



ELSEVIER

Journal of Monetary Economics 44 (1999) 339–377

Journal of
MONETARY
ECONOMICS

www.elsevier.nl/locate/econbase

The effects of irreversibility and uncertainty on capital accumulation[☆]

Andrew B. Abel^a, Janice C. Eberly^{b,*}

^a*The Wharton School of the University of Pennsylvania, Philadelphia, PA 19104, USA*

^b*J.L. Kellogg Graduate School of Management, Northwestern University, 2001 Sheridan Road, Evanston, IL 60208-2001, USA*

Received 30 September 1997; accepted 12 June 1998

Abstract

Irreversibility and uncertainty increase the user cost of capital which tends to reduce the capital stock. Working in the opposite direction is a hangover effect, which arises because irreversibility prevents the firm from selling capital even when the marginal revenue product of capital is low. Neither the user cost effect nor the hangover effect dominates globally, so that irreversibility may increase or decrease capital accumulation. Furthermore, an increase in uncertainty can either increase or decrease the long-run capital stock under irreversibility relative to that under reversibility. Other effects that we consider, however, have unambiguous effects on long-run capital accumulation. © 1999 Published by Elsevier Science B.V. All rights reserved.

[☆]The authors benefited from discussions with Domenico Cuoco, Andrew Lo, and Robert McDonald, and from correspondence with Giuseppe Bertola and Ricardo Caballero. We thank seminar participants at the NBER, Northwestern University, and the Penn Macro Lunch Group for helpful comments, and the National Science Foundation for financial support. We also thank an anonymous referee for comments that led us to write Section 4.2. This work was initiated while Eberly was visiting Harvard University and the NBER, and completed while on the faculty of the Wharton School of the University of Pennsylvania; their support is gratefully acknowledged, as is a Sloan Foundation Research Fellowship.

* Corresponding author. Tel.: + 847-467-1840; fax: 847-491-5719.

E-mail address: eberly@nwu.edu (J.C. Eberly)

JEL classification: E22

Keywords: Investment; Irreversibility; Uncertainty

1. Introduction

The major result of the large literature on irreversible investment under uncertainty is that irreversibility increases the hurdle that projects must clear in order to be profitably undertaken. In addition, the hurdle is increased the higher is uncertainty. Although these results are enormously helpful to managers making capital budgeting decisions, they are silent about the optimal amount of investment and about the long-run accumulation of capital. As described by Hubbard (1994) in his review of Dixit and Pindyck's seminal book *Investment Under Uncertainty*, 'the new view models ... do not offer specific predictions about the *level* of investment. To go this extra step requires the specification of structural links between the marginal profitability of capital and the desired capital stock'. (1994, p. 1828, emphasis original). Our paper takes the extra step by providing those structural links and goes on to explicitly calculate the impacts of irreversibility and uncertainty on the expected long-run capital stock. As Hubbard emphasizes, these results are crucial to empirical evaluation of the model. Since the investment rule itself is typically unobservable, data on investment and the capital stock are more commonly used to evaluate investment models. While data on investment and the capital stock at the *moment* of investment might be used to deduce the investment rule, if the data are time aggregated (arising from the use of discretely observed data) then this is not possible. Observed investment is aggregated over periods of both positive and zero investment. Our results give the implications of irreversibility and uncertainty for such a time-aggregated, or long-run, measure of the capital stock.

Optimal capital budgeting in the standard neoclassical model with reversibility¹ maintains the marginal revenue product of capital equal to the user cost derived by Jorgenson (1963). However, when investment is irreversible, the optimal investment policy is to purchase capital only as needed to prevent the marginal revenue product of capital from rising above an optimally derived hurdle. This hurdle, which is the user cost of capital appropriately defined to take account of irreversibility and uncertainty, is higher than the Jorgensonian user cost. Thus, if the firm currently has no capital and faces a given marginal revenue product schedule (as a decreasing function of the capital stock), the

¹ Throughout this paper we will use the term 'reversibility' to refer to the situation in which the firm can purchase and sell capital at the same price without any transactions costs or adjustment costs.

optimal capital stock under irreversibility is smaller than the optimal capital stock under reversibility.² This result, which we will call the ‘user-cost’ effect, occurs because the firm anticipates that the irreversibility constraint may bind in the future and thus is more reluctant to invest today; this finding has been emphasized by Bertola (1988), Pindyck (1988), Dixit (1989), and Dixit and Pindyck (1994). A related result is that an increase in the variance of the shocks facing the firm tends to increase the user cost under irreversibility without affecting the user cost in the standard reversible case. This increase in the user cost due to increased uncertainty tends to further reduce the optimal capital stock under irreversibility.

The results described above apply to a firm that currently has zero capital, such as a new firm just getting started. But what are the effects of irreversibility and increased uncertainty for an ongoing firm? To address this question we focus on the long-run capital stock. Of course, in the presence of uncertainty the capital stock does not converge to a constant in the long run. Therefore, we focus on the current expectation of the capital stock K_t at a future date t and then examine long-run behavior by letting t approach infinity.

To examine the behavior of the capital stock in the long run, it is important to recognize that a firm will arrive at any future date t with a capital stock representing the cumulation of investment prior to that date (taking account of depreciation). If demand for the firm’s output is unusually low at date t , the firm would like to sell some of its capital at a positive price. However, under irreversibility, the firm cannot sell capital, and it would be constrained by its own past investment behavior to have a capital stock that is higher than it would choose if it could start fresh at date t . This dissonance between the firm’s actual capital stock and the level that it would choose to hold does not reflect any failure of rationality. Instead it reflects the firm’s optimal response to favorable circumstances in the past. We refer to this effect as the ‘hangover’ effect to indicate the dependence of the current capital stock on past behavior, especially behavior that later the firm would like to reverse.³ The hangover effect can lead to a higher capital stock under irreversibility than under reversibility.

² It is important for this result that the marginal revenue product of the firm is a decreasing function of the capital stock, as noted by Pindyck (1993) and Abel and Eberly (1997). If the marginal revenue product does not depend on the capital stock, then current and future marginal revenue products are unaffected by today’s investment, so the link from today’s investment to future returns is broken. The firm is then no more reluctant to invest under irreversibility than with reversible investment. Caballero (1991) also emphasizes the importance of decreasing marginal revenue product of capital in his analysis of the effect of uncertainty on investment.

³ The hangover effect is an example of ‘hysteresis’ in irreversible investment discussed by Dixit (1992). Dixit emphasizes the finding that with irreversibility past events affect current investment behavior.

The user-cost and hangover effects have opposing implications for the current expectation of the long-run capital stock. With irreversibility, the user-cost effect tends to reduce the expected capital stock whereas the hangover effect tends to increase the expected capital stock. The two effects also give opposing answers regarding the effect of increased uncertainty on the expected long-run capital stock. As we have discussed, the user-cost effect implies that increased uncertainty tends to reduce the expected long-run capital stock under irreversibility. However, the hangover effect implies that increased uncertainty tends to increase the expected long-run capital stock. In this paper we analyze the effects of irreversibility and uncertainty on the expected long-run capital stock taking account of the user-cost and hangover effects together.

The literature on irreversible investment under uncertainty has attempted in various ways to assess the long-run or average effects of irreversibility on the capital stock.⁴ Bertola (1988) and Bentolila and Bertola (1990)⁵ examine the long-run distribution of the marginal revenue product of capital and conclude that the mean of this distribution is reduced by the presence of irreversibility — and reduced further the higher is uncertainty. They also show that a particular statistic that resembles the expected value of the capital stock in the long run — but, importantly, is not the expected value of the capital stock (as we show in Section 3) — is increased by the presence of irreversibility. Caballero (1993) describes the opposing effects that we call the user-cost effect and the hangover effect, and asserts without proof that ‘whether the firm holds more or less capital on average than in a frictionless framework is ambiguous’ (p. 88). Bertola and Caballero (1994) state without proof ‘On average, the capital intensity of production under investment irreversibility is actually *higher* [italics in original] than it would be if Eq. (7) [which describes the optimal amount of capital in the case of costless reversibility] applied at all times’ (p. 229, footnote 4). We show in Section 3 that this claim is true when applied to the capital–labor *ratio*, but not to the level of the capital stock.

The literature has also examined the effect of uncertainty on investment. Caballero’s (1991) analysis of the effect of uncertainty on investment concludes that ‘If this effect is sufficiently strong (i.e., the asymmetry of adjustment costs is large and the negative dependence of the marginal profitability of capital on the level of capital is strong), the investment–uncertainty relationship becomes negative. The irreversible-investment arguments analyzed in the literature

⁴ Bertola (1992) considers a deterministic model of employment with hiring and firing costs. He finds that with positive discounting and/or attrition, hiring costs tend to reduce average employment, while firing costs increase average employment. Obviously, he does not consider the effects of uncertainty.

⁵ In Bentolila and Bertola (1990), the factor of production that is costly to adjust is labor, but it is straightforward to interpret this factor as capital and to treat investment as irreversible.

typically correspond to this case' (p. 286). This strong result is obtained by assuming that the firm begins with zero initial capital (p. 283), so that what we call the hangover effect is inoperative. Dixit and Pindyck (1994, pp. 372–373) consider a model similar to ours and focus on the effect of uncertainty on investment. Specifically, they calculate the expected change in the logarithm of the capital stock (rather than the expected value of the long-run capital stock or the expected value of the change (per unit time) in the long-run capital stock). They conclude that in this case a 'larger σ means a lower long-run average growth rate of the capital stock, and thus less investment on average' (p. 373), where σ measures uncertainty. We show in Section 4 that when the expected value of the long-run capital stock is calculated explicitly, the results of Caballero and Dixit and Pindyck can be reversed.

We calculate directly the expected value of the future capital stock for a firm facing irreversibility. For a firm starting with a zero capital stock, the user-cost effect initially causes the firm to accumulate less capital than under reversibility, an effect that is magnified by increased uncertainty. As the firm accumulates capital, however, it becomes more likely that during a low-demand episode the firm would like to sell capital, if it could. The hangover effect is then operative, increasing the capital stock under irreversibility relative to that under reversibility. In the long run, we find cases in which the user-cost effect dominates, and others in which the hangover effect dominates. Increasing uncertainty does not resolve this ambiguity, but instead deepens it. We show that in the long run, increased uncertainty increases the expected capital stock under irreversibility, but can increase it even more under reversibility. Thus, whether the increase in the expected long-run capital stock is larger under reversibility or under irreversibility depends on the choice of parameter values.

In the next section of the paper, we construct a simple model of a firm with an infinite horizon and characterize its optimal investment decision with irreversibility and uncertainty. In Section 3, we calculate the expected value of the capital stock, comparing the irreversible investment case to the reversible investment case and identifying the user-cost and hangover effects. Section 4 focuses on the effects of uncertainty on the expected long-run capital stock, while Section 5 considers the effects of growth, the interest rate, the capital share in production, and the price elasticity of demand. In Section 6 we offer concluding remarks.

2. The firm's optimization problem

We develop a simple model of the firm in order to focus on the key elements of our question: the effects of *irreversibility* and *uncertainty* on capital accumulation in the *long run*. Accordingly, we assume that investment is irreversible, returns to capital are uncertain, and the firm has an infinite horizon. The functional forms

we use are chosen for their tractability and are also used by Bertola (1988), Bentolila and Bertola (1990), Dixit (1991), Bertola and Caballero (1994), and Abel and Eberly (1996).

