Q for the Long Run*

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Abstract

Traditional Q theory relates a firm's investment to its value of Q at all frequencies; weekly or even daily fluctuations in Q should be just as informative for investment decisions as quarterly or annual data. We develop a model in which investment is more responsive to Q at long horizons than at short horizons; instantaneous investment is responsive to contemporaneous cash flow. These effects arise because a firm's value depends on both its existing capital and its available technologies, even if they are not yet installed. In contrast, the firm's current investment depends only on the currently installed technology. Thus, the value of the firm, and hence Tobin's Q, are "too forward-looking" relative to the investment decision. Cash flow, on the other hand, reflects only current technology and demand. The excessively forward-looking information in Tobin's Q, while extraneous to high-frequency investment decisions, does predict future adoptions of the frontier technology. In this way, it is a better predictor of long-run investment than of short-run investment. Short-run investment is better predicted by the firm's cash flow.

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1 Introduction

The value of a firm includes the market value of its current capital and technology in place as well as its expected future technologies. This value of the firm should be informative for investment decisions, as suggested by Brainard and Tobin (1968), who introduced the idea that "the market valuation of equities, relative to the replacement cost of the physical assets they represent, is the main determinant of new investment," (pages 103-4). Tobin (1969) dubbed this ratio "q" - thereafter generally called "Tobin's Q". This idea was introduced without explicit optimization. It is based on Keynes's (1936) notion that when the market value of installed capital exceeds the price of uninstalled capital, the firm has an incentive to invest. This approach, even in its formalizations that include adjustment costs, does not distinguish between investment at different horizons: daily fluctuations in Q should be just as informative for investment decisions as quarterly or annual changes in value.

The theoretical development of Q theory has tended to emphasize the market value of capital currently in place, rather than the value of future technologies. Future technologies are not ignored or omitted, but in most formulations future technologies are simply overgrown versions of current technology. In the current paper, we allow a more flexible view of the future in which the firm chooses when to adopt a frontier technology, which evolves stochastically. Because it is costly to adopt the frontier technology, the firm will upgrade its technology at points of time separated by finite intervals. When the firm upgrades its technology to the frontier, its marginal product of capital jumps upward. Because we assume that the capital stock can be adjusted costlessly, reversibly, and instantaneously, the firm increases its capital stock by a discrete amount when it upgrades its technology. We refer to the jumps in the capital stock that accompany technology upgrades as "gulps" of investment. During intervals between upgrades, the firm undertakes continuous investment in response to factors, such as the demand for its product or the user cost of capital, that vary over time. Investment data that are observed over regular intervals of time, such as quarterly or annually, will exhibit both continuous investment and investment gulps. In the model we introduce here, there is a sharp dichotomy between the response of continuous investment to Q and the response of investment gulps to Q: continuous investment is independent of Q, but investment gulps are positively related to Q.

When a firm has access to an available, but uninstalled, frontier technology, its value depends not only on profits that can be earned with its current technology, but also on the status of the frontier technology. The firm's value includes "growth options,"[1] which reflect the prospect of future upgrades in technology and of the increases in the optimal capital stock that accompany these upgrades. Movements in the frontier technology generate volatility in the growth options component of the firm's value and Tobin's Q that are unrelated to current profitability. While high-frequency movements in Q are accurate reflections of the firm's value, from the viewpoint of current investment, Tobin's Q is too forward-looking. However, the firm's cash flow provides information relevant to current investment.

Because Tobin's Q contains information about future optimal adoptions of the frontier technology and the accompanying investment gulps, it is a good predictor of investment in the long run. The frontier technology is adopted when it surpasses the technology currently in place by an amount sufficient to compensate for the cost of adopting the new technology. Tobin's Q takes account of the quality of the frontier technology relative to that of the currently installed technology, and thus is an indicator of when the firm will upgrade its technology. Because episodic technology adoptions are generally accompanied by capital investment, Tobin's Q will predict investment episodically, even though it does not predict investment well at high frequencies.

Empirical studies of investment typically find that investment is positively related to Tobin's Q and to cash flow. In addition, capital expenditures tend to be temporally concentrated, or "lumpy." At a more casual level, the stock market, and hence Tobin's Q, exhibits sharp high-frequency movements that do not tend to be mirrored by similar sharp high-frequency fluctuations in investment expenditures. Our model is consistent with all of these features of the data. Continuous investment is positively related to cash flow in our model, even though firms do not face any sort of financing constraints. Because continuous investment is independent of Tobin's Q in our model, high-frequency fluctuations in Q, arising from high-frequency fluctuations in the expected present value of future frontier technologies, do not induce

[1] The analysis of investment projects as options was originated by McDonald and Siegel (1986) for the case of discrete projects, and first applied to continuous capital investment by Pindyck (1988) and Bertola (1988). This paper is in the spirit of that earlier work in emphasizing the role of growth options in firm value. Those earlier papers, however, assumed that capital investments were irreversible, whereas in the current paper investment is frictionless and costlessly reversible. Instead of assuming that the firm cannot reduce its capital stock (or that the firm never chooses to reduce its capital stock), as in the irreversibility literature, we develop a model in which the firm will never choose to reduce its level of technology.

high-frequency fluctuations in investment. Though Tobin's Q is irrelevant for continuous investment, we do not conclude, as suggested by Bond and Cummins (2000) for example, that market valuations are a "sideshow" for firm-level investment and may contain persistent measurement error or deviations from fundamentals. Indeed, market valuations, as captured by Tobin's Q are useful for predicting "gulps" of investment. Investment gulps contribute to increased temporal concentration of investment.

In other work (Abel and Eberly (2002a)) we also examine the relationship among investment, Tobin's Q, and cash flow, but without an endogenous technology choice. In that model, installed technology evolves exogenously, costlessly, and continuously, and there are no investment gulps. Continuous investment is positively related to both cash flow and, unlike in the current paper, Tobin's Q. In that paper, the instantaneous drift in the firm's operating profits varies over time. An increase, for example, in this drift increases Tobin's Q and increases the optimal rate of capital accumulation, thereby leading to a positive relationship between continuous investment and Tobin's Q. In the current paper, the instantaneous drift in operating profits is constant over time, and thus the link between Tobin's Q and continuous investment is severed. Instead, variations in Tobin's Q arise from variations in the frontier technology relative to the currently installed technology. These variations in Tobin's Q are positively related to investment gulps.

The paper begins with the static optimization of operating profits and determination of the optimal capital stock, given the level of technology, in Section 2. We then consider the optimal upgrade of technology in Section 3, where we also examine the value of the firm with access to the frontier technology. In Section 4, we use the solution for the value of the firm to calculate Tobin's Q . We examine continuous investment in Section 5 where we show that continuous investment depends on cash flow but is independent of Tobin's Q. In Section 6 we analyze the investment "gulps" that accompany the firm's discrete technology upgrades, and we show that Tobin's Q helps to predict these gulps of investment. We also examine the role of investment gulps in the temporal concentration of investment. Section 7 offers concluding remarks.

