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# OPTIMAL MONITORING DESIGN

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# OPTIMAL MONITORING DESIGN

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This paper considers a Principal–Agent model with hidden action in which the Principal can monitor the Agent by acquiring independent signals conditional on effort at a constant marginal cost. The Principal aims to implement a target effort level at minimal cost. The main result of the paper is that the optimal information-acquisition strategy is a two-threshold policy and, consequently, the equilibrium contract specifies two possible wages for the Agent. This result provides a rationale for the frequently observed *single-bonus wage contracts*.

KEYWORDS: Principal-agent, moral hazard, monitoring.

## 1. INTRODUCTION

A GENERAL LESSON FROM CONTRACT THEORY is that in order to induce a worker to exert effort, he should be rewarded for those output realizations that indicate high effort. Designing such incentive schemes in practice can be challenging for various reasons. For example, if a firm has many employees, profit reflects aggregate performance and it is hard to disentangle an individual worker's contribution from that of her coworkers. Moreover, some aspects of performance (e.g., quality of customer service) are difficult to quantify and measure. In such cases, to monitor their workers, firms must identify variables that are informative about their effort. Then, by constructing performance measures based on these variables and offering wage plans contingent on these measures, firms can reduce agency costs. Indeed, firms devote significant resources to searching for effective ways to evaluate their employees; see, for example, Mauboussin (2012), WorldatWork and Consulting LLP (2014), and Buckingham and Goodall (2015). The goal of this paper is a theoretical investigation of the optimal monitoring structure in the absence of freely available information about the Agent's effort.

In our specific setup, there is a single Agent who chooses one of continuum many effort levels. The Principal can acquire arbitrarily many signals that are independent from one another conditional on the Agent's effort at a constant marginal cost. This is modeled by assuming that the Principal can observe a diffusion process with the drift being the Agent's effort at a cost proportional to the time at which the Principal stops observing this process. A contract specifies a stopping time with respect to this process and a wage scheme that is contingent on the Principal's observation. The Agent is risk-averse and has limited liability. Our goal is to characterize the contract that induces the Agent to choose a target effort level at minimal cost while respecting limited liability.

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We emphasize that our model is static in the sense that neither party discounts the future and the Agent chooses an effort level once and for all. Time is introduced to the information acquisition process only to provide the Principal with a *flexible* monitoring technology. Indeed, the Principal can acquire arbitrarily precise information about the Agent's effort irrespective of the signal she already acquired. Our model is meant to capture a cycle of an employer-worker relationship where the worker is evaluated periodically and the evaluation phase is short compared to the employment cycle.[1]

Our main result shows that, under certain conditions on the Agent's utility function, the optimal contract features a binary wage scheme; that is, the Agent is paid a base wage, plus a fixed bonus if his performance is deemed sufficiently good. This provides a new rationale for single-bonus contracts, which abound in practice (Murphy (1999) and Holmström (2016)). These single-bonus contracts are particularly common for consulting and banking jobs; see Aris (2019) and PageExecutive (2019) for recent surveys.[2] Prendergast (1999) noted that perhaps the most common occurrence of binary wage-contracts involves the threat of being fired; wages vary little with performance but poor performance is punished by dismissal.

Our result also addresses the criticism that canonical Principal–Agent models generate optimal contracts that are sensitive to minutiae of the Principal's exogenously given information (often assumed to be the output). Indeed, in these models, the Agent's wage depends on the likelihood ratios and only very particular distributions yield wage contracts that have any resemblance to the contracts observed in practice (Hart and Holmström (1986)). For example, single-bonus contracts are optimal only if there are exactly two possible values of the likelihood ratio, which is an unlikely feature of a typical output distribution. In contrast, we show that the optimal signal structure has this property if it is endogenously determined by the Principal's information-acquisition strategy.

Our analysis has two main building blocks. First, through a sequence of steps, we reformulate the problem of identifying an optimal contract to a *flexible information design problem*. In this new problem, the Principal's choice set is a set of distributions instead of a set of stopping rules. Second, we show that finding a solution to the information design problem is equivalent to characterizing an equilibrium in a *zero-sum game* played by the Principal and *Nature*. At the end, our main result is stated as an equilibrium characterization of this game. Below, we explain both of these ideas in detail.

*Information Design.* The Principal's problem can be decomposed into two parts: a stopping rule defining the information-acquisition strategy and a wage-function mapping from the Principal's observations to the Agent's monetary compensation. This wage can depend on the whole path of the diffusion. However, we argue that, in a relaxed problem where only local incentive compatibility is required, the optimal wage depends only on a one-dimensional variable, henceforth referred to as the *score*.[3] More precisely, we show that, for *any* given stopping rule, the cost-minimizing wage depends only on the value and the time of the Principal's last observation.[4] Since the optimal contract is incentive compatible and the drift of the diffusion is the target effort level, the optimal wage can be expressed as a function of the (driftless) Brownian motion part of the stochastic process.

[1]An employment cycle often coincides with a calendar year; see, for example, Payscale (2019).

[2]The variation of the size of the bonus is typically small.

[3]We later verify that such a first-order approach is valid under some conditions on the Agent's effort-cost function.

[4]The score is the counterpart of the derivative of the log-density function with respect to the Agent's effort in the canonical Principal–Agent model of Holmström (1979), which is a sufficient statistic for the optimal wage scheme.

We will argue that each stopping rule generates a distribution over the scores with zero mean. Results from Skorokhod embedding theory imply that the converse is also true: for any zero-mean distribution over the scores, there is a stopping time that generates this distribution. Recall that the Principal's information-acquisition cost is the expectation of the stopping time. It turns out that the expectation of the stopping time generating a certain distribution is the variance of the distribution. Therefore, the Principal's contracting problem can be rewritten as an information design problem where she chooses a distribution over scores at a cost equal to its variance (instead of a stopping time) and a wage function defined on scores.

*The Zero-Sum Game.* For any given distribution over scores $F$, the standard approach to solve for the optimal wage is to pointwise minimize the corresponding Lagrangian function; see, for example, Bolton and Dewatripont (2005). Let $\lambda$ denote the Lagrange multiplier corresponding to the incentive constraint and $L(\lambda, F)$ denote the value of the Lagrangian function evaluated at the cost-minimizing wage function. We show that strong duality holds, that is, the Principal's value for a given $F$ is $\sup_\lambda L(\lambda, F)$. Since the Principal chooses $F$ to minimize her overall cost, her problem can be written as $\inf_F \sup_\lambda L(\lambda, F)$. Instead of solving this inf sup problem, we characterize the solution of the corresponding sup inf problem. The key to this characterization is to observe that for each dual multiplier $\lambda$, the problem $\inf_F L(\lambda, F)$ is an unconstrained information design problem.[5] Using the concavification arguments developed in Aumann and Perles (1965) and Kamenica and Gentzkow (2011), we show that there exists a solution among the binary distributions, that is, a distribution supported only on two points.

It remains to argue that the inf sup and sup inf problems are equivalent. This follows from von Neumann's minimax theorem (see von Neumann (1928)), if the following zero-sum game has a Nash equilibrium. The game is played by *Nature*, who chooses a dual multiplier to maximize the Lagrange function, $L$, and the Principal, who chooses a probability distribution over scores to minimize $L$. We prove that, under some conditions on the Agent's utility function, this game indeed has a unique equilibrium.[6] The aforementioned concavification argument implies that the Principal always has a best response that is binary. We argue that the Principal's equilibrium distribution also has this feature, that is, there are only two values of the score that arise with positive probability. Consequently, the Agent's wage is also binary, and hence, a single-bonus wage scheme is optimal.

We believe that considering such a zero-sum game might turn out to be useful to analyze a class of problems where the Principal not only designs the information structure but also determines other policy variables subject to certain constraints. In our model, this policy variable is the wage, and the constraint guarantees incentive compatibility. In such environments, it is unclear how one can use the concavification arguments developed to solve unconstrained problems. However, analyzing the Principal's best responses in the zero-sum game is just an unconstrained information design problem. These best responses might have some robust features, as demonstrated in this paper, which then will be the features of the equilibrium as well as the solution for the original optimization problem.

What is the Principal's stopping rule that induces the optimal binary distribution over scores? This rule is characterized by two lines with intercepts equal to the two scores and

[5]This problem is similar to a Bayesian persuasion problem with a binary state space (see Kamenica and Gentzkow (2011)). If the state space is binary, the space of posteriors is one-dimensional, just like the set of scores in our model.

[6]These conditions are satisfied if, for example, the Agent's utility exhibits constant absolute risk aversion or constant relative risk aversion with coefficient greater than one-half.

slopes equal to the target effort level. The Principal stops observing the diffusion when it hits one of these lines. This rule appears to be similar to an $(s, S)$ policy familiar from dynamic models of lumpy decisions; see, for example, Clark and Scarf (1960). Indeed, for a given wage scheme, which is a mapping from scores to monetary compensations, the Principal's optimal stopping problem is similar to a standard dynamic optimization problem where the state variable is the score except that it is subject to the Agent's incentive constraint. Unfortunately, this is an ex ante constraint and depends on the entire distribution over scores resulting from the implemented stopping rule. Let us explain how the economic reasoning behind the derivation of the $(s, S)$ policy can be applied despite this ex ante constraint. One can Lagrangianize the constraint and replace the Principal's objective function with the integrand of the corresponding Lagrangian function. For each Lagrange multiplier, this new problem becomes a standard stochastic dynamic optimization problem which can be solved using the Bellman approach. Of course, the solution to this problem is binary: the Principal continues to acquire information as long as his value from doing so exceeds the value of his objective function, and stops otherwise. The problem is that the Lagrange multiplier corresponding to the Agent's incentive constraint depends on the solution and, typically, it is different from the one used in the Principal's redefined objective function. In other words, solving the optimization problem with a given multiplier leads to a new multiplier. So, in order to argue that this approach delivers a solution to the Principal's original problem, one must show that this procedure has a fixed point, that is, the two multipliers coincide. This last result follows from the equilibrium existence of our zero-sum game.

We establish an additional result, which holds under different conditions on the Agent's utility function. We show that there exists a sequence of binary distributions and single-bonus wage schemes, which approximates the first-best outcome arbitrarily closely. In other words, the Principal's payoff in the limit is the same as it would be if effort was contractible. A contract in the sequence pays the Agent a base wage, plus a large bonus with a small probability. Intuitively, the condition on the Agent's utility function is satisfied if the Agent is not too risk-averse, and so it is not too expensive to motivate him with a large wage that he receives with a small probability. As an example, this condition is satisfied if the Agent's utility exhibits constant relative risk aversion with coefficient less than one-half.

How general is our main result regarding single-bonus contracts? As mentioned above, our assumption that the marginal cost of information is constant enables us to transform the Principal's problem to an information design problem where the cost of a distribution is its variance. Alternatively, we could have started with the information design problem where the Principal chooses a distribution over scores. We will argue that as long as the cost of a distribution is a general convex moment of the distribution, our main theorem holds, that is, the optimal wage scheme is binary.[7] From this viewpoint, modeling the Principal's information acquisition with a diffusion can be considered as a micro-foundation for specifying the cost of a distribution as a moment.

We emphasize that our main result regarding single-bonus contracts is, at least partially, due to the Principal's ability to design the monitoring structure in a flexible way. In our setup, this is achieved by allowing the stopping time to depend on the already observed path. If, instead, one considers a less flexible, parametric class of monitoring structures,

[7]Note that the information design problem corresponds to the relaxed incentive-compatibility constraint. Therefore, to validate the first-order approach, one must make assumptions on the cost of effort as well as on the probabilities of the optimal scores conditional on each effort.

the optimal contract is unlikely to feature a binary wage scheme. For example, assume that the Principal can observe a normal signal around the Agent's effort at a cost equal to its variance. This would be the case in our model if the stopping rule was restricted to be deterministic and independent of the path of the diffusion. Then, no matter what variance the Principal chooses, the range of her signal will be a continuum and each signal will determine a different wage.

*Related Literature.* First and foremost, this paper is related to the literature on Principal–Agent problems under moral hazard. In the seminal work of Mirrlees (1976) and Holmström (1979), a Principal contracts with a risk-averse Agent. The Principal has access to a contractible signal that is informative about the Agent's effort. The authors characterize the wage contract that maximizes the Principal's profit subject to the Agent's incentive compatibility and participation constraints. Extensions of this model include settings in which the performance measure is not contractible, the Agent's effort is multidimensional and some tasks are easier to measure than others, or the Principal and the Agent interact repeatedly; see Holmström (2017) for a comprehensive treatment. Unlike our paper, this literature typically treats the Principal's signal as free and exogenous.

Dye (1986) analyzed a Principal–Agent model in which, after observing a (costless) signal that is informative of the Agent's effort, the Principal can acquire an additional costly signal. It was shown that, under certain conditions, the Principal acquires the additional signal only if the value of the first signal is sufficiently low.[8] Feltham and Xie (1994) and Datar, Kulp, and Lambert (2001) examined how a set of available performance measures should be weighed in an optimal linear wage scheme. It was shown, for example, that it may be optimal to ignore informative signals of effort.[9] Hoffmann, Inderst, and Opp (2019) studied a model in which the Principal observes signals over time that are informative of the Agent's one-shot effort, and designs a deferred compensation scheme. Deferring compensation enables the Principal to obtain more accurate information and thus reduce agency costs, but because the Agent is less patient than the Principal, doing so entails a cost. Their model is meant to capture a situation in which information about the Agent's effort unfolds over time. In contrast, ours is static and aims to capture a cycle of an employer-worker relationship in which the latter exerts effort during the cycle, followed by a performance evaluation, the outcome of which determines his pay. Li and Yang (forthcoming) considered a game in which the Agent's hidden action generates a signal, and the Principal chooses a partition of the state-space (at a cost that increases in the fineness of the partition) and a wage scheme that specifies the Agent's wage conditional on the cell of the partition in which the signal lies. Their main result shows that the optimal partition comprises convex cells in the space of likelihood ratios. Khalil and Lawarrée (1995) considered a model with moral hazard and adverse selection in which payments to the Agent can be conditioned either on his effort, or on his output. It was shown that the Principal prefers the former if and only if she is the residual claimant of output.

[8]See also Townsend (1979), Baiman and Demski (1980), and Kim and Suh (1992).

[9]Note that Holmström's *informativeness principle*, which asserts that any signal that is informative of the Agent's action should be incorporated into the optimal contract, does not apply if the Principal is restricted to linear contracts. Relatedly, Kim (1995) showed that if, for a given level of effort, one distribution over scores is a mean-preserving spread of another, then the former implements this level of effort at a lower expected cost than the latter.