Consider a firm that faces an isoelastic demand curve

$$Q_t = X_t P_t^{-\varepsilon} \quad (1)$$

where Q_t is the quantity of output demanded, P_t is the price of output, X_t is a stochastic demand shock, and $\varepsilon > 1$ is the price elasticity of demand. The firm produces nonstorable output Q_t according to the Cobb–Douglas production function

$$Q_t = L_t^{1-\beta} K_t^\beta \quad (2)$$

where L_t is labor, K_t is the capital stock, and the capital share β satisfies $0 < \beta < 1$. At each point of time the firm chooses L_t to maximize its operating profit $P_t Q_t - w L_t$ where w is the wage rate, which is assumed constant. The maximized value of operating profit is given by⁶

$$\pi(K, X) = \frac{h}{1-\gamma} X_t^\gamma K_t^{1-\gamma} \quad (3)$$

where

$$h \equiv (1-\gamma) \left(\frac{1}{\gamma\varepsilon} \right)^{\gamma\varepsilon} (\gamma\varepsilon - 1)^{\gamma\varepsilon - 1} w^{1-\gamma\varepsilon} > 0 \text{ and}$$

$$0 < \frac{1}{\varepsilon} < \gamma \equiv \frac{1}{1 + \beta(\varepsilon - 1)} < 1.$$

Because the instantaneous operating profit of the firm depends on the firm's capital stock K_t and on the stochastic component of demand X_t , the evolution of the operating profit depends on the evolution of X_t and K_t over time. Assume that the demand shock X_t evolves exogenously according to a geometric Brownian motion

$$\frac{dX_t}{X_t} = \mu dt + \sigma dz, \quad \sigma > 0 \quad (4)$$

⁶ The operating profit is given by $\pi(K, X) = \max_L [X^{1/\varepsilon} L^{(1-\beta)(1-1/\varepsilon)} K^{\beta(1-1/\varepsilon)} - wL] = \{\max_l [(X/K)^{1/\varepsilon} l^{1-1/(\gamma\varepsilon)} - w] \} K$ where $l = L/K$ and γ is defined in Eq. (3). Solving this maximization yields

$$l = w^{-\gamma\varepsilon} \left(\frac{\gamma\varepsilon}{\gamma\varepsilon - 1} \right)^{-\gamma\varepsilon} \left(\frac{X}{K} \right)^\gamma \quad \text{and} \quad \pi(K, X) = \frac{w/K}{\gamma\varepsilon - 1}$$

which is equivalent to Eq. (3) in the text.

where $X_0 > 0$, and dz is an increment to a standard Wiener process, with $E\{dz\} = 0$ and $(dz)^2 = dt$.

The firm can purchase capital at a constant price $b > 0$ but is unable to sell capital. The capital stock does not depreciate.⁷ Therefore, the evolution of the capital stock K_t depends only on the firm's purchases of capital.⁸

Assume that the firm is risk-neutral and discounts future cash flows at the constant rate $r > 0$, where $r > \mu$.⁹ The investment policy that maximizes the value of the firm is derived in Appendix A. This policy is easily expressed in terms of the marginal revenue product of capital and the user cost of capital, which we present below. The marginal revenue product of capital is hy^γ where $y \equiv X/K$. The user cost of capital is

$$c \equiv \left(1 - \frac{\gamma}{\alpha_N}\right)rb \quad (5a)$$

where α_N is the negative root of the quadratic equation

$$\rho(\eta) \equiv -\frac{1}{2}\sigma^2\eta^2 - \left(\mu - \frac{1}{2}\sigma^2\right)\eta + r = 0. \quad (5b)$$

The user cost of capital in Eq. (5a) is the natural extension of the Jorgensonian user cost of capital to the case of irreversibility under uncertainty. More precisely, it can be shown (see Abel and Eberly, 1996) that c is the sum of: (1) the interest cost rb ; and (2) the expected capital loss on a marginal unit of capital.¹⁰

The optimal investment policy is a 'barrier control' policy according to which the firm purchases capital as necessary to prevent the marginal revenue product of capital from rising above the user cost c . When the marginal revenue product of capital is lower than c , it is optimal not to purchase capital. Only when the marginal revenue product of capital equals the user cost is it optimal to

⁷ This assumption simplifies the calculation of the expected future capital stock. Specifically, for any time t at which the firm does not purchase capital, the capital stock equals the capital stock at date s , where s is the latest time prior to t that the firm purchased capital. If there were depreciation, then the capital stock at time t would have to be adjusted for depreciation since time s , and this adjustment would require information about the length of time $(t - s)$ since the firm's most recent purchase of capital.

⁸ Because the cost of adjustment is linear, the *rate* of investment (i.e., investment per unit of time) can be infinite. The capital stock therefore follows a continuous, but non-differentiable (with respect to time), path.

⁹ The expected present value of operating profits, $E_t\{\int_0^\infty e^{-rs}\pi(K_{t+s}, X_{t+s})ds\}$, is finite if $\mu < r$. Footnote 25 in Appendix A provides a rigorous treatment of this issue.

¹⁰ More generally, the user cost contains a term reflecting the physical depreciation of capital, but in this model we have assumed that capital does not depreciate.

purchase capital. Under this barrier control policy the marginal revenue product of capital, hy^γ , is a regulated geometric Brownian motion, where $hy^\gamma \leq c$, or equivalently $y \leq (c/h)^{1/\gamma} \equiv y_U$.

To describe the behavior of the marginal revenue product of capital more formally, we first characterize the behavior of y^θ where θ is an arbitrary constant. Using Ito's Lemma and recalling that $y \equiv X/K$, the behavior of y^θ for $y < y_U$ is given by

$$\frac{dy^\theta}{y^\theta} = \theta M(\theta)dt + \theta\sigma dz \tag{6a}$$

where

$$M(\theta) \equiv \frac{1}{\theta} \frac{1}{dt} E \left\{ \frac{dX^\theta}{X^\theta} \right\} = \mu + \frac{1}{2}(\theta - 1)\sigma^2. \tag{6b}$$

The behavior of the marginal revenue product of capital, more precisely, $d(hy^\gamma)/(hy^\gamma)$, is given by Eqs. (6a) and (6b) by setting $\theta = \gamma$. Bertola (1988) shows that the marginal revenue product of capital will have a nondegenerate ergodic distribution¹¹ if and only if $\gamma M(\gamma) > (1/2)[\gamma\sigma]^2$. Using the definition of $M(\theta)$ in Eq. (6b) this condition can be written as

$$\mu > \frac{1}{2}\sigma^2. \tag{7}$$

For the remainder of this paper we restrict attention to cases in which the marginal revenue product of capital has a nondegenerate ergodic distribution and therefore we assume that the restriction in Eq. (7) holds.

3. The expected value of the capital stock

The optimal investment policy of the firm is to purchase capital whenever it is needed to keep the marginal revenue product of capital, hy^γ , from rising above

¹¹ Result 2 in Bentolila and Bertola (1990, p. 389) implies that with irreversibility the ergodic distribution of y (which is a regulated geometric Brownian motion that is bounded above by y_U and in the absence of regulation obeys Eq. (6a) and (6b) with $\theta = 1$) is

$$f(y) = \frac{m-1}{y_U^{m-1}} y^{m-2}, \quad \text{for } y \leq y_U,$$

where $m \equiv 2\mu/\sigma^2$. Therefore, for any $\theta > 1 - m$

$$E\{y^\theta\} = \frac{m-1}{m-1+\theta} y_U^\theta,$$

where the expectation is with respect to the ergodic distribution.

the user cost c . Recalling that $y \equiv X/K$, the marginal revenue product of capital is $h(X/K)^\gamma$, and the condition that the marginal revenue product of capital is always less than or equal to c implies

$$X_t \leq (c/h)^{1/\gamma} K_t. \tag{8}$$

Because we have assumed that capital does not depreciate, we have

$$K_t = (c/h)^{-1/\gamma} \max_{s \leq t} X_s. \tag{9}$$

Suppose that the firm is born at time 0 (without any initial capital) and normalize the demand process X_t such that $X_0 = 1$. In this case, the expected value, as of date 0, of the capital stock at any date $t \geq 0$ is

$$E_0\{K_t\} = (c/h)^{-1/\gamma} E\left\{\max_{0 \leq s \leq t} X_s | X_0 = 1\right\}. \tag{10}$$

Calculating the expected value in Eq. (10) yields¹²

$$E_0\{K_t\} = (c/h)^{-1/\gamma} \left[\frac{\mu + \frac{1}{2}\sigma^2}{\mu} \Phi\left(\frac{\mu + \frac{1}{2}\sigma^2}{\sigma} t^{1/2}\right) \exp(\mu t) + \frac{\mu - \frac{1}{2}\sigma^2}{\mu} \Phi\left(-\frac{\mu - \frac{1}{2}\sigma^2}{\sigma} t^{1/2}\right) \right]. \tag{11}$$

where $\Phi(\cdot)$ is the standard normal c.d.f.

In order to focus on the effect of irreversibility, we introduce the case of costlessly reversible investment for comparison. Introducing costless reversibility also simplifies notation because we can express expected future capital stocks under irreversibility relative to the corresponding expected values under the benchmark of reversibility. In the standard case of costlessly reversible investment analyzed by Jorgenson (1963), the firm continuously adjusts its capital

¹² The expected value of $\max_{0 \leq s \leq t} X_s$ given $X_0 = 1$ can be calculated by defining $W \equiv \ln X$ and observing that $\max_{0 \leq s \leq t} X_s = \exp(\max_{0 \leq s \leq t} W_s)$. The distribution of the maximum of W_s (an arithmetic Brownian motion) is taken from Harrison (1985, p. 13, Eq. (8)):

$$F_t(w) \equiv \Pr\left\{\max_{0 \leq s \leq t} W_s \leq w | W_0 = 0\right\} = \Phi\left(\frac{w - \mu_w t}{\sigma_w t^{1/2}}\right) - e^{2\mu_w w / \sigma_w^2} \Phi\left(\frac{-w - \mu_w t}{\sigma_w t^{1/2}}\right).$$

The desired expectation is $E\{\max_{0 \leq s \leq t} X_s | X_0 = 1\} = \int_0^\infty e^{w} F_t'(w) dw$. Tedious calculation shows that this equation is equivalent to Eq. (11) in the text. Alternatively, the desired expectation can be taken from Goldman et al. (1979) where, using their notation, $S(t)$ is a geometric Brownian motion and $E\{\max_{0 \leq \delta \leq t} S(\delta) | S(0) = 1\} = e^{rt} V_{\max}[1, 1, t]$. Setting $S(\tau) = M(\tau) = 1$, and setting r equal to μ in their Eq. (10) on p. 1116 yields our Eq. (11) in the text.

stock to maintain the marginal revenue product of capital equal to the Jorgensonian user cost of capital. In the absence of depreciation, and with a constant purchase price of capital, the Jorgensonian user cost c_R equals rb (where ‘R’ indicates the case of costless reversibility). This equality of marginal revenue product and user cost implies that $K^R = (c_R/h)^{-1/\gamma} X$ always. Therefore,

$$E_0\{K_t^R\} = (c_R/h)^{-1/\gamma} E\{X_t | X_0 = 1\}. \tag{12}$$

Under the geometric Brownian motion in Eq. (4), the expected growth rate of X_t is μ which implies that $E\{X_t | X_0 = 1\} = e^{\mu t}$. Therefore, the expected capital stock under reversibility in Eq. (12) can be written as

$$E_0\{K_t^R\} = (c_R/h)^{-1/\gamma} e^{\mu t}. \tag{13}$$

We now compare the expected capital stock under irreversibility to that obtained with costless reversibility.¹³ Define $\kappa(t)$ as the ratio of the expected value of the capital stock at date t under irreversibility to the expected value of the capital stock at date t under costless reversibility. Using this definition along with Eqs. (10) and (12) we have

$$\kappa(t) \equiv \frac{E\{K_t | X_0 = 1\}}{E\{K_t^R | X_0 = 1\}} = C \times H(t) \tag{14a}$$

where

$$C \equiv \left(\frac{c}{c_R}\right)^{-1/\gamma} = \left(1 - \frac{\gamma}{\alpha_N}\right)^{-1/\gamma} < 1 \tag{14b}$$

and

$$H(t) \equiv \frac{E\left\{\max_{0 \leq s \leq t} X_s | X_0 = 1\right\}}{E\{X_t | X_0 = 1\}} \geq 1. \tag{14c}$$

The inequality in Eq. (14c) holds strictly for $t > 0$.

Eqs. (14a)–(14c) illustrate the two opposing effects of irreversibility on the expected capital stock: the user-cost effect measured by C and the hangover effect measured by $H(t)$. The introduction of irreversibility increases the user cost of capital relative to the user cost in the standard case of costless reversibility. This increase in the user cost tends to reduce the optimal capital stock – as

¹³ We examine the ratio of the expected capital stocks rather than, say, the expectation of the ratio of the capital stocks because we are ultimately interested in comparing the expected value of the capital stock under irreversibility and the expected value of the capital stock under reversibility. Examining the ratio of the expected capital stocks is simply an analytically convenient representation of this comparison.

reflected in the value of C less than one. Working in the opposite direction is the hangover effect. Under irreversibility, the capital stock at any date t , K_t , is proportional to $\max_{0 \leq s \leq t} X_s$ whereas under reversibility the capital stock at any date t is proportional to the contemporaneous value of the demand shock X_t . The historical peak of X is at least as large as the contemporaneous value of X , and this consideration, which is reflected in a value of $H(t)$ greater than (or equal to) one, tends to increase the expected value of the optimal capital stock under irreversibility. To see which effect — the user-cost effect C or the hangover effect $H(t)$ — is dominant, we need to compute the product $C \times H(t)$.