2 Operating Profits and Static Optimization

Since there are no costs of adjusting the capital stock, the optimal capital stock is chosen to maximize static operating profits. The firm's revenue, net of flexible factors other than capital, is given by $(A_tY_t)^{1-\gamma} K_t^{\gamma}$, where A_t is the level of technology, Y_t is the level of demand (which may also represent wages or the prices of other flexible factors), and K_t is the capital stock.[2] We assume that $0 < \gamma < 1$, so that the firm has decreasing returns to scale in production or market power in the output market. Operating profits are net revenue less the user cost of capital, or

$$\pi_t = (A_tY_t)^{1-\gamma} K_t^{\gamma} - u_tp_tK_t \tag{1}$$

where p_t is the price of capital, which grows deterministically at rate μ_p, $u_t \equiv r+\delta_t-\mu_p$ is the user cost factor, where r is the discount rate and δ_t is the depreciation rate of capital at time t,[3] and u_tp_t is the user cost of a unit of capital. Since there are no adjustment costs on capital (and capital investment is completely reversible), the optimal capital stock at each point in time is determined by maximizing operating profits in equation (1) with respect to K_t. It is straightforward to show that[4] the

[2]The fact that that A_t and Y_t are raised to the $1-\gamma$ power in the revenue function reflects a convenient normalization that exploits the fact that if a variable x_t is a geometric Brownian motion, then x_t^{α} is also a geometric Brownian motion.

[3] We allow the depreciation rate to be stochastic to motivate the stochastic user cost of capital. Specifically, since the user cost factor is $u_t \equiv r + \delta_t - \mu_p$, the increment to the user cost factor, u_t, equals the increment to the depreciation rate, $du_t = d\delta_t$.

[4]Differentiating the right-hand side of equation (1) with respect to K_t, and setting the derivative equal to zero yields

$$\gamma \left(\frac{A_tY_t}{K_t}\right)^{1-\gamma} = u_tp_t. \tag{*}$$

Solving this first-order condition for the optimal capital stock yields

$$K_t = A_tY_t \left(\frac{\gamma}{u_tp_t}\right)^{\frac{1}{1-\gamma}}. \tag{**}$$

Substituting equation (**) into the operating profit function in equation (1) yields optimized operating profits

$$\pi_t = u_tp_tK_t \left(\frac{1-\gamma}{\gamma}\right) = A_tY_t \left(\frac{\gamma}{u_tp_t}\right)^{\frac{\gamma}{1-\gamma}} (1-\gamma). \tag{***}$$

Use the definition of X_t in equation (4) to rewrite equation (**) as equation (2) and equation (***) as equation (3).

optimal capital stock is

$$K_t = \frac{A_t X_t}{u_t p_t} \frac{\gamma}{1-\gamma}, \tag{2}$$

and the optimized value of operating profits is

$$\pi_t = A_t X_t, \tag{3}$$

where

$$X_t \equiv Y_t \left(\frac{\gamma}{u_t p_t} \right)^{\frac{\gamma}{1-\gamma}} (1-\gamma) \tag{4}$$

summarizes the sources of non-technology uncertainty about operating profits. We assume that X_t follows a geometric Brownian motion

$$dX_t = mX_t dt + sX_t dz_X, \tag{5}$$

where the drift, m, and instantaneous variance, s^2, depend on the drifts and instantaneous variances and covariances of the underlying processes for Y_t, u_t, and p_t. We assume that the user cost factor, u_t, follows a driftless Brownian motion, with instantaneous variance σ_u^2. As mentioned above, the price of capital, p_t, grows at a constant rate μ_p.[5]

Anticipating the analysis of the firm's cash flow beginning in Section 5, here we calculate the firm's cash flow and relate it to operating profits and to the user cost factor. The firm's cash flow before investment expenditure is given by $C_t \equiv (A_t Y_t)^{1-\gamma} K_t^{\gamma}$. Equations (1), (3) and equation (**) in footnote (4) imply that

$$C_t \equiv \frac{\pi_t}{1-\gamma} = \frac{A_t X_t}{1-\gamma}. \tag{6}$$

When cash flow is used in empirical investment equations, it is usually normalized by the replacement cost of the capital stock, $p_t K_t$. Therefore, we define the cash flow-capital stock ratio as

$$c_t \equiv \frac{C_t}{p_t K_t}. \tag{7}$$

[5] If Y_t, u_t, and p_t are geometric Brownian motions, then the composite term X_t also follows a geometric Brownian motion. Specifically, let the instantaneous drift of the process for Y_t be μ_Y and its instantaneous variance be σ_Y^2. Then given our specification of the processes for u_t and p_t, $m \equiv \mu_Y - \frac{\gamma}{1-\gamma}\left[\mu_p - \frac{1}{2}\frac{\sigma_u^2}{1-\gamma} + \rho_{Yu}\sigma_Y\sigma_u\right]$ and $sdz_X = \sigma_Y dz_Y - \frac{\gamma\sigma_u}{1-\gamma}dz_u$, where $\rho_{Yu} \equiv \frac{1}{dt}E(dz_Y dz_u)$ is the correlation between the shocks to Y_t and u_t. In addition, $s^2 = \sigma_Y^2 + \left(\frac{\gamma\sigma_u}{1-\gamma}\right)^2 - 2\frac{\gamma}{1-\gamma}\rho_{Yu}\sigma_Y\sigma_u$; $s\rho_{Xu} = \rho_{Yu}\sigma_Y - \frac{\gamma}{1-\gamma}\sigma_u$; and $s\rho_{X\widehat{A}} = \rho_{Y\widehat{A}}\sigma_Y - \frac{\gamma}{1-\gamma}\rho_{u\widehat{A}}\sigma_u$, where $\rho_{ij} \equiv \frac{1}{dt}E(dz_i dz_j)$.

This ratio is proportional to the user cost factor, specifically,[6]

$$c_t = \frac{u_t}{\gamma} = \frac{1}{\gamma}\left(r + \delta_t - \mu_p\right). \tag{8}$$

The technology variable, A_t, represents the firm's currently installed technology. The firm also has the choice to upgrade to the available technology, $\widehat{A}_t$, which evolves exogenously according to the geometric Brownian motion

$$d\widehat{A}_t = \mu \widehat{A}_t dt + \sigma \widehat{A}_t dz_{\widehat{A}}. \tag{9}$$

The correlation between the innovations to X_t and $\widehat{A}_t$ is $\frac{1}{dt}E(dz_X dz_{\widehat{A}}) \equiv \rho_{X\widehat{A}}$, and we assume that $\mu > \frac{1}{2}\sigma^2$.[7]

3 Optimal Upgrades and The Value of the Firm

We have shown how the optimal capital stock and the maximized value of operating profits depend on the level of installed technology, A_t. In this section, we analyze the firm's decision about when to upgrade to the frontier technology, $\widehat{A}_t$. We assume that the cost of upgrading to the frontier technology, $\widehat{A}_t$, at time t, is $\theta_t \widehat{A}_t X_t$, where $\theta \geq 0$ is a constant. Since $\widehat{A}_t X_t$ is the firm's operating profit at time t if the frontier technology is adopted at time t, one interpretation is that cost of upgrading is the cost of slowing operations while bringing the new technology on line.[8] Since the cost of upgrading depends only on the exogenous variables $\widehat{A}_t$ and X_t, we will describe this cost as a fixed cost. Because upgrading incurs a fixed cost, it will not be optimal to upgrade continuously. Upgrades occur at discrete times τ_j, $j = 0, 1, 2, ...$, which the firm determines optimally.

In the process of deriving the optimal timing of technology upgrades, we will derive the value of the firm. It is easiest to begin with a firm that does not own any capital.

[6]Use the first equality in equation (***) in footnote 4 to substitute for π_t in the definition of C_t from equation (6). Using the definition of c_t in equation (7), this yields $c_t \equiv \frac{C_t}{p_t K_t} = \frac{u_t p_t K_t}{\gamma p_t K_t} = \frac{u_t}{\gamma}$.