There are some papers that attempt to rationalize single-bonus contracts.[10] For instance, Oyer (2000) analyzed a static Principal–Agent model where the Agent is risk-neutral and the wage is restricted to be increasing in output. The author showed that the optimal wage scheme is binary if the score is a hump-shaped function of output. The reason is that, due to the restriction to monotone wages, the Agent receives the minimum wage, plus a fixed bonus if the output is larger than the one that maximizes the score. Palomino and Prat (2003), Levin (2003), and Herweg, Muller, and Weinschenk (2010) also considered various settings where the Agent has linear preferences for money. As a consequence, the Principal's optimal contract solves a linear programming problem, so it is a corner solution. Then, imposing bounds on the minimum and maximum wages delivers the optimality of single-bonus contracts. In contrast, we consider a risk-averse Agent and impose only the limited liability constraint.

Like our paper, Boleslavsky and Kim (2018) also employed information design techniques to solve a model with hidden actions. The authors considered a Bayesian persuasion problem (without transfers) where the sender determines the distribution over states by taking a costly action. Using techniques similar to Doval and Skreta (2018), they showed that there is an equilibrium where the support of the receiver's signal is either binary or trinary. Similarly, we show that, even if the Agent's utility function violates the hypothesis of our main theorem, there is an optimal contract which induces either binary or trinary distribution over scores.[11]

Finally, the way we model information acquisition is reminiscent of Bolton and Harris (1999) and Morris and Strack (2017) in that the Principal decides when to stop observing a diffusion process. In these models, the Principal uses her observations and Bayes's rule to learn about the drift of the process. In contrast, owing to the Agent's incentive constraint, in equilibrium, the Principal knows the Agent's effort and hence the drift of the process. However, she must commit to acquire information to provide the Agent with incentives to exert the target level of effort.

## 2. MODEL

There is a Principal (she) and an Agent (he). The Agent exerts effort $a \in \mathbb{R}_+$ at cost $c(a)$. The Agent's choice of effort is unobservable, but it generates a diffusion $X_t$ with drift $a$, that is, $dX_t = a\,dt + dB_t$, where $B_t$ is a standard Brownian motion with the corresponding canonical probability space $(\Omega, B, P)$ and $B_0 = 0$.[12] The Principal acquires information about the Agent's effort by observing this process. Information acquisition is costly and the Principal's cost is $t$ if she chooses to observe this process until time $t$. The Agent's payoff is $u(w) - c(a)$, and the Principal's cost is $w + t$ if she pays the Agent wage $w$. The function $u$ is strictly increasing, strictly concave, and $\lim_{w\to\infty} u'(w) = 0$. The function $c$ is strictly increasing and strictly convex. Both $u$ and $c$ are twice differentiable.

The Principal can commit to a path-contingent stopping rule and a path-contingent wage scheme. To be more precise, a *contract* is a pair $(\tau, W)$, where $\tau$ is a stopping time and $W$ is a mapping from *paths* to wages. Formally, $\tau : \Omega \to \mathbb{R}_+$ is a stopping time of the

[10] Other papers provide conditions under which linear contracts are optimal; see, for example, Holmström and Milgrom (1987), Carroll (2015), and Barron, Georgiadis, and Swinkels (2020).

[11] A related but distinct literature studies information design problems under asymmetric information; see, for example, Bergemann and Pesendorfer (2007), Carrasco, Luz, Kos, Messner, Monteiro, and Moreira (2018), and Brooks and Du (2018).

[12] In particular, $\Omega = C([0, \infty))$.

filtration generated by $B_t$, and $W : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ is a measurable function. If the Principal stops information acquisition at time $t$ and observes the path $\boldsymbol{\omega}_t = \{\omega_i\}_{i\le t}$, then the Agent receives wage $W(\boldsymbol{\omega}_t)$.[13] We assume that the Agent has limited liability, that is, the Principal faces a minimum wage constraint, $W \ge \underline{w}$, where $\underline{w} > -\infty$ and $u'(\underline{w}) < \infty$.[14]

The game played by the Principal and the Agent proceeds as follows. First, the Principal offers a contract. After observing this contract, the Agent chooses an effort level.[15] Then the Principal acquires information and pays the wage according to the offered contract. The Principal's objective is to induce the Agent to exert a target level of effort, $a^*$. Our goal is to characterize the contract that achieves this objective at the lowest expected cost. Formally, we analyze the following constrained optimization problem:

$$\inf_{\tau, W} \mathbb{E}_{a^*}\big[W(\boldsymbol{\omega}_\tau) + \tau\big] \tag{1}$$

$$\text{s.t. } a^* \in \arg\max_a \mathbb{E}_a\big[u\big(W(\boldsymbol{\omega}_\tau)\big)\big] - c(a), \tag{2}$$

$$W(\boldsymbol{\omega}_\tau) \ge \underline{w}. \tag{3}$$

If $a^* = 0$, then the contract $(\tau, W) \equiv (0, \underline{w})$ solves this problem, so we assume that $a^* > 0$. If this problem has a solution and its value is finite, then each optimal stopping rule has finite expectation. In what follows, we restrict attention to stopping rules, $\tau$, such that $\mathbb{E}_{a^*}[\tau] < \infty$. Our main theorem implies that the value of this problem is indeed finite and hence, this restriction is made without loss of generality.

## 3. REFORMULATING THE PRINCIPAL'S PROBLEM

This section accomplishes the following three goals:

First, we consider a relaxed version of the Principal's problem in which we replace the incentive constraint with the first-order condition corresponding to the Agent's optimal choice of effort. Then we show that the only determinant of the wage, the so-called *score*, is the value of the Brownian motion at the stopping time of the Principal. In other words, if the Principal stops acquiring information at time $\tau$ and observes the path $\boldsymbol{\omega}_\tau$, then she pays a wage that depends only on $\omega_\tau - a^*\tau$. Note that, since the Agent exerts effort $a^*$ in equilibrium, this quantity is just the realization of $B_\tau (= X_\tau - a^*\tau)$.

As will be explained, each stopping rule of the Principal results in a different distribution of the score with mean zero. Our second objective is to argue that the reverse is also true: the Principal can induce any zero-mean distribution over the scores by appropriately choosing her stopping rule. In addition, the Principal's expected cost is the variance of the distribution. This observation allows us to rewrite the Principal's problem as a flexible information design problem, where the Principal can choose a distribution over scores (instead of determining the stopping rule).

Finally, we analyze the optimal wage scheme for any distribution over scores. In particular, we consider the Lagrangian corresponding to the Principal's information design problem, characterize the optimal wage as a function of the dual multiplier, and show that strong duality holds.

[13]Note that $\boldsymbol{\omega}_t$ is a realization of the path of $X$ until $t$.

[14]This assumption rules out the possibility that the Principal can approximate the first-best outcome by observing the process $X_t$ for an arbitrarily short duration and offering a Mirrlees "shoot the Agent" contract (Bolton and Dewatripont (2005)).

[15]All results continue to hold if the contract must also satisfy a participation constraint. We omit it for simplicity.

### 3.1. *The Score*

Our first objective is to show that, despite the Principal observing the path of the diffusion, the wage of the Agent depends only on the last value of the path. To this end, first observe that the Principal's problem can be decomposed into two parts: finding an optimal stopping rule and determining the wage scheme given the stopping time. In other words, the optimal wage structure minimizes the Principal's cost (subject to incentive compatibility and limited liability) for the optimal stopping rule. In this section, we take the Principal's information acquisition as given and describe some properties of the optimal wage.

In order to better explain the derivation of the score in our setting, let us briefly recall how this object is derived in standard Principal–Agent models with continuous effort. Consider a model where the Agent's effort $a$ generates an observable output distribution defined by the cumulative distribution function (hereafter CDF) $G_a$ and the corresponding probability distribution function (hereafter pdf) $g_a$. A standard approach in the literature is to solve the doubly-relaxed problem where the incentive-compatibility constraint is replaced by the weaker condition,

$$\int u\big(w(z)\big)\frac{\partial g_a(z)}{\partial a}\bigg\rfloor_{a=a^*} dz \geq c'\big(a^*\big);$$

see, for example, Rogerson (1985). This condition guarantees that the Agent prefers to exert $a^*$ to a *local* downward deviation. The Principal's Lagrangian with this weaker constraint becomes

$$\int\left[w(z) - \lambda u\big(w(z)\big)\frac{\partial g_a(z)/\partial a\rfloor_{a=a^*}}{g_{a^*}(z)}\right] dG_{a^*}(z) + \lambda c'\big(a^*\big).$$

Note that, in addition to the endogenously determined $w(z)$, the integrand only depends on $(\partial g_a(z)/\partial a\rfloor_{a=a^*})/g_{a^*}(z)$. Therefore, pointwise maximization of the Lagrangian yields that the optimal wage only depends on the derivative of the log-density. This quantity is often referred to as the (Fisher) *score*.

To see, intuitively, why the score is the value of the Brownian motion at the stopping time in our setting, assume for simplicity that the Principal observes the diffusion up to time $t$, that is, $\tau \equiv t$. If the Agent exerts effort $a$, then the density corresponding to the last observation, $z$, is $g_a(z) = e^{-(z-at)^2/2t}/\sqrt{2\pi t}$. Note that $\partial g_a(z)/\partial a = (z - at)g_a(z)$. So, if the Principal is restricted to making the wage dependent only on the last observation, the score becomes $z - a^*t$. Can the Principal benefit from additional observations? Suppose that the Principal makes the wage dependent on the last observation, as well as the observation at $t/2$, $x$. Then $g_a(x, z) = [e^{-(x-at/2)^2/t}/\sqrt{\pi t}][e^{-(z-x-at/2)^2/t}/\sqrt{\pi t}]$ and $\partial g_a(x, z)/\partial a = (z - at)g_a(x, z)$. Therefore, the score is still $z - a^*t$. In other words, even if the Principal could make the wage depend on her observation at $t/2$ in addition to her last observation, she chooses not to do so.

In our model, given a stopping time, the Principal's observation about the Agent's effort is not a finite-dimensional object and cannot be described by a pdf. Nevertheless, Girsanov's theorem characterizes a Radon–Nikodym derivative of the measure generated by $a^*$ over the Principal's observations with respect to the measure generated by any other $a$. This enables us to express the Agent's deviation payoff for each effort $a$ in terms of the measure generated by $a^*$. More precisely, Girsanov's theorem implies that if the

Agent exerts effort $a$, then her payoff is

$$\mathbb{E}_{a^*}\big[u\big(W(\boldsymbol{\omega}_\tau)\big)e^{(a-a^*)B_\tau-\frac{1}{2}(a-a^*)^2\tau}\big]-c(a), \tag{4}$$

where the expectation is taken according to the measure over $\Omega$ generated by $a^*$. Differentiating this expression with respect to $a$, and evaluating it at $a=a^*$, we obtain the following relaxed incentive-compatibility constraint:

$$\mathbb{E}_{a^*}\big[u\big(W(\boldsymbol{\omega}_\tau)\big)B_\tau(\boldsymbol{\omega}_\tau)\big]\geq c'\big(a^*\big). \tag{5}$$

Using arguments similar to the ones explained in the previous paragraph, we show that for any stopping rule $\tau$, it is without loss of generality to condition wages only on $B_\tau$, or equivalently, on the score $s_\tau := X_\tau - a^*\tau$.

LEMMA 1: *Fix a stopping rule $\tau$, and consider the relaxed constrained optimization problem given by* (1), (3), *and* (5). *In an optimal contract, the Agent's wage only depends on* $s_\tau$.

PROOF: See the Supplemental Material (Georgiadis and Szentes (2020)). *Q.E.D.*

In what follows, we characterize the solution to the relaxed problem, where the incentive-compatibility constraint is replaced by (5). In Section 7, we provide conditions under which the solution to the original problem coincides with this relaxed one.

### 3.2. *Flexible Information Design*

Each stopping rule generates a probability distribution over scores. We aim to rewrite the Principal's problem so that her choice set is a class of distributions over scores instead of the set of stopping rules. To this end, we next characterize the set of distributions which can be generated by a stopping time.

Let $F_\tau$ denote the CDF over scores generated by the stopping time $\tau$. The following lemma shows that if the stopping time, $\tau$, has finite expectation, then $F_\tau$ has zero mean and finite variance.

LEMMA 2: *Let $\tau$ be a stopping rule such that $\mathbb{E}_{a^*}[\tau]<\infty$. Then*

$$F_\tau \in \mathcal{F} = \big\{F\in\Delta(\mathbb{R}):\mathbb{E}_F[s]=0, \mathbb{E}_F\big[s^2\big]<\infty\big\}.$$

PROOF: See the Supplemental Material. *Q.E.D.*

A question that arises now is which distributions can be generated by some stopping rule and what is the corresponding cost. This is known as the Skorokhod embedding problem. The following lemma asserts that the Principal can generate *any* distribution over scores in $\mathcal{F}$ by choosing an appropriate stopping time. Furthermore, the Principal's expected cost is the variance of the distribution. The following lemma is due to Root (1969) (Theorem 2.1) and Rost (1976) (Theorem 2).

LEMMA 3: *For all $F\in\mathcal{F}$,*

(i) *there exists a stopping time $\tau$ such that $F_\tau = F$ and $\mathbb{E}_{a^*}[\tau]=\mathbb{E}_F[s^2]$, and*

(ii) *if $F_{\tau'}=F$ for stopping time $\tau'$, then $\mathbb{E}_{a^*}[\tau]\leq\mathbb{E}_{a^*}[\tau']$.*

The previous two lemmas allow us to reformulate the Principal's problem as an information design problem. Formally,

$$\inf_{F\in\mathcal{F},\widetilde{W}} \mathbb{E}_F\big[\widetilde{W}(s)+s^2\big] \tag{Obj}$$

$$\text{s.t. } \mathbb{E}_F\big[su\big(\widetilde{W}(s)\big)\big]\geq c'\big(a^*\big), \tag{IC}$$

$$\widetilde{W}(s)\geq \underline{w} \quad \text{for all } s\in\mathbb{R}. \tag{LL}$$

If $\widetilde{W}$ was set to be an optimal wage scheme, then finding the optimal $F$ in the previous problem becomes a pure information design problem. Of course, the optimal distribution must still satisfy the two constraints, (IC) and (LL). We intend to use standard techniques in information design developed to analyze *unconstrained* optimization problems. Therefore, our next goal is to eliminate the constraints by considering the corresponding Lagrangian function.

