

Capital Reallocation and Growth

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Heterogeneity is ubiquitous in firm-level and sectoral data. Equilibrium models, however, typically assume a representative firm, as in Andrew B. Abel and Olivier J. Blanchard (1983). The representative firm paradigm leaves no role for the distribution of capital. We model capital reallocation in a general equilibrium model with two sectors. Capital adjustment costs capture illiquidity in our model, similar to Hirofumi Uzawa's (1969) capital installation technology. We follow Fumio Hayashi (1982) in assuming that the production technology is linearly homogeneous, which allows us to focus on the sectoral distribution of capital, separately from the level of total capital. The two sectors may have different levels of productivity, and we show that the distribution of capital between the two sectors is the single state variable governing investment, growth, and valuation in the economy.

We analytically characterize prices and quantities including investment, growth, the interest rate, and the price of capital (Tobin's q) at both aggregate and sectoral levels, along with the effects of sectoral heterogeneity and reallocation in the economy. Without adjustment costs, capital is immediately reallocated to the more productive sector. With adjustment costs, the central planner optimally trades off growth against the cost of reallocating capital. Hence, reallocation to the high productivity sector is not immediate, and reallocation itself expends resources. When the more productive sector is initially small, investment exceeds output in the high productivity sector, so output from the less productive sector finances growth in the more productive sector. Nonetheless, investment and

growth optimally continue in the initially larger, low productivity sector. This occurs because, while the sector is relatively less productive, its output can be re-invested in the other, more productive, sector. This is more efficient than directly un-installing capital from the less productive sector and re-installing in the more productive sector because of adjustment costs. The capital stock in the less productive sector dwindles over time as its growth rate shrinks, and eventually the economy specializes in the more productive technology. As the economy moves toward specialization, the growth rate is non-monotonic. At first, the aggregate growth rate falls, because more resources are expended on reallocation, but eventually the growth rate rises as the economy specializes in the high productivity sector. The interest rate follows this same non-monotonic pattern, first falling and then rising along with the aggregate growth rate because the equilibrium interest rate must rise with the growth rate of aggregate consumption to clear the market.

I. Model

Consider an infinite-horizon continuous-time production economy. There are two productive sectors: 0 and 1. Let K_n , I_n , and Y_n denote the representative firm's capital stock, investment and output processes in sector n where $n = 0, 1$. This firm has an "AK" production technology:

$$(1) \quad Y_n(t) = A_n K_n(t), \quad n = 0, 1,$$

where A_n is constant. We capture sectoral heterogeneity by letting $A_1 > A_0 > 0$. Capital accumulation is given by

$$(2) \quad dK_n(t) = \Phi(I_n(t), K_n(t))dt,$$

where $\Phi(I_n, K_n)$ denotes the effectiveness in converting investment goods into installed capital, as in Uzawa (1969). As in Hayashi (1982)

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and Abel and Blanchard (1983), we assume that the adjustment technology is homogeneous of degree one in I_n and K_n , in that

$$(3) \quad \Phi(I_n, K_n) = \phi(i_n)K_n,$$

where $i_n = I_n/K_n$ is the sector- n investment-capital ratio. To generate interesting economic tradeoffs, let $\phi'(\cdot) > 0$ and $\phi''(\cdot) \leq 0$.

A representative consumer has a log utility:

$$(4) \quad \int_0^\infty e^{-\rho t} \rho \ln C(s) ds,$$

where $\rho > 0$ is the subjective discount rate. The consumer is endowed with financial claims on the aggregate output from both sectors.

We now describe the market equilibrium. Taking the time-varying but deterministic equilibrium interest rate as given, the representative consumer chooses his consumption process to maximize (4), and firms in both sectors maximize their market values. All produced goods are either consumed or invested in either sector, so the goods-market clearing condition holds:

$$(5) \quad C = Y_0 + Y_1 - I_0 - I_1.$$

In equilibrium, the consumer holds his financial claims on aggregate output.

II. Model Results and Analysis

Using the welfare theorem, we obtain the equilibrium allocation by solving a central planner's problem. Let $V(K_0, K_1)$ denote the planner's value function. By dynamic programming, we have the following Hamilton-Jacobi-Bellman (HJB) equation for $V(K_0, K_1)$:

$$(6) \quad \rho V = \max_{I_0, I_1} \rho \ln C + \phi(i_0)K_0 V_0 + \phi(i_1)K_1 V_1,$$

where $V_n = dV/dK_n$. Capital stocks in both sectors are the natural state variables. Exploiting the model's homogeneity properties, we have that the effective state variable is the relative size of capital stocks in the two sectors. Let

$$z \equiv \frac{K_1}{K_0 + K_1}$$

denote the ratio between sector-1 capital K_1 and the aggregate capital $(K_0 + K_1)$. Note $0 \leq z \leq 1$. Using the homogeneity property, we have

$$(7) \quad V(K_0, K_1) = \ln[(K_0 + K_1)N(z)],$$

where $N(z)$ is a function to be determined.

