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FLEXIBLE MORAL HAZARD PROBLEMS

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This paper considers a moral hazard problem where the agent can choose any output distribution with a support in a given compact set. The agent's effort-cost is *smooth* and increasing in first-order stochastic dominance. To analyze this model, we develop a generalized notion of the first-order approach applicable to optimization problems over measures. We demonstrate each output distribution can be implemented and identify those contracts that implement that distribution. These contracts are characterized by a simple first-order condition for each output that equates the agent's marginal cost of changing the implemented distribution around that output with its marginal benefit. Furthermore, the agent's wage is shown to be increasing in output. Finally, we consider the problem of a profit-maximizing principal and provide a first-order characterization of principal-optimal distributions.

KEYWORDS: Principal-agent, moral hazard, contract theory.

1. INTRODUCTION

PERHAPS THE MOST CELEBRATED CONCLUSION of the literature on moral hazard is that optimal compensation schemes are designed to reward the agent for those output realizations that are informative about the target level of effort (see, e.g., Holmström (1979) and (2017)). Because larger outputs are not necessarily more informative than smaller ones, optimal wage schemes are often non-monotone in output.[1] These results are typically derived in models in which the action space of the agent is restricted to be either a binary or a one-dimensional set. In this paper, we put forward a model where the agent can *flexibly* choose any output distribution and re-examine the aforementioned conclusions of the literature. We demonstrate that, in such flexible models, optimal wage schemes are not motivated by the informativeness of the output. Instead, they simply compensate the agent for his marginal cost of choosing the target distribution. More precisely, optimal contracts are constructed so that the target distribution satisfies a generalized first-order condition: the marginal cost and marginal benefit of changing the probability of any given output must be equal. Moreover, wage schemes are always increasing in output as long

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[1] To guarantee wages are increasing, the distributions available to the agent must satisfy the monotone likelihood property.

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as the agent's cost of choosing a distribution is monotone in first-order stochastic dominance.

In the specific model of this paper, there is a single agent. After receiving a wage contract, the agent can choose any output distribution with support in a given compact subset of $\mathbb{R}_+$. The agent's payoff is additively separable in her utility from wage and the (effort-) cost associated with the selected distribution. Moreover, the agent has limited liability, so the wage must be weakly positive. We make two assumptions about the costs of output distributions. First, the cost is monotone in first-order stochastic dominance. That is, if a distribution first-order stochastically dominates another one, then it costs more. Second, this cost is Gateaux differentiable. We explain the notion of Gateaux differentiability in detail below. For most of our results, we do not need to specify the principal's preferences. Indeed, our main objective is to derive predictions regarding the wage contracts that incentivize the agent.

To illustrate our model and results, considering the following example is useful.

EXAMPLE 1: Suppose the agent can choose any distribution with support in $\{0, 1\}$. The cost of choosing the distribution that specifies probability p of the output realization one is $c(p)$, where c is an increasing and convex function. The agent's utility from wage is given by the increasing function $u : \mathbb{R}_+ \to \mathbb{R}$.

The cost function of this example satisfies our monotonicity and smoothness assumptions whenever c is increasing and differentiable. For each distribution p^*, we next describe those contracts that implement p^*. Fix a wage scheme $w : \{0, 1\} \to \mathbb{R}_+$ and let m denote the agent's utility from $w(0)$, that is, $m = u(w(0))$. When presented with w, the agent maximizes $pu(w(1)) + (1 - p)m - c(p)$ with respect to p. The agent chooses p^* if it satisfies the corresponding first-order condition, that is,

$$u\big(w(1)\big) = c'\big(p^*\big) + m. \tag{1}$$

For each constant m, the previous equation characterizes a wage scheme that implements p^*. The agent's limited-liability constraint determines the smallest m for which such a wage scheme is feasible. This observation suggests several implications. First, the principal can implement any distribution p by a wage contract satisfying equation (1). Second, unlike in the classical Holmström model, the cost-minimizing wage scheme is not motivated by the information content of the output. Instead, it simply equates the agent's marginal cost of a distribution with his marginal benefit. Third, the wage scheme is always weakly increasing on the support of the implemented distribution.[2] Our paper demonstrates that all these results generalize to any flexible moral hazard problem as long as the aforementioned two assumptions, monotonicity and smoothness, are satisfied.

Our first main result is that any distribution can be implemented by an appropriate wage schedule. The key to this result is to develop a notion of the first-order approach based on Gateaux differentiability. Roughly speaking, Gateaux differentiability means the difference between the cost of a given distribution, say, μ, and that of another nearby distribution can be well approximated by the difference between the expectations of a function, c_μ, according to the two distributions. Moreover, the function c_μ depends only on the given distribution μ and it is called the Gateaux derivative of c at μ. We show

[2] If the wage is larger at 0 than at 1, the agent chooses $p = 0$, so the value 1 is not in the support of the implemented distribution.

a wage scheme, w, implements a distribution μ^*, if the agent's utility from wage is the sum of the Gateaux derivative at μ^* and a constant at each output realization, x, on the support of the distribution and less elsewhere. That is,

$$u(w(x)) = c_{\mu^*}(x) + m \tag{2}$$

for each $x \in \text{supp}(\mu^*)$. Note this equation generalizes equation (1) of the example. Intuitively, this condition guarantees the agent has no incentive to modify the target distribution μ^* by relocating the probability mass across different output levels. The derivation of equation (2) relies neither on the agent's full flexibility of choosing a distribution nor on the monotonicity of the associated costs. Indeed, it is a necessary condition for implementation as long as the agent can arbitrarily modify the target distribution *locally* and his cost function is smooth around the target distribution.

We make two remarks related to equation (2). First, this condition does not have a counterpart in standard moral hazard models. If the agent cannot choose a distribution flexibly, any change in effort has a global effect on the output distribution. Therefore, the incentive constraint requires the agent's expected utility gain from a different effort not to exceed the associated marginal cost of effort (e.g., equation (6) in Holmström (1979)). This, however, is an ex ante constraint that involves taking expectations according to the target distribution. In sharp contrast, the agent in our model can modify the target distribution around any given output without affecting it elsewhere. Consequently, there is an incentive constraint for each output; see equation (2). Each such constraint requires the agent's marginal benefit from increasing the likelihood of a given output to be the same as the marginal cost of doing so. Our second and related remark is that equation (2) implies that for each distribution μ, the incentive compatibility requirement determines the wage scheme that implements μ up to a constant. In a sense, this trivializes the principal's problem of identifying the optimal (cost-minimizing) wage contract among the incentive-compatible ones: the principal can choose only the aforementioned constant which, in turn, is pinned down by the limited-liability constraint.

We consider the main takeaway from our analysis to be the observation that if the agent can choose distributions flexibly, optimal wage contacts are not motivated by the information content of the output. To explain this, recall that in moral hazard models, incentive compatibility typically follows from ensuring local deviations are not profitable. In standard problems, such deviations can change the relative likelihood of different outputs in a limited way, and informativeness is defined with respect to these limitations. For example, when effort is one-dimensional, there is only one local deviation, and informativeness of an output can be measured by the relative likelihood of that output under the target distribution and under that deviation. By contrast, when the agent can choose output distributions flexibly, he can use local deviations to manipulate the relative likelihood of any collection of outputs in an arbitrary manner. Consequently, in flexible moral hazard problems, there is no useful notion of informativeness around which one can design the agent's contract.[3] Instead, incentive-compatible wage schemes must eliminate the agent's desire to re-allocate probability mass across outputs. To do so, the optimal contract effectively reimburses the agent for the marginal cost of producing each output.

Let us now turn our attention to the monotonicity of the wage schemes. Recall that in standard principal-agent models with hidden action, cost-minimizing wages are monotone

[3]Of course, even in flexible models, observing output x indicates that the agent choice has x in its support. The agent's wage must be sufficiently low whenever the output is not in the support of the target distribution. However, the information conveyed by observing x is not useful except in this very limited way.

in output only under strong assumptions on the feasible output distributions, namely, they must satisfy the monotone likelihood ratio property. By contrast, in our flexible moral hazard model, the monotonicity of the wage scheme follows directly from the agent's incentives whenever the agent's costs are monotone. More precisely, if a wage scheme implements a certain distribution, this wage scheme is (weakly) increasing over the outputs generated by that distribution. This result is rather obvious and can be explained as follows. Suppose the wage is larger at a small output level than at other higher outputs. Then the agent would never choose a distribution that specifies positive probability on those higher outputs. The reason is that the agent can modify the distribution by moving the probability mass from those higher outputs to the low output. On the one hand, this modification increases the agent's expected wage, because the wage conditional on the low output exceeds the wage conditional on any of those high output levels. On the other hand, the modified distribution is first-order stochastically dominated by the original one, so it is cheaper to the agent.

We conclude our analysis by considering the principal's problem of finding the profit-maximizing distribution and the corresponding optimal contract. To extend the aforementioned first-order approach to the principal's profit maximization problem, we need to make a stronger smoothness assumption. Roughly speaking, this assumption requires the agent's cost function to be twice differentiable. We then characterize the first-order condition corresponding to the principal's problem. Finally, we illustrate how this first-order condition can be used to derive properties of the principal-optimal distribution. For example, we provide sufficient conditions under which this distribution is degenerate.

