Optimal Inattention to the Stock Market with Information Costs and Transactions Costs*

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Abstract

Recurrent intervals of inattention to the stock market are optimal if it is costly to observe asset values. When consumers observe the value of their wealth, they decide whether to transfer funds between a transactions account from which consumption must be financed and an investment portfolio of equity and riskless bonds. Transfers of funds are subject to a proportional transactions cost, so the consumer may choose not to transfer any funds on a particular observation date. In general, the optimal adjustment rule—including the size and direction of transfers, and the time of the next observation—is state-dependent. Surprisingly, the consumer's optimal behavior eventually evolves to a situation with a purely time-dependent rule, with a constant interval of time between observations that can be substantial even for tiny observation costs. In the long run, but only in the long run, the standard consumption Euler equation holds between observation dates if the consumer is sufficiently risk averse.

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A pervasive finding in studies of microeconomic choice is that adjustment to economic news tends to be sluggish and infrequent. Individuals rebalance their portfolios and revisit their spending behavior at discrete and potentially infrequent points of time. Between these times, inaction is the rule. Similar sorts of inaction also characterize the financing, investment, and pricing behavior of firms. These observations have led economists to formulate models that are consistent with infrequent adjustment. One question of particular interest is whether, and to what extent, infrequent adjustment at the micro level can help account for certain macroeconomic phenomena. For instance, if firms adjust their prices infrequently, then monetary policy could have important short-run real effects. Similarly, if individuals take several months or even years to adjust their portfolios and their spending plans, the standard predictions of the consumption smoothing and portfolio choice theories might fail. In particular, the standard intertemporal Euler equation relating asset returns and consumption growth may not hold.

Formal models of infrequent adjustment are often described as either time-dependent or state-dependent. In models with time-dependent adjustment, adjustment is triggered simply by calendar time and is independent of the state of the economy. In models with state-dependent adjustment, adjustment takes place only when a particular state variable reaches some trigger value, so the timing of adjustments depends on factors other than, or in addition to, calendar time alone. A classic example of state-dependent adjustment is the (S,s) model. The difference between time-dependent and state-dependent models can have crucial implications for important economic questions. For instance, monetary policy has substantial real effects that persist for several quarters if firms change their prices according to a time-dependent rule. However, if firms adjust their prices according to a state-dependent rule, then monetary policy may have little or no effect on the real economy. (See e.g. Caplin and Spulber (1987) and Golosov and Lucas (2007).)

In this paper we develop and analyze an optimizing model that can generate both time-dependent adjustment and state-dependent adjustment. The economic context is an infinite-horizon continuous-time model of consumption and portfolio choice that builds on the framework of Merton (1971). We augment Merton's model by requiring that consumption must be purchased with the liquid asset and by introducing two sorts of costs – a cost of observing the value of the consumer's wealth and a cost of transferring assets between a transactions account consisting of liquid assets and an investment portfolio consisting of risky equity and riskless bonds. Because it is costly for the consumer to observe the value of wealth, the consumer will choose to observe this value only at discretely-spaced points in time. At these observation times, the consumer will choose when next to observe the value of wealth, and will also execute any transfers between the investment portfolio and the transactions account, as well as choose the path of consumption until the next observation date.

In general, the timing of asset transfers will be state-dependent. The relevant state of the consumer's balance sheet is captured by x_t, defined as the ratio of the balance in the transactions account to the contemporaneous value of the investment portfolio. On any given observation date, the consumer chooses the date of the next observation, but, depending on the value of x_t that is realized on the next observation date, the consumer may or may not transfer assets between the investment portfolio and the transactions account on that date. Because the timing of asset transfers depends on the value of x_t, we describe these transfers as state-dependent. A surprising result of our analysis, however, is that eventually the consumer's holdings in the investment portfolio and the transactions account will evolve to a situation in which the optimal timing of asset transfers is purely time-dependent. Indeed, when asset holdings get to this stage, the optimal time between successive asset transfers is constant.

When x_t has a high value on an observation date, so that the consumer's balance sheet is relatively heavy in the transactions account, the consumer will find it optimal to use some of the transactions account to purchase additional assets in the investment portfolio. Alternatively, when x_t has an intermediate value on an observation date, the consumer may not find it worthwhile to pay the transactions costs associated with either transferring assets into the investment portfolio or transferring assets out of the investment portfolio. This inaction makes the timing of asset transfers state-dependent. Finally, when x_t is low on an observation date, the consumer will sell some assets from the investment portfolio to replenish the transactions account in order to finance consumption until the next observation date.

We show that eventually, though not necessarily immediately, optimal behavior will lead to a low value of x_t on an observation date. Once a low value of x_t is realized on an observation date, the consumer will not transfer more assets to the transactions account than are needed to finance consumption until the next observation date. This behavior is optimal both because it is costly to transfer assets and because the liquid asset in the transactions account earns a lower riskless rate of return than the riskless bond in the investment portfolio. In this case, the consumer will plan to hold a zero balance in the transactions account on the next observation date, so that x_t will equal zero on the next observation date. Thus, on the next observation date, x_t will have a low value and the situation repeats itself: x_t will equal zero on every subsequent observation date and the optimal interval between successive observations will be constant, which is a purely time-dependent rule.

This paper relates to two strands of the literature. The first strand is the large literature on transactions costs. In the models by Baumol (1952) and Tobin (1956), which are the forerunners of the cash-in-advance model used in macroeconomics, consumers can hold two riskless assets that pay different rates of return: money, which pays zero interest, and a riskless bond that pays a positive rate of interest. Consumers are willing to hold money, despite the fact that its rate of return is dominated by the rate of return on riskless bonds, because goods have to be purchased with money. That is, money offers liquidity services. We also adopt the assumption that the rate of return on the liquid asset is lower than the rate of return on riskless bonds in the investment portfolio.

A more recent literature on portfolio transactions costs, including Constantinides (1986) and Davis and Norman (1990) models the costs of transferring assets between stocks and bonds in the investment portfolio as proportional to the size of the transfers. Here we also model transactions costs as proportional to the size of the transactions, but the transactions costs apply only to transfers between the liquid asset in the transactions account on the one hand and the investment portfolio of stocks and bonds on the other. We do not model the costs of reallocating stocks and bonds *within* the investment portfolio. For a retired consumer who finances consumption by withdrawing assets from a tax-deferred retirement account, the transactions cost of withdrawing the assets includes the taxes paid at the time of withdrawal. The marginal tax rate for most consumers in this situation, which is part of the transactions cost of transferring assets from the investment portfolio to the transactions account, is likely to be far greater than any transactions costs associated with reallocating the stocks and bonds within the investment portfolio.