Let $g_n(z)$ denote the growth rate of capital in sector n . Using (2), we have $g_n(z) = \phi(i_n(z))$, which differs from $i_n(z)$. Substituting (7) into (6), we obtain the following ordinary differential equation (ODE):

$$(8) \quad \rho \ln \frac{N(z)}{c(z)} = (1-z) \left[1 - z \frac{N'(z)}{N(z)} \right] g_0(z) + z \left[1 + (1-z) \frac{N'(z)}{N(z)} \right] g_1(z).$$

The first-order conditions (FOCs) for $i_0(z)$ and $i_1(z)$ are given by:

$$(9) \quad \frac{\rho}{\phi'(i_0(z))} = c(z) \left(1 - \frac{zN'(z)}{N(z)} \right),$$

$$(10) \quad \frac{\rho}{\phi'(i_1(z))} = c(z) \left(1 + (1-z) \frac{N'(z)}{N(z)} \right).$$

In addition, we have the goods-market clearing condition in scaled variables:

$$(11) \quad c(z) + (1-z)i_0(z) + zi_1(z) = A(z),$$

where aggregate productivity $A(z)$ is

$$(12) \quad A(z) = (1-z)A_0 + zA_1.$$

The rate of change for $z(t)$, $dz(t)/dt = \mu_z(z(t))$, is given by

$$(13) \quad \mu_z(z) = z(1-z)[g_1(z) - g_0(z)].$$

Intuitively, the larger the wedge $g_1(z) - g_0(z)$ between the *endogenous* capital growth rates in the two sectors, the faster capital reallocates to sector 1, the more productive sector ($A_1 > A_0$).

Let $Q_n(K_n; z)$ denote the firm value in sector n . Using the homogeneity property, we have

$$(14) \quad Q_n(K_n; z) = q_n(z)K_n, \quad n = 0, 1,$$

where Tobin's q in sector n is given by

$$(15) \quad q_n(z) = \frac{1}{\phi'(i_n(z))}.$$

Now consider aggregation. Aggregate investment is $I = I_0 + I_1$. The aggregate capital stock is $K = K_0 + K_1$ and its market value (aggregate wealth) is $Q(z) = Q_0(z) + Q_1(z)$. Therefore, aggregate Tobin's q is

$$(16) \quad q(z) = \frac{Q(z)}{K} = (1-z)q_0(z) + zq_1(z).$$

With log utility, the aggregate consumption-wealth ratio $C(z)/Q(z)$ is equal to the discount rate ρ , a known result. Equivalently stated in "scaled" terms, $c(z) = \rho q(z)$. Let $i(z)$ denote the aggregate investment-capital ratio:

$$(17) \quad i(z) = \frac{I}{K} = (1-z)i_0(z) + zi_1(z).$$

Let $g(z)$ denote the growth rate of aggregate capital $K(t) = K_0(t) + K_1(t)$:

$$(18) \quad g(z) = \frac{d \ln K(t)}{dt} = (1-z)g_0(z) + zg_1(z),$$

Adjustment costs drive a wedge between capital growth rate $g(z)$ and the investment-capital ratio $i(z)$. The equilibrium interest rate is given by the sum of the subjective discount rate ρ and the growth rate of consumption, and can be simplified as follows:

$$(19) \quad r(z) = \rho + \frac{c'(z)}{c(z)}\mu_z(z) + g(z).$$

Note that the growth rate of consumption differs from $g(z)$, the growth rate of aggregate capital K , because $C(t) = c(z(t))K(t)$. In general, consumption-capital ratio $c(z)$ depends on the relative size of sectoral capital z .

One-sector economy: The solution to the one-sector economy is a special case (with $z = 0$ and $z = 1$) of the above two-sector economy. Both $z = 0$ and $z = 1$ are absorbing barriers for the ODE (8). In the one-sector economy, we have $N(0) = v_0$ and $N(1) = v_1$, where

$$(20) \quad v_n = (A_n - i_n^*)e^{\phi(i_n^*)/\rho}, \quad n = 0, 1,$$

with the investment-capital ratio i_n^* maximizing (20) by solving $(A - i)\phi'(i) = \rho$.

Two-sector economy: We now summarize the solution for the two-sector economy. We solve the ODE (8) for $N(z)$ subject to (a) the FOCs (9) and (10), (b) the equilibrium (investing is equal to saving) condition (11), and (c) the boundary conditions (20) for one-sector economies. Next, we perform a quantitative exercise for a two-sector economy.

III. A Parametric Example

For both sectors, we specify

$$(21) \quad \phi(i_n) = \alpha + \Gamma \ln \left(1 + \frac{i_n}{\theta} \right),$$

where $\Gamma, \theta > 0$. The solution for the one-sector economy with productivity A_n is given by

$$(22) \quad c^* = \rho q_n^*, \quad i_n^* = \frac{\Gamma A_n - \rho \theta}{\rho + \Gamma}, \quad q_n^* = \frac{A_n + \theta}{\rho + \Gamma},$$

and $g_n^* = \phi(i_n^*) = \alpha + \Gamma \ln(\Gamma q_n^*/\theta)$, and the interest rate $r_n^* = \rho + g_n^*$.