[7]The assumption that $\mu > \frac{1}{2}\sigma^2$ guarantees that the first passage times we calculate below are finite. We also assume initial conditions X_0, $\widehat{A}_0, u_0, p_0 > 0$.

[8]We specify the cost of upgrading as $\theta \widehat{A}_t X_t$, which is a share of operating profits evaluated at the post-upgrade level of technology. We could alternatively specify the cost as a share of pre-upgrade operating profits. That approach, however, would complicate our solution because the firm would need to take account of the effect that its current choice of technology has on its future costs of upgrading. Since this feature does not contribute to the insights of the model, we employ the simpler specification in equation (10).

This firm rents the services of capital at each point in time, paying a user cost of $u_t p_t$ per unit of capital at time t. The value of this firm is the expected present value of operating profits less the cost of technology upgrades. Let $\Psi\left(A_t, X_t, \widehat{A}_t\right)$ be the expected present value of operating profits, net of upgrade costs, from time t onward, so

$$\Psi\left(A_t, X_t, \widehat{A}_t\right) = \max_{\{\tau_j\}_{j=1}^{\infty}} E_t \left\{ \int_0^{\infty} A_{t+s} X_{t+s} e^{-rs} ds - \sum_{j=1}^{\infty} \theta \widehat{A}_{\tau_j} X_{\tau_j} e^{-r(\tau_j - t)} \right\}, \quad (10)$$

where $\widehat{A}_{\tau_j}$ is the value of the available frontier technology when the upgrade occurs at time τ_j. We require that (1) $r - m > 0$ so that a firm that never upgrades has finite value; (2) $r - m - \mu - \rho_{X\widehat{A}} s\sigma > 0$ so that a firm that continuously maintains $A_t = \widehat{A}_t$ has a value that is bounded from above;[9] and (3) $(r - m)\,\theta < 1$ so that the upgrade cost is not large enough to prevent the firm from ever upgrading.[10]

The value of a firm that does not own capital is $\Psi\left(A_t, X_t, \widehat{A}_t\right)$. Using the Bellman equation, we calculate the value of the firm when it is not upgrading, and then we will examine the boundary conditions that hold when the firm upgrades its technology. The required return on this firm, $r\Psi_t$, must equal current operating profits plus its expected capital gain. When the firm is not upgrading its technology, A_t is constant, so the required return on the firm must equal (omitting time subscripts)

$$\begin{aligned} r\Psi &= \pi + E(d\Psi) \quad (11) \\ &= AX + mX\Psi_X + \frac{1}{2}s^2 X^2 \Psi_{XX} + \mu \widehat{A} \Psi_{\widehat{A}} + \frac{1}{2}\sigma^2 \widehat{A}^2 \Psi_{\widehat{A}\widehat{A}} + \rho_{X\widehat{A}} s\sigma X \widehat{A} \Psi_{X\widehat{A}}. \end{aligned}$$

Direct substitution verifies that the following function satisfies the differential equation in equation (11)

$$\Psi\left(A_t, X_t, \widehat{A}_t\right) = \frac{A_t X_t}{r - m} + B A_t X_t \left(\frac{\widehat{A}_t}{A_t}\right)^{\phi}, \quad (12)$$

where B is an unknown constant and the parameter $\phi > 1$ is the positive root[11,12] of

[9] The condition $r - m - \mu - \rho_{X\widehat{A}} s\sigma > 0$ imples that even if the firm could maintain $A_t = \widehat{A}_t$ for all t without facing any upgrade costs, its value would be finite. Therefore, the value of a firm that faces upgrade costs would be bounded from above if it maintained $A_t = \widehat{A}_t$ for all t.

[10] See footnote 14 for the properties of the upgrade trigger $\overline{a}$.

[11] Notice that $f(0) > 0$, $f(1) > 0$, and $f''(\zeta) < 0$, so that the positive root of this equation exceeds one.

[12] An additional term including the negative root of the quadratic equation also enters the general solution to the differential equation. However, the negative exponent would imply that the firm's value goes to infinity as the frontier technology approaches zero. We set the unknown constant in this term equal to zero and eliminate this term from the solution.

the quadratic equation

$$f(\zeta) \equiv r - m - (\mu + \rho_{X\widehat{A}} s\sigma - \frac{1}{2}\sigma^2)\zeta - \frac{1}{2}\sigma^2\zeta^2 = 0. \tag{13}$$

The boundary conditions imposed at times of technological upgrading determine the constant B and the rule for optimally upgrading to the new technology. The first boundary condition is the value-matching condition, which requires that at the time of the upgrade, the value of the firm increases by the amount of the fixed cost. Formally this requires

$$\Psi\left(\widehat{A}_{\tau_j}, X_{\tau_j}, \widehat{A}_{\tau_j}\right) - \Psi\left(A_{\tau_j}, X_{\tau_j}, \widehat{A}_{\tau_j}\right) = \theta \widehat{A}_{\tau_j} X_{\tau_j}, \tag{14}$$

where $\Psi\left(A_{\tau_j}, X_{\tau_j}, \widehat{A}_{\tau_j}\right)$ is the value of the firm evaluated at the current (pre-upgrade) technology, A_{τ_j}, and $\Psi\left(\widehat{A}_{\tau_j}, X_{\tau_j}, \widehat{A}_{\tau_j}\right)$ is the value of the firm immediately after upgrading to the frontier technology. Substitute the proposed value of the firm from equation (12) into equation (14) and simplify to obtain the boundary condition in terms of the relative technology variable $a \equiv \frac{\widehat{A}}{A}$ and the unknown constant B:

$$\frac{a-1}{r-m} - aB\left(a^{\phi-1} - 1\right) = \theta a \quad \text{for } a = \overline{a}, \tag{15}$$

where $\overline{a}$ is the boundary value of $a \equiv \frac{\widehat{A}}{A}$ associated with a technological upgrade. The value-matching condition thus reduces to a nonlinear equation in a, the ratio of the frontier technology to the installed technology, and the unknown constant B. The value-matching condition holds with equality when $a = \overline{a}$, which is the value of a that will "trigger" a technological upgrade.

The second boundary condition requires that the value of the firm is maximized with respect to the choice of τ_j, the upgrade time. Formally, this requires[13]

$$\Psi_A\left(A_{\tau_j}, X_{\tau_j}, \widehat{A}_{\tau_j}\right) = \frac{X_{\tau_j}}{r-m} + (1-\phi) B X_{\tau_j} \left(\frac{\widehat{A}_{\tau_j}}{A_{\tau_j}}\right)^{\phi} = 0, \tag{16}$$

[13]This boundary condition can be expressed in a more familiar way by noting that the value of the firm, $\Psi\left(A_t, X_t, \widehat{A}_t\right)$, in equation (10) is proportional to X_t and is a linearly homogeneous function of A_t and $\widehat{A}_t$. Therefore, the value of the firm, $\Psi\left(A_t, X_t, \widehat{A}_t\right)$, can be rewritten as $\widehat{A}_t X_t \psi(a_t)$. The value matching condition is $\widehat{A}_t X_t \psi(1) - \widehat{A}_t X_t \psi(\overline{a}) = \widehat{A}_t X_t \theta$, which simplifies to $\psi(1) - \psi(\overline{a}) = \theta$. The second boundary condition is simply $\psi'(\overline{a}) = 0$. Equation (12) implies that $\psi(a) = \frac{a^{-1}}{r-m} + Ba^{\phi-1}$, so $\psi'(\overline{a}) = 0$ implies $-\frac{a^{-2}}{r-m} + (\phi-1) Ba^{\phi-2} = 0$, which is equivalent to equation (17).