A second strand of the literature, which includes Abel, Eberly, and Panageas (2007), Duffie and Sun (1990) and Gabaix and Laibson (2002), analyzes infrequent adjustment of portfolios that arises because it is costly to observe and process information.[1] If these costs are proportional to the value of the entire investment portfolio, then in a continuous-time framework, the consumer will choose not to observe the value of wealth continuously. The two closest antecedents to our current paper are Duffie and Sun (1990) and Abel, Eberly, and Panageas (2007). In our current paper, we show that the consumer's behavior differs depending on the value of the state variable x_t. In particular, we show that when x_t is

[1] Reis (2006) develops and analyzes a model of optimal inattention for a consumer who faces a cost of observing additive income, such as labor income. In that model, the consumer can hold only a single riskless asset so there is no portfolio allocation problem.

low, the consumer will plan to arrive at the next observation date with a zero balance in the transactions account, and that the length of time between subsequent observations is constant. Duffie and Sun derive this result, but they implicitly confine attention to low values of x_t. Here we show that optimal behavior is potentially different for intermediate and high values of x_t, situations not considered by Duffie and Sun. But we go on to show that eventually x_t will indeed become low on an observation date and then will remain low on all subsequent observation dates. In this sense, Duffie and Sun confine attention to the long run, and we consider the transition path to the long run as well as the long run. Importantly, consideration of behavior outside of the long-run situation allows the model to incorporate state-dependent adjustment as well as purely time-dependent adjustment. A second contribution of this paper relative to Duffie and Sun is that we offer an assessment of the length of the interval of time between consecutive observations in the long run. This assessment includes an analytic component based on a quadratic approximation and a quantitative component. Finally, relative to our own earlier paper, this paper explicitly allows separate consideration of observation costs and transactions costs. In addition, our earlier paper assumes that the investment portfolio is continuously re-balanced by a portfolio manager who, on each observation date, charges a fee proportional to the size of the portfolio (and thus the fee is not separately identifiable from an observation cost that is proportional to the size of the portfolio) whereas the current paper does not allow re-balancing of the investment portfolio between observation dates.

In the absence of observation costs and transactions costs, there is a standard Euler equation that holds between any two points of time. Specifically, the product of the intertemporal marginal rate of substitution and the excess rate of return on stocks relative to riskless bonds has a conditional expectation equal to zero. However, in the presence of observation costs, an Euler equation of this sort will hold only between two observation dates. Even restricting attention to observation dates, the relevant Euler equation can differ from the standard Euler equation because of the presence of transactions costs. Nevertheless, in the long run, if consumers are sufficiently risk averse, the standard Euler equation holds between any pair of observation dates.

We set up the consumer's decision problem in Section 1. The consumer lives in continuous time but observes the value of the investment portfolio and makes decisions – about consumption, transfers between the investment portfolio and the transactions account, the share of the investment portfolio to hold in risky equity, and the next date at which to observe the value of the investment portfolio – at discretely spaced points in time. Between consecutive observation dates, the consumer does not react to any unpredictable realizations of the stock price. Hence, the path of consumption from one observation date to the next observation date can be calculated by the consumer at the earlier observation date. Thus the consumer's problem can be expressed as an infinite-horizon discrete-time problem, albeit with a potentially state-dependent length of time intervals. In Section 2, we derive useful characteristics of the value function for this discrete-time problem and use these characteristics to describe optimal transactions between the investment portfolio and the transactions account. In Section 3 we set up the Lagrangian expression for the consumer's problem and analyze whether various circumstances lead the consumer to arrive at the next observation date with a zero balance in the transactions account. These results serve as the basis for the graphical illustration of the dynamic behavior of the transaction account and the investment portfolio in Section 3.1. Then in Section 3.2 we show that once the consumer arrives at an observation date with a zero balance in the transactions account, she will arrive at *all* subsequent observation dates with zero liquid assets. In formal terms, the amount of liquid assets held when arriving at an observation date is a stochastic process, and zero is an absorbing state for this process. We show in Section 3.2 that the absorbing state is reached in finite time. Of course, the consumer does not continuously hold zero liquid assets in the absorbing state. On each observation date, the consumer will sell assets from the investment portfolio to replenish the liquid assets in the transactions account, and then will gradually but completely deplete the transaction account to finance consumption until the next observation date. In Section 4 we examine the impact of observation costs, transactions costs, and portfolio constraints on the Euler equation. In Section 5 we confine attention to the long run, in which the time interval between consecutive observation dates is constant and we analyze the length of this interval. The final section of the paper presents concluding remarks and directions for future research. Lengthy proofs are in the appendix.

1 Consumer's Decision Problem

Consider an infinitely-lived consumer whose objective at time t is to maximize

$$E_t \left\{ \int_0^\infty \frac{1}{1-\alpha} c_{t+s}^{1-\alpha} e^{-\rho s} ds \right\}, \tag{1}$$

where the coefficient of relative risk aversion is $0 < \alpha \neq 1$ and the rate of time preference is $\rho > 0$. The consumer does not earn any labor income but has wealth that consists of risky equity, riskless bonds, and a riskless liquid asset. Consumption must be purchased with the liquid asset, which the consumer holds in a transactions account. Risky equity and riskless bonds are held in an investment portfolio and cannot be used directly to purchase consumption. The consumer is not permitted to hold more than 100% of the investment portfolio in risky equity. That is, the consumer is not permitted to take a leveraged position in equity.

Equity is a non-dividend-paying stock with a price P_t that evolves according to a geometric Brownian motion

$$\frac{dP_t}{P_t} = \mu dt + \sigma dz, \tag{2}$$

where $\mu > 0$ is the mean rate of return and σ is the instantaneous standard deviation. The riskless bond in the investment portfolio has a constant instantaneous rate of return r_f that is positive and less than the mean rate of return on equity, so $0 < r_f < \mu$. The total value of the investment portfolio, consisting of equity and riskless bonds, is S_t at time t.

At time t, the consumer holds X_t in the liquid asset, which pays a riskless instantaneous rate of return r_L, where $r_L < r_f$. The rate of return on the liquid asset, r_L, is lower than the rate of return on the riskless bond in the investment portfolio, r_f, because the liquid asset provides transactions services not provided by the bond in the investment portfolio.

We introduce two types of costs into the consumer's intertemporal decision problem: a cost to observe the value of the stock market and a cost to transfer assets between the investment portfolio and the transactions account. Because it is costly to observe the value of the stock market, the consumer will choose not to observe its value continuously. Instead, the consumer will optimally choose discretely-spaced points in time, t_j, $j = 0, 1, 2, ...$, at which to observe the value of assets; during intervals of time between consecutive observation dates, the consumer will be inattentive to asset values. We assume that the cost of observing the value of the stock market is an increasing function of the consumer's wealth. Specifically, we assume that the cost of observing the stock market at date t_j is $\theta(X_{t_j} + S_{t_j})$, where $0 < \theta < 1$.[2]

Suppose the consumer observes the value of wealth at time t_j and next observes its value at time $t_{j+1} = t_j + \tau_j$. Immediately upon observing the values of S_{t_j} and X_{t_j}, the consumer

[2] A rationale for assuming that the observation cost is proportional to contemporaneous wealth is that the observation cost is the value of foregone leisure, and this value is proportional to the consumer's wealth. If the instantaneous utility function is $u(c, l) = \frac{1}{1-\alpha} c^{1-\alpha} f(l)$, where l is leisure, then the marginal value of leisure is $u_l(c, l) = \frac{1}{1-\alpha} c^{1-\alpha} f'(l)$. Thus, for a given level of leisure, the cost of a unit of leisure foregone to observe the value of assets is proportional to $c^{1-\alpha}$ in units of utility. Therefore, the cost of a unit of foregone leisure, expressed in terms of real goods, is proportional to c, which, in turn, is proportional to wealth.

may transfer assets between the investment portfolio and the liquid asset in the transactions account (at a cost described below) so that at time t_j^+ the value of the investment portfolio is $S_{t_j^+}$. The consumer chooses to hold a fraction $\phi_j \leq 1$ of $S_{t_j^+}$ as risky equity and a fraction $1-\phi_j \geq 0$ in riskless bonds and does not rebalance the investment portfolio before the next observation date.[3] When the consumer next observes the value of the investment portfolio, at time $t_{j+1} \equiv t_j + \tau_j$, its value, *after* paying the observation cost at time t_{j+1}, is

$$S_{t_{j+1}} = (1-\theta)\left[\phi_j \frac{P_{t_{j+1}}}{P_{t_j}} S_{t_j^+} + \left(1-\phi_j\right) R^f\left(\tau_j\right) S_{t_j^+}\right], \tag{3}$$

where

$$R^f\left(\tau_j\right) \equiv e^{r_f \tau_j} \tag{4}$$

is the gross rate of return on the riskless bond in the investment portfolio between times t_j and t_{j+1}, *before* paying the observation cost at time t_{j+1}.