Related Literature. First and foremost, our paper is related to the literature on principal-agent problems under moral hazard (Mirrlees (1976) and Holmström (1979)). In the canonical model, the principal offers a wage contract, and then the agent chooses a (typically) one-dimensional action that determines the distribution of output. The optimal contract is shaped by the information content of the output, as well as a trade-off between incentives and insurance. See Holmström (2017) and Georgiadis (2022) for reviews. Instead, the agent can choose *any* output distribution in our model.

As Example 1 highlights, more restrictive ways of enriching the standard moral hazard model with flexible production have been studied before. An early instance is Holmstrom and Milgrom (1987) who, among other things, considered a locally flexible model in which both the principal and the agent have CARA utility functions and effort costs are monetary.[4] They showed that the set of contracts implementing any interior distribution can be parameterized by the agent's certainty equivalent. A few more recent studies have additively-separable costs and more general preferences, but impose other restrictions, such as a finite output space, mean-measurable costs, or requiring costs to come from the f-divergence family (e.g., Diamond (1998), Mirrlees and Zhou (2006), Bonham (2021), Bonham and Riggs-Cragun (2023), Mattsson and Weibull (2022)).[5] All of these papers derive a version of the first-order condition (2) for their setting. As mentioned above, this

[4]Hellwig (2007) extended Holmstrom and Milgrom's (1987) analysis by allowing for boundary solutions. He explicitly characterized the optimal wage scheme, and showed it is non-decreasing in output.

[5]Another related paper within this vein is Hébert (2018), who studied security design by an entrepreneur who can flexible control output. That paper assumes risk neutrality and that costs belong to the f-divergence family. A few other related papers consider models with partial flexibility. In particular, Barron, Georgiadis, and Swinkels (2020) studied a version of the Holmström (1979) model where the agent can costlessly add risk to the realized output, whereas Palomino and Prat (2003) allowed the agent to control the first two moments of the output distribution.

condition pins down the agent's contract up to a constant. Hence, one could use these studies to conclude that informativeness plays a diminished role in their specialized environments. Our contribution to this literature is the general treatment of the flexible moral hazard problem. Indeed, it is the generality of our model that allows us to conclude that flexible production, rather than any specific restriction, is what results in the optimal contract's shape being determined by the agent's incentives rather than the informativeness of the output. Our generality also enables the analysis of various properties of interest, which may be impeded by imposing particular structures on the underlying environment. This is exemplified by the case of monotonicity, which can be studied in our model, but not by models with cost functions that either require it (e.g., mean-measurable costs) or exclude it (such as with f-divergence). And, indeed, none of the existing models explore the implications of monotonicity on the set of incentive-compatible contracts.

Our paper is also related to the literature on robust contracting; see, for example, Carroll (2015), Carroll (2019) for a review, Antic (2022), and Antic and Georgiadis (2022). Like our paper, this literature imposes only minimal restrictions on the technology available to the agent. Their premise, however, is that the principal has limited knowledge regarding the technology and evaluates contracts according to the worst-case scenario. In contrast, the agent's cost of choosing any distribution is common knowledge in our model.

2. MODEL

There is an agent who can produce any output distribution with support in a compact subset X of $\mathbb{R}_+$.[6] Throughout, we let $\underline{x} := \min X$ and $\bar{x} := \max X$ denote the lowest and highest possible outputs, respectively. Let $\mathcal{M}$ denote the set of Borel probability measures on X. The agent's payoff is additively separable in the utility from wage and the effort cost of producing. The utility function from money, $u : \mathbb{R}_+ \to \mathbb{R}$, is strictly increasing, continuous, unbounded and it is normalized so that $u(0) = 0$. The agent's cost of producing $\mu \in \mathcal{M}$ is $C(\mu)$, where $C : \mathcal{M} \to \mathbb{R}_+$ is a weak*-continuous and convex function. So, if the agent chooses $\mu \in \mathcal{M}$ and receives wage w, then his payoff is $u(w) - C(\mu)$. Moreover, the agent is an expected-payoff maximizer.

Before the agent decides which distribution to produce, he receives a wage contract. A wage contract is a measurable mapping from realized outputs to monetary compensations. The agent has limited liability so every contract must specify weakly positive wages. To ensure the agent's payoff is well-defined, we also require the agent's contract to be bounded from above. Let $\mathcal{W}$ denote the set of such contracts, that is, $\mathcal{W} = \{w \mid w : X \to \mathbb{R}_+, \sup w(X) < \infty\}$.

We next argue that assuming the convexity of C is without loss. Indeed, since the agent may randomize, the cost of any distribution should be evaluated by the expected cost of the cheapest randomization that generates it, resulting in a convex cost function.

We state two further assumptions on the cost of production. First, we assume that producing more in the sense of first-order stochastic dominance costs more.[7]

ASSUMPTION 1—Monotonicity: *If the distribution μ first-order stochastically dominates μ', then $C(\mu) \geq C(\mu')$.*

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[6]The output is assumed to be positive only for the sake of economic interpretation of our model. All our results hold as long as $X \subseteq \mathbb{R}$. As will become apparent, we can allow X to be unbounded if we instead impose that the Gateaux derivative c_μ is bounded for all μ.

[7]If the agent can dispose output freely and privately, his effective cost function satisfies this monotonicity property; see Innes (1990) for a discussion.

Our second assumption ensures that the cost function is smooth.

ASSUMPTION 2—Smoothness: *The function C is* Gateaux differentiable, *which means that every μ admits a continuous function $c_\mu : X \to \mathbb{R}$ such that*

$$\lim_{\epsilon \searrow 0} \frac{1}{\epsilon}\left[C\big(\mu + \epsilon(\mu' - \mu)\big) - C(\mu)\right] = \int c_\mu(x)\big(\mu' - \mu\big)(\mathrm{d}x)$$

for all $\mu' \in \mathcal{M}$. The function c_μ is referred to as the (*Gateaux*) *derivative of C.*[8]

Let us make a few remarks regarding Assumption 2. First, if c_μ is a derivative of C at μ, then so is $c_\mu + k$ for any constant $k \in \mathbb{R}$. It is therefore without loss to normalize $c_\mu(\underline{x}) = 0$. Second, whenever Assumption 2 holds, Assumption 1 is equivalent to c_μ being increasing for all μ (see Cerreia-Vioglio, Maccheroni, and Marinacci (2017), for example). And third, when there are only n outputs, $X = \{x_1, \dots, x_n\}$, C becomes a mapping from the n-dimensional simplex to $\mathbb{R}_+$. In this case, Assumption 2 is equivalent to the usual notion of differentiability, and one can express c_μ in terms of the partial derivatives of C. More specifically, let $C_i'(\mu)$ denote the partial derivative of C with respect to the probability of output i at the distribution μ, and suppose $x_1 = \underline{x}$ is the lowest output. Then one can express the Gateaux derivative of C as $c_\mu(x_i) = C_i'(\mu) - C_1'(\mu)$.[9]

Our goal is to analyze the set of those distributions which can be implemented and characterize the wage contracts which implement them. More formally, for each $w \in \mathcal{W}$, the measure $\mu \in \mathcal{M}$ is called *w-incentive compatible* (w-IC) if the agent finds it optimal to produce μ after he receives the contract w. Note that if the wage contract is w and the agent chooses $\mu \in \mathcal{M}$, then his payoff is

$$U(\mu, w) = \int u \circ w(x)\mu(\mathrm{d}x) - C(\mu).$$

So, the measure μ is w-IC if $U(\mu, w) = \sup_{\mu' \in \mathcal{M}} U(\mu', w)$. We say μ is *implementable* whenever it is w-IC for some $w \in \mathcal{W}$.

We emphasize that for most of our results, we do not need both assumptions above. For example, even if neither of these assumptions holds, the set of implementable distributions is large.

THEOREM 1: *The set of distributions that are implementable is dense.*

PROOF: See the Appendix. *Q.E.D.*

To prove the above theorem, we identify each measure in $\mathcal{M}$ with its corresponding CDF. By equipping the set of CDFs with the L^2-norm, we recast C as a convex and lower-semicontinuous function over a Banach space. To conclude the proof, we show one can

[8]Our definition of Gateaux differentiability comes from the decision-theory literature (e.g., Hong, Karni, and Safra (1987), Cerreia-Vioglio, Maccheroni, and Marinacci (2017)). Our results continue to hold if we require c_μ only to be measurable and bounded. One only needs to modify the proof of Corollary 2, since the cited result of Cerreia-Vioglio, Maccheroni, and Marinacci (2017) no longer applies. Instead, one needs to appeal to Lemma 3 (see the Appendix), which obtains a similar result for the case where c_μ is bounded and measurable.

[9]Consequently, when X is finite, Assumption 2 holds Lebesgue almost everywhere for all cost functions (see Rockafellar (1970), Theorem 25.4).

implement every CDF at which the subdifferential of C is non-empty, a condition which holds over a dense set of the cost function's domain by the Brondsted–Rockafellar theorem (Brøndsted and Tyrrell Rockafellar (1965)).

We conclude this section by providing an example for the agent's cost function, C, which satisfies our assumptions. We will use this example to illustrate many of our results throughout the paper.