which implies that

$$\frac{1}{r-m} + (1-\phi)\, B a^{\phi} = 0 \quad \text{for } a = \overline{a}. \tag{17}$$

The second boundary condition also reduces to a nonlinear equation that is a function of the relative technology a and the unknown constant B. This condition holds when $a = \overline{a}$, that is, when an upgrade from the current value of A_t to the available frontier technology $\widehat{A}_t$ occurs. Solving equation (17) for B yields

$$B = \frac{\overline{a}^{-\phi}}{(\phi-1)(r-m)} > 0. \tag{18}$$

Using equation (18) to eliminate B in equation (15) yields a single nonlinear equation characterizing the solution for optimal upgrades

$$g(a;\theta) \equiv a - 1 - \frac{1-a^{1-\phi}}{\phi - 1} - a\theta\,(r-m) = 0 \quad \text{for } a = \overline{a}. \tag{19}$$

Notice that this expression depends only on the relative technology, $a \equiv \frac{\widehat{A}}{A}$, and constant parameters. Therefore, the relative technology a must have the same value whenever the firm upgrades its technology; we defined this boundary value above as $\overline{a}$, so $g(\overline{a};\theta) = 0$. It is straightforward to verify that $\overline{a} \geq 1$, with strict inequality when $\theta > 0$ and that $\frac{d\overline{a}}{d\theta} > 0$ when $\theta > 0$.[14] The firm upgrades A_t to the available technology when $\widehat{A}_t$ reaches a sufficiently high value; specifically, the firm upgrades when $\widehat{A}_t = \overline{a} \times A_t \geq A_t$. The size of the increase in A_t that is needed to trigger an upgrade is an increasing function of the fixed cost parameter θ.

Substituting equation (18) into the value of the firm in equation (12) yields

$$\Psi\left(A_t, X_t, \widehat{A}_t\right) = \frac{A_t X_t}{r-m}\left[1 + \frac{1}{\phi-1}\left(\frac{a_t}{\overline{a}}\right)^{\phi}\right] > \frac{A_t X_t}{r-m}. \tag{20}$$

The value of the firm in equation (20) can be expressed as the expected present value of operating profits, $\pi_t = A_t X_t$, evaluated along the path of no future upgrades, multiplied by a term that exceeds one and captures the value of expected future

[14] To see that $\overline{a} \geq 1$, use $\phi > 1$ and $(r-m)\,\theta < 1$ to note that $\lim_{a\to 0} g(a;\theta) > 0$, $g(1;\theta) = -\theta(r-m) < 0$, $\lim_{a\to\infty} g(a;\theta) > 0$, and $g''(a;\theta) > 0$. Thus $g(a;\theta)$ is a convex function of a with two distinct positive roots, $0 < \underline{a} < 1 < \overline{a}$, when $\theta > 0$, with $\frac{\partial g(\underline{a};\theta)}{\partial a} < 0$ and $\frac{\partial g(\overline{a};\theta)}{\partial a} > 0$. The smaller root, $\underline{a} < 1$, can be ruled out since it implies that the firm reduces the value of its technology whenever it changes technology. Since $\frac{\partial g(\overline{a};\theta)}{\partial \theta} = -(r-m)\,a < 0$, the implicit function theorem implies that $\frac{d\overline{a}}{d\theta} > 0$ when $\theta > 0$. When $\theta = 0$ there is a unique positive value of a that solves equation (19); specifically, $\overline{a} = 1$ when $\theta = 0$.

technological upgrades. If the frontier technology were permanently unavailable, so that the firm would have to maintain the current level of technology, A_t, forever, then the value of the firm would simply be $\frac{A_t X_t}{r-m}$. Otherwise, the value of firm is increasing in the relative value of the frontier technology, a_t, as well as in current operating profits, $A_t X_t$.

We have calculated $\Psi\left(A_t, X_t, \widehat{A}_t\right)$, which is the value of a firm that never owns capital but rents the services of capital at each point of time. The value of a firm that owns capital K_t at time t is simply equal to the sum of $p_t K_t$ and $\Psi\left(A_t, X_t, \widehat{A}_t\right)$. Thus, if V_t is the value of the firm at time t, equation (20) implies that

$$V_t = p_t K_t + \frac{A_t X_t}{r-m}\left[1 + \frac{1}{\phi - 1}\left(\frac{a_t}{\overline{a}}\right)^{\phi}\right]. \tag{21}$$

We can easily interpret the two components of the firm's value in equation (21) by thinking of the firm as composed of two divisions: a capital-owning division and a capital-using production division. The value of the capital-owning division, which owns capital stock K_t, is simply $p_t K_t$, because this division can instantaneously and costlessly sell this capital at the market price of p_t per unit. The value of the capital-using production division is the expected present value of operating profits, net of upgrade costs, which is $\Psi\left(A_t, X_t, \widehat{A}_t\right)$. We interpret the firm as composed of two divisions simply to aid in understanding the value of the firm in equation (21). In fact, we treat the firm as an integrated unit in which in the capital-owning division chooses to own exactly the amount of capital (K_t in equation (2)), that the capital-using production division wants to rent at each point in time.

4 Tobin's Q

Tobin's Q is the ratio of the value of the firm, V_t, to the replacement cost of the firm's capital stock, $p_t K_t$. To calculate Tobin's Q, first use equation (6) to substitute $(1-\gamma) C_t$ for $A_t X_t$ in equation (21), and then divide both sides of equation (21) by the replacement cost of the capital stock, $p_t K_t$, to obtain

$$Q_t \equiv \frac{V_t}{p_t K_t} = 1 + \frac{(1-\gamma) C_t}{p_t K_t (r-m)}\left[1 + \frac{1}{\phi - 1}\left(\frac{a_t}{\overline{a}}\right)^{\phi}\right]. \tag{22}$$

Next substitute the cash flow-capital stock ratio from equation (7) into equation (22) to obtain

$$Q_t = 1 + \frac{1-\gamma}{r-m} c_t \left[1 + \frac{1}{\phi - 1}\left(\frac{a_t}{\overline{a}}\right)^{\phi}\right]. \tag{23}$$

Tobin's Q exceeds 1 because of the rents represented by the operating profits, π_t. The excess of Tobin's Q over 1 can be decomposed into the product of two terms. The first is the expected present value of operating profits per unit of capital (measured at replacement cost), calculated along the path of no future technological upgrades so that A_t is held fixed indefinitely. The second term reflects the value of the expected future upgrades. It is an increasing function of the value of the frontier technology relative to the installed technology, measured by a_t.