First we study the properties of C and $H(t)$ separately.

Proposition 1. $0.367879 = e^{-1} < e^{1/\alpha_n} < C < 1$.

Proof. See Appendix D. \square

Proposition 2. $H(0) = 1$ and $1 < H(\infty) = 1 + \sigma^2/2\mu < 2$

Proof. Inspect Eqs. (11) and (14c) using the facts that $E\{X_t | X_0 = 1\} = \exp(\mu t)$, $\Phi(\infty) = 1$, and $\Phi(-\infty) = 0$, and the assumption that $\mu > (1/2)\sigma^2$ from Eq. (7). \square

The ratio of the expected capital stocks at time 0, $\kappa(0)$, equals $C \times H(0) = C < 1$. Therefore irreversibility reduces the expected value of the initial capital stock because only the user-cost effect is operative for the initial capital stock; the hangover effect is inoperative because the firm has not yet accumulated any capital in the past. However, as time proceeds, the hangover effect becomes operative. Depending on the parameters of the problem, the expected capital stock under irreversibility may eventually exceed the expected capital stock under reversibility, or it may turn out that even in the long run the expected capital stock is lower under irreversibility than under reversibility. We illustrate various possibilities later in Figs. 1–3.

Our finding that irreversibility may either increase or decrease the expected value of the long-run capital stock stands in contrast to a major result of Bentolila and Bertola (1990).¹⁴ To understand the difference in results, we cast Bentolila and Bertola's analysis in our notation. Let η_t be the marginal revenue

¹⁴ Bentolila and Bertola (1990) focus on labor, rather than capital, as the factor of production that is costly to change, and they allow for a productivity shock as well as a demand shock. Our model is a special case of theirs in which we interpret the factor of production as capital, investment is irreversible, and we ignore uncertainty about productivity.

Subsequent literature has referred to Bentolila and Bertola (1990) and earlier work by Bertola as finding that irreversibility increases the expected value of the long-run capital stock. For instance, Hopenhayn and Rogerson (1993, p. 917) interpret Bentolila and Bertola (1990) as finding that 'a dismissal cost actually increases long-run employment' which in the context of capital would mean that irreversibility increases the [expected] long-run capital stock.

product of capital, hy_t^γ , and use the fact that $y_t \equiv X_t/K_t$ to write $K_t = (\eta_t/h)^{-1/\gamma} X_t$. To calculate the average capital stock, specifically the expected value of K_t , one needs to know the joint distribution of η_t and X_t . Bentolila and Bertola use the ergodic distribution for η_t , but cannot use an ergodic joint distribution for η_t and X_t , because X_t is not stationary. Instead, they calculate $\bar{K}(X_t) \equiv \{E(\eta_\infty/h)^{-1/\gamma}\} X_t$, where the notation η_∞ indicates that the expectation is with respect to the ergodic distribution of η . They describe $\bar{K}(X_t)$ as

the best guess for employment [capital] in the firm (minimizes the mean-square error), if the available information includes the current level of demand and productivity and all the technology, demand and dynamic parameters – but nothing is known of the past history of the firm, except that it has been in operation for a sufficiently long time that the ergodic distribution well approximates the actual probability distribution function of the MRPL [marginal revenue product of capital] process. (p. 390)

A few pages later (pp. 392–393) Bentolila and Bertola show that the ‘steady-state mean of labor demand for a given value’ of the stochastic shock increases as firing costs increase. The implication is that $\bar{K}(X_t)$ is higher under irreversibility than under costless reversibility. But $\bar{K}(X_t)$ is *not* the steady-state mean of the capital stock.¹⁵ Moreover, as we demonstrate in Figs. 1–3 in the next section, irreversibility may either increase or decrease the expected capital stock $E_0\{K_t\}$ for large t .¹⁶

¹⁵ Appendix F demonstrates that using $\bar{K}(X_t)$ to calculate the effect of irreversibility on long-run capital accumulation overstates the actual effect of irreversibility on the expected value of the capital stock in the long run. This overstatement arises because $\bar{K}(X_t)$ ignores the covariance between $(\eta_t/h)^{-1/\gamma}$ and X_t .

¹⁶ The effect of irreversibility on the capital–labor ratio, K/L , is unambiguous. The capital–labor ratio equals ℓ^{-1} where $\ell \equiv L/K$. Recalling that $y \equiv X/K$, Footnote 6 implies that $\ell^{-1} = Ay^{-\gamma}$ where $A \equiv w^{\gamma\epsilon}/(\gamma\epsilon - 1)^{\gamma\epsilon}$. Therefore, using the expression for $E\{y^\theta\}$ in Footnote 11 (setting $\theta = -\gamma$ and assuming that $m - 1 - \gamma > 0$) and the fact that $y\bar{v}^{-\gamma} = (c/h)^{-1}$ yields

$$E\{\ell^{-1}\} = AE\{y^{-\gamma}\} = A \frac{m - 1}{m - 1 - \gamma} (c/h)^{-1}.$$

Under perfect reversibility, $y^{-\gamma} = (c_R/h)^{-1}$ always. Using the fact that $c = (1 - \gamma/\alpha_N)c_R$, the ratio of the expected capital–labor ratio under irreversibility to the capital–labor ratio under reversibility is

$$\psi \equiv E\{\ell^{-1}\}/E\{\ell_R^{-1}\} = \frac{m - 1}{m - 1 - \gamma} (1 - \gamma/\alpha_N)^{-1}.$$

A sufficient condition for $\psi > 1$ is $1 - m + \gamma > \alpha_N$. To see that this condition holds, evaluate the quadratic equation $\rho(\eta)$ in Eq. (5b) at $\eta = 1 - m = 1 - 2\mu/\sigma^2$ to obtain $\rho(1 - m) = r > 0$. Since $\rho(\eta) \leq 0$ for $\eta \leq \alpha_N$, we have $1 - m > \alpha_N$. Since $\gamma > 0$, we have $\gamma + 1 - m > \alpha_N$ which implies $\psi > 1$. Therefore the expected capital–labor ratio is greater under irreversibility than under reversibility.

4. The effects of increased uncertainty

In this section we examine the effects on the expected capital stock of an increase in uncertainty. In order to isolate the effect of uncertainty we would like to focus on increases in uncertainty that preserve a measure of central tendency, but we must decide which measure of central tendency, such as the mean or the median, to hold fixed, and we must decide which variable has its central tendency preserved. As we show below, these choices are not just normalizations; they have both qualitative and quantitative importance for the results. In Section 4.1 we examine changes in the distribution of demand shocks that preserve a measure of central tendency of these shocks. Then in Section 4.2 we focus on changes in the distribution of the average profit of capital, which is an endogenous variable.

4.1. The effects of mean- and median-preserving changes in the distribution of demand shocks

We begin by focusing on mean-preserving increases in uncertainty and consider two different candidate variables for mean-preserving spreads. In the current model, the firm faces the downward-sloping demand curve in Eq. (1). It is evident from Eq. (1) that if we interpret demand shocks as changes in the quantity demanded at any given price, then the relevant shock is X_t ; in this case, a mean-preserving increase in uncertainty leaves the expected value of X_t unchanged. Alternatively, the demand curve in Eq. (1) can be rewritten as $P_t = X_t^{1/\varepsilon} Q_t^{-1/\varepsilon}$. This formulation of the demand curve indicates that if we interpret demand shocks as changes in the price associated with any given quantity of output demanded, then the relevant shock is $X_t^{1/\varepsilon}$. In this case, a mean-preserving increase in uncertainty leaves the expected value of $X_t^{1/\varepsilon}$ unchanged.

To accommodate both of these cases in a more general framework, we will examine changes in σ^2 that leave the expected value of X_t^θ unchanged, recognizing that $\theta = 1$ and $\theta = 1/\varepsilon$ represent the two forms of demand shocks discussed above.¹⁷ In order to implement an increase in uncertainty that leaves the expected value of X_t^θ unchanged, write this expected value as $E\{X_t^\theta | X_0\} = X_0 \exp[\theta M(\theta)t]$, where $M(\theta)$ is defined in Eq. (6b). We require changes in μ and σ that leave $M(\theta)$ unchanged so that $E\{X_t^\theta | X_0\}$ is also

¹⁷ In their analyses of investment under uncertainty, Caballero (1991) and Pindyck (1993) adopt the second formulation discussed above, i.e., $\theta = 1/\varepsilon$.

unchanged. Setting $M(\theta)$ equal to a constant, M_0 , in Eq. (6b) and applying the implicit function theorem yields

$$\frac{d\mu}{d\sigma^2} \Big|_{M(\theta)=M_0} = \frac{1}{2}(1 - \theta). \tag{15}$$

When $\theta = 1$, the expression on the right-hand side of Eq. (15) equals zero, which means that an increase in σ holding μ fixed is a mean-preserving spread on X . However, when $\theta = 1/\varepsilon < 1$, the right hand side of Eq. (15) is positive so that an increase in σ must be accompanied by an increase in μ in order to leave the mean of $X^{1/\varepsilon}$ unchanged.

The above approach indicates how much μ must be adjusted in order to implement a mean-preserving spread on X^θ for any value of θ . However, since X evolves according to a geometric Brownian motion, the conditional distribution of X at any future date is not symmetric (in fact, it is log-normal). In view of this asymmetry, it may be of interest to examine a median-preserving, rather than a mean-preserving, spread. For any value of λ , the median of the conditional distribution of future X^λ is preserved by changes in σ^2 that preserve the value of $M(0) \equiv \mu - (1/2)\sigma^2$. Thus, we can also use the apparatus of a $M(\theta)$ -preserving increase in σ^2 to analyze a median-preserving increase in σ^2 . Specifically, for any value of λ , a median-preserving increase in the variance of X^λ is a $M(\theta)$ -preserving increase in σ^2 for $\theta = 0$.

Using this apparatus, we can now consider the effects on capital accumulation of an increase in uncertainty. For any θ , we can calculate the effect of an increase in uncertainty that leaves $M(\theta)$ (and thus the expected value of X_t^θ) unchanged. Different choices of θ correspond to different choices of which central tendency is preserved. In Appendix B we calculate the response of the root α_N to a $M(\theta)$ -preserving increase in σ^2 , and we show that

$$\frac{d\alpha_N}{d\sigma^2} \Big|_{M(\theta)=M_0} > 0 \quad \text{if } \theta \geq 0. \tag{16}$$

Using this result, we derive the effect of σ^2 on C in Proposition 3.

Proposition 3. If $\theta \geq 0$, then

$$\frac{dC}{d\sigma^2} \Big|_{M(\theta)=M_0} = -\frac{1}{2} \frac{(\theta - \alpha_N)C}{(\gamma - \alpha_N)\rho'(\alpha_N)} < 0.$$

Proof. Observe from the definition of C in Eq. (14b) that σ^2 enters the expression for C only through the root α_N . The product of the partial effect of σ^2 on α_N in (B.6) and the partial effect of α_N on C in (E.2) produces the expression above, where $\rho'(\alpha_N) = -\sigma^2\alpha_N - \mu + \frac{1}{2}\sigma^2 > 0$ from (B.1). \square

According to Proposition 3, a $M(\theta)$ -preserving increase in σ^2 reduces C and hence tends to reduce the expected capital stock under irreversibility relative to the expected capital stock under reversibility. This occurs because an increase in σ^2 increases the user cost of capital under irreversibility but has no effect on the user cost with reversibility. Recall, however, that C captures only the user-cost effect. We must also take into account the effect on the hangover factor $H(t)$. For simplicity we focus on $H(\infty)$.