Define

$$R_{j+1}\left(\tau_j\right) \equiv \phi_j \frac{P_{t_{j+1}}}{P_{t_j}} + \left(1-\phi_j\right) R^f\left(\tau_j\right) \tag{5}$$

as the gross rate of return on the investment portfolio between times t_j and t_{j+1}, *before* paying the observation cost at time t_{j+1}, and observe that

$$S_{t_{j+1}} = (1-\theta) R_{j+1}\left(\tau_j\right) S_{t_j^+}. \tag{6}$$

The second cost is a transaction cost that the consumer must pay whenever transferring assets between the investment portfolio and the transactions account. If the consumer uses some of the liquid asset to purchase additional assets in the investment portfolio at the jth observation date t_j, the transaction cost is $\psi_b \Delta S_j^{\text{buy}} \geq 0$, where $\Delta S_j^{\text{buy}} \geq 0$ is the gross increase in assets in the investment portfolio and $\psi_b \geq 0$ is a proportional transactions cost. If the consumer sells some of the assets in the investment portfolio at time t_j to increase the amount held in the transactions account, the transaction cost is $-\psi_s \Delta S_j^{\text{sell}} \geq 0$, where $\Delta S_j^{\text{sell}} \leq 0$ is the (negative of the) gross decrease of assets in the investment portfolio and $0 \leq \psi_s < 1$ is a proportional transactions cost. We allow, but do not require, ψ_s and ψ_b to

[3]The consumer does not observe any new information between time t_j^+ and time t_{j+1} and hence cannot adjust the portfolio in response to any news that arrives during this interval of inattention. It is possible that the consumer could decide at time t_j^+ to exchange equity for bonds at some time(s) before t_{j+1}, but we do not consider this possibility in this paper. In Abel, Eberly, and Panageas (2007) a portfolio manager continuously rebalances equity and bonds in the investment portfolio.

be equal. Perhaps the most obvious interpretation of the proportional transactions costs, ψ_s and ψ_b, is that they represent brokerage fees. Another interpretation presents itself if we consider the investment portfolio to be a tax-deferred account such as a 401k account. In this case, the consumer must pay a tax on withdrawals from the investment portfolio, and ψ_s would include the consumer's income tax rate, which would be substantially higher than a brokerage fee.[4]

In order to sell a dollar of assets from the investment portfolio, the consumer must first pay θ to observe the value of the stock market and then must pay ψ_s to transfer a dollar from the investment portfolio to the transactions account. The consumer will realize a positive amount from the observation and sale of the dollar of assets if and only if

$$\psi_s + \theta < 1, \tag{7}$$

a condition that we assume holds throughout the paper.

Immediately after observing X_{t_j} and S_{t_j} at time t_j, the consumer transfers assets between the transactions account and the investment portfolio. The new holdings of assets in the investment portfolio and the transactions account, respectively, are

$$S_{t_j^+} = S_{t_j} + \Delta S_j^{\text{buy}} + \Delta S_j^{\text{sell}} \tag{8}$$

and

$$X_{t_j^+} = X_{t_j} - (1 + \psi_b)\, \Delta S_j^{\text{buy}} - (1 - \psi_s)\, \Delta S_j^{\text{sell}}. \tag{9}$$

1.1 Consumption between Observation Dates

The consumer observes new information only at observation dates. That is, once the consumer observes the value of assets at date t_j, he will not observe any new information until the next observation date t_{j+1}. Therefore, the consumer can plan the entire path of consumption from time t_j^+ to time t_{j+1} at time t_j. Let $C(t_j, \tau_j)$ be the present value, discounted at rate r_L, of the flow of consumption over the interval of time from t_j^+ until the

[4]This interpretation of ψ_s as a tax rate is most plausible if the consumer only withdraws money from the investment portfolio and never transfers assets into the investment portfolio. As we will see in Section 3.2, the long run is characterized by precisely this situation in which the consumer never transfers funds into the investment portfolio.

next observation date $t_{j+1} \equiv t_j + \tau_j$. Specifically,

$$C(t_j, \tau_j) = \int_{t_j}^{t_{j+1}} c_s e^{-r_L(s-t_j)} ds, \tag{10}$$

where the path of consumption c_s, $t_j^+ \leq s \leq t_{j+1}$, is chosen to maximize the discounted value of utility over the interval from t_j^+ to t_{j+1}. Let

$$U(C(t_j, \tau_j)) = \max_{\{c_s\}_{s=t_j^+}^{t_{j+1}}} \int_{t_j}^{t_{j+1}} \frac{1}{1-\alpha} c_s^{1-\alpha} e^{-\rho(s-t_j)} ds, \tag{11}$$

subject to a given value of $C(t_j, \tau_j)$ in equation (10). Since the consumer does not observe any new information between time t_j^+ and time t_{j+1}, the maximization in equation (11) is a standard intertemporal optimization under certainty, so we relegate the details to the Appendix. We show that

$$U(C(t_j, \tau_j)) = \frac{1}{1-\alpha} [h(\tau_j)]^\alpha [C(t_j, \tau_j)]^{1-\alpha} \tag{12}$$

and

$$U'(C(t_j, \tau_j)) = c_{t_j^+}^{-\alpha}, \tag{13}$$

where

$$h(\tau_j) \equiv \int_0^{\tau_j} e^{-\omega s} ds = \frac{1 - e^{-\omega \tau_j}}{\omega} \tag{14}$$

and we assume that

$$\omega \equiv \frac{\rho - (1-\alpha) r_L}{\alpha} > 0. \tag{15}$$

Since all of the consumption during the interval of time from t_j^+ to t_{j+1} is financed from the liquid asset in the transactions account, which earns an instantaneous riskless rate of return r_L, we have

$$X_{t_{j+1}} = (1-\theta) R^L(\tau_j) \left(X_{t_j^+} - C(t_j, \tau_j) \right), \tag{16}$$

where

$$R^L(\tau_j) \equiv e^{r_L \tau_j}$$

is the gross rate of return on the liquid asset between times t_j and $t_{j+1} \equiv t_j + \tau_j$, *after* paying the observation cost at time t_{j+1}.