Next, we summarize the analytic results for the two-sector economy. Each sector's investment-capital ratio is affine in Tobin's q :

$$(23) \quad i_n(z) = \Gamma q_n(z) - \theta, \quad n = 0, 1,$$

where sectoral $q_0(z)$ and $q_1(z)$ are given by

$$(24) \quad q_0(z) = q(z) \left[1 - z \frac{N'(z)}{N(z)} \right],$$

$$(25) \quad q_1(z) = q(z) \left[1 + (1-z) \frac{N'(z)}{N(z)} \right],$$

and aggregate Tobin's q is given by

$$(26) \quad q(z) = \frac{A(z) + \theta}{\rho + \Gamma}.$$

Using (23), we obtain the following expressions for the aggregate investment-capital ratio $i(z)$ and consumption-capital ratio $c(z)$:

$$(27) \quad i(z) = \Gamma q(z) - \theta = \frac{\Gamma A(z) - \rho \theta}{\rho + \Gamma},$$

$$(28) \quad c(z) = A(z) - i(z) = \frac{\rho A(z) + \rho \theta}{\rho + \Gamma}$$

With log utility and the log installation function (21), Tobin's q , the investment-capital ratio $i(z)$, and the consumption-capital ratio $c(z)$ at the aggregate level all increase linearly with aggregate productivity $A(z)$.

At the sectoral level, intuitively, investment and q should be lower in the less productive sector. The FOCs imply

$$(29) \quad q_0(z)\phi'(i_0(z)) = q_1(z)\phi'(i_1(z)) = 1.$$

Intuitively, the marginal benefit of a unit of capital is $q_n(z)$ and the marginal investment installs $\phi'(i_n(z))$ units of capital. Therefore, the marginal benefit of investing is given by $q_0(z)\phi'(i_0(z))$ which is a unit in terms of forgone consumption or investment in the other sector. Due to the convexity induced by the adjustment cost specification in our optimization problem, $i_0(z)$ for the less productive sector still has an interior solution. Note that $i_0(z) < i_1(z)$ naturally implies $q_0(z) < q_1(z)$ as we see from (29). Second, production efficiency implies that capital should over time be reallocated either to sector 1, the more productive sector or to consumption (note $\mu_z(z) > 0$ for $0 < z < 1$). For either reason, $i_0(z)$ should fall and hence $q_0(z)$ decreases (from (23)).

Due to the concave installation function $\phi(i)$, investment does not translate into growth one-to-one (even after accounting for depreciation). Sectoral capital growth rates $g_n(z(t)) = dK_n(t)/dt$ are given by

$$(30) \quad g_n(z) = \alpha + \Gamma \ln \left[\frac{\Gamma}{\theta} q_n(z) \right], \quad n = 0, 1.$$

Therefore, the growth rate for aggregate capital $g(z) = (1-z)g_0(z) + zg_1(z)$ satisfies

$$(31) \quad g(z) < \alpha + \Gamma \ln \left[\frac{\Gamma}{\theta} q(z) \right], \quad n = 0, 1.$$

While sectoral growth $g_n(z)$ is a log function of sectoral q_n as in (30), the same relation does not hold in the aggregate. Both sectors incur adjustment costs, so the growth rate of capital aggregate $g(z)$ is lower than implied by the sectoral "investment function" evaluated at aggregate Tobin's q .

The equilibrium interest rate is given by

$$(32) \quad r(z) = \rho + \frac{(1-z)(A_0 + \theta)g_0(z) + z(A_1 + \theta)g_1(z)}{(1-z)(A_0 + \theta) + z(A_1 + \theta)},$$

the sum of the consumer's discount rate ρ and a weighted average of sectoral growth $g_n(z)$, where the weights depend on sectoral productivity and size. Note that $r(z)$ does not increase with z for all values of z . Indeed, we show that $r'(z) < 0$ for z close to zero. Around $z = 0$, aggregate growth $g(z)$ is decreasing in z because of adjustments costs for capital reallocation.

We choose model parameters to generate sensible aggregate predictions and to highlight the impact of endogenous investment and growth on capital reallocation. The annual subjective discount rate is $\rho = 0.02$. The annual productivity parameters are $A_0 = 0.10$ and $A_1 = 0.12$. Finally, we choose $\Gamma = 0.05$, $\alpha = -0.10$, and $\theta = 0.01$ to generate the following aggregate predictions for the one-sector economy: Tobin's $q_0^* = 1.57$ and $q_1^* = 1.86$, investment-capital ratios $i_0^* = 0.069$ and $i_1^* = 0.083$, the capital growth rates $g_0^* = 0.003$ and $g_1^* = 0.011$, and equilibrium interest rates $r_0^* = 0.023$ and $r_1^* = 0.031$.

Sector 1 has higher productivity than sector 0 ($A_1 > A_0$), so the planner would like to specialize in sector 1. However, starting from a capital distribution with $K_0 > 0$, the planner does not immediately reallocate all capital to sector 1 because of the adjustment costs.