EXAMPLE 2: Let $c : X \to \mathbb{R}_+$ be an increasing and continuous function with $c(\underline{x}) = 0$. Furthermore, let $K : \mathbb{R} \to \mathbb{R}$. be an increasing, convex, and differentiable function. If the agent's cost function is defined by

$$C(\mu) = K\left(\int c(x)\mu(\mathrm{d}x)\right), \tag{3}$$

then it satisfies Assumptions 1 and 2. Indeed, by the Chain Rule, this function is Gateaux differentiable, with the derivative given by

$$c_\mu(x) = K'\left(\int c(y)\mu(\mathrm{d}y)\right)c(x). \tag{4}$$

3. MAIN RESULTS

3.1. *Monotone Wages*

Our first result establishes that the monotonicity of C (Assumption 1) implies the monotonicity of any wage scheme on the support of the distribution it implements.

DEFINITION 1: The contract $w \in \mathcal{W}$ is *μ-increasing* if, for all $x \in X$,

$$\mu(\{x' \in X : x < x', w(x') < w(x)\}) = 0.$$

That is, w is μ-increasing if, for every x, the probability that μ generates a higher output that gives the agent a lower wage is zero. We are ready to state our first result.

PROPOSITION 1: *If Assumption 1 holds and $\mu \in \mathcal{M}$ is w-IC, then w is μ-increasing.*

The proof of the proposition is established along the same arguments described in the Introduction. We show that if w is not μ-increasing, then the measure μ can be modified by moving probability from high outputs at which the wage is low to a low output realization at which the wage is high. This new measure is cheaper to the agent and generates higher expected utility, that is, μ was not w-IC.

PROOF: Suppose, by contradiction, that w is not μ-almost increasing. Then there exists $x \in X$ such that $\mu(S_x) > 0$, where

$$S_x = \{x' \in X : x' > x, w(x') < w(x)\}.$$

Let $\mu' \in M$ be a modification of μ so that all the mass from the set S_x is moved to x. Formally, for each Borel set A,

$$\mu'(A) = \begin{cases} \mu(A \backslash S_x) + \mu(S_x) & \text{if } x \in A, \\ \mu(A \backslash S_x) & \text{otherwise.} \end{cases}$$

Since $x < x'$ for all $x' \in S_x$ and $\mu(S_x) > 0$, it follows that μ strictly first-order stochastically dominates μ'. Finally, note that

$$U(\mu, w) \leq \int u \circ w(x)\mu(\mathrm{d}x) - C(\mu') < \int u \circ w(x)\mu'(\mathrm{d}x) - C(\mu') = U(\mu', w),$$

where the weak inequality follows from Assumption 1 and the fact that μ first-order stochastically dominates μ' and the strict inequality follows because $w(x) > w(x')$ for each $x' \in S_x$ and $\mu(S_x) > 0$. This inequality chain implies that $U(\mu, w) < \sup_{\mu' \in \mathcal{M}} U(\mu', w)$, that is, μ is not w-IC, a contradiction. *Q.E.D.*

Note that Proposition 1 does not rule out that the agent's wage is non-monotone over outputs that never arise under the implemented distribution. Nevertheless, it turns out that one can adjust the agent's wage following those outputs so as to make it monotone without impacting incentives. We refer the reader to the Appendix for the formal statement and proof.

3.2. *Implementability*

Next, we explore the consequences of Assumption 2. The following lemma develops a notion of the first-order approach based on Gateaux differentiability. In particular, it proves necessity and sufficiency of a first-order condition for maximization. The first-order approach is then applied to characterize the agent's optimal distribution for a given wage contract. In turn, this leads to our main result: each distribution can be implemented and the corresponding wage scheme is determined by the aforementioned first-order condition.[10]

To understand how the statement of the next lemma is related to the first-order condition familiar from one-dimensional calculus, consider the problem of maximizing $vx - c(x)$ on $[0, 1]$, where $v \in \mathbb{R}_+$, and c is a convex, differentiable function. Then $x^* \in (0, 1)$ solves this problem if and only if it satisfies the first-order condition $v = c'(x^*)$. An equivalent way of stating it is that $x^* \in (0, 1)$ solves the problem if and only if x^* also solves $\max_{x \in [0,1]}[vx - c'(x^*)x]$. In what follows, we generalize this latter condition for Gateaux differentiable cost functions.

LEMMA 1: *For a bounded and measureable $v : X \to \mathbb{R}$, and $\mu^* \in \mathcal{M}$,*

$$\mu^* \in \arg\max_{\mu \in \mathcal{M}} \int v(x)\mu(\mathrm{d}x) - C(\mu)$$

if, and only if,

$$\mu^* \in \arg\max_{\mu \in \mathcal{M}} \int \bigl(v(x) - c_{\mu^*}(x)\bigr)\mu(\mathrm{d}x).$$

We note that the convexity of the function C plays a role only in the "if" part of the proof. That is, the first-order condition would be necessary even if C was not convex.[11]

[10] Recall that Theorem 1 only states that, absent Assumption 2, the set of implementable distributions is dense.

[11] We also note that an identical proof shows the lemma continues to hold if one replaces $\mathcal{M}$ with any convex subset, $\bar{\mathcal{M}} \subseteq \mathcal{M}$.

PROOF: We first prove that the first-order condition is necessary. Fix any $\tilde{\mu} \in \mathcal{M}$. For all $\epsilon \in (0, 1)$, define $\mu_\epsilon := \mu^* + \epsilon(\tilde{\mu} - \mu^*)$, which is in the convex set $\mathcal{M}$. If $\mu^* \in \arg\max_{\mu \in \mathcal{M}}[\int v(x)\mu(\mathrm{d}x) - C(\mu)]$, then

$$0 \geq \frac{1}{\epsilon}\left[\int v(x)\left(\mu_\epsilon - \mu^*\right)(\mathrm{d}x)\right] - \frac{1}{\epsilon}\left[C(\mu_\epsilon) - C\left(\mu^*\right)\right]$$
$$= \int v(x)\left(\tilde{\mu} - \mu^*\right)(\mathrm{d}x) - \frac{1}{\epsilon}\left[C(\mu_\epsilon) - C\left(\mu^*\right)\right],$$

where the inequality follows from μ^* being a maximizer and the equality is implied by the definition of μ_ϵ. Observe that, since C is Gateaux differentiable at μ^*, the last expression of the previous displayed inequality chain converges to

$$\int \left[v(x) - c_{\mu^*}(x)\right]\left(\tilde{\mu} - \mu^*\right)(\mathrm{d}x),$$

as ϵ goes to zero.

We now show that the first-order condition is sufficient when C is convex. To that end, we first claim that

$$C(\mu) - C\left(\mu^*\right) \geq \int c_{\mu^*}(x)\left(\mu - \mu^*\right)(\mathrm{d}x) \tag{5}$$

holds for all μ. To prove this inequality, note that the convexity of C means that

$$\frac{1}{\epsilon}\left[C\left(\mu^* + \epsilon\left(\mu - \mu^*\right)\right) - C\left(\mu^*\right)\right]$$

is decreasing in $\epsilon \in (0, 1)$. Letting $(\epsilon_n)_{n \in \mathbb{N}}$ be a decreasing sequence in $(0, 1)$ converging to zero, we have

$$C(\mu) - C\left(\mu^*\right) \geq \frac{1}{\epsilon_n}\left[C\left(\mu^* + \epsilon_n\left(\mu - \mu^*\right)\right) - C\left(\mu^*\right)\right] \xrightarrow{n \to \infty} \int c_{\mu^*}\,\mathrm{d}\left(\mu - \mu^*\right).$$

Therefore, if μ^* satisfies the first-order condition, the following must hold for every μ:

$$0 \geq \int (v - c_{\mu^*})(x)\left(\mu - \mu^*\right)(\mathrm{d}x) \geq \int v(x)\left(\mu - \mu^*\right)(\mathrm{d}x) - \left[C(\mu) - C\left(\mu^*\right)\right],$$

where the first inequality follows from the fact that μ^* satisfies the first-order condition, that is, $\mu^* \in \arg\max_{\mu \in \mathcal{M}}[\int (v(x) - c_{\mu^*}(x))\mu(\mathrm{d}x)]$. The second inequality is just (5). Finally, the previous inequality chain implies $\mu^* \in \arg\max_{\mu \in \mathcal{M}}[\int v(x)\mu(\mathrm{d}x) - C(\mu)]$. *Q.E.D.*

Next, we apply the previous lemma to the agent's problem of choosing a distribution. To this end, for each $\mu \in \mathcal{M}$, let $m^*(\mu) = \inf\{m : \min_{x \in X} c_\mu(x) + m \geq 0\}$. and for each $m \geq m^*(\mu)$, define

$$w_{\mu,m}(x) := u^{-1}\left(c_\mu(x) + m\right).$$

The next proposition states that the wage contract $w_{\mu,m}$ implements μ for each $m \geq m^*$.

PROPOSITION 2: *Suppose that C satisfies Assumption 2. Then, the measure $\mu \in \mathcal{M}$ is w-IC if, and only if,*

$$w(x) \begin{cases} = w_{\mu,m}(x) & \text{if } x \in Y, \\ \leq w_{\mu,m}(x) & \text{otherwise,} \end{cases}$$

holds for some $m \geq m^*(\mu)$ and some $Y \subseteq X$ with $\mu(Y) = 1$.