### 3.3. *Optimal Wages and Strong Duality*

As mentioned above, the Principal's problem can be decomposed into two parts: finding an optimal distribution over scores and determining the wage scheme given this distribution. This section focuses on the second part: for each $F$, we characterize the wage scheme that minimizes the Principal's cost subject to (local) incentive compatibility. Formally, for all $F\in\mathcal{F}$, we consider

$$\begin{aligned} &\inf_{\widetilde{W}} \quad \mathbb{E}_F\big[\widetilde{W}(s)+s^2\big] \\ &\text{s.t.} \quad \text{(IC) and (LL).} \end{aligned} \tag{6}$$

Note that this is a standard Principal–Agent problem as in Holmström (1979), except that above, we have a limited-liability constraint instead of an individual-rationality constraint, and $F$ is a probability distribution over scores instead of outputs. Let $\Pi(F)$ denote the value of this problem.

The Lagrangian function corresponding to this problem can be written as

$$L(\lambda,F)=\inf_{\widetilde{W}(\cdot)\geq\underline{w}}\int\big[\widetilde{W}(s)-\lambda su\big(\widetilde{W}(s)\big)+s^2\big]\,dF(s)+\lambda c'\big(a^*\big), \tag{7}$$

where $\lambda\geq 0$ is the dual multiplier associated with (IC). To solve this problem, note that the first-order condition corresponding to the pointwise minimization of the integrand is $\lambda su'(w)=1$. If the solution of this equation, $w$, is larger than $\underline{w}$, then this $w$ is the optimal wage at $s$. Otherwise, the optimal wage is $\underline{w}$. To summarize, the wage scheme that minimizes the value of the integral on the right-hand side of (7) is defined by the following equation:

$$w(\lambda,s)=\begin{cases}\underline{w} & \text{if } s\leq s_*(\lambda),\\ u'^{-1}\left(\dfrac{1}{\lambda s}\right) & \text{if } s>s_*(\lambda),\end{cases} \tag{8}$$

where $s_*(\lambda)$ is the critical score at which the solution of the first-order condition is exactly $\underline{w}$, that is, $s_*(\lambda)=1/[\lambda u'(\underline{w})]$. The following lemma shows that strong duality always holds. Moreover, if the incentive constraint, (IC), binds at $w(\lambda,\cdot)$ for some $\lambda\geq 0$, then this wage scheme is uniquely optimal.

LEMMA 4: *For all* $F \in \mathcal{F}$,

(i) *strong duality holds*; *that is*, $\sup_{\lambda\in\mathbb{R}_+} L(\lambda, F) = \Pi(F)$, *and*

(ii) *the problem in* (6) *has a solution if and only if there exists a unique* $\widehat{\lambda} \in \mathbb{R}_+$ *such that* $L(\widehat{\lambda}, F) = \Pi(F)$. *Furthermore*, $\widehat{\lambda}$ *satisfies*

$$\int su\big(w(\widehat{\lambda}, s)\big)\, dF(s) = c'\big(a^*\big),$$

*where* $w(\widehat{\lambda}, s)$ *is given in* (8). *This wage scheme uniquely solves* (6).

PROOF: See the Supplemental Material. *Q.E.D.*

The proof of the uniqueness of the solution to (6) in part (ii) follows the logic of that of Proposition 1 in Jewitt, Kadan, and Swinkels (2008). The difference is that wages are specified as a function of output in their setting, whereas in our model, they depend on the realized score.

Recall that throughout this section, we fixed the distribution over the scores, $F$, and characterized the corresponding optimal wage scheme. Of course, the Principal also chooses this distribution to minimize her cost, that is, she solves $\inf_{F\in\mathcal{F}} \Pi(F)$. Part (i) of this lemma enables us to rewrite this problem as

$$\inf_{F\in\mathcal{F}} \sup_{\lambda\in\mathbb{R}_+} L(\lambda, F). \tag{9}$$

It turns out to be difficult to characterize the $F$ that solves this problem. The reason is that there is a different $\lambda$ corresponding to each possible $F$, and hence, it is hard to identify the change in $\sup_{\lambda\in\mathbb{R}_+} L(\lambda, F)$ due to a change in $F$. In the next section, we show that solving the corresponding sup inf problem is simpler, and we investigate the circumstances under which the two problems are equivalent.

## 4. THE ZERO-SUM GAME

Our next objective is to define a zero-sum game and show that, if there exists an equilibrium in this game, then the inf sup problem in (9) is equivalent to

$$\sup_{\lambda\in\mathbb{R}_+} \inf_{F\in\mathcal{F}} L(\lambda, F). \tag{10}$$

We are able to characterize the solution to this sup inf problem. Indeed, for any $\lambda$, $\inf_{F\in\mathcal{F}} L(\lambda, F)$ is an information design problem akin to that in Kamenica and Gentzkow (2011). We show that for any $\lambda$, if $\inf_{F\in\mathcal{F}} L(\lambda, F)$ has a solution, then there is an optimal $F$ that is either a *two-point distribution* (i.e., its support has two elements), or $F$ is degenerate, specifying an atom of size 1 at zero.

In what follows, we first formally define the zero-sum game. Then, using arguments from the theory of zero-sum games, we prove that equilibrium existence implies the equivalence of (9) and (10). Finally, we explain how *two-point distributions* arise as best-responses in this game. This last observation is crucial to our main result according to which the optimal wage scheme is binary.

*The Game.*—There are two players, the Principal and Nature. The action space of the Principal is $\mathcal{F}$, and the action space of Nature is $\mathbb{R}_+$. Furthermore, Nature's payoff is $L(\lambda, F)$. That is, the Principal chooses a probability distribution $F \in \mathcal{F}$ to minimize $L(\lambda, F)$, whereas Nature chooses the dual multiplier $\lambda \in \mathbb{R}_+$ to maximize $L(\lambda, F)$.

The following lemma shows that a Nash equilibrium of this game corresponds to a solution to both (9) and (10).

LEMMA 5: *Suppose that $\{\lambda^*, F^*\}$ is a Nash equilibrium in the zero-sum game defined above. Then*

$$\sup_{\lambda \geq 0} \inf_{F \in \mathcal{F}} L(\lambda, F) = \inf_{F \in \mathcal{F}} \sup_{\lambda \geq 0} L(\lambda, F),$$

*and $w(\lambda^*, \cdot)$ and $F^*$ solve the problem in* (Obj).

The equality in the lemma follows from von Neumann's minimax theorem; see von Neumann (1928). This theorem also implies that both sides of the equality are $L(\lambda^*, F^*)$. To explain the last part of the statement, note that part (ii) of Lemma 4 implies that $w(\lambda^*, \cdot)$ solves (6) with $F = F^*$ and, since $L(\lambda^*, F^*) = \inf_{F \in \mathcal{F}} \sup_{\lambda \geq 0} L(\lambda, F)$, it follows that $w(\lambda^*, \cdot)$ and $F^*$ solve (Obj).

PROOF: See the Supplemental Material. *Q.E.D.*

One may ponder if it is easier to establish the equivalence between (9) and (10) by applying a minimax theorem such as von Neumann's or Sion's. This does not appear to be the case, because minimax theorems require that at least one of the choice sets satisfies some notion of compactness (see, e.g., Simons (1995)). In our setting, these choice sets are $\mathbb{R}_+$ and $\mathcal{F}$, violating compactness. Moreover, as will be demonstrated, both the support of $F^*$ and $\lambda^*$ can be arbitrarily large. Consequently, neither of these objects can be restricted to be in a compact set.

*Two-Point Distribution.*—Next, we argue that the Principal's best response is either a two-point distribution or the degenerate distribution placing an atom of size 1 at zero. Furthermore, we argue that the latter case cannot arise in equilibrium. To this end, recall from (7) that the payoffs can be expressed as an expectation, that is,

$$L(\lambda, F) = \mathbb{E}_F\big[Z(\lambda, s)\big], \quad \text{where } Z(\lambda, s) = w(\lambda, s) - \lambda\big[su\big(w(\lambda, s)\big) - c'\big(a^*\big)\big] + s^2, \tag{11}$$

and $w(\lambda, s)$ is defined by (8). Then the problem of finding the Principal's best response against $\lambda$ can be written as

$$\inf_{F \in \mathcal{F}} \mathbb{E}_F\big[Z(\lambda, s)\big].$$

The solution to this problem can be characterized as follows by using standard arguments in information design (see Aumann and Perles (1965), and Kamenica and Gentzkow (2011)). First, let $Z^c(\lambda, \cdot)$ denote the convexification of $Z(\lambda, \cdot)$ in $s$; that is,

$$Z^c(\lambda, s) = \inf_{\underline{s}, \overline{s} \in \mathbb{R}, \alpha \in [0,1] \text{ s.t. } \alpha \underline{s} + (1-\alpha)\overline{s} = s} \big\{\alpha Z(\lambda, \underline{s}) + (1-\alpha) Z(\lambda, \overline{s})\big\}. \tag{12}$$

Note that for any $F \in \mathcal{F}$,

$$\mathbb{E}_F\big[Z(\lambda, s)\big] \geq \mathbb{E}_F\big[Z^c(\lambda, s)\big] \geq Z^c(\lambda, 0),$$

where the first inequality follows because $Z(\lambda, s) \geq Z^c(\lambda, s)$, and the second one follows from Jensen's inequality and $\mathbb{E}_F[s] = 0$. This inequality implies that $Z^c(\lambda, 0)$ is a lower

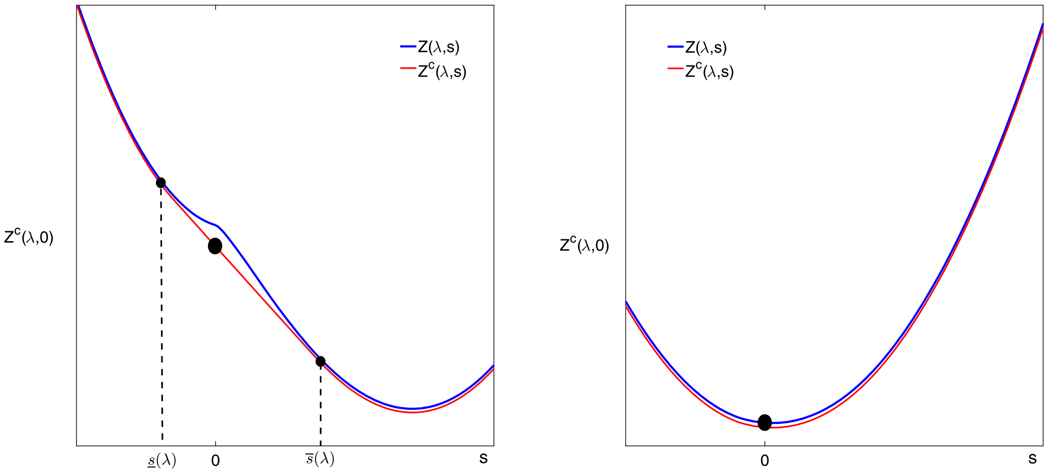


FIGURE 1.—Illustration of the Principal's best response.

bound on the Principal's payoff. Next, we explain that the Principal can achieve this bound by considering the following two cases.

If $Z(\lambda, 0) > Z^c(\lambda, 0)$, then the point $(0, Z^c(\lambda, 0))$ lies on the line segment defining $Z^c$ on the non-convex region around 0, as illustrated in Figure 1(a).[16] The point $(0, Z^c(\lambda, 0))$ is a convex combination of $(\underline{s}, Z(\lambda, \underline{s}))$ and $(\overline{s}, Z(\lambda, \overline{s}))$ for some $\underline{s} < 0 < \overline{s}$, that is, there exists $\alpha \in (0, 1)$ such that

$$\alpha\big(\underline{s}, Z(\lambda, \underline{s})\big) + (1 - \alpha)\big(\overline{s}, Z(\lambda, \overline{s})\big) = \big(0, Z^c(\lambda, 0)\big). \tag{13}$$

Consider now the probability distribution, $\widehat{F}$, defined by the weights in this convex combination over $\{\underline{s}, \overline{s}\}$, that is, an atom of size $\alpha$ at $\underline{s}$ and an atom of size $(1 - \alpha)$ at $\overline{s}$. Equation (13) implies that $\alpha\underline{s} + (1 - \alpha)\overline{s} = 0$, which means that $\widehat{F}$ is feasible for the Principal, that is, $\widehat{F} \in \mathcal{F}$. Equation (13) also implies that

$$\alpha Z(\lambda, \underline{s}) + (1 - \alpha) Z(\lambda, \overline{s}) = \mathbb{E}_{\widehat{F}}\big[Z(\lambda, s)\big] = Z^c(\lambda, 0),$$

which means that the lower bound, $Z^c(\lambda, 0)$, is attained by the distribution $\widehat{F}$. Therefore, $\widehat{F}$ is a best response of the Principal.

If $Z^c(\lambda, 0) = Z(\lambda, 0)$, then the lower bound can be trivially attained by the degenerate distribution that places probability only on zero, as illustrated in Figure 1(b). However, this latter case cannot arise in equilibrium. To see this, suppose that the distribution $F$ is degenerate and specifies an atom of size 1 at zero. Then, by (8), $w(\lambda, 0) = \underline{w}$, and hence, Nature's payoff becomes $\underline{w} + \lambda c'(a^*)$ by (11). Since $c'(a^*) > 0$, this quantity is strictly increasing in $\lambda$, and therefore, Nature does not have a best response. This, in turn, implies that the degenerate distribution cannot arise in an equilibrium.

The observation that the Principal has a two-point distribution best response against Nature's equilibrium multiplier is the key observation for our main result. Indeed, any

[16] To be precise, the infimum need not be attained by any distribution. In this case, the Principal achieves $Z^c(\lambda, 0)$ only as a limit of two-point distributions.

two-point distribution corresponds to an information-acquisition strategy of the Principal that generates only two possible values of the score. As a consequence, the Agent receives only two possible wages. Note, however, that the argument in the previous paragraphs by no means implies that the Principal does not have a best response that is supported on more than two points. Of course, it is possible that the line connecting the points $(\underline{s}, Z(\lambda, \underline{s}))$ and $(\overline{s}, Z(\lambda, \overline{s}))$ contains other points $(s, Z(\lambda, s))$. In such a scenario, the point $(0, Z^c(\lambda, 0))$ is also a convex combination of (weakly) more than three such points on the line, and each of these convex combinations defines a best response of the Principal. We shall provide conditions on the Agent's utility function under which best responses supported on more than two points do not exist.

To summarize, in order to show that the Principal's optimal contract specifies two possible wages, we need to prove that (i) both the Principal's and Nature's best responses are unique, and (ii) an equilibrium in our zero-sum game exists.