Figure 1 plots the rate of reallocation from sector 0 to sector 1, measured by $dz(t)/dt = \mu_z(z(t))$. Note that $\mu_z(z) = z(1-z)(g_1(z) - g_0(z))$ implies that (a) $\mu_z(z)$ is hump-shaped in z (which follows from the quadratic component $z(1-z)$), and (b) $z = 0$ and $z = 1$ are absorbing states (i.e. $\mu_z(0) = \mu_z(1) = 0$). Intuitively, starting with low values of z , the rate of reallocation is initially low and then rises, reaching a maximum at $z < 1/2$, before declining again. The asymmetry of $\mu_z(z)$ around $z = 1/2$ follows from the growth wedge $g_1(z) - g_0(z)$, which is larger than 0 (intuitively, the more productive sector invests more and grows faster, *ceteris paribus*.) Figure 1 also shows that the equilibrium interest rate $r(z)$ tends to rise with z because the interest rate moves with the growth rate of aggregate consumption, which tends to

grow with z . However, there is a non-monotonic relation between $r(z)$ and z near $z = 0$ due to the concavity of the adjustment costs and the inefficiency of incurring adjustment costs in both sectors, as we have noted earlier.

Figure 2 graphs growth rates of capital both at the sectoral level ($g_0(z), g_1(z)$), and at the aggregate level $g(z)$. Both $g_0(z)$ and $g_1(z)$ decrease with z . For sector 0, the low productivity sector, growth is initially positive, but it declines and becomes negative as capital is reallocated to sector 1, the more productive sector. Initially, however, growth remains positive in sector 0, and output from sector 0 is used to finance growth in sector 1. Sector 0 initially shrinks in relative terms and then in absolute terms, as its capital stock dwindles. Growth in sector 1 is initially high, when sector 1 is small, and then its growth decreases and stabilizes as the economy specializes in sector 1. The aggregate growth rate, on the other hand, is non-monotonic in z . Initially aggregate growth falls, as the economy expends resources reallocating capital from sector 0 to sector 1. As noted above, we show that $g'(z) < 0$ at $z = 0$, so that the growth rate always decreases before increasing as the economy shifts toward the high productivity sector. For sufficiently high z , aggregate growth increases with z .

Figure 3 shows that sectoral Tobin's q in both sectors decreases in z , but aggregate Tobin's q increases linearly in z . Capital installed in sector 0 is initially valuable as a source of output to be re-invested in sector 1; note that Tobin's q in sector 1 is very high at this stage. As capital is reallocated to sector 1 and z rises, the value of capital in both sectors falls. Nonetheless, the value of capital in sector 1 always exceeds that in sector 0, so as reallocation occurs into sector 1, aggregate Tobin's q rises. There is no contradiction between decreasing sectoral q and increasing aggregate q in z . Note that $q'(z) = q_1(z) - q_0(z) + zq_1'(z) + (1-z)q_0'(z)$. As long as the wedge between sectoral q s, $q_1(z) - q_0(z)$, is big enough, $q'(z)$ can be positive, while $q_0'(z) < 0$ and $q_1'(z) < 0$.

Since the investment-capital ratio is affine in Tobin's q at both the sectoral and aggregate levels, as shown in (23) and (23), the properties for $i_n(z)$ and $i(z)$ are the same as those for $q_n(z)$ and $q(z)$, respectively.

Figure 4 plots the dynamic evolution of the sector-1 share of capital $z(t)$ over time, and its slope $\mu_z(z(t)) = dz(t)/dt$ over time. For $t \leq 166.78$, the slope $\mu_z(z(t))$ is increasing and hence z is convex in time. For $t \geq 166.78$, the slope $\mu_z(z(t)) = dz(t)/dt$ starts falling and hence z increases at a slower pace, eventually approaching $z = 1$ around $t = 500$. Since there are no shocks, for any given initial value $z(0)$ we simply start with the corresponding calendar time $t(0)$ and the dynamics continue deterministically from then on.

Figure 5 plots the time series dynamics of the growth rate of aggregate capital g and the equilibrium interest rate r . As we have noted, aggregate growth g first decreases to reflect the duplication of adjustment costs in the two sectors, when z is small. When z is high enough, the more productive sector is sufficiently large and hence reallocation increases growth. As a result, equilibrium consumption increases with z . Note that the growth rate of consumption differs from that of capital. Indeed, consumption grows at a faster rate than capital in our model due to the fact that $c(z(t)) = C(t)/K(t)$ is also increasing over time. As a result, the equilibrium interest rate must increase with z earlier in order to discourage consumption and sustain equilibrium. This explains why the interest rate reaches its minimum earlier than the growth rate of aggregate capital. (See (19) for the analytics.) After sufficiently time has elapsed, reallocation to the more productive sector is effectively complete and hence the aggregate capital growth rate and the interest rate approach the levels in the corresponding one-sector economy with high productivity.

Future work. These results emphasize the role of sectoral heterogeneity and adjustment costs in the reallocation of capital. While the economy trends toward specializing in the high productivity sector, convex adjustment costs imply that the reallocation process does not occur immediately and that the resources expended in reallocation cause non-monotonic changes in the growth rate and interest rate. Similarly, the low productivity sector finances the growth of the high productivity sector, until the low productivity sector eventually dwindles away and the economy specializes. Since this setting is deterministic, specialization is a natural outcome.