PROOF: Observe that the agent's objective, $\int u \circ w(x)\mu(\mathrm{d}x) - C(\mu')$, is concave and Gateaux differentiable in μ'. Therefore, Lemma 1 implies that μ is w-IC if and only if μ satisfies the agent's first-order condition, that is, μ solves

$$\max_{\mu'} \int \big(u \circ w(x) - c_\mu(x)\big)\mu'(\mathrm{d}x),$$

which is equivalent to $[u \circ w(x) - c_\mu(x)] \leq \sup_{x \in X}[u \circ w(x) - c_\mu(x)] =: m$ holding with equality μ-almost surely. The proposition follows from rearranging this inequality and noting that u^{-1} is strictly increasing. Finally, note that $m \geq m^*(\mu)$ must hold because of limited liability. *Q.E.D.*

We now show that the previous proposition implies that every $\mu \in \mathcal{M}$ can be implemented, and that the contract

$$w^*_\mu := w_{\mu,m^*(\mu)}$$

is a cost-minimizing contract among those that implement μ.[12]

COROLLARY 1: *Suppose C satisfies Assumption 2, and fix any $\mu \in \mathcal{M}$. Then μ is w^*_μ-IC. Moreover, for any other $w \in \mathcal{W}$ for which μ is w-IC, $w \geq w^*_\mu$ holds μ-almost surely.*

We point out that the cost-minimizing wage scheme implementing any μ is uniquely determined μ-almost everywhere. For sets that arise with zero probability under μ, the cost-minimizing contract can be defined arbitrarily as long as it is weakly smaller than w^*_μ.

PROOF: That μ is w^*_μ-IC follows immediately from Proposition 2. The same proposition also implies that for every $w \in \mathcal{W}$ for which μ is w-IC, there exists some $m \geq m^*(\mu)$ such that $w = w_{\mu,m}$ μ-almost surely. Since $w_{\mu,m}(x) = u^{-1}(c_\mu(x) + m)$, $m \geq m^*(\mu)$, and u^{-1} is strictly increasing, it follows that $w_{\mu,m} \geq w_{\mu,m^*(\mu)} = w^*_\mu$. *Q.E.D.*

To conclude this section, we consider what happens when the cost function C satisfies both Assumptions 1 and 2. We show that, in this case, w^*_μ is increasing, and so one can obtain a more explicit characterization of a cost-minimizing contract.

COROLLARY 2: *Suppose C satisfies Assumptions 1 and 2. Then,*

$$w^*_\mu(x) = u^{-1}\big(c_\mu(x)\big)$$

is a cost-minimizing contract among those that implement μ. Moreover, w^*_μ is increasing everywhere.

[12]Note that w^*_μ is well-defined, because X is compact, c_μ is continuous, and u is a continuous, unbounded, and strictly increasing function satisfying $u(0) = 0$.

PROOF: We first note that Assumption 1 implies that c_μ is increasing (see, e.g., Cerreia-Vioglio, Maccheroni, and Marinacci (2017)). Since u^{-1} is also increasing, $u^{-1}(c_\mu(x)+m)\geq 0$ if and only if this inequality holds at $x=0$. Therefore, the normalizations, $u(0)=0$ and $c_\mu(\underline{x})=0$, imply that $m^*_\mu=0$. Consequently, $w^*_\mu(x)=u^{-1}(c_\mu(x))$ and this function is increasing. *Q.E.D.*

We now revisit the example of Section 2 and compute the cost-minimizing wage contract for each distribution. We also show that in the special case where the agent's marginal cost is constant (i.e., c_μ does not depend on μ), the cost-minimizing wage does not depend on the implemented distribution.

EXAMPLE 2—continued: Recall that the agent's cost function is defined by (3) and its Gateaux derivative is given by (4). Moreover, this cost function is monotone. Therefore, an immediate consequence of the previous corollary is that, for each μ, the cost-minimizing wage is given by the following equation:

$$w^*_\mu(x)=u^{-1}\biggl(K'\biggl(\int c(y)\mu(\mathrm{d}y)\biggr)c(x)\biggr). \tag{6}$$

This equation has some interesting implications in the special case where the agent's marginal cost is constant and equal to 1, that is, the function K is the identity function,

$$K(x)=x.$$

In this case, $K'=1$, and so equation (6) simplifies to $w^*_\mu(x)=u^{-1}(c(x))$ for *all* output distributions. In other words, this wage scheme is the cost-minimizing one for each distribution. Notice this contract results in the agent getting a net utility of zero regardless of the output, since

$$u\bigl(w^*_\mu(x)\bigr)-c(x)=u\bigl(u^{-1}\bigl(c(x)\bigr)\bigr)-c(x)=0.$$

More generally, the above indifference holds whenever K is affine. For non-affine K, the cost-minimizing contract w^*_μ depends on μ through equation (4), and gives the agent positive rents. To see why the agent's rents are positive, suppose $K(0)=0$, which is without loss (one can always subtract $K(0)$ from K without changing the analysis). For every distribution μ, let $I_\mu:=\int c_\mu(x)\mu(\mathrm{d}x)$. Then the agent's expected utility from distribution μ under the contract w^*_μ defined in equation (4) can be written as

$$\int u\bigl(w^*_\mu(x)\bigr)\mu(\mathrm{d}x)-C(\mu)=K'(I_\mu)I_\mu-K(I_\mu)=\int_0^{I_\mu}\bigl[K'(I_\mu)-K'(z)\bigr]\,\mathrm{d}z,$$

which is strictly larger than zero whenever K is non-affine on the interval $[0,I_\mu]$.

4. PROFIT MAXIMIZATION

In this section, we turn our attention to the principal's problem of finding the profit-maximizing distribution and the corresponding contract. We assume the principal's payoff is $x-w$ if output is x and she pays wage w to the agent, and that she is an expected payoff-maximizer. We first make a further assumption on the cost function C which roughly requires it to be twice differentiable. Then we show that a consequence of this assumption is that the principal's profit as a function of the implemented distribution μ is also Gateaux differentiable, and characterize a first-order condition corresponding to the principal's problem. Finally, we illustrate how this first-order condition can be used to make meaningful statements about the principal-optimal distribution and contract.

Let us now state the aforementioned assumption which essentially requires the Gateaux derivative of C to be Gateaux differentiable.

ASSUMPTION 3: *The cost function is Gateaux differentiable, with $\mu \mapsto c_\mu(\cdot)$ being weak*-to-supnorm continuous. Moreover, for every μ, a continuous function $h_\mu : X \times X \to \mathbb{R}$ exists such that for all $\tilde{\mu} \in \mathcal{M}$,*

$$\lim_{\epsilon \searrow 0} \frac{1}{\epsilon}\left[c_{\mu+\epsilon(\tilde{\mu}-\mu)}(\cdot) - c_\mu(\cdot)\right] = \int h_\mu(\cdot, y)(\tilde{\mu} - \mu)(\mathrm{d}y),$$

where convergence is according to the sup norm, $\|\cdot\|_\infty$.

We now describe the problem of a profit-maximizing principal. In order to maximize her profit, the principal chooses an output distribution and a wage contract which implements it. Formally, the principal's program can be written as

$$\max_{\mu \in \mathcal{M}, w \in \mathcal{W}} \int \left[x - w(x)\right]\mu(\mathrm{d}x), \quad \text{subject to } \mu \text{ is } w\text{-IC}.$$

Of course, if a pair (μ, w) solves this problem, then the wage scheme w is cost-minimizing among those that implement μ. For each $\mu \in \mathcal{M}$, let $W(\mu)$ be the expected cost-minimizing wage implementing μ.[13] Then, the principal's program can be rewritten as

$$\max_{\mu \in \mathcal{M}} \left[\int x\mu(\mathrm{d}x) - W(\mu)\right]. \tag{7}$$

We call a distribution μ^* *principal-optimal* if it solves this maximization problem.

We aim to provide a partial characterization of a principal-optimal distribution in two steps. First, we compute the Gateaux derivative of the function W. And second, we appeal to Lemma 1 to derive a necessary first-order condition for μ to be principal-optimal.

To this end, suppose u is a continuously differentiable function with a strictly positive derivative, and define the function $\kappa^*_\mu : X \to \mathbb{R}$ as follows:

$$\kappa^*_\mu(x) = \int \frac{h_\mu(y, x)}{u'\big(w^*_\mu(y)\big)}\mu(\mathrm{d}y).$$

To interpret κ^*_μ, note that $h_\mu(y, x)$ represents the change in the marginal cost of producing output y associated with a slight increase in the probability of output x. Multiplying $h_\mu(y, x)$ by the ratio $1/u'(w^*_\mu(y))$ converts the change in the agent's marginal cost to a change in the agent's monetary wage. Thus, $\kappa_\mu(x)$ gives the marginal change in the agent's expected compensation associated with an increase in the probability of output x.

The next theorem describes the Gateaux derivative of the principal's expected-wage payments under the cost-minimizing contract as a function of the induced output distribution.

[13]Recall that, by Corollary 2, $W(\mu) = \int [w^*_\mu(x)]\mu(\mathrm{d}x)$.

LEMMA 2: *Suppose C satisfies Assumptions 1 and 3 and u is a continuously differentiable function with a strictly positive derivative. Then the function W is continuous and Gateaux differentiable with derivative*

$$w^*_\mu(x) + \kappa^*_\mu(x).$$

Each term in the Gateaux derivative, $w^*_\mu(x) + \kappa^*_\mu(x)$, expresses a different force that impacts the principal's expected payments when she shifts the implemented output distribution away from μ. The first term, $w^*_\mu(x)$, is the wage the agent receives when generating an output of x. The second term, $\kappa^*_\mu(x)$, expresses the impact on the agent's expected compensation due to the change in the cost-minimizing contract that arises from changing the probability of output x.