## 5. THE MAIN THEOREM

Our main theorem provides sufficient conditions on the Agent's utility function for the existence of a unique equilibrium in the zero-sum game described in the previous section. In order to state these conditions, let $\rho(z) = u(u'^{-1}(1/z))$ for all $z$. This function is familiar from the literature on moral hazard problems with continuous effort. Indeed, one of the sufficient conditions guaranteeing that the first-order approach is valid (i.e., that the global incentive constraint can be replaced by a local one) is that the function $\rho$ is concave (see Jewitt (1988)). This assumption ensures that the Agent's indirect utility function is concave in the score. To see this, note that, by (8), the Agent's indirect utility from $s$, $u(w(\lambda, s))$, is $\rho(\lambda s)$ if $s > s_*(\lambda)$ and it is $\rho(1/u'(\underline{w}))$ otherwise. Therefore, the Agent's indirect utility is concave on $[1/u'(\underline{w}), \infty)$ if and only if $\rho$ is concave on the same interval. We make slightly stronger assumptions by requiring the function $\rho$ to be strictly concave and to satisfy the Inada condition at infinity. Formally, we state the following:

ASSUMPTION 1:
(i) *$\rho$ is strictly concave on* $[1/u'(\underline{w}), \infty)$, *and*
(ii) $\lim_{z\to\infty} \rho'(z) = 0$.

Finally, we are ready to state the main theorem of the paper.

THEOREM 1: *Suppose that the Agent's utility function, $u$, satisfies Assumption 1. Then there exists a unique equilibrium $(\lambda^*, F^*) \in \mathbb{R}_+ \times \mathcal{F}$ in the zero-sum game, and $F^*$ is a two-point distribution.*

Let us explain the implication of this theorem to the original contracting problem. By Lemma 5, the equilibrium $(\lambda^*, F^*)$ defines the solution for the constrained information design problem in (Obj). That is, the optimal distribution over scores is $F^*$, and the wage scheme is $w(\lambda^*, \cdot)$. By Lemma 3, this wage scheme and the optimal stopping rule generating $F^*$ over the scores solve the Principal's relaxed problem (1) subject to (3) and (5). According to this theorem, $F^*$ is a two-point distribution. Let $\{\underline{s}, \overline{s}\}$ denote its support such that $\underline{s} < 0 < \overline{s}$. Then the Principal's optimal information-acquisition strategy can be defined by the stopping rule in which she observes the diffusion process $X_t = at + B_t$ until the first time it hits $a^*t + \underline{s}$ or $a^*t + \overline{s}$. Thus, the value of the Brownian motion is either $\underline{s}$ or $\overline{s}$ at the stopping time, and the Principal observes two possible scores. Finally, the Principal pays the Agent $\underline{w}$ if she observes $\underline{s}$, and pays him $w(\lambda^*, \overline{s})$ if she observes $\overline{s}$. In other words, the Agent receives a base wage of $\underline{w}$ and bonus of $w(\lambda^*, \bar{s}) - \underline{w}$ if the information gathered is favorable, which is just a single-bonus contract.[17] We state this result formally in the following:

COROLLARY 1: *Suppose that $u$ satisfies Assumption 1. Then there is a unique contract, $(\tau^*, W^*)$, which solves the problem in (1) subject to (3) and (5). The stopping rule $\tau^*$ is defined by*

$$\tau^* = \min_t \{X_t = a^* t + \underline{s} \text{ or } X_t = a^* t + \bar{s}\},$$

*where $\{\underline{s}, \bar{s}\} = \operatorname{supp}(F^*)$. The wage scheme $W^*$ is defined by*

$$W^*(\boldsymbol{\omega}_{\tau^*}) = \begin{cases} \underline{w} & \text{if } \omega_{\tau^*} = a^*\tau^* + \underline{s}, \\ w(\lambda^*, \bar{s}) & \text{if } \omega_{\tau^*} = a^*\tau^* + \bar{s}. \end{cases}$$

The formal proof of Theorem 1 is presented in Section 5.1. Here, we only provide a sketch of the proof. Before doing so, we discuss how restrictive Assumption 1 is by examining whether it is satisfied by familiar parametric classes of utility functions. To this end, we first observe that part (i) of Assumption 1 is equivalent to $[u']^3/u''$ being strictly increasing.[18] Similarly, part (ii) of Assumption 1 is equivalent to $\lim_{w\to\infty}[u'^3(w)/u''(w)] = 0$.

*Commonly Used Utility Function.*—Consider first utility functions exhibiting constant absolute risk aversion (CARA), that is,

$$u(w) = -e^{-\alpha w}, \tag{14}$$

where $\alpha (> 0)$ is the coefficient of absolute risk aversion. In this case, $[u'(w)]^3/u''(w) = -\alpha e^{-\alpha w}$, so Assumption 1 is satisfied for all $\alpha$.

Suppose now that the Agent's utility function exhibits constant relative risk aversion (CRRA), that is,

$$u(w) = \frac{w^{1-\gamma}}{1-\gamma}, \tag{15}$$

where $\gamma \in (0, 1)$ is the coefficient of relative risk aversion. In this case, $[u'(w)]^3/u''(w) = -[w^{1-2\gamma}]/\gamma$. Therefore, Assumption 1 is satisfied if and only if $\gamma > 1/2$. What happens if $\gamma < 1/2$? Observe that the Principal's cost is bounded from below by $\underline{w}$. In Section 6.2, we show that the Principal can induce the Agent to exert $a^*$ at the minimum cost of $\underline{w}$. To be more precise, for any $\varepsilon > 0$, we construct an incentive-compatible single-bonus contract such that the Principal's payoff from this contract is less than $\underline{w} + \varepsilon$.

If the Agent has a logarithmic utility function, $u(w) = \log w$, then $[u'(w)]^3/u''(w) = -1/w$, so Assumption 1 is satisfied. More generally, if the Agent's utility function exhibits hyperbolic absolute risk aversion (HARA), and so is of the form

$$u(w) = \frac{\gamma}{1-\gamma}\left(\frac{\alpha w}{\gamma} + \beta\right)^{1-\gamma}, \tag{HARA}$$

then Assumption 1 is satisfied if $\alpha > 0$, $\gamma > 1/2$, and $\beta > -\alpha\underline{w}/\gamma$.

[17] Note that, in equilibrium, the Agent always exerts the target effort level, $a^*$. Nevertheless, both his wage and the Principal's information acquisition cost are random variables.

[18] To see this, note that, denoting $u'^{-1}(1/z)$ by $f(z)$, $\rho'(z)$ can be expressed as $-[u'(f(z))]^3/u''(f(z))$. Since $f$ is strictly increasing, $\rho'$ is strictly decreasing if and only if $[u']^3/u''$ is strictly increasing.

*Proof-Sketch.*—Let us first explain the last statement of the theorem, that is, provided that an equilibrium exists, the Principal's equilibrium distribution is a two-point distribution. Applying the convexification argument described in the previous section, we show that the Principal's best response is either a two-point distribution or the degenerate one. This is the part of the proof where we use Assumption 1. As we will show, a consequence of this assumption is that the function $Z(\lambda, \cdot)$ looks like either the one depicted in Figure 1(a) or the one in Figure 1(b). More precisely, if $\lambda$ is large, then the function $Z(\lambda, \cdot)$ is convex-concave-convex, whereas otherwise, this function is convex. This observation implies that the Principal's best response is essentially unique, and there are no best-response distributions of the Principal that are supported on at least three points. Since the equilibrium distribution cannot be degenerate, this implies that if an equilibrium exists, then it is a two-point distribution.

Let us now explain the two main steps of the equilibrium-existence result in Theorem 1. The first step is to characterize some properties of Nature's best responses. Using standard arguments from Lagrangian optimization, we show that, unless $F$ is the degenerate distribution, the best response $\lambda$ against $F$ is defined by the incentive constraint (IC), that is, Nature chooses the multiplier so that the incentive constraint binds. The second step is to show that if $\lambda$ is small, then the incentive constraint evaluated at the Principal's best response is violated. In contrast, if $\lambda$ is large, then the incentive constraint evaluated at the Principal's best response is slack. Then we use the intermediate value theorem to conclude that there exists a unique $\lambda^*$ at which the incentive constraint at the Principal's best response, say $F^*$, binds. As mentioned above, Nature's best response is characterized precisely by this binding incentive constraint. Therefore, $\lambda^*$ is also the best response against $F^*$.

### 5.1. *Proof of Theorem 1*

Towards proving Theorem 1, we establish a series of lemmas, which enable us to construct a unique equilibrium in the zero-sum game described in Section 4.

#### 5.1.1. *Nature's Best Response*

First, we show that Nature best-responds to a distribution $F$ by choosing $\lambda$ so that the Agent's incentive constraint (IC) binds, that is,

$$\int su\big(w(\lambda, s)\big)\, dF(s) = c'\big(a^*\big). \tag{16}$$

Formally, we state the following:

LEMMA 6:
(i) *If* (16) *has a solution*, *then it is unique and defines Nature's best response*, $\lambda_F$.
(ii) *If* (16) *does not have a solution*, *then Nature does not have a best response*.

Regarding part (ii), we point out that Nature has no best response if her objective function is strictly increasing in $\lambda$, so she can improve on any $\lambda$ by choosing a larger one. This is the case, for example, if the Principal's distribution is degenerate; that is, $F(s) = \mathbb{I}_{\{s\geq 0\}}$. It can be shown that if $F$ is not the degenerate distribution and $\lim_{w\to\infty} u(w) = \infty$, then (16) always has a solution.

PROOF: Recall that if the Principal chooses $F$, then Nature's problem is

$$\sup_{\lambda\geq 0}\mathbb{E}_F\big[Z(\lambda,s)\big], \tag{17}$$

where $Z(\lambda,s)$ is defined in (11). Since the wage scheme $w(\lambda,\cdot)$ is optimally chosen for $\lambda$ (see (8)), the Envelope Condition implies that

$$\frac{\partial\mathbb{E}_F\big[Z(\lambda,s)\big]}{\partial\lambda}=-\int su\big(w(\lambda,s)\big)\,dF(s)+c'\big(a^*\big). \tag{18}$$

Note that the first-order condition corresponding to this derivative is just (16).

The second-order condition corresponding to (17) is

$$\frac{\partial^2\mathbb{E}_F\big[Z(\lambda,s)\big]}{\partial\lambda^2}=\int_{s_*(\lambda)}^{\infty}s^2\frac{\big[u'\big(w(\lambda,s)\big)\big]^3}{u''\big(w(\lambda,s)\big)}\,dF(s)\leq 0,$$

where the inequality follows from $u$ being strictly increasing and strictly concave. Notice that the inequality is strict whenever $F(s_*(\lambda))<1$, that is, $s>s_*(\lambda)$ with positive probability.

The previous displayed inequality implies that Nature's objective function, $\mathbb{E}_F[Z(\lambda,s)]$, is concave in $\lambda$. Therefore, if (17) has an interior solution, then it is defined by the first-order condition (16). This completes the proof of part (i).

To see part (ii), note that

$$\begin{aligned}\lim_{\lambda\to 0}\frac{\partial\mathbb{E}_F\big[Z(\lambda,s)\big]}{\partial\lambda}&=-\lim_{\lambda\to 0}\int su\big(w(\lambda,s)\big)\,dF(s)+c'\big(a^*\big)\\&=u(\underline{w})\int s\,dF(s)+c'\big(a^*\big)=c'\big(a^*\big),\end{aligned}$$

where the second equality follows from the facts that $w(\lambda,s)$ is increasing in $\lambda$ and $\lim_{\lambda\to 0}w(\lambda,s)=\underline{w}$ for all $s$ and from Lebesgue's monotone convergence theorem. The last equality follows from $F\in\mathcal{F}$, that is, $F$ has zero expectation. The previous equation implies that Nature's objective function, $\mathbb{E}_F[Z(\lambda,s)]$, is strictly increasing in $\lambda$ at zero. This means that if (16) does not have a solution, Nature's objective function is strictly increasing for all $\lambda\geq 0$, and hence, Nature has no best response. *Q.E.D.*

### 5.1.2. *The Principal's Best Response*

This section is devoted to the characterization of the Principal's best response. Recall that for a given $\lambda$, the Principal's problem is $\inf_{F\in\mathcal{F}}\mathbb{E}_F[Z(\lambda,s)]$. As mentioned above, this is an information design problem, and we solve it by using standard convexification arguments. Of course, the solution depends on the shape of the function $Z(\lambda,\cdot)$. The next lemma shows that if Assumption 1 is satisfied, then this function is strictly convex for small values of $\lambda$, whereas it is convex-concave-convex for larger values of $\lambda$. Then we show that, in the former case, the Principal's best response is degenerate, placing all the probability mass at zero, and in the latter case, the best response is a binary distribution.

In this section, we maintain Assumption 1.

LEMMA 7: *There exists a* $\lambda_c>0$ *such that*

(i) *if* $\lambda \leq \lambda_c$, *then* $Z(\lambda, s)$ *is strictly convex in* $s$, *and*
(ii) *if* $\lambda > \lambda_c$, *then there exists a* $\widetilde{s} > s_*(\lambda)$ *such that*[19]

$$Z_{22}(\lambda, s) = \begin{cases} > 0 & \text{if } s < s_*(\lambda), \\ < 0 & \text{if } s_*(\lambda) < s < \widetilde{s}, \text{ and} \\ > 0 & \text{if } s > \widetilde{s}. \end{cases} \tag{19}$$

Let us explain the proof of this lemma and the role Assumption 1 plays in it. From (11), note that the sum of the first two terms of $Z(\lambda, s)$, $w(\lambda, s) - \lambda[su(w(\lambda, s)) - c'(a^*)]$, reflects the cost of compensating the Agent (given a realization of $s$), while the third term, $s^2$, reflects the cost of information acquisition. Using the Envelope Condition, it is not hard to show that the sum of the first two terms is concave and the last term is convex in $s$. If $\lambda$ is small, then the last term dominates and $Z$ is convex in $s$. If $\lambda$ is large and $|s|$ is small, the concavity of the sum of the first two terms also implies the concavity of $Z$ in $s$. As will be shown, Assumption 1 ensures that as $|s|$ becomes larger, the $s^2$ term will dominate and $Z$ becomes convex in $s$.