However, specialization disallows the potential resurgence of the low productivity sector, because the household has no incentive to diversify in deterministic settings. These issues lead us to introduce uncertainty into the model. In Janice C. Eberly and Neng Wang (2009), we analyze a two-sector model with adjustment costs and uncertainty. We focus on the trade-off between diversification and growth, in addition to the trade-off between reallocation and growth emphasized here. This approach is fundamentally different from existing two-sector stochastic models, which either assume that capital is frictionless, as in John C. Cox, Jonathan E. Ingersoll, and Stephen A. Ross (1985) and Larry Jones and Rodolfo E. Manuelli (2005), or that capital is fixed, as in John H. Cochrane, Francis A. Longstaff, and Pedro Santa-Clara (2008). When capital is perfectly liquid, Tobin's q is one at all times and heterogeneity plays no role in equilibrium. When capital is completely illiquid (as in "two trees"), investment is zero at all times. In our model, investment drives the dynamics of Tobin's q and the distribution of capital, as well as risk-return asset pricing relations. Furthermore, in this paper we model illiquid capital via a concave installation function (or equivalently, convex adjustment costs). In future work we extend our notion of the illiquidity of capital to include non-convex adjustment costs via an augmented adjustment cost function.

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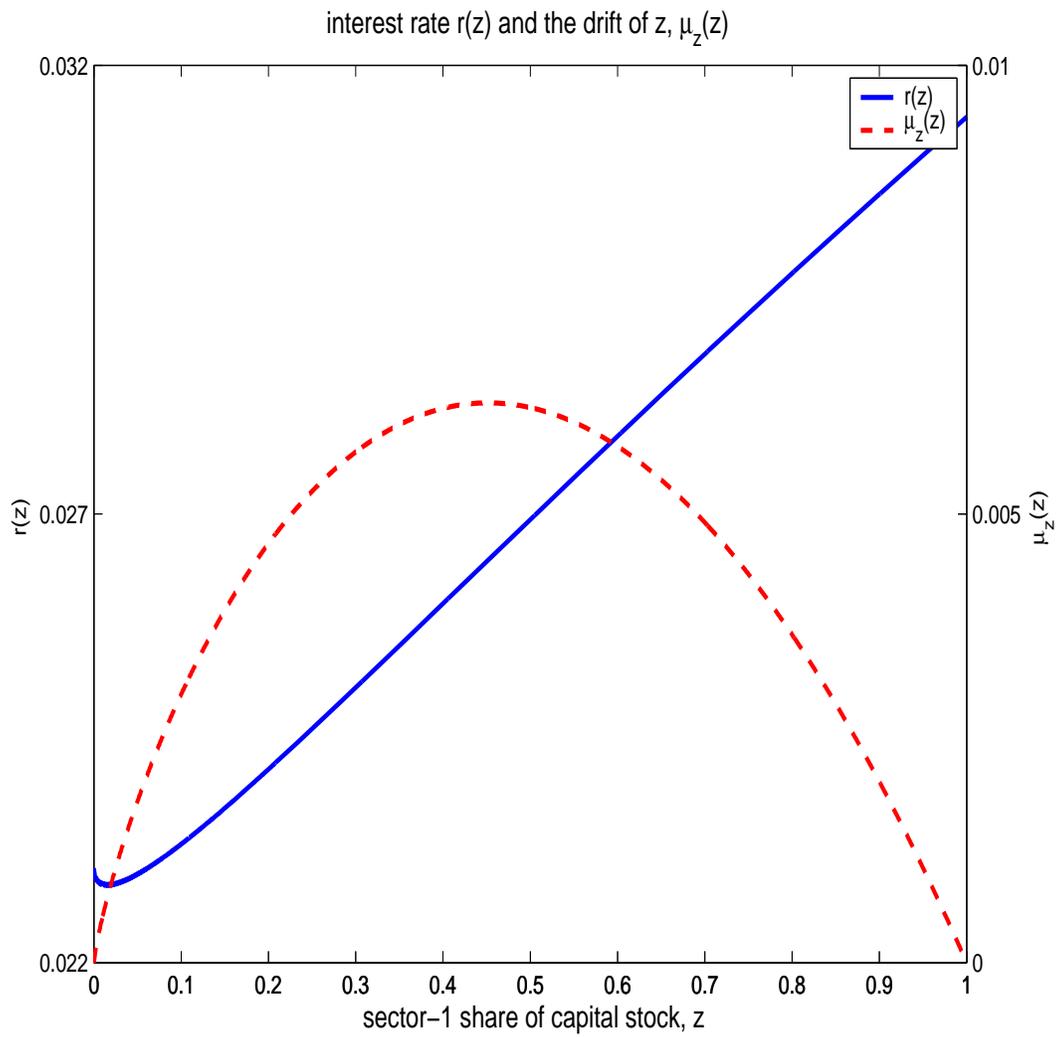