We are now ready to characterize the first-order condition describing a principal-optimal distribution. Recall that Lemma 1 developed a first-order approach for a class of maximization problems. Substituting $v(x) = x$ and $C(\mu) = W(\mu)$ into the statement of the lemma, and noting that W is Gateaux differentiable (by Lemma 2), it becomes clear that the "if" part of the statement of Lemma 1 is applicable to the principal's profit maximization problem (7). Since W may not be convex, the "only if" part is not applicable, so the following theorem provides only a necessary condition for optimality.

THEOREM 2: *Suppose C satisfies Assumptions 1 and 3, and that u is a continuously differentiable function with a strictly positive derivative. Then, a principal-optimal μ^* exists and*

$$\operatorname{supp}\mu^* \subseteq \arg\max_{x\in X}\left[\pi_{\mu^*}(x)\right], \tag{8}$$

where $\pi_\mu(x) := x - w^*_\mu(x) - \kappa^*_\mu(x)$.

PROOF: Note that, by Lemma 2, the principal's objective function in (7) is continuous. Since the domain $\mathcal{M}$ is compact, the existence of a principal-optimal distribution follows. As mentioned above, equation (8) is implied by the "if" part of the statement by Lemma 1. *Q.E.D.*

Let us return to Example 2 to illustrate how to compute the derivative of the principal's expected profit for each distribution.

EXAMPLE 2—continued: Recall that C is given by (3) and that we have already characterized w^*_μ in equation (6). By the previous theorem, in order to derive the derivative of the principal's expected profit, it remains to compute κ^*_μ. To this end, assume that K is twice continuously differentiable. Then C also satisfies Assumption 3, where

$$h_\mu(x, y) = K''\left(\int c(z)\mu(\mathrm{d}z)\right)c(x)c(y).$$

Furthermore, whenever u is continuously differentiable with a strictly positive derivative,

$$\kappa^*_\mu(x) = K''\left(\int c(z)\mu(\mathrm{d}z)\right)\left[\int \frac{c(y)}{u'\left(w^*_\mu(y)\right)}\mu(\mathrm{d}y)\right]c(x). \tag{9}$$

Let us now return to the general analysis and demonstrate that the condition in (8) can be used to deduce properties of the principal's optimal distribution and the corresponding wage contract. Observe that this condition depends on the function π_μ, which we characterized in terms of the agent's utility function u and cost function C. The next corollary establishes relationships between the shape of π_μ and the support of the optimal distribution.

COROLLARY 3: *Suppose Assumptions 1 and 3 hold*, $X = [\underline{x}, \overline{x}]$, *and that u is a continuously differentiable function with a strictly positive derivative.*

(i) *If π_μ is strictly quasiconcave for every μ with more than one output, the principal-optimal distribution has at most one output in its support.*
(ii) *If π_μ is strictly quasiconvex for every μ that includes some non-extreme output $x \in (\underline{x}, \bar{x})$ in its support, the principal-optimal distribution is supported on $\{\underline{x}, \bar{x}\}$.*
(iii) *If $w_\mu + \kappa_\mu$ is a non-affine analytic function whenever μ is not discrete, the principal-optimal distribution is discrete.*

PROOF: As explained above, if μ is principal-optimal, it must be supported on the set of outputs that maximize π_μ. Part (i) then follows from observing that this set can have at most one output whenever π_μ is strictly quasiconcave. Part (ii) follows from noting that a strictly quasiconvex function over a compact interval is maximized at the interval's end points. For part (iii), observe that $w_\mu + \kappa_\mu$ being a non-affine analytic function means the function $x \mapsto [\pi_\mu(x) - \max \pi_\mu(X)]$ is a non-zero analytical function. Therefore, by the identity theorem, the set

$$\arg\max_{x\in X}\big[\pi_\mu(x)\big] = \big\{x \in X : \pi_\mu - \max \pi_\mu(X) = 0\big\}$$

cannot have any accumulation points in $(\underline{x}, \bar{x})$. The conclusion follows. *Q.E.D.*

Let us illustrate each part of the previous corollary by considering various specifications of Example 2.

EXAMPLE 2—continued: Note that Theorem 2 implies the derivative of the principal's expected profit is $\pi_\mu(x) = x - w^*_\mu(x) - \kappa^*_\mu(x)$, where w^*_μ and κ^*_μ are given by (6) and (9), respectively. Suppose X is an interval, the agent is risk neutral, and $u(x) = x$, so $u'(\cdot) = 1$. Then if c is strictly convex, π_μ is strictly concave (hence strictly quasiconcave), and so part (i) implies it is always optimal to induce a single output. If c is strictly concave instead, part (ii) implies the principal-optimal distribution has at most two outputs, because π_μ is strictly (quasi-)convex. Finally, if we replace the convexity or concavity assumptions with the postulate that c is a non-affine analytic function, the same holds for π_μ, in which case the principal-optimal distribution must be discrete by part (iii).

Corollary 3 is particularly useful when either part (i) or part (ii) holds. In these cases, the principal's program reduces to a one-dimensional optimization problem. To see this, suppose first that Corollary 3's assumptions hold and that π_μ is strictly quasiconcave for every μ. By Corollary 3, the principal-optimal distribution has only one output. Letting δ_x be the distribution generating output x with probability 1, it follows that the principal-optimal output distribution must solve

$$\max_{x\in X}\big[x - w_{\delta_x}(x)\big]. \tag{10}$$

Suppose now instead that π_μ is strictly quasiconvex for every μ. Applying Corollary 4, the principal-optimal distribution takes the form $\mu_p := p\delta_{\bar{x}} + (1-p)\delta_{\underline{x}}$ for some $p \in [0,1]$, and so one can write the principal's problem as

$$\max_{p\in[0,1]}\big[(1-p)\underline{x} + p\bar{x} - pw_{\mu_p}(\bar{x})\big].^{14} \tag{11}$$

Hence, the principal's problem reduces to finding the optimal probability p with which to generate the highest output, just as in the binary output example.

Finally, we reconsider our running example with a risk-averse agent.

EXAMPLE 2—continued: Suppose $X = [0, \bar{x}]$, $c(x) = x^\gamma$, $K(a) = a^{1+\lambda}/(1+\lambda)$, and $u(y) = y^\rho$ for γ, λ, and ρ all strictly positive, and $\rho < 1$. In this case, simple algebra reveals that $w^*_\mu(x)$ equals a positive constant times $x^{\gamma/\rho}$, whereas $\kappa^*_\mu(x)$ is some positive constant times x^γ, with both constants being strictly positive whenever $\mu \neq \delta_0$. Hence, if $\gamma \leq \rho$, $\pi_\mu(x) = x - (w^*_\mu(x) + \kappa^*_\mu(x))$ is strictly convex for all $\mu \neq \delta_0$, and so (by Corollary 3, part (ii)) the optimal distribution takes the form $\mu_p = p\delta_{\bar{x}} + (1-p)\delta_0$, where p solves the program detailed in (11). If $\bar{x} = 1$, then $w_{\mu_p}(0) = 0$ and $w_{\mu_p}(1) = p^{\lambda/\rho}$, and so the principal's program becomes

$$\max_{p\in[0,1]}\big[p - p^{\frac{\lambda+\rho}{\rho}}\big].$$

Clearly, the above objective is concave, and so one can solve for the optimal p using the principal's first-order condition, the solution to which is

$$p^* = \left[\frac{\rho}{\lambda+\rho}\right]^{\frac{\rho}{\lambda}}.$$

If $\gamma \geq 1 > \rho$, π_μ is strictly concave, and so part (i) of Corollary 3 implies the optimal distribution induces a single output x^*, which is determined by the program in (10). The objective in this program is given by $x - x^{\frac{\gamma}{\rho}(1+\lambda)}$. Since $\gamma > \rho$ and $\lambda > 0$, this objective is strictly concave, and so the optimal x^* is equal to the lower of $\bar{x}$ and the solution to the first-order condition; that is, $x^* = \min\{\bar{x}, [\rho/\gamma(1+\lambda)]^{\rho/[\gamma(1+\lambda)-\rho]}\}$.

Finally, if $\rho < \gamma < 1$, π_μ is neither always concave nor always convex. However, it is apparent that the function $w^*_\mu + \kappa^*_\mu$ is a non-affine analytical function whenever μ assigns positive probability to any output strictly larger than 0. As such, one can apply part (iii) of Corollary 3 to obtain that, in this case, the principal-optimal distribution must be discrete.

5. CONCLUSION

Our goal in this paper was to explore the consequences of the agent's flexibility in generating output distributions in moral hazard problems. We emphasize that our model is stylized and abstracts from many constraints agents may face in applications. We recognize that, in practice, agents may have only limited flexibility of generating output. In fact, some output distributions may not be feasible even if it is first-order stochastically dominated by a feasible one. That is, such distributions would be infinitely costly, so even our monotonicity assumption would not necessarily hold.

[14]Recall $w_\mu(\underline{x}) = u^{-1}(c_\mu(\underline{x})) = u^{-1}(0) = 0$.

