - \underline{y}) f_a(y|a)\, dy > 0.$$

Therefore, $s_L(\cdot)$ implements some effort level $a' > a$, which implies that $\beta > c'(a)$.

Observe that $s_a^{\mathrm{L}}(y) < s_L(y)$ for all $y > \underline{y}$, because $s_a^{\mathrm{L}}(\underline{y}) = -M$ by assumption and $c'(a) < \beta$. Moreover, $s_a^{\mathrm{L}}(\cdot)$ implements a and satisfies both the individual rationality and the limited liability constraints. Therefore, $s_a^{\mathrm{L}}(\cdot)$ implements effort a at strictly higher profit than $\hat{s}(\cdot)$. So $\lambda_{\hat{s}}(\cdot)$ must be constant and $\sigma_{\hat{s}}(\underline{y}) = -M$, in which case $s_a^{\mathrm{L}}(\cdot)$ is also optimal. □

D.6 *Comparative static of optimal contract with respect to $\underline{y}$*

This appendix considers how a^* changes with the lower bound $\underline{y}$ on output. A decrease in $\underline{y}$ implies that the agent can take on more severe left-tail risk by gambling over worse outcomes. We prove that a lower $\underline{y}$ makes it costlier for the principal to induce any nonzero effort level. As $\underline{y}$ approaches $-\infty$, inducing any positive effort becomes arbitrarily expensive and so the agent exerts no effort in the optimal contract.

Corollary 3. *Consider a decreasing sequence $\{\underline{y}_k\}_{k=0}^{\infty}$ with $\lim_{k\to\infty} \underline{y}_k = -\infty$. For each $k \geq 0$, consider $\mathcal{Y} = [\underline{y}_k, \overline{y}]$ and some output distribution $F_k(\cdot|a)$ that satisfies our assumptions (i.e., has full support on $[\underline{y}_k, \overline{y}]$, satisfies $\mathbb{E}_{F_k(\cdot|a)}[x] = a$, etc.), and let a_k^* be the corresponding optimal effort. Then $\lim_{k\to\infty} a_k^* = 0$, and if $\pi(y) \equiv y$, then a_k^* is decreasing in k.*

Proposition 2 implies that the principal's expected payment from inducing $a^* \geq 0$ equals $E_{F(\cdot|a^*)}[\pi(y - c'(a^*)(y - \underline{y}) + w)]$. For small enough $\underline{y}$, $s_{a^*}^{\mathrm{L}}(\underline{y}) = -M$. But then implementing $a^* > 0$ becomes arbitrarily costly as $\underline{y} \to -\infty$, in which case the principal is better off not motivating the agent at all. If the principal is risk-neutral, then we can show that the principal's profit under $s_{a^*}^{\mathrm{L}}(\cdot)$ is supermodular in a^* and $\underline{y}$, so that a^* is increasing in $\underline{y}$.

Proof of Corollary 3. Fix $\hat{a} > 0$. Define

$$y_1 \equiv \min_{a \in [\hat{a}, a^{\mathrm{FB}}]} \left\{ a - \frac{c(a) + u_0 + M}{c'(a)} \right\}$$

and

$$y_2 \equiv \min_{a \in [\hat{a}, a^{\mathrm{FB}}]} \left\{ \frac{u^{-1}(u_0) - \big(1 - c'(a) a\big) - M}{c'(\hat{a})} \right\},$$

and note that since $c'(a) \geq c'(\hat{a}) > 0$ for all $a \geq \hat{a}$, $y_{\min} \equiv \min\{0, y_1, y_2\} > -\infty$.

Let $\underline{y} < y_{\min}$ and suppose toward a contradiction that there exists a distribution $F(\cdot|a)$ on $[\underline{y}, \overline{y}]$ such that effort $a^* \geq \hat{a}$ is optimal under $F(\cdot|a)$. Note first that Proposition 2 implies that the principal's expected payoff equals

$$\mathbb{E}_{F(\cdot|a^*)}\big[\pi\big(y - s_{a^*}^{\mathrm{L}}(y)\big)\big] = \mathbb{E}_{F(\cdot|a^*)}\big[\pi\big(y - c'\big(a^*\big)(y - \underline{y}) + \min\big\{M, c'\big(a^*\big)\big(a^* - \underline{y}\big) - c\big(a^*\big) - u_0\big\}\big)\big].$$

Since $\underline{y} < y_1$, $c'(a^*)(a^* - \underline{y}) - c(a^*) - u_0 > M$. Furthermore, the principal's payoff is bounded above by

$$\pi\big(\big(1 - c'\big(a^*\big)\big)a^* + c'\big(a^*\big)\underline{y} + M\big)$$

by Jensen's inequality. Since $\underline{y} < \min\{0, y_2\}$, $(1 - c'(a))a + c'(a)\underline{y} + M < u^{-1}(u_0)$ for any $a \in [\hat{a}, a^{\mathrm{FB}}]$. But then $a^* \geq \hat{a}$ cannot be optimal because it is strictly dominated by $a^* = 0$ and $s(\cdot) \equiv u^{-1}(u_0)$, a contradiction. Hence, for $\underline{y} < y_{\min}$, any distribution $F(\cdot|a)$, and any optimal a^*, it must be that $a^* < \hat{a}$. Since $\hat{a} > 0$ is arbitrary, $\lim_{\underline{y} \to -\infty} a^* = 0$.

Suppose $\pi(y) \equiv y$. To prove that a^* is increasing in $\underline{y}$, it suffices to show that the principal's payoff from implementing a in an optimal contract, $\Pi(a, \underline{y}) = a - c'(a)(a - \underline{y}) + w$, is supermodular in a and $\underline{y}$.

Recall that $w = \min\{M, c'(a)(a - \underline{y}) - c(a) - u_0\}$ is a function of $(a, \underline{y})$. Therefore,

$$\frac{\partial \Pi}{\partial a} = 1 - c''(a)(a - \underline{y}) - c'(a) + \frac{\partial w}{\partial a}$$

and so

$$\frac{\partial^2 \Pi}{\partial \underline{y}\, \partial a} = c''(a) + \frac{\partial^2 w}{\partial \underline{y}\, \partial a}.$$

But $\frac{\partial^2 w}{\partial \underline{y}\, \partial a} = 0$ if $M < c'(a)(a - \underline{y}) - c(a) - u_0$ and $\frac{\partial^2 w}{\partial \underline{y}\, \partial a} = -c''(a)$ otherwise. In either case, $\frac{\partial^2 \Pi}{\partial \underline{y}\, \partial a} \geq 0$ and so optimal effort a^* is increasing in $\underline{y}$, as desired. □

References

Borell, Christer (1975), "Convex set functions in d-space." *Periodica Mathematica Hungarica*, 6, 111–136. [1]

Moroni, Sofia and Jeroen Swinkels (2014), "Existence and non-existence in the moral hazard problem." *Journal of Economic Theory*, 150, 668–682. [6]

Prékopa, András (1973), "On logarithmic concave measures and functions." *Acta Scientiarum Mathematicarum*, 34, 335–343. [1]

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