FIGURE 1: **Equilibrium interest rate $r(z)$ and the rate of change of sector-1 share of capital stock $z(t)$: $\mu_z(z(t)) = dz(t)/dt$.**

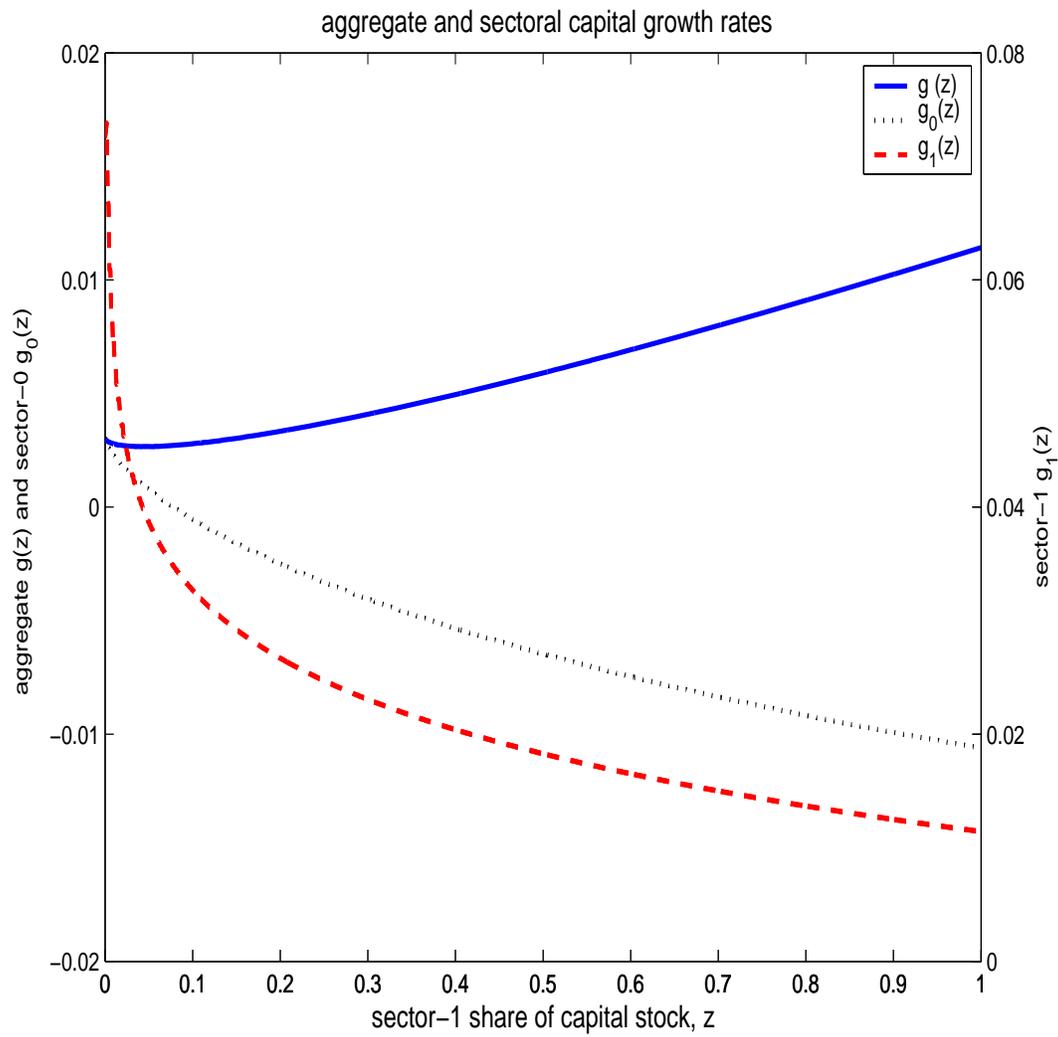


FIGURE 2: Growth rates of capital: Aggregate $g(z)$ and Sectoral ($g_0(z), g_1(z)$).