2 The Value Function and Its Properties

At observation date t_j the value function is

$$V\left(X_{t_j}, S_{t_j}\right) = \max_{C(t_j,\tau_j),\Delta S_j^{\text{sell}},\Delta S_j^{\text{buy}},\phi_j,\tau_j} \frac{1}{1-\alpha}\left[h\left(\tau_{t_j}\right)\right]^{\alpha}\left[C\left(t_j,\tau_j\right)\right]^{1-\alpha} + \beta^{\tau_j} E_{t_j}\left\{V\left(X_{t_{j+1}}, S_{t_{j+1}}\right)\right\}, \tag{17}$$

where the maximization in equation (17) is subject to

$$S_{t_{j+1}} = (1-\theta) R_{j+1}(\tau_j)\left[S_{t_j} + \Delta S_j^{\text{buy}} + \Delta S_j^{\text{sell}}\right] \tag{18}$$

$$X_{t_{j+1}} = (1-\theta) R^L(\tau_j)\left[X_{t_j} - (1+\psi_b)\Delta S_j^{\text{buy}} - (1-\psi_s)\Delta S_j^{\text{sell}} - C(t_j,\tau_j)\right] \tag{19}$$

$$C(t_j,\tau_j) \leq X_{t_j} - (1+\psi_b)\Delta S_j^{\text{buy}} - (1-\psi_s)\Delta S_j^{\text{sell}} \tag{20}$$

and $\phi_j \leq 1$, $\Delta S_j^{\text{buy}} \geq 0$, and $\Delta S_j^{\text{sell}} \leq 0$.

The value function is homogeneous of degree $1-\alpha$ in X_{t_j} and S_{t_j} so that

$$V\left(X_{t_j}, S_{t_j}\right) = \frac{1}{1-\alpha} S_{t_j}^{1-\alpha} v\left(x_{t_j}\right) \tag{21}$$

where

$$x_t \equiv \frac{X_t}{S_t}. \tag{22}$$

The homogeneity of the problem implies that given X_{t_j} and S_{t_j} on an observation date t_j, the optimal values of $\frac{C(t_j,\tau_j)}{S_{t_j}}$, $\frac{\Delta S_j^{\text{sell}}}{S_{t_j}}$, $\frac{\Delta S_j^{\text{buy}}}{S_{t_j}}$, ϕ_j, and τ_j are homogeneous of degree zero in X_{t_j} and S_{t_j} and thus can be written as functions of x_{t_j}. In addition, homogeneity implies that the marginal rate of substitution between X_{t_j} and S_{t_j}, which we define as $m\left(x_{t_j}\right) \equiv \frac{V_S\left(X_{t_j},S_{t_j}\right)}{V_X\left(X_{t_j},S_{t_j}\right)}$, is simply a function of x_{t_j}. Next we derive bounds on $m\left(x_{t_j}\right)$.

2.1 Bounds on $m(x)$

Since all of the variables are contemporaneous, for the remainder of this subsection, we will suppress the time subscripts, with the understanding that the results apply at any observation date. On any observation date, the consumer can instantaneously buy additional assets in the investment portfolio by using $1+\psi_b$ dollars from the transactions account for each dollar of additional assets purchased for the investment portfolio. Since the consumer has the option to move instantaneously from (X,S) to $\left(X-(1+\psi_b)\Delta S^{\text{buy}}, S+\Delta S^{\text{buy}}\right)$, we have

$$V(X,S) \geq V\left(X-(1+\psi_b)\Delta S^{\text{buy}}, S+\Delta S^{\text{buy}}\right), \text{ for all } X,\ S, \text{ and } \Delta S^{\text{buy}} \geq 0. \tag{23}$$

Therefore,

$$\frac{dV\left(X-(1+\psi_b)\Delta S^{\text{buy}}, S+\Delta S^{\text{buy}}\right)}{d\Delta S^{\text{buy}}}\Big|_{\Delta S^{\text{buy}}=0} \leq 0. \tag{24}$$

Evaluating the derivative in equation (24) yields

$$\frac{dV\left(X-(1+\psi_b)\Delta S^{\text{buy}}, S+\Delta S^{\text{buy}}\right)}{d\Delta S^{\text{buy}}}\Big|_{\Delta S^{\text{buy}}=0} = [m(x)-(1+\psi_b)]\, V_X(X,S) \leq 0. \tag{25}$$

We can use a similar line of analysis to study the consumer's option to sell assets. On any observation date, the consumer can instantaneously sell assets from the investment portfolio and obtain $1-\psi_s$ dollars of liquid assets in the transactions account for each dollar of assets sold from the investment portfolio. Since the consumer has the option to move instantaneously from (X,S) to $\left(X+(1-\psi_s)|\Delta S^{\text{sell}}|, S-|\Delta S^{\text{sell}}|\right)$, we have

$$V(X,S) \geq V\left(X+(1-\psi_s)|\Delta S^{\text{sell}}|, S-|\Delta S^{\text{sell}}|\right), \text{ for all } X,\ S, \text{ and } \Delta S^{\text{sell}} \leq 0. \tag{26}$$

Therefore,

$$\frac{dV\left(X+(1-\psi_s)|\Delta S^{\text{sell}}|, S-|\Delta S^{\text{sell}}|\right)}{d|\Delta S^{\text{sell}}|}\Big|_{\Delta S^{\text{sell}}=0} \leq 0. \tag{27}$$

Evaluating the derivative in equation (27) yields

$$\frac{dV\left(X+(1-\psi_s)|\Delta S^{\text{sell}}|, S-|\Delta S^{\text{sell}}|\right)}{d|\Delta S^{\text{sell}}|}\Big|_{\Delta S^{\text{sell}}=0} = [1-\psi_s-m(x)]\, V_X(X,S) \leq 0. \tag{28}$$

Equations (25) and (28), along with $V_X(X,S) > 0$, prove that on any observation date

$$1-\psi_s \leq m(x) \equiv \frac{V_S(X,S)}{V_X(X,S)} \leq 1+\psi_b \text{ for all } x \geq 0, \tag{29}$$

or, equivalently,

$$(1-\psi_s)\, V_X(X,S) \leq V_S(X,S) \leq (1+\psi_b)\, V_X(X,S). \tag{30}$$

2.2 Trigger Values for x

We will define a restricted value function $\widetilde{V}\left(X_{t_j}, S_{t_j}\right)$ at observation date t_j as the maximized expected value of utility over the infinite future subject to the restriction that the consumer does not transfer any assets between the transactions account and the investment portfolio at time t_j (but optimally transfers assets between the transactions account and the investment portfolio at all future observation dates). Therefore,

$$\widetilde{V}\left(X_{t_j}, S_{t_j}\right) = \max_{C(t_j,\tau_j),\phi_j,\tau_j} \frac{1}{1-\alpha}\left[h\left(\tau_{t_j}\right)\right]^{\alpha}\left[C\left(t_j,\tau_j\right)\right]^{1-\alpha} + \beta^{\tau_j} E_{t_j}\left\{V\left(X_{t_{j+1}}, S_{t_{j+1}}\right)\right\}, \quad (31)$$

subject to equations (18) through (20) and $\phi_j \leq 1$. For the remainder of this subsection, we will suppress the time subscripts, with the understanding that the results apply at any observation date.