The maximum value of Q_t, for a given value of c_t, is attained when a_t reaches its trigger value $\overline{a}$. We denote this value of Q_t as $Q_{trigger}$, where

$$Q_{trigger} \equiv 1 + \frac{1-\gamma}{r-m}\frac{\phi}{\phi-1}c_t. \tag{24}$$

Note that $Q_{trigger}$ does not depend on $\overline{a}$, which is the trigger for the relative technology variable, a_t. Thus, for instance, an increase in upgrade cost parameter θ increases $\overline{a}$, but has no effect on $Q_{trigger}$. When the relative technology variable a_t reaches its trigger value $\overline{a}$, Q_t reaches $Q_{trigger}$ and the firm adopts the frontier technology. Upon adoption of the frontier technology and payment of the adoption cost, a_t jumps downward to 1 and (since $c_t = u_t/\gamma$ moves continuously and does not jump) Q_t jumps downward to

$$Q_{return} \equiv 1 + \frac{1-\gamma}{r-m}\left[1 + \frac{1}{\phi-1}\overline{a}^{-\phi}\right]c_t. \tag{25}$$

5 Continuous Investment and Cash Flow

Since its introduction by Brainard and Tobin in the late 1960s, Q theory has provided a framework for the analysis of investment data. The conventional version of the underlying theoretical model assumes quadratic adjustment costs, as well as linearly homogeneous operating profit and adjustment cost functions. These assumptions imply a linear relationship between the rate of investment, $\frac{I_t}{K_t}$, and Tobin's Q. Empirical implementation of this model, while confirming a positive role for Tobin's Q in explaining investment, also typically finds that cash flow plays a significant role, both economically and statistically, in investment behavior. The effect of cash flow on investment is often interpreted as evidence of financing constraints facing the firm. The model we have developed, while departing significantly from the traditional assumptions underlying Q theory, also implies a positive role for both Tobin's Q and cash flow in explaining investment, even though there are no financing constraints.

Over finite intervals of time, investment consistents of two components: continuous investment and gulps of investment. Continuous investment refers to the continuous variation in the optimal capital stock (equation (2)) that arises from continuous variation in X_t, p_t, and u_t. We will show in this section that continuous investment is positively related to contemporaneous cash flow and is independent of Q_t, which is too forward-looking for this component of investment. Investment gulps are the jumps in the optimal capital stock induced by jumps in the level of installed technology when the firm upgrades its technology. We will show in Section 6 that Q_t provides information about future technology upgrades and their accompanying investment gulps.

To calculate continuous investment, first calculate the change in the capital stock between technology upgrades by applying Ito's Lemma to the expression for the optimal capital stock in equation (2). When no upgrade occurs, $dA_t = 0$ so the growth in the capital stock is given by

$$\frac{dK_t}{K_t} = \frac{dX_t}{X_t} - \frac{du_t}{u_t} + (\sigma_u^2 - \rho_{Xu} s \sigma_u - \mu_p)dt. \tag{26}$$

Equation (26) shows the ratio of net investment to the capital stock when no upgrade occurs. Gross investment, I_t, is net investment, dK_t, plus depreciation, $\delta_t K_t dt$. Using the definition of gross investment and applying Ito's Lemma to equation (26) yields the gross investment rate

$$\frac{I_t}{K_t} = \delta_t dt + \frac{dK_t}{K_t} = (\delta_t + m + \sigma_u^2 - \rho_{Xu} s \sigma_u - \mu_p)dt + s dz_X - \sigma_u dz_u. \tag{27}$$

To relate the drift term in equation (27) to cash flow per unit of capital, use equation (8) to substitute $\gamma c_t - r$ for $\delta_t - \mu_p$ to obtain

$$\frac{I_t}{K_t} = (\gamma c_t - \Gamma)dt + s dz_X - \sigma_u dz_u, \tag{28}$$

where $\Gamma \equiv r - m - \sigma_u^2 + \rho_{Xu} s \sigma_u$ is constant. Notice that the expected rate of investment, $\gamma c_t - \Gamma$, is an increasing linear function of the cash flow-capital stock ratio, c_t, and is independent of Q_t.

The finding that continuous investment depends on cash flow but not on Q_t is consistent with the empirical results of Bond and Cummins (2000), who use short-run earnings forecasts of company analysts to calculate an estimate of Tobin's Q. They find that this estimated Q is more highly correlated with current investment than is Tobin's Q calculated in the conventional way, using equity market valuations. They argue that their finding is evidence of measurement error, or a persistent deviation of firms' equity values from fundamentals. Our model, while consistent with the fact that a component of firms' valuation is a "sideshow" for investment, does not require any such deviation from fundamentals. In our model, the value of the frontier technology, measured by a_t, is extraneous to the current investment decision, and hence Q_t does not predict continuous investment. Nonetheless, the frontier technology is not extraneous to the value of the firm and is legitimately part of its market value. As we show in Section 6, Tobin's Q helps predict investment gulps that occur when the firm adopts the frontier technology.

6 Technology Upgrades and Investment Gulps

In Section 5 we showed that continuous investment between upgrades depends on cash flow but is independent of Tobin's Q. In this section, we show that Tobin's Q is useful for predicting investment gulps that accompany technological upgrades. We analyze three aspects of investment gulps. First, we examine the size of gulps. Second, we examine the timing of upgrades and their accompanying gulps. We show that Tobin's Q can be used to predict the expected time until the next gulp as well as the probability that an investment gulp will occur during an upcoming interval of time. Finally, we discuss how these gulps are related to empirical evidence on investment "spikes".

6.1 The Size of Investment Gulps

When a firm upgrades its technology, its capital stock jumps upward. The jump in the capital stock that accompanies the upgrade at time τ_j is calculated using the expression for the optimal capital stock in equation (2) to obtain

$$\frac{K_{\tau_j}^{+}}{K_{\tau_j}^{-}} = \frac{A_{\tau_j}^{+}}{A_{\tau_j}^{-}} = \frac{\widehat{A}_{\tau_j}}{\widehat{A}_{\tau_{j-1}}} = \overline{a}. \tag{29}$$

Therefore, the amount of capital that is added in the investment gulp at time τ_j is

$$K_{\tau_j}^{+} - K_{\tau_j}^{-} = (\overline{a} - 1)\, K_{\tau_j}^{-}. \tag{30}$$

The increase in the capital stock in equation (30) occurs instantaneously and is a component of investment for any interval of time that contains τ_j.

6.2 Tobin's Q and the Timing of Investment Gulps

Technological upgrades occur when the level of the frontier technology, $\widehat{A}_t$, becomes high enough relative to the installed technology, A_t, to compensate for the cost of upgrading to the frontier. The ratio of the frontier technology to the installed technology, a_t, is a sufficient statistic for the upgrade decision. If a_t is below the threshold value, $\overline{a}$, the firm does not upgrade. When a_t reaches $\overline{a}$, the firm upgrades its technology to the frontier. However, the frontier technology, and hence a_t, is unobservable to an outside observer. As we show below, Tobin's Q provides an observable measure of a_t that can help predict the timing of upgrades and gulps.

We assess the relationship between Q_t and expected future upgrades by calculating the expected time until an upgrade and examining the effect of Q_t on this statistic. Let τ be the time of the next upgrade. Conditional on the current value of a_t, the expected time of the next upgrade satisfies

$$E_t(\tau) - t = \frac{\ln(\overline{a}) - \ln(a_t)}{\mu - \frac{1}{2}\sigma^2}. \tag{31}$$

The right-hand side of equation (31) is the expected first passage time of a_t from its current value, $a_t < \overline{a}$, to the trigger value $\overline{a}$.[15]

Let Ω denote the expected time between consecutive upgrades. Immediately after an upgrade, $a_t = 1$, so setting $a_t = 1$ in equation (31) yields the expected length of time between consecutive upgrades

$$\Omega = \frac{\ln(\overline{a})}{\mu - \frac{1}{2}\sigma^2} > 0. \tag{32}$$

Since the value of the frontier technology relative to the installed technology, a_t, is typically unobservable, we rearrange equation (23) to produce an expression for a_t as a function of the observable variable Q_t

$$a_t = \left[\frac{Q_t - 1}{c_t}\frac{r-m}{1-\gamma} - 1\right]^{\frac{1}{\phi}} (\phi - 1)^{\frac{1}{\phi}} \overline{a} > 0. \tag{33}$$

Conditional on the cash flow ratio, c_t, Q_t is an indicator of a_t and hence predicts technology upgrades and the corresponding investment gulps. Since Q_t is a forward-looking variable, its value includes an assessment of the value of the frontier technology, in addition to the current technology.