PROOF: Suppose first that $s < s_*(\lambda)$. Then $Z_2(\lambda, s) = -\lambda u(\underline{w}) + 2s$, and

$$Z_{22}(\lambda, s) = 2 > 0,$$

yielding the convexity of $Z(\lambda, s)$ for $s \leq s_*(\lambda)$ and, in particular, the first line of (19). If $s \geq s_*(\lambda)$, then $Z_2(\lambda, s) = -\lambda u(w(\lambda, s)) + 2s$ and

$$Z_{22}(\lambda, s) = \lambda^2 \frac{\left[u'\big(w(\lambda, s)\big)\right]^3}{u''\big(w(\lambda, s)\big)} + 2. \tag{20}$$

We argue that $Z_{22}(\lambda, s)$ is strictly increasing in $s$. To see this, recall that part (i) of Assumption 1 is equivalent to $[u']^3/u''$ being strictly increasing. Furthermore, $w(\lambda, s)$ is also strictly increasing in $s$.

Observe that at $s = s_*(\lambda)$, this second derivative is

$$Z_{22}(\lambda, s_*) = \lambda^2 \frac{\left[u'(\underline{w})\right]^3}{u''(\underline{w})} + 2.$$

Let us define $\lambda_c$ by

$$\lambda_c = \sqrt{\frac{-2u''(\underline{w})}{\left[u'(\underline{w})\right]^3}},$$

and note that $Z_{22}(\lambda_c, s_*(\lambda_c)) = 0$.

If $\lambda \leq \lambda_c$, then $Z_{22}(\lambda, s_*(\lambda)) \geq 0$. Since $Z_{22}(\lambda, s)$ is increasing in $s$ if $s \geq s_*(\lambda)$, the statement of the lemma follows.

If $\lambda > \lambda_c$, then $Z_{22}(\lambda, s_*(\lambda)) < 0$. As we mentioned before, $Z_{22}(\lambda, s)$ is strictly increasing in $s$. Recall that part (ii) of Assumption 1 is equivalent to $\lim_{w\to\infty}[u'^3(w)/u''(w)] = 0$.

[19]We use $Z_i(\lambda, s)$ and $Z_{ii}(\lambda, s)$ to denote the first and second derivative of $Z(\lambda, s)$ with respect to its $i$th argument, respectively.

Since $w(\lambda, s)$ goes to infinity as $s$ goes to infinity, we conclude that $Z_{22}(\lambda, s)$ is positive if $s$ is large; see (20). Since $Z_{22}(\lambda, s)$ is strictly increasing and continuous in $s$ at $s > s_*(\lambda)$, there exists a unique $\widetilde{s}$ at which $Z_{22}(\lambda, \widetilde{s}) = 0$. *Q.E.D.*

We will argue that the value of the Principal's problem, $\inf_{F\in\mathcal{F}} \mathbb{E}_F[Z(\lambda, s)]$, is just the convexified $Z(\lambda, \cdot)$ evaluated at $s = 0$ (see Figure 1). Recall that this convexification, $Z^c(\lambda, \cdot)$, is defined by (12). If $\lambda \leq \lambda_c$, then, by part (i) of Lemma 7, $Z(\lambda, \cdot)$ is convex, so $Z^c(\lambda, s) = Z(\lambda, s)$ for all $s$. If $\lambda > \lambda_c$, then, by part (ii) of Lemma 7, $Z$ is convex-concave-convex. The function $Z^c(\lambda, \cdot)$ is linear on an interval around the concave region of $Z(\lambda, \cdot)$, and otherwise, it coincides with $Z(\lambda, \cdot)$, as illustrated in the left panel of Figure 1. Formally, there exist $\underline{s}(\lambda)$ and $\overline{s}(\lambda) \geq \underline{s}(\lambda)$ such that

$$Z^c(\lambda, s) = \begin{cases} Z(\lambda, s) & \text{if } s \notin \big[\underline{s}(\lambda), \overline{s}(\lambda)\big], \\ Z\big(\lambda, \underline{s}(\lambda)\big) + \big(s - \underline{s}(\lambda)\big) Z_2\big(\lambda, \underline{s}(\lambda)\big) & \text{if } s \in \big[\underline{s}(\lambda), \overline{s}(\lambda)\big]. \end{cases} \tag{21}$$

Next, we show that the Principal's best response is degenerate if either $\lambda \leq \lambda_c$ or if $Z$ is not affected by the convexification around zero; that is, $0 \notin [\underline{s}(\lambda), \overline{s}(\lambda)]$. Otherwise, the Principal's best response is a binary distribution.

LEMMA 8: *For any* $\lambda(\geq 0)$, $\min_{F\in\mathcal{F}} \mathbb{E}_F[Z(\lambda, s)] = Z^c(\lambda, 0)$. *In addition, the Principal's best response*, $F_\lambda$, *is unique, and*:

(i) *If* $Z^c(\lambda, 0) = Z(\lambda, 0)$, *then* $F_\lambda(s) = \mathbb{I}_{\{s\geq 0\}}$.

(ii) *If* $Z^c(\lambda, 0) < Z(\lambda, 0)$, *then* $\operatorname{supp}(F_\lambda) = \{\underline{s}(\lambda), \overline{s}(\lambda)\}$.

Observe that in part (ii), the Principal's best response can be explicitly expressed in terms of its support as follows:

$$F_\lambda(s) = \begin{cases} 0 & \text{if } s < \underline{s}(\lambda), \\ \dfrac{\overline{s}(\lambda)}{\overline{s}(\lambda) - \underline{s}(\lambda)} & \text{if } s \in [\underline{s}(\lambda), \overline{s}(\lambda)), \\ 1 & \text{if } s \geq \overline{s}(\lambda). \end{cases} \tag{22}$$

This CDF has two jumps, at $\underline{s}(\lambda)$ and at $\overline{s}(\lambda)$, so only these two points occur with positive probabilities. The size of each jump is determined by the requirement that $F_\lambda$ has zero expectation. Observe that since $Z^c(\lambda, 0) < Z(\lambda, 0)$ in this case, it must be that $\underline{s}(\lambda) < 0 < \overline{s}(\lambda)$.

PROOF: Fix some $\lambda \geq 0$. By construction, $Z(\lambda, s) \geq Z^c(\lambda, s)$, so for all $F \in \mathcal{F}$, we have $\mathbb{E}_F[Z(\lambda, s)] \geq \mathbb{E}_F[Z^c(\lambda, s)] \geq Z^c(\lambda, \mathbb{E}_F[s]) = Z^c(\lambda, 0)$, where the last inequality follows from the fact that $Z^c(\lambda, \cdot)$ is convex and from Jensen's inequality. Therefore, $Z^c(\lambda, 0)$ poses a lower bound on $\mathbb{E}_F[Z(\lambda, s)]$ for any $F \in \mathcal{F}$.

Part (i) follows trivially by noting that if $Z^c(\lambda, 0) = Z(\lambda, 0)$, then $\mathbb{E}_F[Z(\lambda, s)] = Z^c(\lambda, 0)$ for $F(s) = \mathbb{I}_{\{s\geq 0\}}$. If $Z^c(\lambda, 0) < Z(\lambda, 0)$, then it follows from Lemma 7 and the definition of $Z^c(\lambda, \cdot)$ that there exist $s_L, s_H$ and $p \in (0, 1)$ such that $(1 - p)s_L + ps_H = 0$ and $Z^c(\lambda, 0) = (1 - p)Z(\lambda, s_L) + pZ(\lambda, s_H) = \mathbb{E}_{F_\lambda}[Z(\lambda, s)]$, where $F_\lambda$ is of the form given in (22) with $\overline{s}(\lambda) = s_H$ and $\underline{s}(\lambda) = s_L$. Thus, we have shown that $\min_{F\in\mathcal{F}} \mathbb{E}_F[Z(\lambda, s)] = Z^c(\lambda, 0)$ and established part (ii). *Q.E.D.*

The following lemma shows that $\underline{s}(\lambda)$ and $\overline{s}(\lambda)$ are continuous in $\lambda$, and both $\underline{s}(\lambda)$ and $\overline{s}(\lambda)$ converge to $s_*(\lambda_c)$ as $\lambda$ goes to $\lambda_c$. These results will be useful for establishing the existence of an equilibrium.

LEMMA 9: *Assume that* $\lambda > \lambda_c$. *Then*:

(i) *The functions* $\underline{s}(\lambda), \overline{s}(\lambda)$ *are continuous in* $\lambda$.

(ii) *Consider a sequence* $\{\lambda_n\}_{n\in\mathbb{N}} > \lambda_c$ *such that* $\lim_{n\to\infty} \lambda_n = \lambda_c$. *Then* $\lim_{n\to\infty} \underline{s}(\lambda_n) = \lim_{n\to\infty} \overline{s}(\lambda_n) = s_*(\lambda_c)$.

PROOF: See the Supplemental Material. *Q.E.D.*

### 5.1.3. *Equilibrium Existence and Uniqueness*

This section proves Theorem 1. First, we show that if $\lambda$ is large enough, then the Principal best-responds by choosing a distribution, $F_\lambda$, such that the Agent's incentive constraint is slack at $(\lambda, F_\lambda)$. Then we consider the infimum of such $\lambda$'s, $\lambda^*$, and we show that the incentive constraint binds at $(\lambda^*, F_{\lambda^*})$. Therefore, we can use Lemma 6 to conclude that $\lambda^*$ is a best response to $F_{\lambda^*}$. Since $F_{\lambda^*}$ is a best response to $\lambda^*$, the action profile $(\lambda^*, F_{\lambda^*})$ is an equilibrium. To prove uniqueness, we show that the incentive constraint cannot bind at $(\lambda, F_\lambda)$ unless $\lambda = \lambda^*$. Then uniqueness follows from Lemma 6.

To this end, note that for any given $\lambda$, the constraint (IC) evaluated at the optimal wage scheme defined by (8) and the Principal's best-response, $F_\lambda$, can be written as

$$\int su\big(w(\lambda, s)\big)\, dF_\lambda(s) \geq c'\big(a^*\big). \tag{23}$$

The following lemma shows that (23) is slack if $\lambda$ is sufficiently large.

LEMMA 10: *There exists a* $\Lambda > 0$ *such that the constraint* (23) *at* $(\lambda, F_\lambda)$ *is slack whenever* $\lambda > \Lambda$.

The proof of this lemma is based on considering binary distributions supported on $\{-s, s\}$ for some $s \in \mathbb{R}_+$. We show that if both $\lambda$ and $s$ are sufficiently large, then the Principal's payoff generated by such a distribution is less than $-\underline{w}$. Consequently, the Principal's best response to such a $\lambda$ must also induce a payoff of less than $-\underline{w}$. Then the statement follows from the observation that the Principal's payoff is bounded by $-\underline{w}$ whenever the constraint (23) is not slack.

PROOF: See the Supplemental Material. *Q.E.D.*

Next, we define a threshold value of $\lambda$ above which the incentive-compatibility constraint is slack at the Principal's best-response:

$$\lambda^* = \inf\big\{\lambda : (23) \text{ is slack}\big\}. \tag{24}$$

The following lemma shows that (23) binds at $\lambda^*$.

LEMMA 11: *If Nature chooses* $\lambda^*$, *then the constraint* (23) *binds at* $(\lambda^*, F_{\lambda^*})$.

The logic is as follows: Recall that $Z(\lambda, \cdot)$ is strictly convex if $\lambda \leq \lambda_c$, and so the constraint (23) at $(\lambda, F_\lambda)$ is violated, whereas it is slack for $\lambda > \Lambda$ (see Lemmas 7 and 10). Moreover, for $\lambda > \lambda_c$, the left-hand side of (23) is continuous in $\lambda$ by Lemma 9. We show that by continuity, the constraint (23) binds at $\lambda^*$.

PROOF: See the Supplemental Material. *Q.E.D.*

Recall from Lemma 8 that whenever $Z^c(\lambda, 0) < Z(\lambda, 0)$, the Principal's best response, $F_\lambda$, is binary and is supported on $\{\underline{s}(\lambda), \overline{s}(\lambda)\}$. So, we can rewrite (23) as

$$\underline{p}(\lambda)\underline{s}(\lambda)u(\underline{w}) + \overline{p}(\lambda)\overline{s}(\lambda)u\big(w\big(\lambda, \overline{s}(\lambda)\big)\big) \geq c'\big(a^*\big), \tag{IC'}$$

where $\underline{p}(\lambda)$ and $\overline{p}(\lambda)$ denote the probability that $\overline{s}(\lambda)$ and $\underline{s}(\lambda)$ are realized, respectively, according to $F_\lambda$. By, (22), $\underline{p}(\lambda) = \overline{s}(\lambda)/(\overline{s}(\lambda) - \underline{s}(\lambda))$ and $\overline{p}(\lambda) = 1 - \underline{p}(\lambda)$. Again by Lemma 8, if $Z^c(\lambda, 0) = Z(\lambda, 0)$, then $F_\lambda$ is degenerate and the left-hand side of (23) is zero. So, by setting $\underline{p}(\lambda) = \overline{p}(\lambda) = 0$, (IC') coincides with (23).

Finally, we are in a position to prove Theorem 1.

PROOF OF THEOREM 1: If Nature chooses $\lambda^*$, then by Lemmas 7 and 8, the Principal's unique best response $F_{\lambda^*}$ is the two-point distribution as given in (22). It remains to show that Nature's best response to $F_{\lambda^*}$ (or equivalently $\{\underline{s}(\lambda^*), \overline{s}(\lambda^*)\}$) is to choose $\lambda^*$. If the Principal chooses $\{\underline{s}(\lambda^*), \overline{s}(\lambda^*)\}$, then Nature's problem is

$$\max_{\lambda \in \mathbb{R}_+} \big\{\underline{p}\big(\lambda^*\big)Z\big(\lambda, \underline{s}\big(\lambda^*\big)\big) + \overline{p}\big(\lambda^*\big)Z\big(\lambda, \overline{s}\big(\lambda^*\big)\big)\big\}.$$

This problem is concave in $\lambda$, and the corresponding first-order condition is

$$\overline{p}\big(\lambda^*\big)\overline{s}\big(\lambda^*\big)u\big(w\big(\lambda, \overline{s}\big(\lambda^*\big)\big)\big) + \underline{p}\big(\lambda^*\big)\underline{s}\big(\lambda^*\big)u(\underline{w}) = c'\big(a^*\big),$$

which is satisfied at $\lambda = \lambda^*$ by Lemma 11. Therefore, $\{\lambda^*, F_{\lambda^*}\}$ is an equilibrium for the zero-sum game described in Section 4, where $\lambda^*$ is given in (24) and $F_\lambda$ is given in (22).