We now discuss the degree to which our analysis applies to models where the agent is restricted to using a convex set $\bar{\mathcal{M}} \subseteq \mathcal{M}$ of distribution. As noted in footnote 11, replacing $\mathcal{M}$ with $\bar{\mathcal{M}}$ does not alter the validity of Lemma 1. Therefore, one can still use the lemma to characterize the wage-schemes that implement any given distribution μ. The applicability of the rest of our logic, however, depends on $\bar{\mathcal{M}}$ and the target distribution μ. For example, for the results of Section 3, we only need local flexibility.[15] Our logic also applies (with some minor modifications) to some cases without local flexibility. For a demonstration, suppose the agent faces a lower bound on the probability of each event.[16] In this case, one can apply our logic by reformulating the problem: instead of viewing the agent as choosing from a constrained set of distributions, think of the agent as flexibly choosing how to allocate the excess probability above the lower bound. With this formulation in hand, one can replicate our analysis with the obvious modifications. Nevertheless, there are many restrictions on which our analysis remains silent. For instance, one cannot use our tools in most specifications of the Holmström (1979) model.

Throughout the paper, we assumed the agent has limited liability. Instead, the literature on moral hazard often considers an outside option. In what follows, we explain how our results change if contracts are subject to a participation constraint but the agent has deep pockets.[17] Recall that the characterization of incentive-compatible contracts, Proposition 2, does not depend on such constraints and it states that these contracts differ only by a constant. The limited-liability constraint then pinned down the value of this constant at the cost-minimizing wage scheme; see Corollary 2. If the agent has an outside option but no limited liability, then the constant is determined by the binding participation constraint. Therefore, a key determinant of optimal contracts is still the Gateaux derivative of the implemented distribution.

Our analysis is based on a generalized notion of the first-order approach. We demonstrated that, unlike in the classical model, the cost-minimizing contract is not motivated by the information content of the output regarding the target distribution. Instead, optimal contracts are constructed so that the target distribution satisfies a simple first-order condition which equates the agent's marginal cost of changing the distribution locally with its marginal benefit. We also showed that optimal wage contracts are monotone whenever the agent's cost function is increasing in first-order stochastic dominance. Finally, we applied our first-order approach to the principal's profit maximization problem and provided a partial characterization of principal-optimal output distributions.

APPENDIX: PROOFS APPENDIX

PROOF OF THEOREM 1: We begin with some notation. Let $\bar{\mathcal{F}}$ be the set of CDFs over $\bar{X} = \text{co}X = [\underline{x}, \bar{x}]$, endowed with the topology of convergence in distribution, and $\bar{\mathcal{M}}$ the set of Borel measures over $\bar{X}$, endowed with its weak* topology. It is well known that the mapping taking every $F \in \mathcal{F}$ to its induced measure μ_F—that is, the measure such that $\mu_F[0, x] = F(x)$ for all x—is a linear homeomorphism between $\bar{\mathcal{F}}$ and $\bar{\mathcal{M}}$. By Theorem 1 of Wang (1993), $\bar{\mathcal{F}}$ can be viewed as a subspace of the Banach space $L^2(\bar{X}, \lambda)$, where λ is

[15]That is, μ and $\bar{\mathcal{M}}$ must be such that, for every distribution $\mu' \in \mathcal{M}$, some $\epsilon > 0$ exists for which $\mu + \epsilon(\mu' - \mu) \in \bar{\mathcal{M}}$ is feasible.

[16]Specifically, there is some $\underline{\mu} \in \mathcal{M}$ and some $b \in (0, 1)$ such that $\bar{\mathcal{M}} = \{\mu \in \mathcal{M} : \mu \geq b\underline{\mu}\}$.

[17]To formally accommodate this change, one must redefine the agent's utility u to take negative wages, and assume it is unbounded both from above and from below.

the Lebesgue measure. Let $\mathcal{F}$ be the set of CDFs whose support is contained in X, and define the function

$$\hat{C} : L^2(\bar{X}, \lambda) \to \mathbb{R} \cup \{\infty\},$$

$$\phi \mapsto \begin{cases} C(\mu_\phi) & \text{if } \phi \in \mathcal{F}, \\ \infty & \text{otherwise.} \end{cases}$$

Given a CDF $F \in \mathcal{F}$, define the subdifferential of $\hat{C}$ at F as

$$\partial\hat{C}(F) = \left\{ \phi \in L^2(\bar{X}, \lambda) : \hat{C}(\varphi) \geq \hat{C}(F) + \int \phi(x)(\varphi - F)(x)\lambda(\mathrm{d}x) \forall \varphi \in L^2(\bar{X}, \lambda) \right\}.$$

In general, $\partial\hat{C}$ might be empty. Let $\mathcal{F}_I = \{F \in \bar{\mathcal{F}} : \partial\hat{C}(F) \neq \emptyset\}$ be the set of all CDFs at which $\partial\hat{C}$ is non-empty. Since $\mathcal{F}$ is convex, and C is convex and continuous, it follows $\hat{C}$ is convex and lower semicontinuous. Noting $\hat{C}$ is also proper, it follows from the Brondsted–Rockafellar theorem (Brøndsted and Tyrrell Rockafellar (1965)) that $\mathcal{F}_I$ is dense in $\mathcal{F}$.

Given $\mu \in \mathcal{M}$, define F_μ to be the CDF such that $\mu_{F_\mu} = \mu$. To conclude the proof, we argue μ is implementable whenever $F_\mu \in \mathcal{F}_I$ (observe this set is dense due to $\mathcal{F}$ and $\mathcal{M}$ being homeomorphic). Indeed, let $\phi \in \partial\hat{C}(F_\mu)$, and define $\Phi(x) := \int_0^x \phi(\tilde{x})\mathrm{d}\tilde{x}$, where the integral is viewed as a Riemann–Stieltjes integral. Note

$$w(x) := u^{-1}\big(\max\Phi(X) - \Phi(x)\big)$$

is well-defined because $\max\Phi(X) - \Phi(x) \in u(\mathbb{R}_+)$ for all x. Then, for every $\mu' \in \mathcal{M}$,

$$\begin{aligned} C(\mu') = \hat{C}(F_{\mu'}) &\geq \hat{C}(F_\mu) + \int \phi(x)(F_{\mu'} - F_\mu)(x)\,\mathrm{d}x \\ &= \hat{C}(F_\mu) - \int (F_{\mu'} - F_\mu)(x)\Phi(\mathrm{d}x) \\ &= \hat{C}(F_\mu) - \int \Phi(x)(F_{\mu'} - F_\mu)(\mathrm{d}x) \\ &= C(\mu) + \int (-\Phi)(x)\big(\mu' - \mu\big)(\mathrm{d}x), \end{aligned}$$

where the inequality follows from $\phi \in \partial\hat{C}(F_\mu)$, and the penultimate equality follows from integration by parts. Thus, we have

$$\begin{aligned} \mu &\in \arg\max_{\mu' \in \mathcal{M}} \int (-\Phi)(x)\mu'(\mathrm{d}x) - C(\mu') \\ &= \arg\max_{\mu' \in \mathcal{M}} \int \big(\max\Phi(X) - \Phi\big)(x)\mu'(\mathrm{d}x) - C(\mu') \\ &= \arg\max_{\mu' \in \mathcal{M}} \int u\big(w(x)\big)\mu'(\mathrm{d}x) - C(\mu'), \end{aligned}$$

as required. *Q.E.D.*

PROOF OF LEMMA 2: Observe first that, by Corollary 2, $W(\mu) = \int [w^*_\mu(x)]\mu(\mathrm{d}x)$. We begin by showing that

$$\mu \mapsto \int w^*_\mu(x)\mu(\mathrm{d}x) = \int u^{-1} \circ c_\mu(x)\mu(\mathrm{d}x)$$

is continuous. To this end, take any sequence $(\mu_n)_{n\in\mathbb{N}}$ that converges to some limit μ_∞. We first claim that

$$\lim_{n\to\infty} \left\| w^*_{\mu_n} - w^*_{\mu_\infty} \right\|_\infty = 0. \tag{12}$$

To prove this claim, fix some $\epsilon > 0$, take $T := [\min c_{\mu_\infty}(X) - \epsilon, \max c_{\mu_\infty}(X) + \epsilon]$, and let $S := u^{-1}(T) = [u^{-1}(\min c_{\mu_\infty}(X) - \epsilon), u^{-1}(\max c_{\mu_\infty}(X) + \epsilon)]$. Note that because u has a continuous and strictly positive derivative, $\bar{b} := \min_{s\in S}[u'(s)]$ is well-defined and strictly positive, and so one can apply the Inverse Function Theorem to obtain that, for all $t \in T$, $\frac{\mathrm{d}u^{-1}}{\mathrm{d}t}(t) = \frac{1}{u'(u^{-1}(t))}$ is well-defined, strictly positive, and bounded from above by $\bar{b}$. Therefore, the Mean Value Theorem implies that

$$\left| u^{-1}(t) - u^{-1}(t') \right| \leq \bar{b}\left| t - t' \right| \quad \text{for all } t, t' \in T. \tag{13}$$

To conclude the proof of the claim, fix some $\eta < \min\{\epsilon, \epsilon/\bar{b}\}$. By Assumption 3, an $N \in \mathbb{N}$ exists such that $\|c_{\mu_n} - c_{\mu_\infty}\|_\infty < \eta$ for all $n > N$. Therefore, all such n, $c_{\mu_n}(x)$ must be in T for all $x \in X$, and

$$\left\| w^*_{\mu_n} - w^*_{\mu_\infty} \right\|_\infty = \left\| u^{-1}(c_{\mu_n}) - u^1(c_{\mu_\infty}) \right\|_\infty \leq \bar{b}\|c_{\mu_n} - c_{\mu_\infty}\| \leq \bar{b}\eta < \epsilon,$$

where the first inequality follows from (13), and the last from choice of η. Since ϵ was arbitrary, we have proven the claim that (12) holds.