The restricted value function is also homogeneous of degree $1-\alpha$ and can be written as

$$\widetilde{V}\left(X, S\right) = \frac{1}{1-\alpha} S^{1-\alpha} \widetilde{v}\left(x\right). \quad (32)$$

Note that on any observation date $\widetilde{V}\left(X, S\right) \leq V\left(X, S\right)$, with equality only if the optimal values of ΔS^{sell} and ΔS^{buy} are both zero.

Define

$$\pi_1 \equiv \inf x \geq 0 : \widetilde{v}\left(x\right) = v\left(x\right) \quad (33)$$

and

$$\pi_2 \equiv \sup x \geq 0 : \widetilde{v}\left(x\right) = v\left(x\right). \quad (34)$$

The proposition below shows that π_1 and π_2 are trigger values for x in the sense that if x exceeds π_2 on an observation date, the consumer will transfer assets to the investment portfolio, and if x is less than π_1 on an observation date, the consumer will transfer assets to the transactions account.

Proposition 1 *1.* $0 < \pi_1 \leq \pi_2 < \infty$.

2. If $x > \pi_2$, *then the optimal value of* ΔS^{buy} *is positive.*

3. For any (X, S) *such that* $x \equiv \frac{X}{S} \geq \pi_2$, $V\left(X, S\right) = \frac{1}{1-\alpha}\left[\frac{X+(1+\psi_b)S}{1+\psi_b+\pi_2}\right]^{1-\alpha} v\left(\pi_2\right)$.

4. If $x < \pi_1$, *then the optimal value of* ΔS^{sell} *is negative.*

5. *For any* (X,S) *such that* $x \equiv \frac{X}{S} \leq \pi_1$, $V(X,S) = \frac{1}{1-\alpha}\left[\frac{X+(1-\psi_s)S}{1-\psi_s+\pi_1}\right]^{1-\alpha} v(\pi_1)$.

The proof of this proposition is in the Appendix; here we simply describe the outline of the proof. According to statement 1, both π_1 and π_2 are positive and finite. To prove that $\pi_1 > 0$, we argue that if $x = 0$ on an observation date, the consumer must sell some assets from the investment portfolio to avoid zero consumption. Therefore, $\widetilde{v}(0) \neq v(0)$ and, in fact, $\widetilde{v}(x) \neq v(x)$ for positive x in a neighborhood of $x = 0$, so π_1 must be positive. To show that $\pi_2 < \infty$, we show that if $S = 0$ on an observation date, so that $x^{-1} = 0$, the consumer will choose to use some of the liquid assets in the transactions account to purchase assets in the investment portfolio to take advantage of the higher rates of return on the assets in the investment portfolio. Therefore, $\widetilde{v}(\infty) \neq v(\infty)$, so π_2 must be finite Statements 2 and 4 are based on the fact that whenever $\widetilde{v}(x) \neq v(x)$, it is optimal for the consumer to transfer assets between the transactions account and the investment portfolio. When the transactions account is large relative to the investment portfolio, specifically, when $x > \pi_2$, it is optimal for the consumer to use some liquid assets to buy assets in the investment portfolio. When the transactions account is small relative to the investment portfolio, specifically, when $x < \pi_1$, it is optimal for the consumer to sell some of the assets in the investment portfolio. Statements 3 and 5 are essentially value-matching conditions. Statement 3 is based on the fact that if $x > \pi_2$, the consumer will immediately buy assets in the investment portfolio and jump to a combination of X and S for which $x = \pi_2$. The value function has the same value at the original combination of X and S as at the new combination. A similar analysis applies for statement 5.

The following corollary follows directly from differentiating statements 3 and 5 in Proposition 1 and using the definition $m(x) \equiv \frac{V_S(X,S)}{V_X(X,S)}$.

Corollary 1 1. *If* $x > \pi_2$, *then* $m(x) = 1 + \psi_b$.

2. *If* $0 \leq x < \pi_1$, *then* $m(x) = 1 - \psi_s$.

3 Dynamic Behavior of Optimal Asset Holdings

In this section we examine the evolution of the consumer's asset holdings. We show that when the state variable x_{t_j} is less than the lower trigger π_1 on an observation date, the consumer will spend all of the liquid assets in the transactions account before the next observation date. We use this result in Section 3.1 to illustrate graphically, that once x_{t_j} is less than

π_1, the interval between observation dates will be constant thereafter; thus, the optimal adjustment rule becomes purely time-dependent. Section 3.2 shows analytically that the consumer will arrive at such a purely time-dependent adjustment rule in finite expected time, but not necessarily immediately.

At observation date t_j, the consumer chooses $C(t_j, \tau_j)$, $\Delta S_j^{\text{sell}} \leq 0$, $\Delta S_j^{\text{buy}} \geq 0$, $\phi_j \leq 1$ and time τ_j until the next observation date to maximize $U(C(t_j, \tau_j)) + \beta^{\tau_j} E_{t_j} \left\{ V\left(X_{t_{j+1}}, S_{t_{j+1}}\right) \right\}$, subject to equations (18) through (20). The consumer's optimization problem at time t_j can be solved with the following Lagrangian

$$\begin{aligned} \mathcal{L}\left(X_{t_j}, S_{t_j}\right) &= \max_{\tau_j, C(t_j,\tau_j), \Delta S_j^{\text{sell}}, \Delta S_j^{\text{buy}}, \phi_j} U\left(C\left(t_j, \tau_j\right)\right) \\ &\quad + \beta^{\tau_j} E_{t_j} \left\{ V \left(\begin{array}{c} (1-\theta) R^L(\tau_j) \left(X_{t_j} - (1-\psi_s) \Delta S_j^{\text{sell}} - (1+\psi_b) \Delta S_j^{\text{buy}} - C(t_j, \tau_j)\right), \\ (1-\theta) \left[\phi_j \frac{P_{t_{j+1}}}{P_{t_j}} + \left(1-\phi_j\right) R^f(\tau_j)\right] \left[S_{t_j} + \Delta S_j^{\text{sell}} + \Delta S_j^{\text{buy}}\right] \end{array} \right) \right\} \\ &\quad + \lambda_j^{\text{buy}} \Delta S_j^{\text{buy}} - \lambda_j^{\text{sell}} \Delta S_j^{\text{sell}} + \xi_j \left(1 - \phi_j\right) \\ &\quad + \eta_j \left[X_{t_j} - (1-\psi_s) \Delta S_j^{\text{sell}} - (1+\psi_b) \Delta S_j^{\text{buy}} - C(t_j, \tau_j)\right], \end{aligned} \tag{35}$$

where $\lambda_j^{\text{buy}} \geq 0$, $\lambda_j^{\text{sell}} \geq 0$, $\xi_j \geq 0$, and $\eta_j \geq 0$ are Lagrange multipliers on the constraints $\Delta S_j^{\text{buy}} \geq 0$, $\Delta S_j^{\text{sell}} \leq 0$, $\phi_j \leq 1$, and $X_{t_j^+} - C(t_j, \tau_j) \geq 0$, respectively.