Towards showing that this equilibrium is unique, first, recall that by Lemma 8, for any $\lambda$, the Principal's best response $F_\lambda$ is unique. Therefore, in any equilibrium, say $\{\lambda', F_{\lambda'}\}$, (IC') must bind. Moreover, by the definition of $\lambda^*$ in (24), it must be the case that $\lambda' < \lambda^*$. We will show that there does not exist any $\lambda' < \lambda^*$ such that (IC') binds.

First, we show that the Principal's payoff is strictly increasing in $\lambda$ on $[\lambda_c, \lambda^*]$. Pick any $\lambda$ and $\lambda'$ such that $\lambda_c < \lambda < \lambda' \leq \lambda^*$. Then

$$\begin{aligned}
&\overline{p}(\lambda)\big[w\big(\lambda, \overline{s}(\lambda)\big) - \lambda\big[su\big(w\big(\lambda, \overline{s}(\lambda)\big)\big) - c'\big(a^*\big)\big] + \overline{s}^2(\lambda)\big] \\
&\quad + \underline{p}(\lambda)\big[\underline{w} - \lambda\big[su(\underline{w}) - c'\big(a^*\big)\big] + \underline{s}^2(\lambda)\big] \\
&< \overline{p}\big(\lambda'\big)\big[w\big(\lambda', \overline{s}\big(\lambda'\big)\big) - \lambda\big[su\big(w\big(\lambda', \overline{s}\big(\lambda'\big)\big)\big) - c'\big(a^*\big)\big] + \overline{s}^2\big(\lambda'\big)\big] \\
&\quad + \underline{p}\big(\lambda'\big)\big[\underline{w} - \lambda\big[su(\underline{w}) - c'\big(a^*\big)\big] + \underline{s}^2\big(\lambda'\big)\big] \\
&\leq \overline{p}\big(\lambda'\big)\big[w\big(\lambda', \overline{s}\big(\lambda'\big)\big) - \lambda'\big[su\big(w\big(\lambda', \overline{s}\big(\lambda'\big)\big)\big) - c'\big(a^*\big)\big] + \overline{s}^2\big(\lambda'\big)\big] \\
&\quad + \underline{p}\big(\lambda'\big)\big[\underline{w} - \lambda'\big[su(\underline{w}) - c'\big(a^*\big)\big] + \underline{s}^2\big(\lambda'\big)\big],
\end{aligned} \tag{25}$$

where the first inequality follows from the fact that the second expression corresponds to the Principal's payoff if Nature chooses $\lambda$ but the Principal uses the suboptimal information-acquisition policy $\{\underline{s}(\lambda'), \overline{s}(\lambda')\}$ instead of $\{(\underline{s}(\lambda), \overline{s}(\lambda))\}$ and pays the suboptimal wage $w(\lambda', \overline{s}(\lambda'))$ instead of $w(\lambda, \overline{s}(\lambda))$. This inequality is strict because $w(\lambda', \overline{s}(\lambda')) \neq w(\lambda, \overline{s}(\lambda))$. The second inequality follows from $\lambda < \lambda'$ and the fact that (IC') is not slack if $\lambda' \leq \lambda^*$ (see 24).

Suppose, by contradiction, that there exists a $\lambda < \lambda^*$ such that (IC') binds at $\lambda$, and let $\lambda' \in (\lambda, \lambda^*]$. It must be that $\lambda > \lambda_c$, for otherwise (IC') would have been violated. Then

$$\begin{aligned}
&\overline{p}(\lambda')\big[w(\lambda', \overline{s}(\lambda')) - \lambda'\big[su\big(w(\lambda', \overline{s}(\lambda'))\big) - c'(a^*)\big] + \overline{s}^2(\lambda')\big] \\
&\quad + \underline{p}(\lambda')\big[\underline{w} - \lambda'\big[su(\underline{w}) - c'(a^*)\big] + \underline{s}^2(\lambda')\big] \\
&\leq \overline{p}(\lambda)\big[w(\lambda, \overline{s}(\lambda)) - \lambda'\big[su\big(w(\lambda, \overline{s}(\lambda))\big) - c'(a^*)\big] + \overline{s}^2(\lambda)\big] \\
&\quad + \underline{p}(\lambda)\big[\underline{w} - \lambda'\big[su(\underline{w}) - c'(a^*)\big] + \underline{s}^2(\lambda)\big] \\
&= \overline{p}(\lambda)\big[w(\lambda, \overline{s}(\lambda)) - \lambda\big[su\big(w(\lambda, \overline{s}(\lambda))\big) - c'(a^*)\big] + \overline{s}^2(\lambda)\big] \\
&\quad + \underline{p}(\lambda)\big[\underline{w} - \lambda'\big[su(\underline{w}) - c'(a^*)\big] + \underline{s}^2(\lambda)\big],
\end{aligned}$$

where the inequality follows from the fact that the second expression corresponds to the Principal's payoff if Nature chooses $\lambda'$ but the Principal uses the suboptimal information-acquisition policy $\{\underline{s}(\lambda), \overline{s}(\lambda)\}$ instead of $\{(\underline{s}(\lambda'), \overline{s}(\lambda'))\}$ and pays the suboptimal wage $w(\lambda, \overline{s}(\lambda))$ instead of $w(\lambda', \overline{s}(\lambda'))$ and the equality follows from the hypothesis that (IC') binds at $\lambda$. Finally, note that this inequality chain contradicts (25). Therefore, we conclude that there does not exist any $\lambda' < \lambda^*$ such that (IC') binds, which completes the proof. *Q.E.D.*

## 6. FURTHER EXTENSIONS

Assumption 1 played an important role in proving that an equilibrium exists in the zero-sum game defined in Section 4 and the support of the Principal's equilibrium distribution is binary. A natural question to ask is: What happens if the Agent's utility function does not satisfy Assumption 1? To provide a partial answer to this question, this section accomplishes the following two goals. First, we show that, as long as an equilibrium exists and hence, an optimal contract also exists, there must be an equilibrium which features a distribution with either binary or trinary support. Second, we provide a different assumption under which equilibria in the zero-sum game do not exist. However, we construct a sequence of binary wage-contracts which achieves the first-best outcome in the limit, that is, satisfying the Agent's incentive constraint, (5), comes for free. We will also argue that this alternative assumption is satisfied if the Agent has CRRA utility with coefficient less than 1/2.

### 6.1. *Binary or Trinary Wages*

In this section, we do not make any assumption on the Agent's utility function in addition to those stated in Section 2.

THEOREM 2: *Suppose that $(\lambda^*, F_1^*)$ is an equilibrium of the zero-sum game. Then there exists an $F_2^* \in \mathcal{F}$ such that*

(i) $(\lambda^*, F_2^*)$ *is also an equilibrium in the zero-sum game*, *and*
(ii) $|\operatorname{supp}(F_2^*)| \in \{2, 3\}$.

PROOF: See the Supplemental Material. *Q.E.D.*

Let us provide an explanation for this result. The convexification argument implies that, since the distribution $F_1^*$ is a best response to $\lambda^*$, the function $Z(\lambda^*, \cdot)$ must coincide with the line defining its convexification around zero at each point in $\operatorname{supp}(F_1^*)$. In fact, any distribution in $\mathcal{F}$ whose support is a subset of $\operatorname{supp}(F_1^*)$ is also a best response to $\lambda^*$. Therefore, we only need to argue that there is such a distribution with either binary or trinary support such that $\lambda^*$ is a best response to it. In order to do so, we first show that the function $Z(\lambda^*, \cdot)$ is convex when $s$ is negative and hence, there is exactly one negative score at which $Z(\lambda^*, \cdot)$ coincides with the line defining the convexification of $Z(\lambda^*, \cdot)$ around zero. Consequently, the support of $F_1^*$ contains a single negative score.

It remains to argue that $\lambda^*$ is a best response to a distribution whose support contains this negative score and either one or two positive scores from $\operatorname{supp}(F_1^*)$. Recall from Lemma 6 that Nature's best response is characterized by the binding incentive constraint, (16), and, in particular, it must be satisfied at $(\lambda^*, F_1^*)$. We show that this binding constraint can be written as the *average* of incentive constraints corresponding to binary distributions whose support consists of the negative score and a positive score in the support of $F_1^*$. If there is such a binary distribution at which the incentive constraint binds, then $\lambda^*$ is a best response to it and the proof is complete. Otherwise, there must be a binary distribution at which the constraint is violated and another one at which the constraint is slack. Then there is a convex combination of these two binary distributions at which the incentive constraint binds. Furthermore, since the smaller scores in the supports of these distributions are the same, the support of this convex combination is trinary.

### 6.2. *A First-Best Result*

If effort was contractible, then the optimal contract would require the Agent to exert effort $a^*$ in exchange for a wage $\underline{w}$. Of course, the Agent exerts $a^*$ and the Principal does not acquire any information. The Principal's cost and the Agent's payoff would be $\underline{w}$ and $u(\underline{w}) - c(a^*)$, respectively. We refer to this outcome as first-best.

The following theorem describes a condition on the Agent's utility function under which a single-bonus contract can approximate the first-best outcome arbitrarily well.

THEOREM 3: *Suppose that there exists a* $\zeta > 1$ *such that*

$$\lim_{w\to\infty} \frac{\left[u'(w)\right]^3}{u''(w)}\left[u(w)\right]^{-\frac{\zeta-1}{\zeta}} = -\infty. \tag{26}$$

*Then for every* $\epsilon > 0$, *there exists a single-bonus wage scheme and a two-point distribution that satisfy* (IC) *and* (LL), *and the Principal's expected cost is no greater than* $\underline{w} + \epsilon$.

PROOF: See the Supplemental Material. *Q.E.D.*

We point out that Assumption 1 and condition (26) are mutually exclusive. As mentioned before, part (ii) of Assumption 1 is equivalent to $\lim_{w\to\infty}[u'(w)]^3/u''(w) = 0$, whereas condition (26) implies that this limit is minus infinity.

In the proof of this theorem, we show that the binary distribution in the contract that approximates the first-best outcome is defined by a score close to zero, $\underline{s}$ and a very large score, $\overline{s}$. This implies that the Agent is paid the minimum wage $\underline{w}$ with near certainty, whereas with a small probability, the Agent is generously rewarded. More precisely, we construct a sequence of distributions and contracts so that the Principal's information-acquisition cost goes to zero and the expected wage goes to $\underline{w}$. Such contracts can be incentive compatible as long as the Agent is not too risk-averse, so it is not too expensive to motivate him with a large wage that he receives with a small probability. In a sense, this contract is the reverse of the *"Mirrlees shoot the Agent"* contract which prescribes a harsh punishment with a small probability and provides the Agent with a small reward with probability close to 1.

The condition of the theorem, (26), is satisfied if the Agent's utility function is of the form of (HARA) with parameters $\alpha > 0$, $\gamma < 1/2$, and $\beta > -\alpha\underline{w}/\gamma$. In particular, it is satisfied if the Agent has CRRA utility with coefficient less than 1/2, that is, $u(w) = w^{1-\gamma}/(1-\gamma)$ with $\gamma < 1/2$.

## 7. VALIDATING THE FIRST-ORDER APPROACH

Throughout the analysis, we have considered a relaxed problem, in which the Principal restricts attention to discouraging local downward deviations from the target effort $a^*$. In this section, assuming that the Agent's utility function satisfies Assumption 1, we consider the optimal binary distribution over scores and the corresponding optimal wage scheme that satisfies the relaxed incentive-compatibility constraint given in (5), and we provide sufficient conditions such that this contract also satisfies the global incentive-compatibility constraint given in (2).[20]

First, we express the global incentive constraint of the Agent for the optimal binary distribution. We note that given $u$ and $\underline{w}$, the optimal binary distribution characterized based on the first-order approach depends only on $c'(a^*)$. In what follows, we define $\delta^* = c'(a^*)$, and we let $\{\underline{s}(\delta^*), \overline{s}(\delta^*)\}$ denote the support of the optimal distribution such that $\underline{s}(\delta^*) < \overline{s}(\delta^*)$.[21] Next, we compute the probability of each score conditional on any deviation, $a(\neq a^*)$. This distribution is implemented by the stopping time $\tau = \inf\{t : s_t \notin (\underline{s}(\delta^*), \overline{s}(\delta^*))\}$, where $ds_t = (a-a^*)\,dt + dB_t$. Let $p(a, \delta^*)$ denote the probability that $\overline{s}(\delta^*)$ is realized given the Agent's effort $a$. Using Ito's lemma, it is not hard to show that[22]

$$p(a, \delta^*) = \frac{e^{-2(a-a^*)\underline{s}(\delta^*)} - 1}{e^{-2(a-a^*)\underline{s}(\delta^*)} - e^{-2(a-a^*)\overline{s}(\delta^*)}}. \tag{27}$$

[20]In the canonical Principal–Agent model (e.g., Holmström (1979)), to ensure that the first-order approach is valid, it is typically assumed that either the transition probability function that maps each effort level into contractible output is convex in effort, or conditions are imposed on that transition probability function and the Agent's utility function; see Bolton and Dewatripont (2005) and Jewitt (1988) for details. In our setting, this distribution is endogenous, and hence we must impose conditions on the Agent's effort-cost function.

[21]In Section 5.1, $\{\underline{s}(\lambda), \overline{s}(\lambda)\}$ denoted the support of the Principal's best response to $\lambda$. The equilibrium value of the multiplier can be written as a function of $\delta^*$, $\lambda^*(\delta^*)$. Then the equilibrium support is $\{\underline{s}(\lambda^*(\delta^*)), \overline{s}(\lambda^*(\delta^*))\}$. Our somewhat abusive notation suppresses the indirect dependency of $\lambda^*$.

[22]To derive (27), fix an effort level $a \neq a^*$, and let $\overline{q}(s)$ denote the probability that $s_\tau = \overline{s}$ given the current score $s$. Applying Ito's lemma on $ds_t = (a-a^*)\,dt + dB_t$, it follows that $\overline{q}$ satisfies $0 = 2(a-a^*)\overline{q}'(s) + \overline{q}''(s)$ subject to the boundary conditions $\overline{q}(\overline{s}) = 1$ and $\overline{q}(\underline{s}) = 0$. This boundary value problem can be solved analytically, and evaluating its solution at $s_0 = 0$ yields (27). Moreover, it follows from L'Hospital's rule that $(1 - e^{-2(a-a^*)\overline{s}})/(e^{-2(a-a^*)\underline{s}} - e^{-2(a-a^*)\overline{s}}) \to \overline{s}/(\overline{s} - \underline{s})$ as $a \to a^*$, and so (27) corresponds to (22) when $a = a^*$.