Armed with (12), one can prove $\int w^*_\mu(x)\mu_n(\mathrm{d}x) \xrightarrow{n\to\infty} \int w^*_{\mu_\infty}(x)\mu_\infty(\mathrm{d}x)$ using the following inequality chain:

$$\begin{aligned}
&\left| \int w^*_{\mu_n}(x)\mu_n(\mathrm{d}x) - \int w^*_{\mu_\infty}(x)\mu_\infty(\mathrm{d}x) \right| \\
&\quad \leq \left| \int \left[w^*_{\mu_n}(x) - w^*_{\mu_\infty}(x)\right]\mu_\infty(\mathrm{d}x) \right| + \left| \int w^*_{\mu_\infty}(x)(\mu_n - \mu_\infty)(\mathrm{d}x) \right| \\
&\qquad + \left| \int \left[w^*_{\mu_n}(x) - w^*_{\mu_\infty}(x)\right](\mu_n - \mu_\infty)(\mathrm{d}x) \right| \\
&\quad \leq \int \left|w^*_{\mu_n}(x) - w^*_{\mu_\infty}(x)\right|\mu_\infty(\mathrm{d}x) + \int \left|w^*_{\mu_\infty}(x)\right|(\mu_n - \mu_\infty)(\mathrm{d}x) \\
&\qquad + \int \left|w^*_{\mu_n}(x) - w^*_{\mu_\infty}(x)\right|(\mu_n - \mu_\infty)(\mathrm{d}x) \\
&\quad \xrightarrow{n\to\infty} 0,
\end{aligned}$$

where convergence of the first and third term follows from (12), and convergence of the middle term follows from $\mu_n \to \mu_\infty$ and $|w^*_{\mu_\infty}(\cdot)|$ being continuous.

Next, we prove that $\mu \mapsto \int w^*_\mu(x)\mu(\mathrm{d}x)$ is a Gateaux differentiable function admitting $w^*_\mu + \kappa^*_\mu(x)$ as its derivative. To this end, fix some $\tilde{\mu} \in \mathcal{M}$, and let $\mu_\epsilon = \mu + \epsilon(\tilde{\mu} - \mu)$.

Observe

$$\begin{aligned}
&\frac{1}{\epsilon}\left[\int w^*_{\mu_\epsilon}(x)\mu_\epsilon(\mathrm{d}x) - \int w^*_\mu(x)\mu(\mathrm{d}x)\right] \\
&\quad = \int w^*_\mu(x)(\tilde{\mu}-\mu)(\mathrm{d}x) + \int \frac{1}{\epsilon}\left[w^*_{\mu_\epsilon} - w^*_\mu\right](x)\mu(\mathrm{d}x) \\
&\qquad + \int \left[w^*_{\mu_\epsilon} - w^*_\mu\right](x)(\tilde{\mu}-\mu)(\mathrm{d}x).
\end{aligned}$$

Since the last term converges to zero as $\epsilon \searrow 0$ by continuity of w^*_μ, it is enough to show that

$$\lim_{\epsilon\searrow 0}\int \frac{1}{\epsilon}\left[w^*_{\mu_\epsilon} - w^*_\mu\right](x)\mu(\mathrm{d}x) = \int \kappa^*_\mu(y)(\tilde{\mu}-\mu)(\mathrm{d}y).$$

We now argue that, to show the above equality, it is sufficient to find a function $\phi : X \to \mathbb{R}$ that is integrable with respect to $(\tilde{\mu}-\mu)$, and an $\bar{\epsilon} \in (0, 1)$ such that $|w^*_{\mu_\epsilon} - w^*_\mu| \leq \phi$ for all $\epsilon \in (0, \bar{\epsilon})$. To see why, note $\lim_{\epsilon\searrow 0} \|c_{\mu_\epsilon}(x) - c_\mu(x)\|_\infty = 0$ holds by Assumption 3, and so

$$\lim_{\epsilon\searrow 0}\left(\frac{u^{-1}\big(c_{\mu_\epsilon}(x)\big) - u^{-1}\big(c_\mu(x)\big)}{c_{\mu_\epsilon}(x) - c_\mu(x)}\right) = \frac{1}{u' \circ u^{-1}\big(c_\mu(x)\big)} = \frac{1}{u'\big(w^*_\mu(x)\big)}.$$

It follows that, for every x,

$$\begin{aligned}
\lim_{\epsilon\searrow 0}\frac{1}{\epsilon}\big(w^*_{\mu_\epsilon}(x) - w^*_\mu(x)\big) &= \lim_{\epsilon\searrow 0}\frac{1}{\epsilon}\big(c_{\mu_\epsilon}(x) - c_\mu(x)\big)\left(\frac{u^{-1}\big(c_{\mu_\epsilon}(x)\big) - u^{-1}\big(c_\mu(x)\big)}{c_{\mu_\epsilon}(x) - c_\mu(x)}\right) \\
&= \int \frac{h_\mu(x,y)}{u'\big(w^*_\mu(x)\big)}(\tilde{\mu}-\mu)(\mathrm{d}y).
\end{aligned}$$

Therefore, if a function ϕ as described above exists, the Lebesgue Dominated Convergence Theorem would imply that

$$\begin{aligned}
\lim_{\epsilon\searrow 0}\int \frac{1}{\epsilon}\left[w^*_{\mu_\epsilon} - w^*_\mu\right](x)\mu(\mathrm{d}x) &= \int \lim_{\epsilon\searrow 0}\frac{1}{\epsilon}\left[w^*_{\mu_\epsilon} - w^*_\mu\right](x)\mu(\mathrm{d}x) \\
&= \int \frac{h_\mu(x,y)}{u'\big(w^*_\mu(x)\big)}(\tilde{\mu}-\mu)(\mathrm{d}y)\mu(\mathrm{d}x) \\
&= \int \frac{h_\mu(x,y)}{u'\big(w^*_\mu(x)\big)}\mu(\mathrm{d}x)(\tilde{\mu}-\mu)(\mathrm{d}y) = \int \kappa^*_\mu(y)(\tilde{\mu}-\mu)(\mathrm{d}y),
\end{aligned}$$

as required.

We now find such a ϕ. Fix some $\eta > 0$, and note that Assumption 3 implies there is some $\bar{\epsilon} \in (0, 1)$ such that, for all $\epsilon \in (0, \bar{\epsilon})$ and all x,

$$\big|c_{\mu_\epsilon}(x) - c_\mu(x)\big| \leq \left|\int h_\mu(x,y)(\tilde{\mu}-\mu)(\mathrm{d}y)\right| + \eta.$$

Let

$$\bar{c} = \max_{x \in X}\left[c_\mu(x) + \left|\int h_\mu(x, y)(\tilde{\mu} - \mu)(\mathrm{d}y)\right|\right],$$

and take

$$b = \max_{y \in [0, \bar{c}+\eta]} \left(u^{-1}\right)'(y) = \max_{y \in [0, \bar{c}+\eta]} \frac{1}{u' \circ u^{-1}(y)},$$

which is finite and strictly positive, because u^{-1} is continuous and u' is strictly positive and continuous. Observe that, for every $\epsilon < \bar{\epsilon}$, and every x, the Mean Value Theorem implies there is some $a \in \operatorname{co}\{c_\mu(x), c_{\mu_\epsilon}(x)\} \subseteq [0, \bar{c} + \eta]$ such that

$$\frac{u^{-1}\left(c_{\mu_\epsilon}(x)\right) - u^{-1}\left(c_\mu(x)\right)}{c_{\mu_\epsilon}(x) - c_\mu(x)} = \left(u^{-1}\right)'(a) \leq b.$$

Therefore, for all $\epsilon < \bar{\epsilon}$ and every x,

$$\begin{aligned}
\left|\frac{1}{\epsilon}\left(w^*_{\mu_\epsilon}(x) - w^*_\mu(x)\right)\right| &= \left|\frac{1}{\epsilon}\left(c_{\mu_\epsilon}(x) - c_\mu(x)\right)\left(\frac{u^{-1}\left(c_{\mu_\epsilon}(x)\right) - u^{-1}\left(c_\mu(x)\right)}{c_{\mu_\epsilon}(x) - c_\mu(x)}\right)\right| \\
&\leq \frac{1}{\epsilon}\left|c_{\mu_\epsilon}(x) - c_\mu(x)\right|\left|\frac{u^{-1}\left(c_{\mu_\epsilon}(x)\right) - u^{-1}\left(c_\mu(x)\right)}{c_{\mu_\epsilon}(x) - c_\mu(x)}\right| \\
&\leq \frac{b}{\epsilon}\left|c_{\mu_\epsilon}(x) - c_\mu(x)\right| \leq b\left(\int h_\mu(x, y)(\tilde{\mu} - \mu)(\mathrm{d}y)\right) + \eta.
\end{aligned}$$

Thus, setting $\phi(x) = \eta + \int b h_\mu(x, y)(\tilde{\mu} - \mu)(\mathrm{d}y)$ gives the desired function. This concludes the proof. *Q.E.D.*

Monotonicity of Gateaux Derivative. Next, we prove a result that is required for generalizing Corollary 2 for the case in which Assumption 2 is relaxed to make c_μ bounded and measurable (see footnote 8 in the main text).