[15] This equation uses the fact that the expected first passage time of a geometric Brownian motion from its current value x_t to a value $x_0 > x_t$ is given by $\frac{\ln(x_0/x_t)}{\mu_x - \frac{1}{2}\sigma_x^2}$, where the instantaneous drift and variance of x_t are μ_x and σ_x^2, respectively.

Substitute the expression for a_t from equation (33) into equation (31) to obtain

$$E_t(\tau) - t = -\frac{1}{\phi}\frac{\ln\left[\frac{Q_t - 1}{c_t}\frac{r-m}{1-\gamma} - 1\right] + \ln(\phi - 1)}{\mu - \frac{1}{2}\sigma^2}. \tag{34}$$

Equation (34) shows that the expected time until an upgrade is decreasing in Q_t. High values of Tobin's Q, specifically, values close to $Q_{trigger}$ in equation (24), thus predict imminent technology upgrades and the associated gulps of investment.

To derive another measure of the imminence of an upgrade to technology, suppose that we observe data over intervals of length T. Let $\Pi(a_t, T)$ denote the probability, as of time t, that the firm upgrades its technology at least once during the time interval $(t, t+T]$. This probability equals $prob\left\{\max_{t<s\leq T}\frac{\widehat{A}_s}{A_t} \geq \overline{a}\right\}$. Using Harrison's (1985, p. 14, equation 11) characterization of the distribution of the maximum of Brownian motion, generalized to allow for an arbitrary (rather than unitary) starting point, yields

$$\Pi(a_t, T) = 1 - \Phi\left(\frac{\ln(\overline{a}) - \ln(a_t) - gT}{\sigma T^{\frac{1}{2}}}\right) + \left(\frac{\overline{a}}{a_t}\right)^{\frac{2g}{\sigma^2}}\Phi\left(\frac{-\ln(\overline{a}) + \ln(a_t) - gT}{\sigma T^{\frac{1}{2}}}\right) \tag{35}$$

where $g \equiv \mu - \frac{1}{2}\sigma^2 > 0$ and $\Phi()$ is the standard normal c.d.f. The probability of at least one upgrade during an interval of length T, $\Pi(a_t, T)$, is an increasing function of a_t. Equation (33) implies that, for a given value of c_t, a_t is an increasing function of Q_t. Therefore, given c_t, the probability of an upgrade, and the accompanying investment gulp, is an increasing function of Q_t. We illustrate the relationship between Q_t and the probablity of an investment gulp in Figure 1, which we discuss later.

To illustrate the magnitudes of various aspects of the model, we present a numerical example, which is summarized in Table 1. The parameter values listed in the top part of the table (Assumptions) are the inputs to the example. The discount rate, $r = 0.16$, and the depreciation rate, δ,= 0.10, are chosen with annual units of time in mind. An annual discount rate of 16% is, of course, high for a riskless rate of return, but we have chosen such a high value to correspond to the risk-adjusted hurdle rates of return used by firms. The curvature parameter, $\gamma = 0.8$, and the upgrade cost parameter, $\theta = 0.5$, do not have time units associated with them. The model specifies four geometric Brownian motions, which are mutually independent in the numerical example: the frontier technology $\widehat{A}_t$; the random variable Y_t, which captures the demand facing the firm; the user cost factor u_t, which is a driftless geo-

Table 1: Numerical Illustration			
Assumptions			
Discount rate, r	0.16	Depreciation rate, δ	0.10
Curvature parameter, γ	0.80	Upgrade cost, θ	0.50
g.b.m. for $\widehat{A}$			
Drift, μ	0.03	Standard deviation, σ	0.15
g.b.m. for Y			
Drift, μ_Y	0.065	Standard deviation, σ_Y	0.20
Std. dev. of user cost, σ_u	0.06	Drift in p, μ_p	0.02
Correlations: $\rho_{Y\widehat{A}} = \rho_{u\widehat{A}} = \rho_{Yu} = 0$			
Implications			
g.b.m. for X			
Drift, m	0.021	Standard deviation, s	0.312
User cost factor, u	0.24	Cash flow-capital stock ratio, c	0.30
Technology trigger ratio, $\overline{a}$	1.30	Expected time between upgrades, Ω	14.0
Q_{return}	1.55	$Q_{trigger}$	1.67

metric Brownian motion and has positive variance; and the price of capital, p_t, which is deterministic and grows at a constant rate.

The results of the model are listed in the bottom part of the table (Implications). The variable X_t is a geometric average of Y_t, u_t, and p_t, and follows a geometric Brownian motion that is implied by the geometric Brownian motions for the individual variables. The user cost factor, u_t, in this example is 24% per year, and the cash flow-capital stock ratio is 30% per year. The trigger technology ratio, $\overline{a}$, is 1.30, which means that the firm waits until the frontier technology is 30% more productive than its currently installed technology before upgrading and undertaking a gulp of investment. The average time between upgrades and their accompanying investment gulps is 14.0 years. Immediately before the firm upgrades, $Q_t = Q_{trigger} = 1.67$, and immediately after it upgrades, $Q_t = Q_{return} = 1.55$. It is important to note that the value of Q_t is not confined to the range $[1.55, 1.67]$, for two reasons. First, Q_t can fall below $Q_{return} = 1.55$ if the frontier technology technology wanders below the currently installed technology so that $a_t \equiv \frac{\widehat{A}_t}{A_t} < 1$. (For example, a new regulation may prohibit future installation of a technology or technique that is deemed to be dangerous or otherwise undesirable, while allowing existing users of the technology