Differentiate the right hand side of equation (35) with respect to $C(t_j, \tau_j)$, ϕ_j, ΔS_j^{buy}, and ΔS_j^{sell}, respectively, set the derivatives equal to zero, and use equations (5) and (8) to obtain

$$U'\left(C\left(t_j, \tau_j\right)\right) = \beta^{\tau_j} E_{t_j} \left\{V_X\left(X_{t_{j+1}}, S_{t_{j+1}}\right)\right\} (1-\theta) R^L(\tau_j) + \eta_j \tag{36}$$

$$(1-\theta) E_{t_j} \left\{ V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right) \left[\frac{P_{t_{j+1}}}{P_{t_j}} - R^f(\tau_j)\right] \right\} = \frac{\xi_j}{\beta^{\tau_j} S_{t_j}^+} \tag{37}$$

$$\beta^{\tau_j} (1-\theta) \left[\begin{array}{c} -(1+\psi_b) E_{t_j} \left\{V_X\left(X_{t_{j+1}}, S_{t_{j+1}}\right)\right\} R^L(\tau_j) \\ + E_{t_j} \left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right) R_{j+1}(\tau_j)\right\} \end{array} \right] = -\lambda_j^{\text{buy}} + (1+\psi_b) \eta_j \tag{38}$$

and

$$\beta^{\tau_j} (1-\theta) \left[\begin{array}{c} -(1-\psi_s) E_{t_j} \left\{V_X\left(X_{t_{j+1}}, S_{t_{j+1}}\right)\right\} R^L(\tau_j) \\ + E_{t_j} \left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right) R_{j+1}(\tau_j)\right\} \end{array} \right] = \lambda_j^{\text{sell}} + (1-\psi_s) \eta_j. \tag{39}$$

Multiply both sides of equation (37) by ϕ_j, add and subtract $\left(1-\phi_j\right) R^f(\tau_j)$ inside the square brackets on the left hand side of the equation, use the definition $R_{j+1}(\tau_j) \equiv \phi_j \frac{P_{t_{j+1}}}{P_{t_j}} +$

$\left(1-\phi_j\right) R^f\left(\tau_j\right)$, and recognize that[5] $\xi_j \phi_j=\xi_j$ to obtain

$$(1-\theta) E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)\left[R_{j+1}\left(\tau_j\right)-R^f\left(\tau_j\right)\right]\right\}=\frac{\xi_j}{\beta^{\tau_j} S_{t_j}^{+}}. \tag{40}$$

If the optimal risky share, ϕ_j, is strictly less than one, then the Lagrange multiplier ξ_j equals zero, and equation (40) implies that the product of the excess rate of return and the marginal valuation of assets in the investment portfolio has a conditional expectation equal to zero. That is, the expected marginal-valuation-weighted rate of return on the investment portfolio equals the expected marginal-valuation-weighted riskless rate of return. However, if $\xi_j>0$, the constraint $\phi_j \leq 1$ is binding and equation (40) implies that the expected marginal-valuation-weighted return on the investment portfolio exceeds the expected marginal-valuation-weighted riskless return. This difference reflects the consumer's desire to shift assets in the investment portfolio from riskless bonds to equity, which is prevented by the binding constraint on ϕ_j.

Lemma 1 *If $\psi_b+\psi_s>0$, then $\Delta S_j^{sell}=0$ or $\Delta S_j^{buy}=0$.*

Lemma 1 implies that if it is costly to transfer assets between the investment portfolio and the transactions account, the consumer will not simultaneously buy and sell assets in the investment portfolio.

Define

$$G\left(\tau_j\right) \equiv(1-\theta)\left[\frac{1-\psi_s}{1+\psi_b} R^f\left(\tau_j\right)-R^L\left(\tau_j\right)\right] \tag{41}$$

as the net gain to the consumer from a round-trip transaction from the liquid asset in the transactions account to the riskless asset in the investment portfolio at time t_j^{+}, and then back to the liquid asset in the transactions account on the next observation date, $t_{j+1} \equiv t_j+\tau_j$. Specifically, the consumer reduces the transactions account by one dollar and uses this dollar to buy $\frac{1}{1+\psi_b}$ dollars of the riskless bond in the investment portfolio at time t_j^{+}. This amount will grow to $\frac{1}{1+\psi_b} R^f\left(\tau_j\right)$ dollars at time t_{j+1} and can be converted to $(1-\theta) \frac{1-\psi_s}{1+\psi_b} R^f\left(\tau_j\right)$ dollars of liquid assets in the transactions account at time t_{j+1}^{+}. $G\left(\tau_j\right)$ is the excess return on this round-trip transaction compared to leaving the dollar in the transactions account from time t_j^{+} to time t_{j+1}^{+} and growing to $(1-\theta) R^L\left(\tau_j\right)$ dollars. This excess return can be negative, zero, or positive depending on the value of τ_j, the length of time until the next observation date.

[5] If $\phi_j<1$, then $\xi_j=0$, so $\phi_j \xi_j=\xi_j$. If $\phi_j=1$, then $\phi_j \xi_j=\xi_j$.

Lemma 2 *If $G(\tau_j) > 0$, then $C(t_j, \tau_j) = X_{t_j^+}$ and hence $X_{t_{j+1}} = 0$.*

The intuition underlying Lemma 2 is straightforward. As we have explained earlier, if $G(\tau_j) > 0$, the consumer can earn a positive riskless return from a round-trip transaction from the riskless liquid asset in the transactions account to the riskless asset in the investment portfolio, and then back to the liquid asset in the transactions account on the next observation date. Because of this opportunity for a riskless gain by transferring assets from the transactions account to the investment portfolio, the consumer will transfer as many liquid assets as possible from the transactions account, while leaving only enough liquid assets in the transactions account to finance consumption until the next observation date. Since the consumer will hold only enough liquid assets in the transactions account to finance consumption until the next observation date, the transactions account will be completely depleted on the next observation date. Thus, if the next observation date is time t_{j+1}, then $X_{t_{j+1}}$ will be zero.

Next we will show that, while sufficient, it is not necessary for $G(\tau_j) > 0$ in order for $C(t_j, \tau_j) = X_{t_j^+}$ and hence $X_{t_{j+1}} = 0$.

Lemma 3 *If $\Delta S_j^{sell} < 0$, then $C(t_j, \tau_j) = X_{t_j^+}$ and $X_{t_{j+1}} = 0$.*

The intuition for Lemma 3 is that the consumer will sell only enough of the investment portfolio to obtain enough of the liquid asset to finance consumption, $C(t_j, \tau_j)$, until the next observation date. He will not want to acquire additional liquid assets because he knows that he would arrive at time t_{j+1} with a positive holding of the liquid asset. Instead of paying a transaction cost to acquire an additional dollar of the liquid asset and earn $R^L(\tau_j)$ over the interval from time t_j^+ to time t_{j+1}, the consumer could leave the dollar in the investment portfolio and hold the dollar in riskless bonds earning $R^f(\tau_j)$ over the interval from t_j^+ to t_{j+1}. Since $R^f(\tau_j) > R^L(\tau_j)$, the consumer will choose not to acquire the additional dollar of liquid asset at time t_j^+. That is, the consumer will acquire only enough of the liquid asset at time t_j^+ to finance $C(t_j, \tau_j)$, the present value of the consumption stream from t_j^+ to t_{j+1}. Note that this result holds regardless of the sign of $G(\tau_j)$.[6]

If $x_{t_j} < \pi_1$, the consumer transfers assets from the investment portfolio to the liquid asset in the transactions account at time t_j^+, so $\Delta S_j^{\text{sell}} < 0$, which proves the following corollary.