LEMMA 3: *Suppose $C : \mathcal{M} \to \mathbb{R}$ is such that, for every μ, there is a bounded measurable function $c_\mu : [0, 1] \to \mathbb{R}$ and*

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon}\left[C\left(\mu + \epsilon\left(\mu' - \mu\right)\right) - C(\mu)\right] = \int c_\mu(x)\left(\mu' - \mu\right)(\mathrm{d}x)$$

for all $\mu' \in \mathcal{M}$. If C also satisfies Assumption 1, then c_μ is increasing.

PROOF: Fix any $y, z \in X$ such that $y > z$. Observe that, for any $\epsilon \in [0, 1]$, the distribution $\mu_{\epsilon,y} := \mu + \epsilon(\delta_y - \mu)$ first-order dominates $\mu_{\epsilon,z} := \mu + \epsilon(\delta_z - \mu)$, where δ_y and δ_z are the distributions that respectively generate the outputs y and z for sure. We therefore obtain the following inequality chain:

$$
\begin{aligned}
0 &\leq \frac{1}{\epsilon}\big[C(\mu_{\epsilon,y}) - C(\mu_{\epsilon,z})\big] \\
&= \frac{1}{\epsilon}\big[C(\mu_{\epsilon,y}) - C(\mu)\big] + \frac{1}{\epsilon}\big[C(\mu) - C(\mu_{\epsilon,z})\big] \\
&\xrightarrow{\epsilon\searrow 0} \int c_\mu(x)(\delta_y - \mu)(\mathrm{d}x) - \int c_\mu(x)(\delta_z - \mu)(\mathrm{d}x) \\
&= \int c_\mu(x)(\delta_y - \delta_z)(\mathrm{d}x) = c_\mu(y) - c_\mu(z).
\end{aligned}
$$

The result follows. *Q.E.D.*

Increasing Wages Without Differentiability. Proposition 1 shows that, under Assumption 1, (w, μ) is IC only if the wage w is μ-increasing. This result leaves open the possibility that the wage is non-increasing in outputs that never arise under μ. Corollary 2 shows one can close this gap if C also satisfies Assumption 2. This part of the Appendix closes this gap without using Assumption 2. Specifically, we prove the following theorem.

THEOREM 3: *Suppose (w, μ) is IC and C satisfies Assumption 1. Then a $\bar{w} \in \mathcal{W}$ exists such that $\bar{w}$ is increasing, $\bar{w} = w$ μ-almost surely, and $(\bar{w}, \mu)$ is IC.*

Before proving the theorem, we present the following lemma, which generalizes Proposition 1. In what follows, all measurability statements are made with respect to the Borel σ-algebra.

LEMMA 4: *Suppose Assumption 1 holds and (w, μ) is IC. Let $f : X \to X$ be a measurable function such that $f(x) \leq x$ for all x. Then $w(x) \geq w(f(x))$ μ-almost surely.*

PROOF: By way of contradiction, suppose $\mu\{x : w(x) < w(f(x))\} > 0$. Define

$$
g(x) = \begin{cases} x & \text{if } w(x) \geq w\big(f(x)\big), \\ f(x) & \text{if } w(x) < w\big(f(x)\big). \end{cases}
$$

Note that w and f are measurable, and so g is measurable as well. Let $\nu := \mu \circ g^{-1}$ be the push-forward measure of g—that is, for every Borel $Y \subset X$, $\nu(Y) = \mu(g^{-1}(Y))$. Since μ first-order stochastically dominates ν, $C(\nu) \leq C(\mu)$. Moreover,

$$
\int w(x)(\nu - \mu)(\mathrm{d}x) = \int \big(w\big(g(x)\big) - w(x)\big)\mu(\mathrm{d}x) > 0,
$$

where the strict inequality follows from our contradiction assumption. Thus, we have that

$$
\int w(x)\nu(\mathrm{d}x) - C(\nu) > \int w(x)\mu(\mathrm{d}x) - C(\mu),
$$

which contradicts (w, μ) being IC. *Q.E.D.*

Define the function $\bar{w}: X \to \mathbb{R}_+$ via

$$\bar{w}(x) := \sup w(\{y \in X : y \le x\}).$$

Note $\bar{w}$ is increasing, and therefore measurable. Moreover, because w is bounded, $\bar{w}$ is bounded as well.

LEMMA 5: *For every $\nu \in \mathcal{M}$ and every $\epsilon > 0$, a measurable function $f: X \to X$ exists such that $f(x) \le x$ for all x, and $w \circ f(x) \ge [\bar{w}(x) - \epsilon]$ ν-almost surely.*

PROOF: Fix $\epsilon > 0$. Define $g_1 : X \times X \to \mathbb{R}^2$ via $g_1(x, y) = (\bar{w}(x), w(y))$, and $g_2 : \mathbb{R}^2 \to \mathbb{R}$ via $g(a, b) = \mathbf{1}_{b \ge (a-\epsilon)}$. Note g_1 and g_2 are both (Borel) measurable, and so $h(x, y) = g_2 \circ g_1(x, y)$ is a measurable function from $X \times X$ to $\mathbb{R}$.

Define the correspondence $H : X \rightrightarrows X$ via

$$H(x) = \{y : w(y) \ge \bar{w}(x) - \epsilon\} = \{y : (x, y) \in h^{-1}(1)\}.$$

Notice that graph$(H) = h^{-1}(1)$, which is a measurable subset of $X \times X$. Therefore, H has a measurable graph. By Aliprantis and Border (2006), Corollary 18.26, a measurable function $f : X \to X$ exists such that $f(x) \in H(x)$ on a ν-almost sure set $\tilde{X} \subset X$. Editing $f(x)$ such that $f(x) = x$ on the complement of $\tilde{X}$ delivers the result. *Q.E.D.*

PROOF OF THEOREM 3: Since $\bar{w}$ is obviously increasing, we need only to show that $\bar{w} = w$ μ-almost surely, and that $(\bar{w}, \mu)$ is IC. We first claim that $\bar{w} = w$ μ-almost surely. Since $\bar{w} \ge w$ by definition, to prove the claim it suffices to show that $\bar{w} \le w$ μ-almost surely. Suppose $\mu\{x : \bar{w}(x) > w(x)\} > 0$. Then some $\epsilon > 0$ exists such that $\mu\{x : w(x) + \epsilon < \bar{w}(x)\} > 0$. Let $f : X \to X$ be the measurable function from Lemma 5 that satisfies $f(x) \le x$ for all x, and $w \circ f(x) \ge [\bar{w}(x) - 0.5\epsilon]$ μ-almost surely. Then,

$$\begin{aligned} 0 < \mu\{x : w(x) + \epsilon < \bar{w}(x)\} &= \mu\{x : w(x) < \bar{w}(x) - \epsilon < w \circ f(x)\} \\ &\le \mu\{x : w(x) < w \circ f(x)\} = 0, \end{aligned}$$

where the first equality follows from $w \circ f(x) \ge [\bar{w}(x) - 0.5\epsilon]$ μ-almost surely, and the last equality from Lemma 4. It follows $\mu\{x : \bar{w}(x) > w(x)\} = 0$, meaning $\bar{w} = w$ μ-almost surely.

Next, we claim that $(\bar{w}, \mu)$ is IC. Suppose by way of a contradiction some $\nu \in \mathcal{M}$ and $\epsilon > 0$ exist such that

$$\int \bar{w}(x)\nu(\mathrm{d}x) - C(\nu) > \int \bar{w}(x)\mu(\mathrm{d}x) - C(\mu) + \epsilon.$$

Let $g : X \to X$ be such that $g(x) \le x$ for all x, and that $w \circ g \ge [\bar{w} - \epsilon]$ ν-almost surely (such a g exists by Lemma 5). Define $\tilde{\nu} = \nu \circ g^{-1}$ to be the push-forward measure defined by ν and g—that is, $\tilde{\nu}(Y) = \nu \circ g^{-1}(Y)$ for all Borel Y. We claim $\tilde{\nu}$ delivers the agent strictly higher utility under w than μ does, thereby contradicting that (w, μ) is IC. This claim is implied by the following inequality chain:

$$\begin{aligned} \int w(x)\tilde{\nu}(\mathrm{d}x) - C(\tilde{\nu}) &\ge \int [\bar{w}(x) - \epsilon]\nu(\mathrm{d}x) - C(\tilde{\nu}) \ge \int \bar{w}(x)\nu(\mathrm{d}x) - C(\nu) - \epsilon \\ &> \int \bar{w}(x)\mu(\mathrm{d}x) - C(\mu) = \int \bar{w}(x)\mu(\mathrm{d}x) - C(\mu), \end{aligned}$$

where the first inequality comes from $w \circ g \geq \bar{w} - \epsilon$ holding ν-almost surely, the second inequality from ν first-order stochastically dominating $\tilde{\nu}$, the third inequality from the contradiction assumption, and the equality from $\bar{w} = w$ μ-almost surely. Thus, we have shown (w, μ) is IC. The proof is now complete. *Q.E.D.*

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