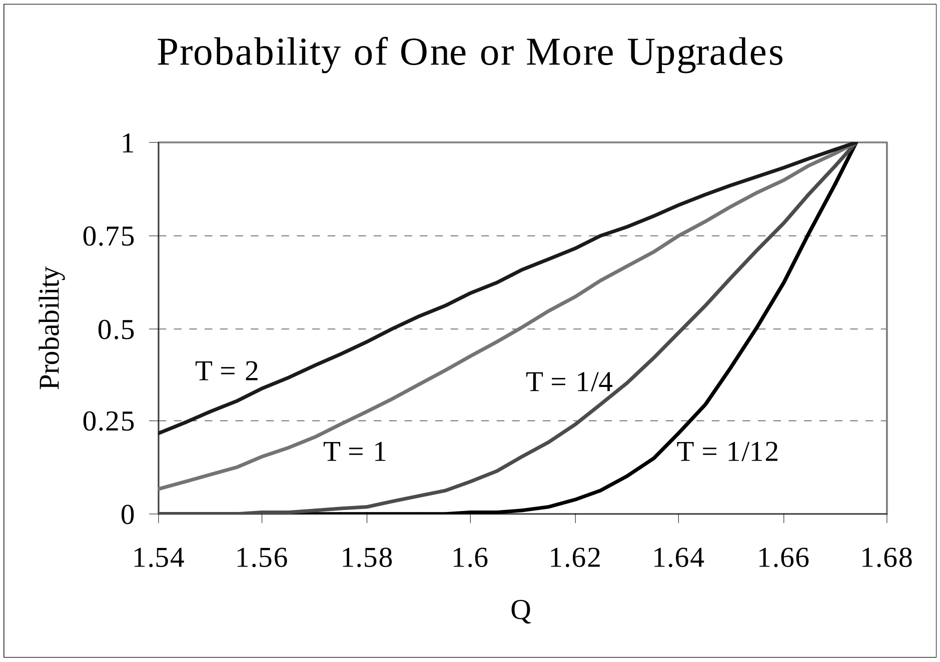

Figure 1: Probability of One or More Upgrades in Period of Length T

to continue using it for a while.) Second, and probably more important, is that the range $[Q_{return}, Q_{trigger}]$ is calculated for given values of the cash flow-capital stock ratio, c_t. But, as shown in equations (23) to (25), $Q_t - 1$, $Q_{trigger} - 1$, and $Q_{return} - 1$ are proportional to c_t for given a_t. Therefore, an increase in the user cost factor that causes c_t to increase above 0.30 will increase $Q_{trigger}$ and allow Q_t to increase above 1.67. Similarly, a decrease in the user cost factor that causes c_t to fall below 0.30 will reduce Q_{return} and allow Q_t to fall below 1.55.

Figure 1 shows the probability of an upgrade during an interval of length T as a function of the value of Tobin's Q at the beginning of the interval. The four curves in this figure correspond to four different values of T: $T = 1/12$ corresponds to monthly observation intervals, $T = 1/4$ corresponds to quarterly observation intervals, $T = 1$ corresponds to annual observation intervals, and $T = 2$ corresponds to bi-annual observation intervals. For each of these observation intervals, the probability of one or more investment gulps is an increasing function of Q_t at the beginning of the observation interval. Also, for a given value of Q_t, the probability of one or more gulps is an increasing function of the length of the observation interval.

6.3 The Size and Importance of Investment Gulps

In this subsection we compare the quantitative importance of continuous investment and gulps of investment, and we illustrate the implications of episodic investment gulps for the temporal concentration of investment. The dichotomy between continuous investment and gulps of investment can be highlighted by rewriting equation (2) as

$$K_t = \frac{\gamma}{1-\gamma}\lambda_t A_t, \tag{36}$$

where $\lambda_t \equiv \frac{X_t}{u_t p_t}$ summarizes all of the non-technology uncertainty about the optimal capital stock and is a geometric Brownian motion with drift[16] $\mu_\lambda \equiv \mu_Y - \frac{1}{1-\gamma}\left(\mu_p + \rho_{Yu}\sigma_Y\sigma_u\right) + \frac{1}{2}\frac{2-\gamma}{(1-\gamma)^2}\sigma_u^2$. Continuous variation in λ_t generates continuous variation in the optimal capital stock, which generates continuous (net) investment. The installed technology, A_t, in equation (36) jumps when the firm upgrades its technolgy, thereby generating a jump in the optimal capital stock, which gives rise to a gulp of investment.

Now consider investment over the interval of time that begins immediately following the upgrade at time τ_{j-1} and ends immediately following the upgrade at time τ_j. During this interval, there is an investment gulp at time τ_j. The size of this gulp is

$$K_{\tau_j}^+ - K_{\tau_j}^- = (\overline{a}-1)\,K_{\tau_j}^- = (\overline{a}-1)\,\frac{\gamma}{1-\gamma}\lambda_{\tau_j}\widehat{A}_{\tau_{j-1}}. \tag{37}$$

Cumulative continuous net investment over this interval of time is

$$K_{\tau_j}^- - K_{\tau_{j-1}}^+ = \left(\lambda_{\tau_j} - \lambda_{\tau_{j-1}}\right)\frac{\gamma}{1-\gamma}\widehat{A}_{\tau_{j-1}}. \tag{38}$$

Total net investment over this interval of time, $K_{\tau_j}^+ - K_{\tau_{j-1}}^+$, is the sum of the investment gulp in equation (37) and cumulative continuous net investment in equation (38).

Define $\chi_{net,j} \equiv \frac{K_{\tau_j}^+ - K_{\tau_j}^-}{K_{\tau_j}^+ - K_{\tau_{j-1}}^+}$ as the ratio of capital added during a gulp to the total capital added over the interval. This ratio is a measure of the temporal concentration of net investment. Use equations (37) and (38) and $K_{\tau_j}^+ - K_{\tau_{j-1}}^+ = \left(K_{\tau_j}^+ - K_{\tau_j}^-\right) +$

[16] Since X_t is a function of p_t and u_t, it is helpful to substitute the definition of X_t from equation (4) into the definition of λ_t to obtain $\lambda_t = (1-\gamma)\,\gamma^{\frac{\gamma}{1-\gamma}} Y_t\,(u_t p_t)^{-\frac{1}{1-\gamma}}$. Apply Ito's Lemma to this expression for λ_t and recall that u_t has zero drift and p_t has zero variance to obtain the expression for μ_λ in the text.

$\left(K^-_{\tau_j} - K^+_{\tau_{j-1}}\right)$ to calculate

$$\chi_{net,j} = \frac{\overline{a} - 1}{\overline{a} - \frac{\lambda_{\tau_{j-1}}}{\lambda_{\tau_j}}}. \tag{39}$$

Equation (39) is a measure of the temporal concentration of net investment over the interval preceding the j-th upgrade. If, for instance, λ_t is constant over this interval, then continuous net investment is zero, $\frac{\lambda_{\tau_{j-1}}}{\lambda_{\tau_j}} = 1$ in equation (39), and $\chi_{net,j} = 1$ so that the temporal concentration of investment is 100%. In this case, all net investment takes place in gulps. Alternatively, if λ_t grows over this interval of time, the optimal capital stock grows, so continuous net investment is positive. Because λ_t grows over time, $\frac{\lambda_{\tau_{j-1}}}{\lambda_{\tau_j}} < 1$, which implies that $\chi_{net,j} < 1$, and the temporal concentration of investment smaller than 100%.[17]

For the purpose of illustration, suppose that the interval of time between upgrades has length Ω, which is the expected length of time between upgrades in equation (32). Also suppose that λ_t follows its expected path throughout this interval, i.e.,

$$\lambda_t = E_{\tau_{j-1}}\left\{\lambda_t\right\} = \lambda_{\tau_{j-1}} e^{\mu_\lambda (t - \tau_{j-1})} \quad \text{for } \tau_{j-1} \leq t \leq \tau_j. \tag{40}$$

Equation (40) implies that $\lambda_{\tau_{j-1}}/\lambda_{\tau_j} = e^{-\mu_\lambda \Omega}$, so the temporal concentration of net investment in equation (39) is

$$\chi_{net,j} = \frac{\overline{a} - 1}{\overline{a} - e^{-\mu_\lambda \Omega}}. \tag{41}$$

For the parameter values in Table 1, $\mu_\lambda = 0.019$, and $\chi_{net,j} = 0.56$, so that 56% of net investment takes place during gulps.