Let $\{\underline{w}, \widetilde{W}(\overline{s}(\delta^*))\}$ denote the optimal wage scheme. Then the Agent's problem is

$$\max_a u(\underline{w}) + p(a, \delta^*)\big[u\big(\widetilde{W}\big(\overline{s}\big(\delta^*\big)\big)\big) - u(\underline{w})\big] - c(a). \tag{28}$$

The first-order condition at $a^*$ is

$$p_1\big(a^*, \delta^*\big)\big[u\big(\widetilde{W}\big(\overline{s}\big(\delta^*\big)\big)\big) - u(\underline{w})\big] = c'\big(a^*\big)\big(= \delta^*\big),$$

that is, wages must satisfy $u(\widetilde{W}(\overline{s}(\delta^*))) - u(\underline{w})) = c'(a^*)/p_1(a^*, \delta^*)$. Plugging this into (28), the Agent's global incentive-compatibility constraint can be expressed as

$$a^* \in \arg\max_{a\geq 0}\left\{u(\underline{w}) + p\big(a, \delta^*\big)\frac{c'\big(a^*\big)}{p_1\big(a^*, \delta^*\big)} - c(a)\right\}.$$

This constraint is satisfied if the maximand is single-peaked at $a^*$. Note that the maximand's derivative is $p_1(a, \delta^*)[c'(a^*)/p_1(a^*, \delta^*)] - c'(a)$. So, the first-order approach is valid if

$$\frac{p_1\big(a, \delta^*\big)}{p_1\big(a^*, \delta^*\big)} \geq \frac{c'(a)}{c'\big(a^*\big)} \quad \text{if and only if} \quad a \leq a^*. \tag{29}$$

In what follows, we consider a parametric family of effort-cost functions, $\{c_\kappa\}_{\kappa\in\mathbb{R}_+}$, and show that, if this sequence satisfies certain properties, then $c_\kappa$ satisfies (29) whenever $\kappa$ is large enough. We assume that $c_\kappa$ satisfies our assumptions on the effort-cost function $c$ for each $\kappa$ and $c_\kappa'(a)$ is twice-differentiable in both $a$ and $\kappa$. Next, we state the two key properties.

CONDITION 1: For each $a > 0$, $c_\kappa''(a)/c_\kappa'(a)$ is increasing in $\kappa$ and $\lim_{k\to\infty}[c_\kappa''(a)/c_\kappa'(a)] = \infty$.

Since $c_\kappa$ is convex, its derivative is increasing and the fraction $c_\kappa''(a)/c_\kappa'(a)$ measures the rate of increase. Condition 1 requires this rate to be increasing in $\kappa$ and to be converging to infinity; that is, that the $c_k$'s are progressively *more convex* in the same sense that a utility function with a larger coefficient of absolute risk aversion is more concave.

CONDITION 2: There exist an $\overline{a}, \underline{d}, \overline{d} \in \mathbb{R}_{++}$ such that $c_\kappa'(\overline{a}) \in [\underline{d}, \overline{d}]$ for all $\kappa$.

To illustrate a consequence of this condition, suppose that the target effort level is $\overline{a}$, that is, $a^* = \overline{a}$. Since the marginal cost of effort, $\delta_\kappa^* = c_\kappa'(\overline{a})$, depends on $\kappa$, the Principal's optimal contract also varies with $\kappa$. So, in order to show that $\overline{a}$ is implementable for any large enough $\kappa$, one has to check the incentive compatibility of continuum many contracts. However, the condition requires $\delta_\kappa^* \in [\underline{d}, \overline{d}]$, so this set of contracts is contained in a compact space.

Finally, we are ready to state the main result of this section.

PROPOSITION 1: *Suppose that the sequence of convex effort-cost functions*, $\{c_\kappa\}_{\kappa\in\mathbb{R}_+}$, *satisfies Conditions 1 and 2. Then, for each* $\underline{a} \in (0, \overline{a})$, *there exists a* $K \in \mathbb{R}_{++}$ *such that the first-order approach is valid for any target level of effort* $a^* \in [\underline{a}, \overline{a}]$ *and cost function* $c_\kappa$ *whenever* $\kappa > K$.

PROOF: See the Supplemental Material. *Q.E.D.*

To see how this proposition can be applied, consider the family of cost functions given by $c'_\kappa(a) = a^\kappa$ for each $\kappa \in \mathbb{R}_+$. In this case, $c''_\kappa(a)/c'_\kappa(a) = \kappa/a$ so this family satisfies Condition 1. Moreover, $c'_\kappa(1) = 1$ for all $\kappa$, and hence, Condition 2 is also satisfied with $\overline{a} = \underline{d} = \overline{d} = 1$. Therefore, the proposition implies that the first-order approach is valid for any $a^* \in (0, 1]$ and $c_\kappa$ if $\kappa$ is sufficiently large.[23] Similarly, the family of cost functions, $c'_\kappa(a) = e^{\kappa(a-1)}$, also satisfies the hypothesis of Proposition 1 for $\overline{a} = 1$. To see this, observe that $c''_\kappa(a)/c'_\kappa(a) = \kappa$ so Condition 1 holds. Moreover, $c'_\kappa(1) = 1$ so Condition 2 is also satisfied with $\underline{d} = \overline{d} = 1$.[24]

Let us explain the main steps of the proof of the proposition. In order to guarantee that a contract implements a target effort level, the global incentive constraint must be satisfied for all possible deviations in $\mathbb{R}_+$. The first step of the proof is to show that one can restrict attention to a compact set of deviations. More precisely, suppose that the effort-cost function is given by $c_\kappa$ and the Principal wants to implement $a^* \in (0, \overline{a}]$, so offers the contract defined by the scores $\{\underline{s}(\delta^*_\kappa), \overline{s}(\delta^*_\kappa)\}$ and wages $\{\underline{w}, \widetilde{W}(\overline{s}(\delta^*_\kappa))\}$, where $\delta^*_\kappa = c'_\kappa(a^*)$. We argue that Condition 2 implies the existence of an effort level $\widehat{a}(> \overline{a})$ such that the Agent is better off exerting effort zero than any $a > \widehat{a}$ and, importantly, $\widehat{a}$ depends neither on $\kappa$ nor on $a^*$. In other words, if the incentive constraint is satisfied at zero, then it is also satisfied at any effort level above $\widehat{a}$. Observe first that the Agent's wage is at most $\widetilde{W}(\overline{s}(\delta^*_\kappa))$ if he exerts $a$ and it is at least $\underline{w}$ when his effort level is zero. Hence, the utility gain from exerting a positive effort instead of zero is less than $u(\widetilde{W}(\overline{s}(\delta^*_\kappa))) - u(\underline{w})$, which is bounded since $\delta^* \in (0, \overline{d}]$. In contrast, the cost of exerting $a$ converges to infinity uniformly in $\kappa$ as $a$ goes to infinity. The reason is that Condition 2 and the convexity of $c_\kappa$ imply that $c'_\kappa(a) \geq \underline{d}$ for all $a \geq \overline{a}$, and hence, the cost of exerting $a$ is at least $\underline{d}(a - \overline{a})$. As a consequence, there is indeed an $\widehat{a}$ above which the effort cost is larger than the utility gain relative to zero, irrespective of $\kappa$ and $a^*(\in (0, \overline{a}])$.

The second step is to show that, for a large enough $\kappa$, the condition in (29) with the additional restriction $a \leq \widehat{a}$ is satisfied for any $a^* \in [\underline{a}, \overline{a}]$, with $c \equiv c_\kappa$ and $\delta^* = c'_\kappa(a^*)$. Instead of showing this, we argue that, for a large enough $\kappa$, for each $a^* \in [\underline{a}, \overline{a}]$,

$$\begin{aligned} &\inf_{\delta \in (0, \overline{d}]} \frac{p_1(a, \delta)}{p_1(a^*, \delta)} \geq \frac{c'_\kappa(a)}{c'_\kappa(a^*)} \quad \text{if } a \leq a^* \quad \text{and} \\ &\sup_{\delta \in (0, \overline{d}]} \frac{p_1(a, \delta)}{p_1(a^*, \delta)} \leq \frac{c'_\kappa(a)}{c'_\kappa(a^*)} \quad \text{if } a^* \leq a \leq \widehat{a}. \end{aligned} \tag{30}$$

Since $a^* \in [\underline{a}, \overline{a}]$ and $c'_\kappa(\overline{a}) \leq \overline{d}$ for all $\kappa$, the convexity of $c_\kappa$ implies that $c'_\kappa(a^*) \in (0, \overline{d}]$. Therefore, the conditions in (30) imply the one in (29) with $c = c_\kappa$ and $\delta^* = c'_\kappa(a^*)$.

Towards proving (30), we intend to appeal to the theorem of the maximum and argue that the left-hand sides of the two inequalities are continuous functions of $a$. Unfortunately, the variable $\delta$ does not lie in a compact set because the fraction $p_1(a, \delta)/p_1(a^*, \delta)$ is not defined for $\delta = 0$. However, we demonstrate that this function can be continuously extended to the compact interval $[0, \overline{d}]$ by showing that the limit of this fraction exists and

[23]For example, simulations indicate that if the Agent's utility exhibits CRRA with coefficient $\gamma = 0.7$, $\underline{w} = 0.1$, and $c'_\kappa(a) = a^\kappa$, then the first-order approach is valid for any $a^* \in (0, 1]$ as long as $\kappa \geq 5$.

[24]More generally, Proposition 1 is applicable if the sequence of cost functions is either given by $c'_\kappa(a) = d(a/\overline{a})^\kappa$ or $c'_\kappa(a) = de^{\kappa(a-\overline{a})}$, where $d, \overline{a} \in \mathbb{R}_{++}$.

it is not zero as $\delta$ converges to zero. From this observation, it follows that the left-hand sides of the two inequalities in (30) are continuous in $a$ and bounded away from zero. In order to establish that these inequalities are satisfied for a sufficiently large $\kappa$, we next explore a consequence of Condition 1.

LEMMA 12: *Condition 1 implies that for all $a > \widetilde{a}$, $c'_\kappa(a)/c'_\kappa(\widetilde{a})$ is increasing in $\kappa$ and* $\lim_{\kappa\to\infty}[c'_\kappa(a)/c'_\kappa(\widetilde{a})] = \infty$.

This lemma implies that the right-hand side of the inequalities in (30), $c'_\kappa(a)/c'_\kappa(a^*)$, converges to zero if $a < a^*$ and to infinity if $a > a^*$ as $\kappa$ goes to infinity. Therefore, for all $a^*$ and $a$, there is a critical value of $\kappa$ at which (30) holds. Moreover, the monotonicity property in the statement of Lemma 12 implies that the (30) is also satisfied for any $\kappa$ above this critical value.

The last step of the proof is to show that the aforementioned critical value of $\kappa$ can be chosen independently of $a^*$ and $a$. To guarantee the existence of a uniform threshold, we first show that Lemma 12 implies that, for each $a \neq a^*$, there is an open neighborhood of the point $(a^*, a) \in \mathbb{R}^2_+$, $N^a_{a^*}$, and $K^a_{a^*} \in \mathbb{R}_+$ such that (30) is satisfied for all $(a^*, a) \in N^a_{a^*}$ whenever $\kappa \geq K^a_{a^*}$. Second, we argue that it follows from Condition 1 that there also exists such a neighborhood of $(a^*, a^*)$, $N^{a^*}_{a^*}$. Recall that the set of points, $(a^*, a)$, at which (30) must hold is $[\underline{a} \times \overline{a}] \times [0, \widehat{a}]$, which is compact. Moreover, we have constructed a covering of it consisting of open sets, $\{N^a_{a^*}\}_{a^*,a}$, and hence, there is also a finite covering, say $N^{a_1}_{a^*_1}, \dots, N^{a_n}_{a^*_n}$. Finally, we show that (30) holds whenever $\kappa > \max\{K^{a_1}_{a^*_1}, \dots, K^{a_n}_{a^*_n}.\}$.

## 8. COMPARATIVE STATICS

In this section, we use simulations to investigate how the optimal contract depends on the parameters of the problem.

Figure 2 provides comparative statics when the Agent's utility function exhibits CRRA, and so it is of the form

$$u(w) = \frac{w^{1-\gamma}}{1-\gamma},$$

and we vary the coefficient of relative risk aversion $\gamma$ from 1/2 to 1, while setting $\underline{w} = 0.1$ and $c'(a^*) = 1$. The left panel illustrates the scores in the support of the equilibrium distribution, $\overline{s}$ and $\underline{s}$, while the right panels illustrate the size of the bonus, $w(\lambda^*, \overline{s}) - \underline{w}$, and the probability that it is paid, $\overline{p}(\lambda^*)$, as a function of $\gamma$. As this figure illustrates, if $\gamma$ is close to one-half so the Agent is moderately risk-averse, then the optimal contract specifies an $\underline{s}$ close to zero and a large $\overline{s}$. This implies that the Agent only receives the bonus if the acquired information is overwhelmingly favorable. Of course, this event occurs with only a small probability as shown on the top-right panel of Figure 2. Therefore, in order to motivate the Agent, the bonus must be large, confirmed by the bottom-right panel. Note that the equilibrium contract in this case is similar to the ones used in the proof of Theorem 3: the Agent receives a large bonus with a small probability if he exerts the target effort. Recall that this theorem implies that if $\gamma < 1/2$, then it is possible to approximate the first-best outcome arbitrarily closely using such single-bonus wage schemes.