Proposition 4. If $\theta \geq 0$, then

$$\left. \frac{dH(\infty)}{d\sigma^2} \right|_{M(\theta)=M_0} = \frac{1}{2\mu} \left[1 - \frac{\sigma^2}{2\mu} + \theta \frac{\sigma^2}{2\mu} \right] > 0.$$

Proof. Differentiate the expression for $H(\infty)$ in Proposition 2 with respect to σ^2 using Eq. (15). The sign of the resulting derivative is obtained by applying the inequality in Eq. (7). \square

Recall that $\kappa(\infty) = C \times H(\infty)$. We have shown that $C < 1$ under uncertainty and that increases in the instantaneous variance σ^2 decrease C . Working in the opposite direction, however, are the facts that $H(\infty) > 1$ under uncertainty and that $H(\infty)$ is an increasing function of the instantaneous variance σ^2 . Whether $\kappa(\infty)$ is less than or greater than one, and whether $\kappa(\infty)$ is an increasing or decreasing function of σ^2 depends on whether the user-cost effect operating through C or the hangover effect operating through $H(\infty)$ is dominant. Figs. 1–3 illustrate that in some cases the user-cost effect is dominant while in other cases the hangover effect is dominant.

All three figures present results for an example in which the interest rate, r , equals 0.05, the capital share in the Cobb–Douglas production function, β , equals 0.33, and the elasticity of demand, ε , equals 10. Together the values of β and ε imply that $\gamma = 0.251889$. These values are not chosen necessarily for their realism, but to illustrate that for admissible parameter values, a wide variety of results is possible.

In Fig. 1, we examine changes in the instantaneous standard deviation σ that leave the mean of X unchanged (the expected growth rate of demand, μ , is fixed at 0.029). Recall that holding the mean of X fixed corresponds to a mean-preserving spread on the shocks to the quantity of output demanded at any given price. The behavior of $\kappa(\infty)$ is not monotonic in uncertainty; also, depending on the value of σ , $\kappa(\infty)$ may be greater than, less than, or equal to 1.

Fig. 2 presents results for the case in which the mean of $X^{1/\varepsilon}$ is held fixed, which corresponds to a mean-preserving spread on shocks to the price at any given quantity of output demanded. In this case, $\mu = 0.029$ when $\sigma = 0.01$, but as σ increases, μ increases in order to keep $M(\theta)$ constant. As in Fig. 1, $\kappa(\infty)$ is

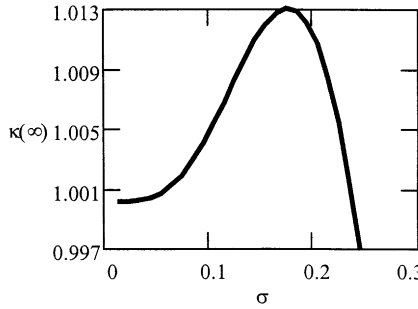


Fig. 1. The long-run effect of uncertainty on $\kappa(\theta = 1)$.

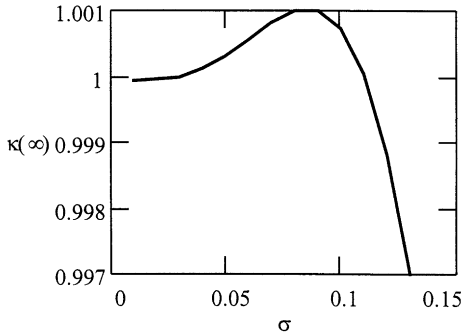


Fig. 2. The long-run effect of uncertainty on $\kappa(\theta = 1/\epsilon)$.

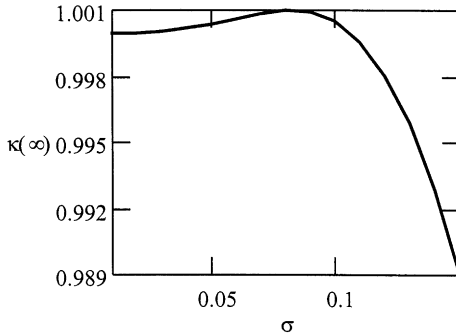


Fig. 3. The long-run effect of uncertainty on $\kappa(\theta = 0)$.

not monotonic in the instantaneous standard deviation σ , and $\kappa(\infty)$ may be greater than, equal to, or less than one.

Fig. 3 presents results for the case in which the median of X^λ is held fixed (for any λ). Again we have that $\mu = 0.029$ when $\sigma = 0.01$, but as σ increases,

μ increases in order to keep $M(0)$ constant. As above, $\kappa(\infty)$ is not monotonic in the instantaneous standard deviation σ , and $\kappa(\infty)$ may be greater than, equal to, or less than one.

Figs. 1–3 suggest that there is considerable ambiguity both in the magnitude and the sign of the long-run effects of irreversibility and uncertainty.¹⁸ As discussed in the previous section, this finding contradicts earlier work suggesting that irreversibility increases capital accumulation (Bertola (1988) and Bentolila and Bertola (1990)) and that uncertainty decreases capital accumulation (Dixit and Pindyck (1994) in reference to the log of the capital stock). Our results, by focusing on the long-run expectation of the level of the capital stock in an explicitly stochastic environment, reveal that depending on the choice of parameter values, either the user-cost effect or the hangover effect may dominate. Thus, relative to the case with reversibility, irreversibility may either increase or decrease the expected long-run capital stock, and increased uncertainty may either increase or decrease the expected long-run capital stock.

Moreover, our results focus attention on the fact that the type of increase in uncertainty has considerable impact on conclusions about the relationship between investment and uncertainty. Table 1 shows the effect on $\kappa(\infty)$ of increasing uncertainty while preserving three different central tendencies: the mean of ‘horizontal demand shocks’ ($\theta = 1$), the mean of ‘vertical demand shocks’ ($\theta = 1/\varepsilon$), and the median of shocks to demand ($\theta = 0$). Clearly, the choices of the type of shock and type of central tendency are not just normalizations; they affect both the sign and the magnitude of the effect of uncertainty on capital accumulation — as is evident by comparing the entries in the last three columns. In addition, the values in any column indicate that the effect of a (marginal) increase in uncertainty depends on the values of μ and σ^2 .

Our analysis focuses on $\kappa(t)$, the *ratio* of the expected capital stock under irreversibility to the expected capital stock under reversibility. In the case of a $M(\theta)$ -preserving spread with $0 \leq \theta < 1$, we can derive unambiguous results about the long-run effect of uncertainty on the *levels* of the expected capital stock under irreversibility and under reversibility. Specifically, when $0 \leq \theta < 1$, a $M(\theta)$ -preserving increase in σ^2 also increases μ , the drift in X . Eventually, the increase in the drift dominates any other effects and increases the expected capital stock. This result is based on the following lemma.

Lemma 1. Let $\chi_j(t) \equiv a_j(t)e^{\mu_j t} + b_j(t)$ where $a_j(t) > 0$ and $b_j(t)$ are finite for all t , and $j = 1, 2$. If $\mu_1 > \mu_2 \geq 0$, then for sufficiently large t , $\chi_1(t) > \chi_2(t)$.

¹⁸ In all three of these figures, $\kappa(\infty)$ does not differ substantially from one, though it can be greater or less than one depending on the value of the standard deviation, σ . For other parameter values, however, $\kappa(\infty)$ can be much greater or much less than one. For example, suppose that $r = 0.1$, $\varepsilon = 10$, and $\beta = 0.33$. If $\mu = 0.025$ and $\sigma = 0.2$, then $\kappa(\infty) = 1.204$, and if $\mu = 0.09$ and $\sigma = 0.42$, then $\kappa(\infty) = 0.858$.

Table 1
The effect of uncertainty on capital accumulation

Uncertainty σ	Growth rate μ	$d\kappa(\infty)/d\sigma^2$ $\theta = 1$	$\theta = 1/\varepsilon$	$\theta = 0$
0.05	0.02	2.28	2.00	1.97
0.10	0.02	3.78	1.67	1.44
0.15	0.02	3.06	-2.04	-2.60
0.10	0.03	0.52	0.02	-0.04
0.15	0.03	0.13	-1.34	-1.50
0.20	0.03	-0.59	-3.26	-3.55
0.10	0.035	-0.10	-0.35	-0.38
0.15	0.035	-0.54	-1.35	-1.44
0.20	0.035	-1.14	-2.71	-2.88

The derivative is calculated using Propositions 3 and 4 to obtain

$$\frac{d\kappa(\infty)}{d\sigma^2} = \frac{dC}{d\sigma^2}H + \frac{dH}{d\sigma^2}C = \frac{1}{2} \frac{\kappa(\alpha_N - \theta)}{\rho'(\alpha_N)(\gamma - \alpha_N)} + \frac{C}{2\mu} \left[1 - \frac{\sigma^2}{2\mu} + \theta \frac{\sigma^2}{2\mu} \right].$$

The other parameter values used in the calculations are $r = 0.05$, $\varepsilon = 10$, and $\beta = 0.33$.

Proof. $\chi_1(t) - \chi_2(t) = a_1(t)e^{\mu_1 t} - a_2(t)e^{\mu_2 t} + b_1(t) - b_2(t) = e^{\mu_1 t} [a_1(t) - a_2(t)e^{(\mu_2 - \mu_1)t} + (b_1(t) - b_2(t))e^{-\mu_1 t}]$. For sufficiently large t , the term in square brackets is positive. \square

Proposition 5. If $0 \leq \theta < 1$, then

$$\left. \frac{dE_0 \{K_t^R\}}{d\sigma^2} \right|_{M(\theta) = M_0} > 0$$

and, for sufficiently large t ,

$$\left. \frac{dE_0 \{K_t\}}{d\sigma^2} \right|_{M(\theta) = M_0} > 0.$$

Proof. Observe from Eq. (15) that a $M(\theta)$ -preserving spread increases μ . The effect on the expected capital stock in the reversible case then follows directly from Eq. (13), and the effect on the expected capital stock in the irreversible case follows applying Lemma 1 to Eq. (11). \square

Proposition 5 applies to a mean-preserving spread in the distribution of shocks to the price associated with any given quantity of output demanded ($\theta = 1/\varepsilon$) and to a median-preserving increase in σ^2 ($\theta = 0$), but it does not apply to a mean-preserving spread in the distribution of shocks to the quantity of output demanded at any given price ($\theta = 1$). For the cases covered by Proposition 5, an increase in uncertainty unambiguously increases the expected long-run capital stock under irreversibility. This result is not due to irreversibility alone, however, since the expected capital stock increases under reversibility as well. Because an increase in uncertainty can either increase or decrease $\kappa(\infty)$, we cannot determine in general whether the expected long-run capital stock increases by more under irreversibility or under reversibility.

4.2. The effects of mean-preserving changes in the distribution of the average profit of capital¹⁹

The analysis in Section 4.1 focused on the distribution of demand shocks. Though much of the literature analyzes the effects of uncertainty by focusing on shocks to the demand curve (or to the production function), these shocks are not directly observable.²⁰ In this section we focus attention on the distribution of shocks to the average profit of capital, π/K , which is potentially observable. It follows directly from the operating profit function in Eq. (3) and the definition of $y \equiv X/K$ that the average product of capital is $hy^\gamma/(1-\gamma)$ where $0 < 1/\varepsilon < \gamma < 1$. We will examine the effect of increases in uncertainty that leave the unconditional expected value of the average profit of capital unchanged.

The concept of $M(\theta)$ -preserving changes in the distribution of X was developed to examine cases in which the expected value of X^θ remains unchanged when the variance of the distribution of X , which is a geometric Brownian motion, is unchanged. However, this concept is not helpful in studying changes in uncertainty that leave the unconditional mean of y^θ unchanged because y is not a geometric Brownian motion. With irreversibility, y is a *regulated* geometric Brownian motion.²¹ Unlike X , which is exogenous to the firm, the variable y is an endogenous variable that falls whenever the firm purchases capital.

¹⁹ We thank an anonymous referee for suggesting that we consider shocks to a potentially observable measure of the profitability of capital.

²⁰ Abel (1983) and Hartman (1972) focused on demand shocks facing perfectly competitive firms. Under perfect competition, firms are price-takers and demand shocks take the form of shocks to the observable price of output.

²¹ With costlessly reversible investment, y is a constant. We consider the implications of this distinction below.

Indeed, optimality requires the firm facing irreversible investment to purchase capital to prevent y from exceeding $y_U \equiv (c/h)^{1/\gamma}$. Thus, the expected value of y cannot exceed y_U , whereas the expected value of X drifts upward without bound.

Any change in the distribution of X will directly change the distribution of the numerator of $y \equiv X/K$ and will also change the distribution of the denominator of y as the optimal value of y_U responds. It might seem that the additional complications introduced by the endogenous adjustment of y_U , and hence the endogenous adjustment of the path of K , would further muddy the already ambiguous result of the effect of uncertainty on capital accumulation. However, it turns out that we can derive unambiguous effects on $\kappa(\infty)$ and the expected value of the capital stock of changes in the distribution of X that leave the unconditional expected value of the average profit of capital unchanged.