Data on investment expenditures are generally reported as gross investment rather than net investment. To compute gross investment over an interval of time, we need to know the path of depreciation, $\delta_t K_t$, throughout the interval, which means that we need to know the path of the capital stock throughout the interval of time. Suppose that λ_t follows the path in equation (40) and that the depreciation rate is a constant, δ. Assuming again that the interval has length Ω, the temporal concentration of

[17] It is possible for $\chi_{net,j}$ to exceed one. If λ_t falls over time, so that cumulative continuous net investment is negative, then the gulp of investment will account for more than 100% of the growth in the capital stock.

gross investment is[18]

$$\chi_{gross,j} = \frac{\overline{a}-1}{\overline{a}-e^{-\mu_\lambda\Omega}+\frac{\delta}{\mu_\lambda}\left(1-e^{-\mu_\lambda\Omega}\right)}, \quad \text{if } \mu_\lambda \neq 0 \qquad (42)$$
$$= \frac{\overline{a}-1}{\delta\Omega+\overline{a}-1}, \quad \text{if } \mu_\lambda = 0.$$

For the numerical example in Table 1, $\chi_{gross,j} = 0.17$, so that 17% of gross investment takes place during gulps of investment.

Equation (42) can be used to interpret empirical results regarding the temporal concentration of investment in plant-level data. Starting with Doms and Dunne (1998), the "lumpiness" of investment has been measured by the share of investment over a fixed period of time that occurs in a single year (or in consecutive years). For example, using plant-level data from the Census of Manufacturing, they report that "the average plant experiences a 1-year investment episode that accounts for 24.5% of its total real investment spending over the 16 year interval" (page 417). If this period of time corresponds to the time from one "gulp" to another, then equation (42) corresponds to the share of a gulp in cumulative investment over this period.[19] The degree of temporal concentration in plant-level capital accumulation has been used as an indicator of capital lumpiness, potentially arising from nonconvex costs of adjusting capital. Our model indicates that a high degree of temporal concentration in investment can be attained even when there are no fixed costs of adjusting the capital stock. The fixed cost associated with technology upgrades (as opposed to a fixed cost of adjusting the capital stock *per se*) and the complementarity of technology

[18] Cumulative depreciation over the interval is $\int_{\tau_{j-1}}^{\tau_j} \delta\frac{\gamma}{1-\gamma}\lambda_t \widehat{A}_{\tau_{j-1}} dt = \delta\frac{\gamma}{1-\gamma}\widehat{A}_{\tau_{j-1}} \int_{\tau_{j-1}}^{\tau_{j-1}+\Omega} \lambda_t dt = \delta\frac{\gamma}{1-\gamma}\widehat{A}_{\tau_{j-1}}\lambda_{\tau_{j-1}} \int_0^{\Omega} e^{\mu_\lambda s} ds$, which equals $= \frac{\delta}{\mu_\lambda}\left(e^{\mu_\lambda\Omega}-1\right)\frac{\gamma}{1-\gamma}\lambda_{\tau_{j-1}}\widehat{A}_{\tau_{j-1}}$ if $\mu_\lambda \neq 0$ and equals $\delta\Omega\frac{\gamma}{1-\gamma}\widehat{A}_{\tau_{j-1}}\lambda_{\tau_{j-1}}$ if $\mu_\lambda = 0$. Adding cumulative depreciation to cumulative continuous net investment in equation (38) plus the investment gulp in equation (37) yields $\left[-\frac{\lambda_{\tau_{j-1}}}{\lambda_{\tau_j}} + \frac{\delta}{\mu_\lambda}\left(e^{\mu_\lambda\Omega}-1\right)\frac{\lambda_{\tau_{j-1}}}{\lambda_{\tau_j}} + \overline{a}\right]\frac{\gamma}{1-\gamma}\lambda_{\tau_j}\widehat{A}_{\tau_{j-1}}$ if $\mu_\lambda \neq 0$, and equals $[\delta\Omega+\overline{a}-1]\frac{\gamma}{1-\gamma}\lambda_{\tau_j}\widehat{A}_{\tau_{j-1}}$ if $\mu_\lambda = 0$. Therefore, $\chi_{gross,j} = \frac{\overline{a}-1}{\overline{a}-\frac{\lambda_{\tau_{j-1}}}{\lambda_{\tau_j}}+\frac{\delta}{\mu_\lambda}\left(e^{\mu_\lambda\Omega}-1\right)\frac{\lambda_{\tau_{j-1}}}{\lambda_{\tau_j}}}$ if $\mu_\lambda \neq 0$, and equals $\frac{\overline{a}-1}{\delta\Omega+\overline{a}-1}$ if $\mu_\lambda = 0$. Since λ_t grows at the constant rate μ_λ for a period of time Ω, $\frac{\lambda_{\tau_{j-1}}}{\lambda_{\tau_j}} = e^{-\mu_\lambda\Omega}$, so $\chi_{gross,j} = \frac{\overline{a}-1}{\overline{a}-e^{-\mu_\lambda\Omega}+\frac{\delta}{\mu_\lambda}\left(1-e^{-\mu_\lambda\Omega}\right)}$, if $\mu_\lambda \neq 0$.

[19] The Doms and Dunne measure is calculated over a fixed, exogenous sample size, so their results depend on the number of years per firm in the data. If more than one "gulp" occurs during the time period, then measured temporal concentration will be reduced. If the time period includes a gulp, but not the full interval between gulps, then measured temporal concentration increases. Also, since data are measured discretely (typically annually) then there is no guarantee that the year of highest investment contains the gulp, rather than an accumulation of large, but continuous, shocks.

and capital give rise to apparent lumpiness in capital, even when capital itself is infinitely divisible and frictionlessly adjustable.[20]

"Induced lumpiness" is more consistent with the data than is a fixed cost of adjusting the capital stock itself. With a fixed cost of adjusting the capital stock, the "gulps" of investment represent the only form of capital accumulation, and investment would be zero between "gulps". Thus, temporal concentration would be 100%. Doms and Dunne (1998) report that their measure of average temporal concentration is 24.5%, and that "plants still invest in every period" (page 417), so 100% temporal concentration is clearly too extreme. By allowing nonzero investment between gulps, "induced nonconvexities" produce potentially more realistic values of temporal concentration in investment than does imposing fixed costs directly on capital adjustment.[21]

7 Conclusions

The results in this paper emphasize the horizon of capital accumulation. Tobin's Q is a forward-looking measure of the average value of the firm relative to the replacement cost of its current capital. The value of the firm includes the value of available, but uninstalled, technologies. Tobin's Q therefore fluctuates with variation in the value of the frontier technology, but this uninstalled frontier technology is irrelevant to current investment. In the parametric specification of our model, in fact, Tobin's Q is irrelevant to current continuous investment. However, because it contains forward-looking information, Q predicts when the frontier technology will be adopted and the corresponding gulp of investment will occur.

We have modeled the growth options of the firm as arising from a simple, exogenous frontier technology. Two features of this specification are important for the results. First, the frontier technology is not a constant function of the existing, installed technology. This implies that the firm's growth options fluctuate (at least partially) independently of the firm's current operating profits. Second, the horizon of the technology adoption decision exceeds that of the investment decision. In the current framework we dispense completely with capital adjustment costs. Thus, in-

[20]A similar phenomenon arises in Abel and Eberly (1998), where fixed costs of adjustment on capital induce "lumpy" employment adjustment, even though employment adjustment is frictionless.

[21]Aggregation over heterogeneous capital may also help to explain this observation. We explore this hypothesis empirically in Abel and Eberly (2002b).

vestment is a static decision that depends only on current variables and not on the expected future technology. Therefore Q, which is infinitely forward-looking, contains information that is extraneous to current investment. At the opposite extreme, if capital is fixed once and for all, then the initial investment decision also becomes infinitely forward-looking and all of the information in Q is relevant. A natural, though challenging, extension of the model is to consider instead realistic investment lags, so the investment decision incorporates some expectations of future returns to capital.

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