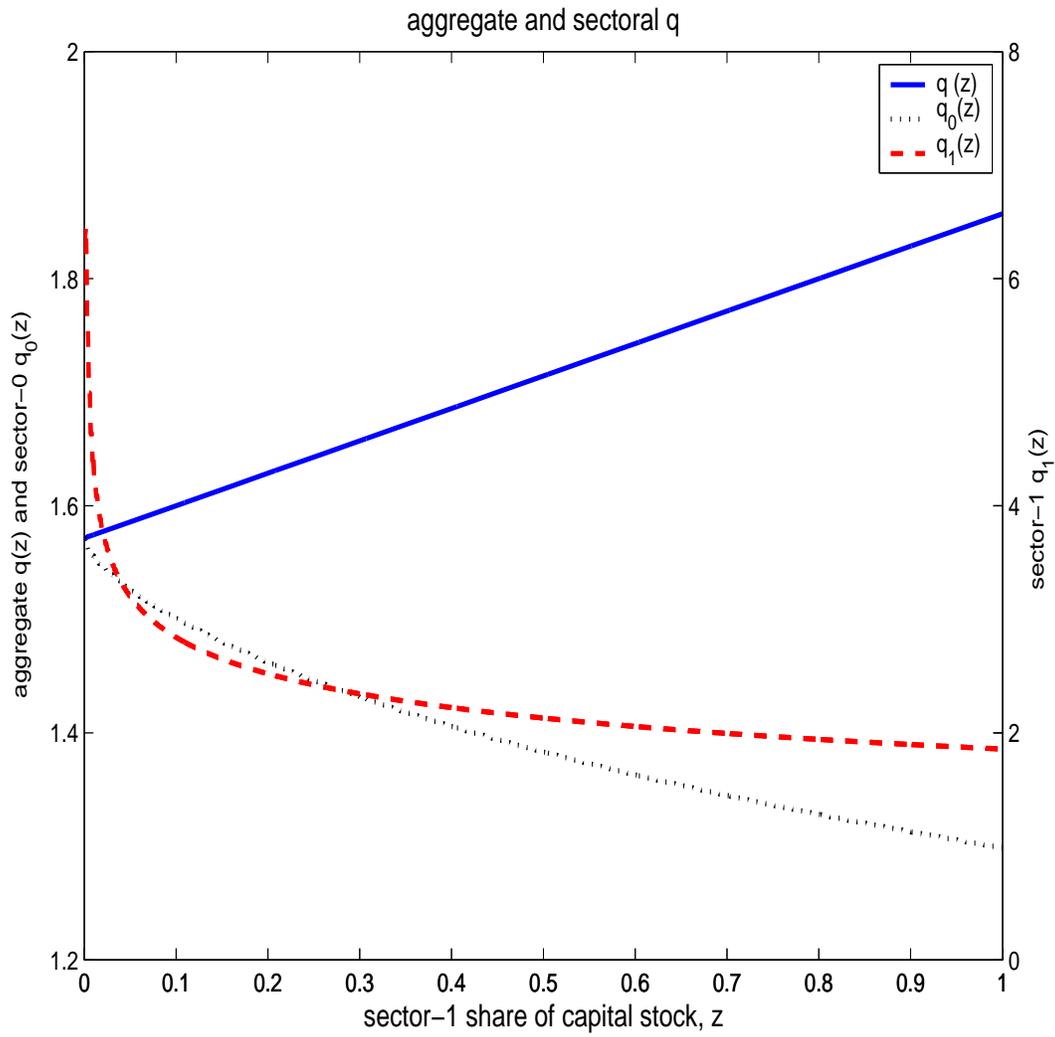


FIGURE 3: Tobin's q : Aggregate $q(z)$ and sectoral $(q_0(z), q_1(z))$.