Recall from Footnote 11 the definition $m \equiv 2\mu/\sigma^2$ and continue to assume that $m > 1$ so that there is a well-defined ergodic distribution for y . Consider the behavior of y^θ , which is proportional to the average profit of capital if $\theta = \gamma$.

Proposition 6. If $\theta > 0$, then

$$\kappa(\infty) = \left(\frac{m+1}{m}\right) \left(\frac{m-1}{m-1+\theta}\right)^{1/\theta} \left(\frac{rb}{h}\right)^{1/\gamma} (\mathbb{E}\{y^\theta\})^{-1/\theta}.$$

Proof. See Appendix C. \square

Proposition 6 shows that for a given expectation of the average profit of capital, the effect of the parameters μ and σ^2 on $\kappa(\infty)$ is completely summarized by the single parameter $m \equiv 2\mu/\sigma^2$.

Lemma 2. If $0 < \theta < 2$, then

$$\left. \frac{dm}{d\sigma^2} \right|_{d\mathbb{E}\{y^\theta\}=0} < 0.$$

Proof. See Appendix C. \square

Lemma 2 states that, in cases of interest, an increase in the variance σ^2 that leaves the expected value of y^θ unchanged will lead to a decrease in m . This finding allows us to derive unambiguous results about the effect on $\kappa(\infty)$ of increases in uncertainty that leave the expected value of the average profit of capital unchanged.

Proposition 7. If $0 < \theta < 2$ then

$$\left. \frac{d\kappa(\infty)}{dm} \right|_{dE\{y^\theta\}=0} > 0 \quad \text{and} \quad \left. \frac{d\kappa(\infty)}{d\sigma^2} \right|_{dE\{y^\theta\}=0} < 0.$$

Proof. See Appendix C. \square

According to Proposition 7, an increase in uncertainty that does not change the unconditional expected value of average profit of capital leads to a decrease in $\kappa(\infty)$. This increase in uncertainty reduces the parameter m and the reduction in m reduces $\kappa(\infty)$.

Notice that this finding requires an equivalent increase in μ in the calculation of both $E_0\{K_t^I\}$ (the numerator of κ) and $E_0\{K_t^R\}$ (the denominator of κ). However, no increase in drift is required in order to hold constant the unconditional expectation of the average profit of capital with costless reversibility (as it is with irreversibility), yet in the above calculation, we increased μ in both the numerator and denominator of $\kappa(\infty)$ enough to hold constant $E\{y^\theta\}$ under irreversibility. Suppose that instead of calculating

$$\left. \frac{dE_0\{K_t^R\}}{d\sigma^2} \right|_{dE\{y^\theta\}=0}$$

using the increase in μ necessary to maintain the expected value of the average profit of capital with irreversibility, we instead consider the effect of the increase in μ necessary to hold constant the unconditional mean of y^θ with costless reversibility. Recall that under costless reversibility, $y \equiv X/K^R = (c_R/h)^{1/\gamma}$ always, where c_R is the Jorgensonian user cost rb . The ratio y , and hence y^θ , is therefore a constant and is independent of the distributional parameters of the process for X . Since (trivially) the unconditional expectation of the average profit of capital does not depend on σ^2 or μ , any change (or none) in these parameters is consistent with preserving the mean of y^θ under reversibility. Hence, the value of the change in μ necessary to hold constant $E\{y^\theta\}$ under reversibility is not well defined. This indeterminacy strains the interpretation of $\kappa(\infty)$ in Proposition 7. However, we can obtain a clear result by focusing directly on the expected value of the capital stock under irreversibility, without taking its ratio to the expected value under reversibility.

Lemma 3. If $0 < \theta < 2$, then

$$\left. \frac{d\mu}{d\sigma^2} \right|_{dE\{y^\theta\}=0} > 0.$$

Proof. See Appendix C. \square

Lemma 3 states that in order to hold constant the expected average profit of capital under irreversibility, an increase in σ^2 implies an increase in μ .

Proposition 8. If $0 < \theta < 2$, then for sufficiently large t ,

$$\left. \frac{dE_0\{K_t\}}{d\sigma^2} \right|_{dE\{y^0\}=0} > 0.$$

Proof. See Appendix C. \square

Since an increase in σ^2 implies an increase in drift, the expected value of the capital stock under irreversibility is increasing in uncertainty. Therefore, an increase in uncertainty, holding constant the unconditional expectation of the average profit of capital, unambiguously increases the capital stock under irreversibility. However, whether this increase is larger or smaller than would be obtained under costless reversibility depends crucially on how drift is treated in the reversible case. Unfortunately, the choice of drift is indeterminate, since the expectation of the average profit of capital with costless reversibility depends on neither μ nor σ^2 .

5. The effects of demand growth, the interest rate, the capital share, and the price elasticity of demand

While the literature has focused on the effects of irreversibility and uncertainty on capital accumulation – which have ambiguous effects – changing other aspects of the firm's environment leads to clear changes in capital accumulation. Specifically, we show that a higher expected growth rate of demand, a higher capital share, and a higher price elasticity of demand all tend to reduce the expected long-run capital stock under irreversibility relative to that under reversibility, while a higher interest rate has the opposite effect.

The expected growth rate of the demand shock, X_t , is given by the drift parameter μ in Eq. (4). We first show that an increase in the expected growth rate of demand increases the expected capital stock under irreversibility and under reversibility. However, this effect is weaker under irreversibility, so that an increase in the expected growth rate of demand decreases the ratio of the long-run expected capital stock with irreversible investment to that under reversible investment.

Proposition 9.

$$\frac{dE_0\{K_t^R\}}{d\mu} > 0$$

and, for sufficiently large t ,

$$\frac{dE_0\{K_t\}}{d\mu} > 0.$$

Proof. Inspect Eq. (13) and apply Lemma 1 to Eq. (11). \square

According to Proposition 9, an increase in the expected growth rate of demand increases the expected long-run capital stock whether investment is reversible or irreversible. To see whether the effect is larger under reversibility or irreversibility, we must examine the effect of expected growth on the user-cost factor C and on the hangover factor $H(t)$.

Proposition 10.

$$\frac{dC}{d\mu} = \frac{C}{(\gamma - \alpha_N)\rho'(\alpha_N)} > 0$$

and

$$\frac{dH(\infty)}{d\mu} = -\frac{\sigma^2}{2\mu^2} < 0.$$

Proof. See Appendix E. \square

Notice that an increase in the growth rate of demand tends to reduce the strength of both the user-cost effect and the hangover effect; that is, both C and $H(\infty)$ become closer to unity as μ rises. In this case, however, the sign of the net effect can be determined, as shown in the following corollary.

Corollary.

$$\frac{d\kappa(\infty)}{d\mu} < 0.$$

Proof. See Appendix E. \square

A higher expected growth rate of demand therefore increases the expected long-run capital stock by more in the case of reversibility than in the case of irreversibility. This means that in a firm with high expected demand growth, we are more likely to find that irreversibility reduces its capital stock in the long run than in a firm with low expected demand growth. This result may seem surprising since the higher is demand growth, the more similar are the user costs calculated in the Jorgensonian and irreversible investment cases.²² That is, the decision rules governing investment become more alike as μ rises, so it may seem that the irreversibility constraint should become less important in a high growth environment. This demonstrates the importance of the hangover effect, though, since this effect is weakened *even more* than the user-cost effect as μ rises. Thus, even while the irreversible investment rule becomes more like the reversible rule when expected demand growth is high, the expected irreversible capital stock is nonetheless driven down compared to the expected reversible capital stock.

The effect of the interest rate on the long-run capital stock is determined solely by its effect on the user-cost effect, since the hangover effect (at any horizon) is independent of the interest rate. Thus, we obtain the effect of the interest rate on the relative expected capital stocks directly from the following proposition.

Proposition 11.

$$\frac{d\kappa(t)}{dr} = \frac{dC}{dr} H(t) = \frac{dC}{d\alpha_N} \frac{d\alpha_N}{dr} H(t) = \frac{-\kappa(t)}{\alpha_N(\gamma - \alpha_N)\rho'(\alpha_N)} > 0.$$

Proof. See Appendix E. \square

This proposition means that a higher interest rate tends to increase the long-run expected capital stock under irreversibility relative to that under reversibility by weakening the user-cost effect. Note that while the user cost under both irreversibility and reversibility rises with the interest rate – which tends to reduce the long-run capital stock in both cases – this effect is weaker

²² From Appendix B, Eq. (B.3), an increase in μ reduces the value of the root, $\alpha_N < 0$, which from the definition of the user cost in Eq. (5a), reduces the user cost in the irreversible case, driving it closer to rb , the Jorgensonian user cost.

under irreversibility than under reversibility.²³ Since firms discount the costs associated with the inability to sell capital in the future, greater discounting tends to reduce the strength of the user-cost effect. This finding is consistent with the ‘discounting effect’ emphasized by Bertola (1992) in a model of employment under certainty.

Together, the capital share, β , and the price elasticity of demand, ε , determine the concavity of the profit function, measured by the coefficient γ . Here we examine the effect of changing γ on the expected long-run capital stock under irreversibility compared to reversibility. Notice that the hangover effect, $H(t)$, in Eq. (14c) is independent of γ , so we need only examine the user-cost effect, C , to determine the effect of γ on the expected capital stock at any horizon.

Proposition 12.

$$\frac{d\kappa(t)}{d\gamma} = \frac{dC}{d\gamma} H(t) = \frac{\kappa(t)}{\gamma^2} \left[\ln \left(1 - \frac{\gamma}{\alpha_N} \right) - \frac{\gamma}{\gamma - \alpha_N} \right] > 0.$$

Proof. See Appendix E. \square

Thus, an increase in γ tends to increase the capital stock under irreversibility relative to that under reversibility. This occurs because as γ rises, the profit function becomes more concave (observe that with $\gamma = 0$ the profit function is linear in K) and thus deviations from the optimal reversible capital stock are

²³ From the definition of the Jorgensonian user cost in Section 2, the elasticity of c_R with respect to r is one. Differentiating Eq. (5a) and using Eq. (B.4), the corresponding elasticity in the irreversible case is given by

$$\frac{dc}{dr} \frac{r}{c} = 1 - \frac{r\gamma}{\left(1 - \frac{\gamma}{\alpha_N}\right) \alpha_N^2 \rho'(\alpha_N)} = 1 - \frac{r\gamma}{(\alpha_N - \gamma) \alpha_N \rho'(\alpha_N)}.$$

Using Eq. (B.1) and the fact that $\alpha_N \rho'(\alpha_N) = \rho(\alpha_N) - (r + \frac{1}{2}\sigma^2 \alpha_N^2) = -(r + \frac{1}{2}\sigma^2 \alpha_N^2)$, this can be simplified to

$$\frac{dc}{dr} \frac{r}{c} = 1 - \frac{r}{r + \frac{1}{2}\sigma^2 \alpha_N^2} \frac{\gamma}{\gamma - \alpha_N},$$

which implies

$$0 < \frac{dc}{dr} \frac{r}{c} < 1.$$

Thus, while in both cases the user cost increases with the interest rate, the elasticity in the irreversible case is less than that in the reversible case.

more costly to the firm.²⁴ This effect dominates the increase in the user cost, c , relative to the Jorgensonian user cost, that occurs with an increase in γ . Thus the value of C increases toward 1 and the expected capital stock under irreversibility increases relative to that under reversibility. Notice from the definition of γ in Eq. (3) that γ is negatively related to the capital share β and the price elasticity of demand ε . Thus, a reduction in either the elasticity of demand or the capital share will be associated with an increase in the expected capital stock under irreversibility relative to the expected capital stock under reversibility. This result occurs because with relatively inelastic demand or a low capital share, it is more costly for the firm to deviate from the optimal reversible capital stock, so the user-cost factor C increases toward one.

6. Concluding remarks

When investment is irreversible, firms cannot disinvest even when the marginal profitability of capital is low. Anticipating that this constraint may bind in the future, firms apply a higher user cost of capital to current investment decisions. Our analysis of the expected long-run capital stock demonstrates that both the ‘feared event’ (inability to disinvest, summarized by the hangover effect) and the firm’s reaction to it (a high user cost, summarized by the user-cost effect) are important features of capital accumulation. In the long run, either effect can dominate, so that the expected capital stock may be higher or lower under irreversibility than under reversibility.