[6] To show that it is possible for $\Delta S_j^{\text{sell}} < 0$ while $G(\tau_j) < 0$, consider the situation in which $x_{t_j} = 0$ (we show in subsection 3.2 that $x_{t_j} = 0$ in the long run). In this situation, ΔS_j^{sell} will be negative regardless of the sign of $G(\tau_j)$, because the consumer must acquire some of the liquid asset to finance consumption over the next interval of time. If the observation cost is sufficiently small, then τ_j will be sufficiently small to make $G(\tau_j)$ negative.

Corollary 2 *If* $x_{t_j} < \pi_1$*, then* $C(t_j, \tau_j) = X_{t_j^+}$ *and* $X_{t_{j+1}} = 0$.

Lemmas 2 and 3 and Corollary 2 describe conditions under which the transactions account balance equals zero on the next observation date. Next we present a condition under which the transaction account balance will not equal zero on the next observation date.

Lemma 4 *Suppose that* $x_{t_j} > \pi_2$*. If* $E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)(1-\theta)\left[\frac{1-\psi_s}{1+\psi_b}R_{j+1}(\tau_j) - R^L(\tau_j)\right]\right\} < 0$*, then* $C(t_j, \tau_j) < X_{t_j^+}$ *and* $X_{t_{j+1}} > 0$.

If $x_{t_j} > \pi_2$, the consumer has a relatively large balance of liquid assets in the transactions account and uses some of the liquid assets to purchase assets for the investment portfolio. Observe that $(1-\theta)\left[\frac{1-\psi_S}{1+\psi_b}R_{j+1}(\tau_j) - R^L(\tau_j)\right]$ is the net gain to the consumer from a round-trip transaction from the liquid asset in the transactions account to the investment portfolio at time t_j^+, and then back to the liquid asset in the transactions account at time t_{j+1}^+. If the expected marginal-value-weighted expectation of this gain is negative, the return on the investment portfolio is not sufficiently greater than the return on the liquid asset to justify transferring the maximum amount of liquid assets to the investment portfolio. Thus, the consumer will plan to arrive at the next observation date with a positive balance of liquid assets in the investment portfolio.

We can develop a simpler form for the condition in Lemma 4 using the lemma below.

Lemma 5 *If (i)* $G(\tau_j) < 0$ *and (ii) the constraint* $\phi_j \leq 1$ *is not binding, then* $E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)(1-\theta)\left[\frac{1-\psi_s}{1+\psi_b}R_{j+1}(\tau_j) - R^L(\tau_j)\right]\right\} < 0$.

Lemma 5 proves the following corollary to Lemma 4.

Corollary 3 *Suppose that* $x_{t_j} > \pi_2$*. If (i)* $G(\tau_j) < 0$ *and (ii) the constraint* $\phi_j \leq 1$ *is not binding, then* $C(t_j, \tau_j) < X_{t_j^+}$ *and* $X_{t_{j+1}} > 0$.

The condition in Lemma 4 involves the difference in returns $(1-\theta)\left[\frac{1-\psi_s}{1+\psi_b}R_{j+1}(\tau_j) - R^L(\tau_j)\right]$, which is random and is weighted by the marginal valuation $V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)$. Corollary 3 presents a simpler condition for the case in which the constraint that $\phi_j \leq 1$ is not binding. When this constraint is not binding, the consumer is indifferent between adding a marginal dollar to risky equity and adding a marginal dollar to the riskless bond in the investment portfolio. Thus, we can simply replace $R_{j+1}(\tau_j)$, the rate of return on the investment portfolio, by $R^f(\tau_j)$, the rate of return on the riskless bond in the investment portfolio. In this case, the crucial difference in rates of return is the non-stochastic variable $(1-\theta)\left[\frac{1-\psi_S}{1+\psi_b}R^f(\tau_j)-R^L(\tau_j)\right]=G(\tau_j)$, and there is no need to weight this difference by $V_S\left(X_{t_{j+1}},S_{t_{j+1}}\right)$ to determine the sign of the relevant expected marginal-value-weighted difference in returns. The condition that $G(\tau_j)<0$ will hold for sufficiently small observation costs.[7]

3.1 A Graphical Illustration of the Evolution of Asset Holdings

In this section we present a graph to illustrate the dynamic behavior implied by Lemmas 2 through 5 and Corollaries 2 and 3. Suppose that time $t_0=0$ is an observation date and that τ_1 is the next observation date. Assume that $E_0\left\{V_S(X_{\tau_1},S_{\tau_1})(1-\theta)\left[\frac{1-\psi_s}{1+\psi_b}R_1(\tau_1)-R^L(\tau_1)\right]\right\}<0$ so that x_{τ_1}will not necessarily equal 0.

The positive quadrant in Figure 1 is divided into three regions by the two half-lines with slopes π_1 and π_2. Region I, which lies below the half-line with slope equal to π_1, consists of points for which $x\equiv X/S$ is less than π_1. Region II, which lies between the two half-lines with slopes π_1 and π_2, consists of points for which $\pi_1<x<\pi_2$. Region III, which lies above the half-line with slope equal to π_2, consists of points for which $x>\pi_2$. Suppose that $x_0\equiv\frac{X_0}{S_0}>\pi_2$, so that (X_0,S_0) is represented by the point labelled "time 0", which is in Region III. The consumer will instantaneously use some of the liquid assets in the transactions account to buy additional assets in the investment portfolio, reducing X_{0^+} by $1+\psi_b$ dollars for every dollar that S_{0^+} is increased. That is, the consumer moves instantaneously from the point labeled "time 0" to the point labeled "time 0^+", which lies on the half-line with slope equal to π_2. The liquid assets in the transactions account earn a known rate of return, r_L, and since the consumer knows the entire consumption path from time 0^+ to time τ_1, the consumer at time 0 knows the value of the transactions account at time τ_1, X_{τ_1}. Since the value of X_{τ_1} is known at time 0, we know at time 0 that at time τ_1, (X_{τ_1},S_{τ_1}) will lie somewhere along the horizontal dashed in line in Figure 1. If the stock market performs very poorly between time 0 and time τ_1, so that S_{τ_1} is small, then (X_{τ_1},S_{τ_1}) will be at a point such as that labeled "A" in Figure 1. That is, if the stock market performs poorly, (X_{τ_1},S_{τ_1}) will be in Region III. Alternatively, if the stock market performs moderately well between time 0 and time τ_1, then (X_{τ_1},S_{τ_1}) will be represented by a point such as that labeled "B" in Figure 1, which is in Region II. Finally, if the stock

[7] If the observation costs are arbitrarily small, then the optimal value of τ_j will be arbitrarily small. Since $G(\tau_j)=(1-\theta)\left[\frac{1-\psi_s}{1+\psi_b}\exp\left[(r_f-r_L)\tau_j\right]-1\right]R^L(0,\tau_j)$, and since $\frac{1-\psi_s}{1+\psi_b}<1$, $G(\tau_j)$ will be negative for arbitrarily small τ_j.