In contrast, if $\gamma$ is close to 1 so the Agent is very risk-averse, then both $\underline{s}$ and $\overline{s}$ are relatively small (see the left panel of Figure 2). This implies that the Agent receives a bonus with high probability, as shown on the top-right panel. Since the Agent receives the bonus frequently, its size is small. The fact that the optimal contract specifies small

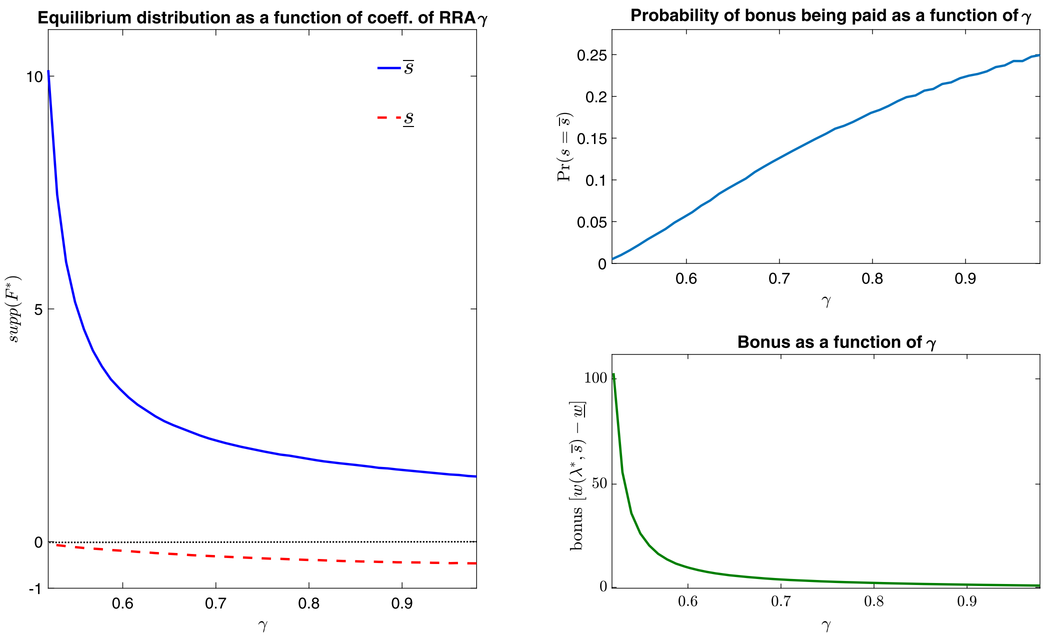


FIGURE 2.—Comparative statics of the optimal contract as the (constant) coefficient of relative risk aversion $\gamma$ varies.

rewards with large probability if $\gamma$ is large is not surprising given that a very risk-averse Agent values income-smoothing more. Note that between the two extreme values of $\gamma$, all the functions are monotone: $\underline{s}$, $\overline{s}$, and the bonus are decreasing, and the probability of the bonus increases. Simulations indicate that these comparative statics are similar when the Agent's utility function exhibits CARA, and we vary the coefficient of absolute risk aversion.

Figure 3 illustrates how the scores in the support of the equilibrium distribution, $\overline{s}$ and $\underline{s}$, the size of the bonus, and the probability that it is paid vary with the minimum wage $\underline{w}$, when the Agent's utility exhibits CRRA or CARA. In the former case, we set $\gamma = 0.9$ and $c'(a^*) = 1$, and vary $\underline{w}$ from 0 to 5. In the latter case, we set the coefficient of absolute risk aversion to 1 and $c'(a^*) = 1$, and vary $\underline{w}$ from $-5$ to 0. In both cases, the size of the bonus increases in the minimum wage. The reason is that, as $\underline{w}$ increases, the marginal utility of the Agent for a given bonus decreases. Therefore, in order to incentivize the Agent to exert the target effort, the Principal must increase the size of the bonus. The comparative statics pertaining to the optimal information-acquisition strategies appear to be quite different in the two cases examined. In particular, if the Agent's utility exhibits CRRA, then the probability of paying the bonus is decreasing in $\underline{w}$, whereas this probability is increasing in the case of CARA utility. The reason is that in the former case, as $\underline{w}$ increases, the Agent becomes less and less risk-averse regarding gambles involving transfers above $\underline{w}$. As a consequence, the optimal contract specifies the familiar small-probability, large-bonus wage scheme. In the case of CARA, such an effect is not present. Recall that $\mathbb{E}_{F^*}[s^2] = -\underline{s}\overline{s}$. In the case of CARA utility, both $\overline{s}$ and $-\underline{s}$ increase in $\underline{w}$, so we can conclude that the Principal's expected information-acquisition cost increases in $\underline{w}$. Simulations indicate that this is also the case if the Agent's utility exhibits CRRA.

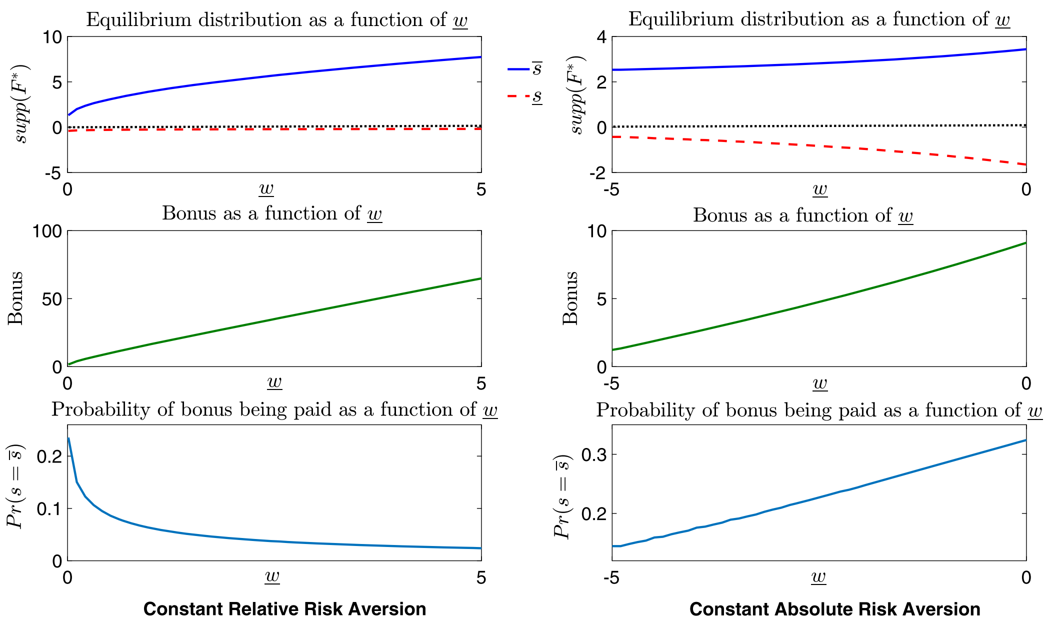


FIGURE 3.—Comparative statics of the optimal contract as the minimum wage $\underline{w}$ varies. The left (right) panel illustrates the case in which the Agent's utility exhibits CRRA (CARA).

It follows immediately from Lemma S1 (given in the Supplemental Material) and the convexity of $c$ that $\bar{s}$, the size of the bonus, and the expected information-acquisition cost are increasing in $a^*$, while $\underline{s}$ is decreasing in $a^*$. This implies that monitoring and monetary incentives are complements: to motivate a larger effort level, the Principal both acquires more information and uses higher powered incentives (in the form of a bigger bonus in case $\bar{s}$ is realized).

How does the optimal contract depend on the cost of information acquisition? Recall that, throughout the paper, we have normalized the Principal's marginal cost of information acquisition to be 1. Suppose instead, that if the Principal monitors the process $X$ until $t$, then she incurs cost $\mu t$. Figure 4 provides comparative statics when we vary $\mu$ from 0.1 to 5, the Agent's utility function exhibits CRRA with coefficient $\gamma = 0.9$, and we set $\underline{w} = 0.1$ and $c'(a^*) = 1$. As information becomes more expensive (i.e., as $\mu$ increases), the Principal acquires less of it, as reflected by the decreasing $\bar{s}$ and $|\underline{s}|$ in the left panel, and instead of monitoring the Agent, she provides incentives by paying a bigger bonus with larger probability as illustrated in the right panels. Finally, simulations also indicate that the Principal's expected information acquisition cost, $\mu\mathbb{E}_{F^*}[s^2] = -\mu\underline{s}\bar{s}$, increases in $\mu$.

## 9. DISCUSSION

We analyze a contracting problem under moral hazard in which the Principal designs both the Agent's wage scheme and the underlying performance measure. In our model, a performance measure is a strategy for sequentially acquiring signals that are informative of the Agent's costly effort, and a wage scheme specifies the Agent's remuneration conditional on the acquired signals. Under a pair of conditions on the Agent's utility function, and provided that the first-order approach is valid, we show that a single-bonus contract is optimal; that is, the Principal chooses a two-point distribution over scores and a binary

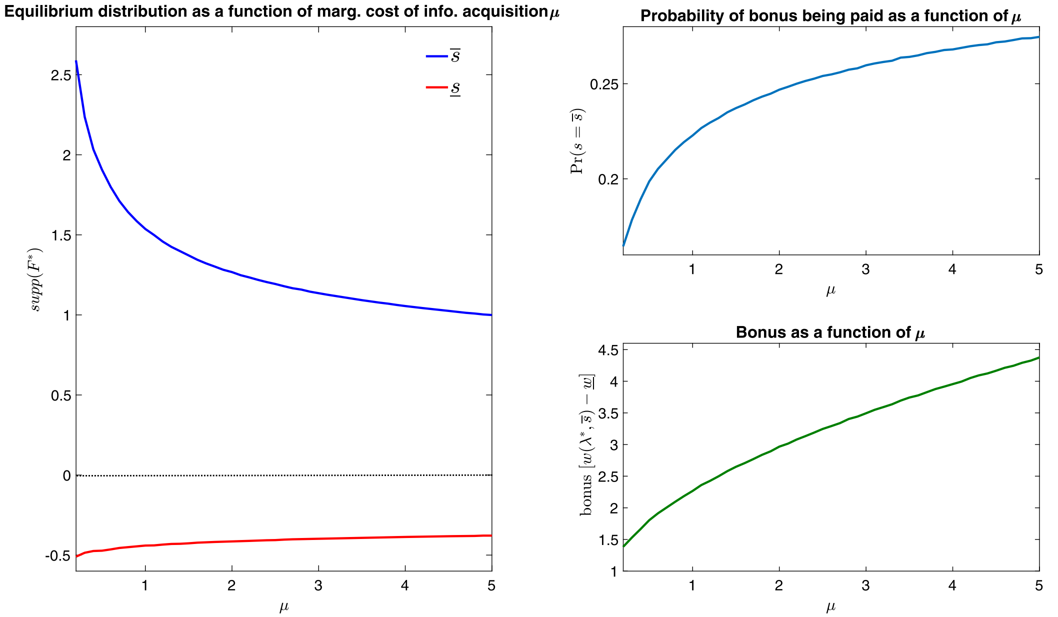


FIGURE 4.—Comparative statics of the optimal contract as the marginal cost of information acquisition, $\mu$, varies.

wage scheme. These conditions are satisfied if, for example, the Agent's utility exhibits CARA or CRRA with coefficient greater than one-half. Under an alternative condition on the Agent's utility, which is satisfied if, for instance, it exhibits CRRA with coefficient less than one-half, we show that the Principal can approximate the first-best outcome arbitrarily closely with a single-bonus contract. More generally, we show that for any utility function, if the zero-sum game has an equilibrium, then there exists an optimal contract which features either a binary or a trinary wage scheme.

Throughout the paper, we have assumed that the Agent has limited liability but does not make a participation decision. It is not hard to incorporate a participation constraint into the Principal's optimization problem and show that the optimal contract still involves binary wages. In this case, even the lower wage might be strictly larger than the minimum wage $\underline{w}$. We chose not to add this constraint because it has little to do with the main arguments of our analysis and involves heavy notational burden. Our results are also valid if the Agent's effort is binary. In this case, there is only a single incentive constraint and the optimality of a single-bonus contract does not require those conditions on the Agent's effort cost that we imposed to validate the first-order approach.

We have considered a particular information-acquisition mechanism, which is equivalent to the Principal choosing any zero-mean distribution over scores at a cost equal to its variance. Alternatively, we could have started with an information design problem in which the Principal chooses a distribution over scores $F \in \mathcal{F}$ at some cost. Our main theorem holds as long as this cost is a general convex moment, that is, it can be expressed as $\mathbb{E}_F[\varphi(s)]$ for some strictly convex function $\varphi$ with $\varphi'''(s) \geq 0$ for all $s \geq 0$.[25] Recall that in

[25]In our model, $\varphi(s) = s^2$. The shape of $\varphi$ is only used in the proof of Lemma 7, which requires that $\varphi$ be strictly convex and $\varphi'''(s) \geq 0$ if $s \geq 0$, so that the threshold $\lambda_c$ is unique and well-defined.

our model, by choosing a distribution corresponding to the target effort level, the Principal is implicitly choosing a distribution corresponding to every other effort level. This alternative approach starts with a relaxed incentive constraint and does not require to specify how the distributions corresponding to different effort levels are linked. However, in order to validate the first-order approach, one must make assumptions not only on the agent's cost of effort but also on probabilities of reaching the two optimal scores conditional on each effort level.

In our model, prior to acquiring costly information, the Principal is completely oblivious to the Agent's effort. Our results can be generalized for the case where, prior to acquiring information, the Principal observes a costless signal about the Agent's effort. For example, public firms are obligated to report various accounting measures such as revenue and operating profits, which are likely to be informative of some employees' actions and not acquired during a monitoring process. Provided that the first-order approach is valid, the Principal's problem can be expressed as choosing a family of distributions over scores, one for each realization of the (costless) signal, and a wage scheme conditional on the realized score. Using the same techniques as in Sections 3–4, one can show that an optimal contract corresponds to an equilibrium of a zero-sum game played by Nature and the Principal. Under the conditions of Theorem 1, if an equilibrium exists, the optimal contract is characterized by an interval. If the score corresponding to the costless signal lies in this interval, the Principal chooses a two-point distribution over scores with mean equal to the score of the costless signal, and pays the Agent according to the realization of the score. If the value of the costless signal is outside of this interval, the Principal does not acquire further information, that is, she chooses a degenerate distribution over scores. In this case, the Agent's wage is based on the costless signal.[26] Interestingly, this wage scheme resembles what Murphy (1999) (Figure 5) and Jensen (2001) argued is a *typical executive incentive plan.*

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[26]Formally, given a costless signal $x_0 \sim H(\cdot|a)$ for some CDF $H(\cdot|a)$, let $s_0 = dH_a(x_0|a^*)/dH(x_0|a^*)$ denote the corresponding score. By the same logic as in Section 3, the Principal chooses, for each realization of $s_0$, a distribution over scores with mean $s_0$, and a wage scheme conditional on the realized (final) score. If an equilibrium in the zero-sum game exists, it is characterized by a multiplier $\lambda^*$, and two thresholds, say $s_l$ and $s_h$. These thresholds depend on $H(\cdot|a^*)$, but not on its realization, $s_0$. If $s_0 \in (s_l, s_h)$, the Principal chooses the two-point distribution over scores with mean $s_0$ and support $\{s_l, s_h\}$, and the Agent is paid $w(\lambda^*, s)$ if $s \in \{s_l, s_h\}$ is realized. If $s_0 < s_l$, the Principal chooses the degenerate distribution (at $s_0$) and pays the Agent the minimum wage $\underline{w}$. Finally, if $s_0 > s_h$, then again, the Principal chooses the degenerate distribution and the Agent is paid $w(\lambda^*, s_0)$.

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