Uncertainty does not ease the ambiguity regarding the long-run effect of irreversibility, but rather deepens it. Higher uncertainty strengthens both the hangover and user-cost effects, but which effect becomes *relatively* stronger depends on characteristics of the firm and its environment. Higher uncertainty increases the level of the expected long-run capital stock under irreversibility (for $0 \leq \theta < 1$), but can increase it even more under reversibility. Thus, in which case the expected capital stock increases more is ambiguous.

While in general it is not possible to determine whether irreversibility increases or decreases the expected long-run capital stock of a firm, other characteristics of the firm provide some clues. We find that a high growth rate of demand decreases the expected long-run capital stock under irreversibility relative to that under reversibility. On the other hand, a high interest rate, a low capital share in production, and a low elasticity of demand have the opposite effect — tending to increase the expected long-run capital stock with

²⁴ This does not imply that the firm does not choose its capital stock optimally, but that it chooses its optimal capital stock conditional on the irreversibility constraint.

irreversibility relative to that under reversibility. Together, these results suggest that a firm’s long-run expected capital stock is most likely to be decreased by the presence of irreversibility in an environment with high demand growth, low interest rates, a high capital share, and highly elastic demand.

These results are obtained in a specific framework, using particular functional forms and assumptions. They may not be general, in that the introduction of equilibrium considerations, different specifications for uncertainty, operating profits, or investment costs, or other generalizations may affect our findings. Simple models such as the one we use, however, have been employed to illustrate the importance of irreversibility and uncertainty for investment and hiring dynamics and to draw conclusions about the long-run effects of irreversibility and uncertainty. Our results demonstrate that even in such models, where changes in the user cost due to irreversibility and uncertainty change investment and employment flows in sometimes dramatic fashion, these effects may be dampened or even reversed when translated into the expected long-run stock. Thus, the effect on the expected long-run stock of capital or labor may be smaller or even of the opposite sign than the effect of irreversibility and uncertainty on the user cost would indicate.

Appendix A. Optimal investment of the firm

Assume that the firm is risk-neutral and discounts future cash flows at the constant rate $r > 0$, where $r > \mu$.²⁵ The fundamental value of the firm is

$$V(K_t, X_t) \equiv \max_{\{I_{t+s}\}} E_t \left\{ \int_0^\infty e^{-rs} [\pi(K_{t+s}, X_{t+s}) ds - b dK_{t+s}] \right\}. \tag{A.1}$$

Since K_t is not differentiable, the last term in Eq. (A.1) is to be interpreted as a Stieltjes integral.

The value function in Eq. (A.1) is homogeneous of degree one in K_t and X_t . Therefore, the marginal valuation of capital, $V_K(K_t, X_t)$, is homogeneous of degree 0 in K_t and X_t and can be written as a function of $y_t \equiv X_t/K_t$. Define

$$q(y_t) \equiv V_K(K_t, X_t) \tag{A.2}$$

²⁵ Under optimal policy the marginal revenue product of capital, $hX^\gamma K^{-\gamma}$, satisfies $hX^\gamma K^{-\gamma} \leq c$ where c is the user cost of capital. Recalling that $0 < \gamma < 1$, so that $(\gamma - 1)/\gamma < 0$, this inequality is equivalent to $h^{(\gamma-1)/\gamma} X^{\gamma-1} K^{1-\gamma} \geq c^{(\gamma-1)/\gamma}$. Multiplying this inequality by $(h^{1/\gamma}/(1-\gamma))X$ and recalling that $\pi(K, X) = (h/(1-\gamma))X^\gamma K^{1-\gamma}$, we obtain $\pi(X, K) \geq (h^{1/\gamma}/(1-\gamma))c^{(\gamma-1)/\gamma}X$. Since the expected growth rate of X is μ , the operating profit of the firm is bounded below by a process with expected growth rate equal to μ . Therefore, in order for the expected present value of operating profits, $E_t \{ \int_0^\infty e^{-rs} \pi(K_{t+s}, X_{t+s}) ds \}$, to be finite we must assume that $\mu < r$.

as the marginal valuation of capital. Optimality requires

$$rq(y) = hy^\gamma + \frac{1}{dt}E\{dq\} \quad (\text{A.3})$$

where the left-hand side is the required return on the marginal unit of capital, and the right-hand side is the expected return on the marginal unit of capital, which consists of the marginal revenue product, hy^γ , and the expected capital gain $(1/dt)E\{dq\}$.

When the marginal valuation of capital, $q(y)$, is less than the purchase price of capital b , it is optimal not to purchase capital. It is optimal to purchase capital only if $q(y)$ reaches the boundary b . The value of y at this boundary, y_U , is given by the smooth-pasting condition²⁶

$$q(y_U) = b. \quad (\text{A.4})$$

In addition, $q(y)$ and y_U must satisfy the high-contact condition

$$q'(y_U) = 0. \quad (\text{A.5})$$

To calculate the expected change in q apply Ito's Lemma to $q(y)$ and use the high-contact condition in Eq. (A.5) to obtain²⁷

$$\frac{1}{dt}E\{dq\} = \mu yq'(y) + \frac{1}{2} \sigma^2 y^2 q''(y). \quad (\text{A.6})$$

Substituting Eq. (A.6) into the optimality condition in Eq. (A.3) yields a second-order differential equation that $q(y)$ must satisfy

$$rq(y) = hy^\gamma + \mu yq'(y) + \frac{1}{2} \sigma^2 y^2 q''(y). \quad (\text{A.7})$$

²⁶ See Dumas (1991) for a clear presentation of the smooth-pasting and high-contact conditions used below.

²⁷ Note that

$$\frac{1}{dt}E\{dq\} = \mu yq'(y) - yq'(y) \frac{dK}{K} + \frac{1}{2} \sigma^2 y^2 q''(y).$$

When $y \neq y_U$, $dK = 0$; when $y = y_U$, $q'(y) = 0$ by the high-contact condition. Therefore, $q'(y)dK = 0$ always, and we obtain Eq. (A.6).

The solution to the differential Eq. (A.7) is²⁸

$$q(y) = Ay^\gamma + Jy^{\alpha_p} \quad \text{where} \quad A \equiv h/\rho(\gamma) > 0 \tag{A.8}$$

and α_p is the positive root of the quadratic equation in Eq. (5b), and J is a constant that will be determined by the boundary conditions. Using Eq. (A.8) in the boundary conditions, Eqs. (A.4) and (A.5), we obtain

$$hy_U^\gamma = c \equiv \left(1 - \frac{\gamma}{\alpha_N}\right)rb \tag{A.9}$$

and

$$q(y) = Ay^\gamma - \frac{\gamma}{\alpha_p - \gamma} b \left(\frac{y}{y_U}\right)^{\alpha_p}. \tag{A.10}$$

Appendix B. Properties of the negative root of $\rho(\eta) = 0$ in Eq. (5b)

Recall that $\rho(\eta) \equiv -\frac{1}{2}\sigma^2\eta^2 - (\mu - \frac{1}{2}\sigma^2)\eta + r = 0$. Note that $\rho(\eta)$ is strictly concave, $\rho(0) = r > 0$, and $\rho(1) = r - \mu > 0$ so that $\rho(\gamma) > 0$. Also, note that $\rho(\eta) = 0$ has two distinct roots, $\alpha_p > 0$ and $\alpha_N < 0$, which satisfy $\alpha_N < 0 < \gamma < 1 < \alpha_p$. The concavity of $\rho(\eta)$ implies that

$$\rho'(\alpha_N) = -\sigma^2\alpha_N - \mu + \frac{1}{2}\sigma^2 > 0. \tag{B.1}$$

Totally differentiating $\rho(\eta)$ with respect to η, μ, σ^2 , and r , and evaluating the derivatives at $\eta = \alpha_N$ yields

$$\rho'(\alpha_N)d\alpha_N - \alpha_N d\mu + \frac{1}{2}\alpha_N(1 - \alpha_N)d\sigma^2 + dr = 0. \tag{B.2}$$

Eq. (B.2) implies

$$d\alpha_N/d\mu = \alpha_N/\rho'(\alpha_N) < 0 \tag{B.3}$$

and

$$d\alpha_N/dr = -1/\rho'(\alpha_N) < 0. \tag{B.4}$$

Now we calculate the effect on the root α_N of a $M(\theta)$ -preserving increase in σ^2 . Observe from Eq. (15) that for a $M(\theta)$ -preserving change

$$d\mu = \frac{1}{2}(1 - \theta)d\sigma^2. \tag{B.5}$$

²⁸ The general solution to the differential equation (A.7) contains a third term Ly^{α_N} , where L is a constant. Note that as y approaches zero, this term approaches infinity because $\alpha_N < 0$. However, the marginal valuation of capital remains finite as $y \equiv X/K$ approaches zero. Therefore the constant L must equal zero.

Substituting Eq. (B.5) into Eq. (B.2) yields

$$\frac{d\alpha_N}{d\sigma^2} \Big|_{M(\theta)=M_0} = \frac{1}{2} \frac{\alpha_N(\alpha_N - \theta)}{\rho'(\alpha_N)}. \tag{B.6}$$

Note from Eq. (B.6) that if $\theta \geq 0$, then

$$\frac{d\alpha_N}{d\sigma^2} \Big|_{M(\theta)=M_0} > 0.$$

Recall from Footnote 11 that $m \equiv 2\mu/\sigma^2$ so that $\mu = m\sigma^2/2$. Totally differentiating this expression with respect to m, σ^2 , and μ yields

$$d\mu = \frac{1}{2}\sigma^2 dm + \frac{1}{2}m d\sigma^2. \tag{B.7}$$

Substitute (B.7) into (B.2) to obtain

$$\rho'(\alpha_N)d\alpha_N - \frac{1}{2}\alpha_N\sigma^2 dm - \frac{1}{2}\alpha_N m d\sigma^2 + \frac{1}{2}\alpha_N(1 - \alpha_N) d\sigma^2 + dr = 0. \tag{B.8}$$

Holding r constant and simplifying Eq. (B.8) yields the following equation which will be useful in proving Lemma 2.

$$d\alpha_N = \frac{1}{2}\alpha_N \frac{\sigma^2}{\rho'(\alpha_N)} dm + \frac{1}{2}\alpha_N \frac{(m + \alpha_N - 1)}{\rho'(\alpha_N)} d\sigma^2. \tag{B.9}$$

Lemma B.1. If $r > \mu > (1/2)\sigma^2$, then $\alpha_N < -m < -1$.