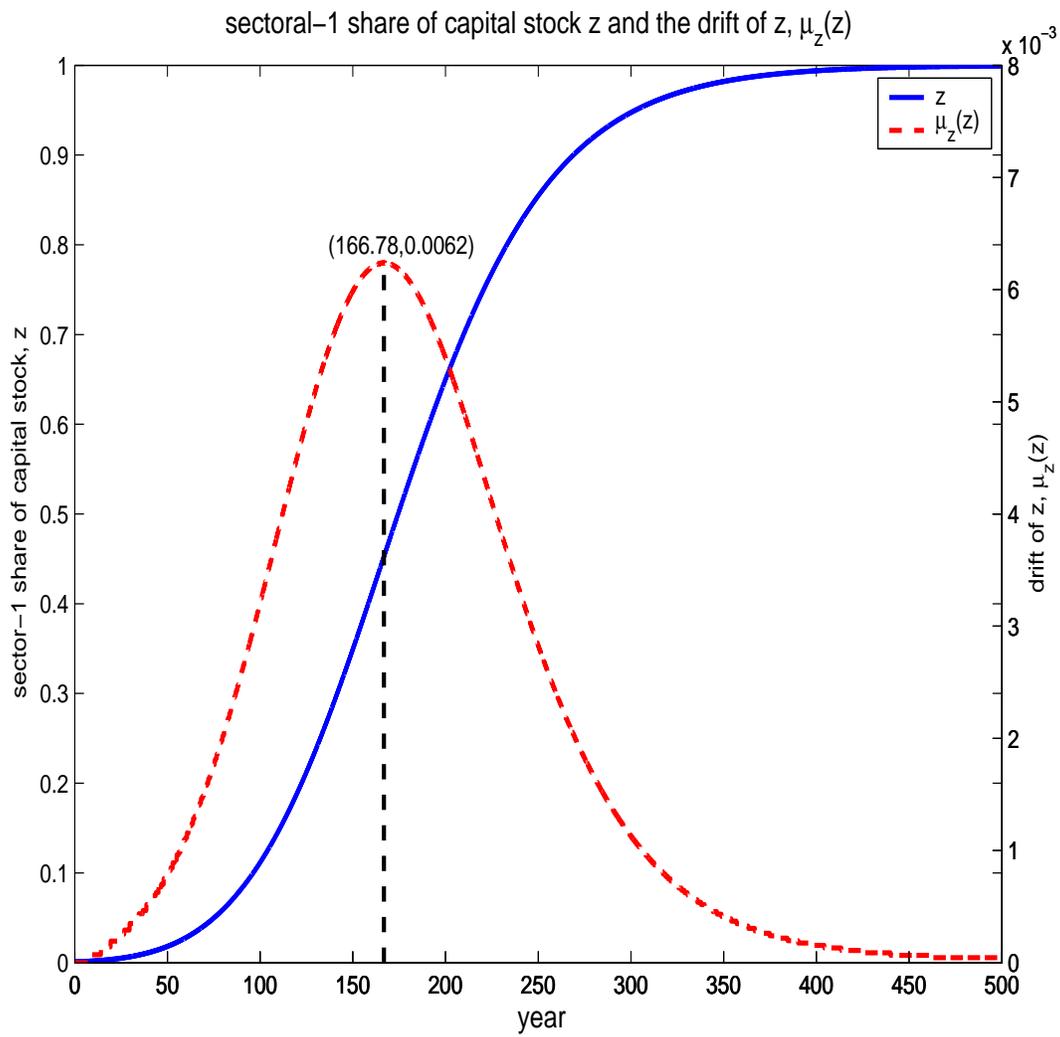


FIGURE 4: Time series dynamics for the sector-1 share of capital stock $z(t)$ and the drift of this share $\mu_z(z(t)) = dz(t)/dt$.

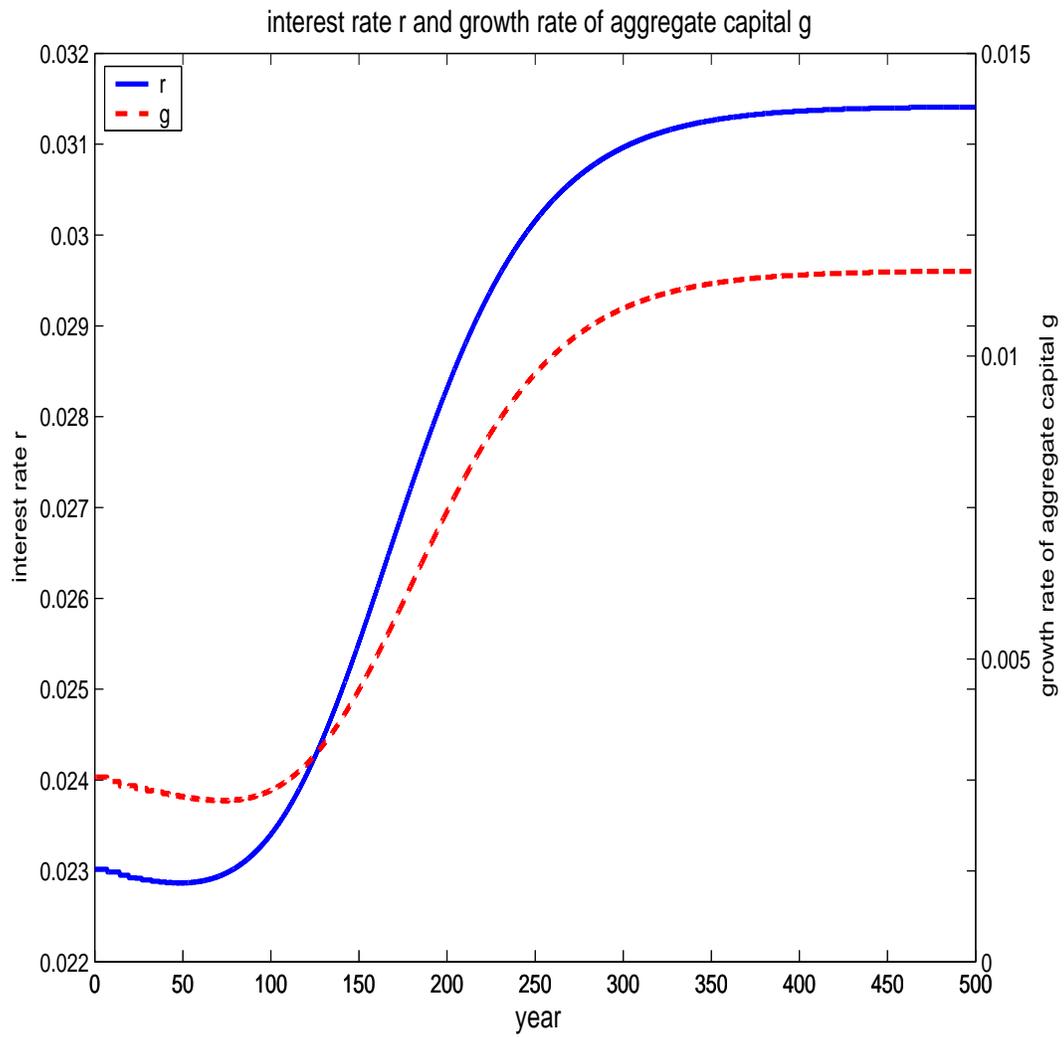


FIGURE 5: Time series dynamics for the equilibrium interest rate $r(z(t))$ and growth rate of aggregate capital $g(z(t)) = d \ln K(t) / dt$.