Proof. $\rho(-m) = -\frac{1}{2}\sigma^2 m^2 + \left(\mu - \frac{1}{2}\sigma^2\right)m + r = -\frac{1}{2}\sigma^2(m^2 + (-m + 1)m) + r = r - \mu > 0$. Therefore, $\alpha_N < -m < -1$. \square

Corollary 1. If $r > \mu > (1/2)\sigma^2$ and $\gamma > -1$, then

$$\frac{\gamma - \alpha_N}{m - 1} > 1.$$

Corollary 2. If $r > \mu > (1/2)\sigma^2$ and $\gamma > -1$, then

$$\frac{m}{1 - \alpha_N} \frac{\gamma - \alpha_N}{m - 1} > 1$$

Lemma B.2. If $r > \mu > (1/2)\sigma^2$ and $0 < \theta < 2$, then

$$\frac{2\rho'(\alpha_N)}{\sigma^2} \frac{1}{m - 1 + \theta} > 1.$$

Proof. Eq. (B.1) implies that $2\rho'(\alpha_N)/\sigma^2$ is positive, and $\mu > (1/2)\sigma^2$ and $\theta > 0$ imply that $1/(m - 1 + \theta)$ is positive. Eq. (B.1) implies

$$\begin{aligned} \frac{2\rho'(\alpha_N)}{\sigma^2} - (m - 1 + \theta) &= -2\alpha_N - m + 1 - (m - 1 + \theta) \\ &= -2(\alpha_N + m) + 2 - \theta > 0. \quad \square \end{aligned}$$

Appendix C. The effects of mean-preserving changes in the distribution of the average profit of capital

Proof of Proposition 6. Recall from Footnote 11 that

$$E\{y^\theta\} = \frac{m - 1}{m - 1 + \theta} y_U^\theta$$

and recall that $y_U = (c/h)^{1/\gamma}$ where c is defined in Eq. (5a) as $c \equiv (1 - \gamma/\alpha_N)rb$. Therefore,

$$E\{y^\theta\} = \frac{m - 1}{m - 1 + \theta} \left(1 - \frac{\gamma}{\alpha_N}\right)^{\theta/\gamma} \left(\frac{rb}{h}\right)^{\theta/\gamma}. \tag{C.1}$$

Use the definition of C in Eq. (14b) to rewrite Eq. (C.1) as

$$C = \left(\frac{m - 1}{m - 1 + \theta}\right)^{1/\theta} \left(\frac{rb}{h}\right)^{1/\gamma} (E\{y^\theta\})^{-1/\theta}. \tag{C.2}$$

Using $m \equiv 2\mu/\sigma^2$, rewrite $H(\infty)$ in Proposition 2 as

$$H(\infty) = \frac{m + 1}{m}. \tag{C.3}$$

Since $\kappa(\infty) = C \times H(\infty)$, Eqs. (C.2) and (C.3) imply

$$\kappa(\infty) = \left(\frac{m + 1}{m}\right) \left(\frac{m - 1}{m - 1 + \theta}\right)^{1/\theta} \left(\frac{rb}{h}\right)^{1/\gamma} (E\{y^\theta\})^{-1/\theta}. \quad \square \tag{C.4}$$

Proof of Lemma 2. Totally logarithmically differentiate Eq. (C.1) with respect to m and α_N , holding $E\{y^\theta\}$ constant to obtain

$$\left(\frac{1}{m - 1}\right) \left(\frac{1}{m - 1 + \theta}\right) dm + \left(\frac{1}{\alpha_N - \gamma}\right) \left(\frac{1}{\alpha_N}\right) d\alpha_N = 0. \tag{C.5}$$

Now substitute Eq. (B.9) into Eq. (C.5) to obtain

$$\left(\frac{1}{m-1}\right)\left(\frac{1}{m-1+\theta}\right)dm + \frac{1}{2}\left(\frac{1}{\alpha_N-\gamma}\right)\frac{\sigma^2}{\rho'(\alpha_N)}dm + \frac{1}{2}\left(\frac{1}{\alpha_N-\gamma}\right)\frac{(m+\alpha_N-1)}{\rho'(\alpha_N)}d\sigma^2 = 0. \tag{C.6}$$

Rewrite Eq. (C.6) to obtain

$$\left[\left(\frac{\gamma-\alpha_N}{m-1}\right)\left(\frac{2\rho'(\alpha_N)}{\sigma^2}\frac{1}{m-1+\theta}\right)-1\right]dm = -\frac{1}{\sigma^2}(1-m-\alpha_N)d\sigma^2. \tag{C.7}$$

Corollary 1 and Lemma B.2 together imply that the term in brackets multiplying dm is positive. Lemma B.1 implies that the coefficient of $d\sigma^2$ is negative. \square

Proof of Proposition 7. Take logarithms of both sides of Eq. (C.4) and then differentiate with respect to m , holding $E\{y^\theta\}$ constant, to obtain

$$\frac{1}{\kappa(\infty)}\frac{d\kappa(\infty)}{dm}\Big|_{dE\{y^\theta\}=0} = -\frac{1}{m(m+1)} + \frac{1}{(m-1+\theta)(m-1)} > 0 \tag{C.8}$$

where the inequality follows from the fact that $m+1 > m-1+\theta$. Eq. (C.8) and Lemma 2 imply that

$$\frac{d\kappa(\infty)}{d\sigma^2}\Big|_{dE\{y^\theta\}=0} < 0. \quad \square$$

Proof of Lemma 3. The definition $m \equiv 2\mu/\sigma^2$ implies $dm/m = d\mu/\mu - d\sigma^2/\sigma^2$ which implies

$$dm = \frac{m}{\mu}d\mu - \frac{m}{\sigma^2}d\sigma^2. \tag{C.9}$$

Eq. (B.2) implies

$$d\alpha_N = \frac{\alpha_N}{\rho'(\alpha_N)}d\mu - \frac{1}{2}\frac{\alpha_N(1-\alpha_N)}{\rho'(\alpha_N)}d\sigma^2. \tag{C.10}$$

Substitute (C.9) and (C.10) into Eq. (C.5) and recognize that $m/\mu = 2/\sigma^2$ to obtain

$$\left(\frac{1}{m-1}\right)\left(\frac{1}{m-1+\theta}\right)\left[\frac{2}{\sigma^2}d\mu - \frac{m}{\sigma^2}d\sigma^2\right] + \left(\frac{1}{\alpha_N-\gamma}\right)\frac{1}{\rho'(\alpha_N)}\left[d\mu - \frac{1}{2}(1-\alpha_N)d\sigma^2\right] = 0. \tag{C.11}$$

Rearrange Eq. (C.11) to obtain

$$\begin{aligned} & \left(\frac{1}{\gamma - \alpha_N}\right) \left(\frac{1}{\rho'(\alpha_N)}\right) \left[\left(\frac{\gamma - \alpha_N}{m - 1}\right) \left(\frac{2\rho'(\alpha_N)/\sigma^2}{m - 1 + \theta}\right) - 1 \right] d\mu \\ & = \left(\frac{1}{\gamma - \alpha_N}\right) \left(\frac{1}{2\rho'(\alpha_N)}\right) (1 - \alpha_N) \left[\frac{m}{1 - \alpha_N} \frac{\gamma - \alpha_N}{m - 1} \frac{2\rho'(\alpha_N)/\sigma^2}{m - 1 + \theta} - 1 \right] d\sigma^2. \end{aligned} \tag{C.12}$$

Corollary 1 and Lemma B.2 imply that the coefficient of $d\mu$ is positive. Corollary 2 and Lemma B.2 imply that the coefficient of $d\sigma^2$ is positive.

Proof of Proposition 8. This is direct implication of Lemmas 1 and 3. \square

Appendix D. Proof of Proposition 1

Lemma D.1. If $\omega(\gamma) \equiv \ln(1 - \gamma/\alpha_N)^{-1/\gamma} = -(1/\gamma)\ln(1 - \gamma/\alpha_N) < 0$, then $\omega(0) = 1/\alpha_N$.

Proof. Apply L'Hopital's Rule to the ratio $N(\gamma)/D(\gamma)$ where $N(\gamma) \equiv -\ln(1 - \gamma/\alpha_N)$ and $D(\gamma) \equiv \gamma$. Observe that

$$N'(\gamma) = \frac{1/\alpha_N}{1 - \gamma/\alpha_N}$$

so that $N'(0) = 1/\alpha_N$. Also $D'(0) = 1$. Therefore, $\omega(0) = N'(0)/D'(0) = 1/\alpha_N$. \square

Lemma D.2. If $\omega(\gamma) \equiv \ln(1 - \gamma/\alpha_N)^{-1/\gamma} = -(1/\gamma)\ln(1 - \gamma/\alpha_N)$, then $\omega'(\gamma) > 0$.

Proof. $\omega'(\gamma) = \frac{1}{\gamma^2}\ln(1 - \gamma/\alpha_N) + \frac{1}{\gamma} \frac{1/\alpha_N}{1 - \gamma/\alpha_N} = \frac{1}{\gamma^2} \left[\ln\left(1 - \frac{\gamma}{\alpha_N}\right) + \frac{\gamma/\alpha_N}{1 - (\gamma/\alpha_N)} \right]$.

Therefore,

$$\omega'(\gamma) = \frac{1}{\gamma^2} v\left(\frac{-\gamma}{\alpha_N}\right)$$

where $v(z) \equiv \ln(1 + z) - z/(1 + z)$.

Note that $v(0) = 0$ and $v'(z) = z/(1 + z)^2$ so that $v(z) > 0$ for $z > 0$. Therefore, $\omega'(\gamma) > 0$ because $\gamma > 0$. \square

Proof of Proposition 1. It follows from the definition of C in Eq. (14b) and the assumption that $\gamma > 0$ that $C < 1$. Lemma D.2 and the assumption that $\gamma > 0$

imply that $C = e^{\omega(\gamma)} > e^{\omega(0)}$. Using the expression for $\omega(0)$ in Lemma D.1 yields $C > e^{1/\alpha_N}$. Lemma B.1 implies that $e^{1/\alpha_N} > e^{-1} = 0.367879$. \square

Appendix E. The effect of demand growth, the interest rate, the capital share, and the price elasticity of demand on the expected long-run capital stock

Proof of Proposition 10. Differentiating the expression for $H(\infty)$ in Proposition 2 with respect to μ yields

$$\frac{dH(\infty)}{d\mu} = -\frac{\sigma^2}{2\mu^2} < 0. \tag{E.1}$$

Turning now to $dC/d\mu$, note that μ enters the expression for C in Eq. (14b) only through its effect on the root α_N . Differentiating Eq. (14b) with respect to α_N yields

$$\frac{dC}{d\alpha_N} = -\frac{1}{\gamma} \frac{C}{1 - \frac{\gamma}{\alpha_N}} \frac{\gamma}{\alpha_N^2} = \frac{C}{\alpha_N(\gamma - \alpha_N)} < 0. \tag{E.2}$$

Differentiating Eq. (14b) with respect to μ , and using Eqs. (E.2) and (B.3) yields

$$\frac{dC}{d\mu} = \frac{dC}{d\alpha_N} \frac{d\alpha_N}{d\mu} = \frac{C}{(\gamma - \alpha_N)\rho'(\alpha_N)} > 0. \quad \square \tag{E.3}$$

Proof of Corollary to Proposition 10. Using Eqs. (E.1) and (E.3) and the fact that $H(\infty) = 1 + \sigma^2/(2\mu)$ yields

$$\frac{d\kappa(\infty)}{d\mu} = H(\infty) \frac{dC}{d\mu} + C \frac{dH(\infty)}{d\mu} = \frac{\left(1 + \frac{\sigma^2}{2\mu}\right)C}{(\gamma - \alpha_N)\rho'(\alpha_N)} - C \frac{\sigma^2}{2\mu^2} \tag{E.4}$$

which can be rearranged as

$$\frac{d\kappa(\infty)}{d\mu} = \frac{C}{(\gamma - \alpha_N)\rho'(\alpha_N)} \left[1 + \frac{\sigma^2}{2\mu} - \frac{\sigma^2}{2\mu^2}(\gamma - \alpha_N)\rho'(\alpha_N) \right]. \tag{E.5}$$

Use the definition $m = 2\mu/\sigma^2$ to rewrite Eq. (E.5) as

$$\frac{d\kappa(\infty)}{d\mu} = \frac{C}{(\gamma - \alpha_N)\rho'(\alpha_N)} \left[1 + \frac{1}{m} + \frac{\alpha_N}{m} \frac{\rho'(\alpha_N)}{\mu} - \frac{\gamma}{m} \frac{\rho'(\alpha_N)}{\mu} \right]. \tag{E.6}$$

Recall that $\rho'(\alpha_N) > 0$. Thus, to show that $d\kappa(\infty)/d\mu < 0$ it suffices to show that $1 + m^{-1} + m^{-1}\alpha_N\rho'(\alpha_N)/\mu < 0$. Lemma B.1 implies

$$\frac{\alpha_N}{m} < -1. \tag{E.7}$$

Observe from Eqs. (B.1) and (E.7) that

$$\frac{\rho'(\alpha_N)}{\mu} = -2\frac{\alpha_N}{m} - 1 + \frac{1}{m} > 1 + \frac{1}{m}. \tag{E.8}$$

It follows from Eqs. (E.7) and (E.8) that

$$1 + \frac{1}{m} + \frac{\alpha_N}{m} \frac{\rho'(\alpha_N)}{\mu} < 1 + \frac{1}{m} - \frac{\rho'(\alpha_N)}{\mu} < 0. \tag{E.9}$$

As discussed below Eq. (E.6), the condition in Eq. (E.9) is sufficient to prove that $d\kappa(\infty)/d\mu < 0$. \square

Proof of Proposition 11. Notice from the definition of $H(t)$ in Eq. (14c) that it is independent of r for all t . Also, from Eq. (14b) r affects C only through its effect on the root, α_N . Thus, using Eqs. (B.4) and (E.2) we obtain

$$\frac{d\kappa(t)}{dr} = \frac{dC}{dr} H(t) = \frac{dC}{d\alpha_N} \frac{d\alpha_N}{dr} H(t) = \frac{-\kappa(t)}{\alpha_N(\gamma - \alpha_N)\rho'(\alpha_N)} > 0. \tag{E.10}$$

Proof of Proposition 12. Notice from the definition of $H(t)$ in Eq. (14c) that it is independent of γ for all t . Also note that $\rho(\eta)$ is independent of γ so that α_N is independent of γ . Thus, using the definition of C in Eq. (14b) and differentiating with respect to γ , we obtain

$$\frac{d\kappa(t)}{d\gamma} = \frac{dC}{d\gamma} H(t) = \frac{\kappa(t)}{\gamma^2} \left[\ln\left(1 - \frac{\gamma}{\alpha_N}\right) - \frac{\gamma}{\gamma - \alpha_N} \right]. \tag{E.11}$$

The expression in Eq. (E.11) will be positive if and only if $D(\alpha_N) > 0$ where

$$D(\alpha_N) \equiv \ln\left(1 - \frac{\gamma}{\alpha_N}\right) - \frac{\gamma}{\gamma - \alpha_N}. \tag{E.12}$$

Observe that

$$D'(\alpha_N) = -\frac{\gamma^2}{\alpha_N(\alpha_N - \gamma)^2} > 0$$

which implies that

$$D(\alpha_N) > D(-\infty) \equiv \ln(1) - 0 = 0. \tag{E.13}$$

Appendix F. A comparison of $\bar{K}(X_t)$ and the expected capital stock

The marginal revenue product of capital, $\eta \equiv hy^\gamma$, follows a regulated geometric Brownian motion with drift $\gamma M(\gamma)$ and instantaneous variance $\gamma^2 \sigma^2$. Using Bentolila and Bertola, (p. 389), Result 2 (and recognizing that the upper bound on η is the user cost c and with irreversibility the lower bound is zero) yields the ergodic distribution of η

$$f(\eta) = \frac{(\phi - 1)\eta^{\phi-2}}{c^{\phi-1}} \text{ for } \eta \leq c \text{ where } \phi \equiv \frac{2\gamma M(\gamma)}{\gamma^2 \sigma^2}. \tag{F.1}$$

The unconditional expectation of $K/X \equiv y^{-1} = (\eta/h)^{-1/\gamma}$ is calculated using the ergodic distribution in Eq. (F.1) to obtain

$$E\left\{\frac{K}{X}\right\} = \frac{(\phi - 1)h^{1/\gamma}}{c^{\phi-1}} \int_0^c \eta^{-1/\gamma} \eta^{\phi-2} d\eta = \frac{\phi - 1}{\phi - 1 - 1/\gamma} \left(\frac{c}{h}\right)^{-1/\gamma}. \tag{F.2}$$

The definition of ϕ implies that

$$\frac{\phi - 1}{\phi - 1 - 1/\gamma} = \frac{2\mu - \sigma^2}{2\mu - 2\sigma^2} \tag{F.3}$$

so that

$$E\left\{\frac{K}{X}\right\} = \frac{2\mu - \sigma^2}{2\mu - 2\sigma^2} \left(\frac{c}{h}\right)^{-1/\gamma} \tag{F.4}$$

and

$$\bar{K}(X_t) = E\left\{\frac{K}{X}\right\} X_t = \frac{2\mu - \sigma^2}{2\mu - 2\sigma^2} \left(\frac{c}{h}\right)^{-1/\gamma} X_t. \tag{F.5}$$

The expressions in Eqs. (F.3)–(F.5) are positive if and only if $\mu > \sigma^2$ and here we restrict attention to this case.

In the case of costless reversibility, hy^γ is always equal to the user cost c_R and thus K/X is always equal to $(c_R/h)^{-1/\gamma}$ which implies that

$$K_t^R = (c_R/h)^{-1/\gamma} X_t. \tag{F.6}$$

Suppose that one wanted to quantify the effect of irreversibility on capital accumulation by comparing $\bar{K}(X_t)$ to K_t^R , or by comparing $E_0\{\bar{K}(X_t)\}$ to $E_0\{K_t^R\}$. Dividing Eq. (F.5) by Eq. (F.6) yields

$$\begin{aligned} \frac{\bar{K}(X_t)}{K_t^R} &= \frac{\frac{2\mu - \sigma^2}{2\mu - 2\sigma^2} \left(\frac{c}{h}\right)^{-1/\gamma} X_t}{\left(\frac{c_R}{h}\right)^{-1/\gamma} X_t} = \left(\frac{c}{c_R}\right)^{-1/\gamma} \frac{2\mu - \sigma^2}{2\mu - 2\sigma^2} > C \left(1 + \frac{\sigma^2}{2\mu}\right) \\ &= \kappa(\infty). \end{aligned} \tag{F.7}$$

Inspection of Eq. (F.7) indicates that the ratio $E_0\{\bar{K}(X_t)\}/E_0\{K_t^R\}$ equals the ratio $\bar{K}(X_t)/K_t^R$ and that both of these ratios are independent of t . Moreover, both of these ratios overstate the expected long-run capital stock under irreversibility relative to the expected long-run capital stock under reversibility as measured by $\kappa(\infty)$.

To see why these ratios overstate the expected long-run capital stock under irreversibility, observe that

$$E_0\{K_t\} = E_0\left\{\frac{K_t}{X_t} X_t\right\} = E_0\left\{\frac{K_t}{X_t}\right\} E_0\{X_t\} + \text{Cov}_0\left(\frac{K_t}{X_t}, X_t\right). \tag{F.8}$$

Dividing Eq. (F.5) by Eq. (F.8) yields

$$\frac{\bar{K}(X_t)}{E_0\{K_t\}} = \frac{\frac{2\mu - \sigma^2}{2\mu - 2\sigma^2} \left(\frac{c}{h}\right)^{-1/\gamma} X_t}{E_0\left\{\frac{K_t}{X_t}\right\} E_0\{X_t\} + \text{Cov}_0\left(\frac{K_t}{X_t}, X_t\right)}. \tag{F.9}$$

Clearly $\bar{K}(X_t)$ is not equal to $E_0\{K_t\}$ because the ratio $\bar{K}(X_t)/E_0\{K_t\}$ is proportional to the random variable X_t and thus is not identically equal to one. Moreover, the expected value of $\bar{K}(X_t)$ is not equal to $E_0\{K_t\}$ as we demonstrate below. Taking the expected values of both sides of Eq. (F.9) yields

$$\frac{E_0\{\bar{K}(X_t)\}}{E_0\{K_t\}} = E_0\left\{\frac{\bar{K}(X_t)}{E_0\{K_t\}}\right\} = \frac{\frac{2\mu - \sigma^2}{2\mu - 2\sigma^2} \left(\frac{c}{h}\right)^{-1/\gamma} E_0\{X_t\}}{E_0\left\{\frac{K_t}{X_t}\right\} E_0\{X_t\} + \text{Cov}_0\left(\frac{K_t}{X_t}, X_t\right)}. \tag{F.10}$$

Next substitute $e^{m t}$ for $E_0\{X_t\}$, and take the limits of both sides of Eq. (F.10) using the ergodic distribution in Eq. (F.1) to calculate $\lim_{t \rightarrow \infty} E_0\{K_t/X_t\}$ to obtain

$$\lim_{t \rightarrow \infty} \frac{E_0\{\bar{K}(X_t)\}}{E_0\{K_t\}} = \left[1 + \lim_{t \rightarrow \infty} \frac{\text{Cov}_0\left(\frac{K_t}{X_t}, X_t\right)}{e^{m t} E_0\left\{\frac{K_t}{X_t}\right\}} \right]^{-1}. \tag{F.11}$$

To calculate the covariance in Eq. (F.11) divide Eq. (F.8) by $E_0\{K_t/X_t\} E_0\{X_t\}$ to obtain

$$\frac{E_0\{K_t\}}{E_0\left\{\frac{K_t}{X_t}\right\} E_0\{X_t\}} = 1 + \frac{\text{Cov}_0\left(\frac{K_t}{X_t}, X_t\right)}{E_0\left\{\frac{K_t}{X_t}\right\} E_0\{X_t\}}. \tag{F.12}$$

Next use Eq. (11) to substitute for $E_0\{K_t\}$, substitute $e^{\mu t}$ for $E_0\{X_t\}$, and take the limits of both sides of Eq. (F.12) using the ergodic distribution in Eq. (F.1) to calculate $\lim_{t \rightarrow \infty} E_0\{K_t/X_t\}$ to obtain

$$\lim_{t \rightarrow \infty} \frac{\text{Cov}_0\left(\frac{K_t}{X_t}, X_t\right)}{e^{\mu t} E_0\left\{\frac{K_t}{X_t}\right\}} = -\frac{\sigma^4}{\mu(2\mu - \sigma^2)} < 0. \quad (\text{F.13})$$

Substituting Eq. (F.13) into Eq. (F.11) yields

$$\lim_{t \rightarrow \infty} \frac{E_0\{\bar{K}(X_t)\}}{E_0\{K_t\}} = \left[1 - \frac{\sigma^4}{\mu(2\mu - \sigma^2)}\right]^{-1} > 1, \quad (\text{F.14})$$

so that the expectation of $\bar{K}(X_t)$ exceeds $E_0\{K_t\}$ because of the negative covariance between K_t/X_t and X_t in the long run.

References

- Abel, A.B., 1983. Optimal investment under uncertainty. *American Economic Review* 73, 228–233.
- Abel, A.B., Eberly, J.C., 1997. An exact solution for the investment and market value of a firm facing uncertainty, adjustment costs, and irreversibility. *Journal of Economic Dynamics and Control* 21, 831–852.
- Abel, A.B., Eberly, J.C., 1996. Optimal investment with costly reversibility. *Review of Economic Studies* 63, 581–593.
- Bentolila, S., Bertola, G., 1990. Firing costs and labor demand: how bad is eurosclerosis?. *Review of Economic Studies* 57, 381–402.
- Bertola, G. 1988. Adjustment costs and dynamic factor demands: investment and employment under uncertainty. Ph.D. Dissertation, MIT.
- Bertola, G., 1992. Labor turnover costs and average labor demand. *Journal of Labor Economics* 10, 389–411.
- Bertola, G., Caballero, R.J., 1994. Irreversibility and aggregate investment. *Review of Economic Studies* 61, 223–246.
- Caballero, R.J., 1991. On the sign of the investment-uncertainty relationship. *American Economic Review* 81, 279–288.
- Caballero, R.J., 1993. On the dynamics of aggregate investment. In: Serven, L., Solimano, A. (Eds.), *Striving for Growth after Adjustment, The Role of Capital Formation*. The World Bank, Washington, DC, pp. 81–106.
- Dixit, A., 1989. Entry and exit decisions under uncertainty. *Journal of Political Economy* 97, 620–638.
- Dixit, A., 1991. Irreversible investment with price ceilings. *Journal of Political Economy* 99, 541–557.
- Dixit, A., 1992. Investment and hysteresis. *Journal of Economic Perspectives* 6, 107–132.
- Dixit, A., Pindyck, R.S., 1994. *Investment under uncertainty*. Princeton University Press, Princeton, NJ.
- Dumas, B., 1991. Super contact and related optimality conditions. *Journal of Economic Dynamics and Control* 15, 675–685.
- Goldman, M.B., Sosin, H.B., Gatto, M.A., 1979. Path dependent options: buy at the low, sell at the high. *Journal of Finance* 34, 1111–1127.

- Harrison, J.M., 1985. *Brownian Motion and Stochastic Flow Systems*. Krieger Publishing Company, Malabar, FL.
- Hartman, R., 1972. The effects of price and cost uncertainty on investment. *Journal of Economic Theory* 5, 258–266.
- Hopenhayn, H., Rogerson, R., 1993. Job turnover and policy evaluation: a general equilibrium analysis. *Journal of Political Economy* 101, 915–938.
- Hubbard, R.G., 1994. Investment under uncertainty: keeping one's options open. *Journal of Economic Literature* 32, 1816–1831.
- Jorgenson, D.W., 1963. Capital theory and investment behavior. *American Economic Review Papers and Proceedings* 53, 247–259.
- Pindyck, R.S., 1988. Irreversible investment, capacity choice, and the value of the firm. *American Economic Review* 78, 969–985.
- Pindyck, R.S., 1993. A note on competitive investment under uncertainty. *American Economic Review* 83, 273–277.