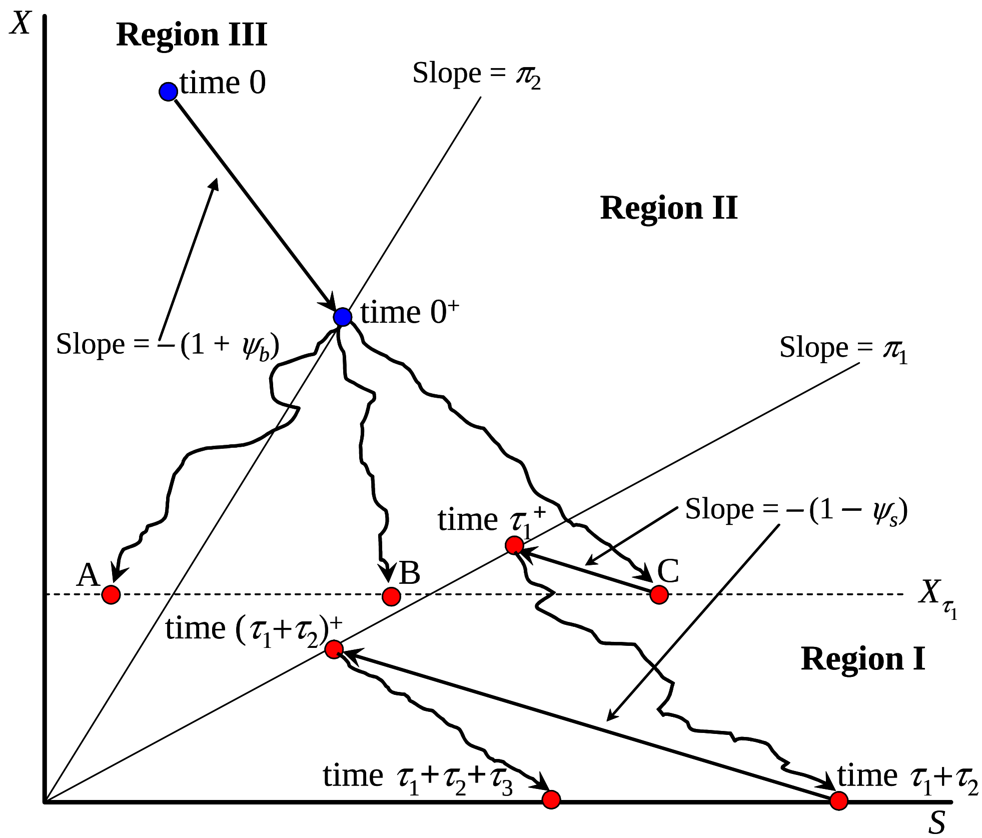

FIGURE 1: Dynamic Behavior of (X, S)

market performs very well between time 0 and time τ_1, then (X_{τ_1}, S_{τ_1}) will be represented by a point such as that labeled "C" in Figure 1, which is in Region I.

If the consumer is in Region III at observation date τ_1, then the situation is the same as at the point labelled "time 0" in Region III. That is, on the following observation date, $\tau_1 + \tau_2$, $(X_{\tau_1+\tau_2}, S_{\tau_1+\tau_2})$ could be in any of the three regions. If the consumer is in Region II at observation date τ_1, then depending on various parameter values, the consumer could potentially be in any of the three regions on the following observation date, $\tau_1 + \tau_2$. Finally, if the consumer is in Region I on observation date τ_1, then the situation is more definitive at date $\tau_1 + \tau_2$, as we will now show.

Suppose that (X_{τ_1}, S_{τ_1}) is in Region I, so that $x_{\tau_1} < \pi_1$. Since $x_{\tau_1} < \pi_1$, the consumer instantaneously sells some of the assets in the investment portfolio, and transfers the proceeds, net of transactions costs, to the transactions account. For each dollar of assets in the investment portfolio that the consumer sells, the transactions account will increase by

$1-\psi_s$ dollar. Thus, the consumer moves instantaneously from the point labeled "C" to the point labeled "time τ_1^+", which lies along the half-line with slope equal to π_1 in Figure 1. Corollary 2 implies that since $x_{\tau_1} < \pi_1$, the value of the transactions account at the following observation date, $X_{\tau_1+\tau_2}$, will be zero. Thus, $(X_{\tau_1+\tau_2}, S_{\tau_1+\tau_2})$ will lie somewhere along the horizontal axis, regardless of the performance of the stock market. At time $(\tau_1+\tau_2)^+$ the consumer will instantaneously sell assets from the investment portfolio and return to the half-line with slope π_1 in Figure 1. Then the process repeats itself over and over again. Once the consumer is in Region I on an observation date, the consumer will be along the horizontal axis in Figure 1, with a zero balance in the transactions account, on all subsequent observation dates.

3.2 Long-Run Behavior

This section formalizes the dynamic behavior of the transactions account and the investment portfolio, which were illustrated graphically in Section 3.1. Recall that j indexes the observation dates t_j, so x_{t_j} is the value of x on an observation date.

Lemmas 2 and 4 and Corollary 2 prove the following proposition.

Proposition 2 *(i) If $G(\tau_j) > 0$ or if $x_{t_j} \leq \pi_1$, then $x_{t_{j+1}} = 0$. (ii) If $x_{t_j} > \pi_2$ and if $E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)(1-\theta)\left[\frac{1-\psi_s}{1+\psi_b}R_{j+1}(\tau_j) - R^L(\tau_j)\right]\right\} < 0$, then $x_{t_{j+1}} > 0$.*

Corollary 4 *If $x_{t_j} \leq \pi_1$, then $x_{t_{j+i}} = 0$ for all $i = 1, 2, 3, \ldots$.*

Corollary 5 *If $G(\tau_j) > 0$ at observation date t_j, then $x_{t_{j+i}} = 0$ for all $i = 1, 2, 3, \ldots$.*

Proposition 2 and its corollaries imply that 0 is an absorbing state for the stochastic process x_{t_j}, $j = 1, 2, 3, \ldots$. The following proposition implies that the absorbing state is not necessarily reached on the next observation date.[8]

Proposition 3 *If $x_{t_j} \geq \pi_2$ and if $E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)(1-\theta)\left[\frac{1-\psi_s}{1+\psi_b}R_{j+1}(\tau_j) - R^L(\tau_j)\right]\right\} < 0$ at observation time t_j, then $\Pr\left\{x_{t_{j+1}} \leq \pi_1\right\} < 1$.*

Lemma 5 implies the following corollary to Proposition 3.

[8] Duffie and Sun (1990) assume as an initial condition that on the first observation date, which is normalized to time 0, the transactions balance is $X_0 = 0$. By making this assumption, they guarantee that absorption is immediate. That is, since they assume $X_0 = 0$, they start in Region I. Therefore, $X_{t_j} = 0$ on all subsequent observation dates t_j.

Corollary 6 *If (i) $x_{t_j} \geq \pi_2$, (ii) $G(\tau_j) < 0$ at observation time t_j, and (iii) the constraint $\phi_j \leq 1$ does not bind, then* $\Pr\left\{x_{t_{j+1}} \leq \pi_1\right\} < 1$.

Recall that the consumer observes the value of the investment portfolio at dates t_j, $j = 0, 1, 2, \ldots$. Let $x_{t_j^+}$ be the value of x immediately after the consumer observes the value of the investment portfolio at date t_j and optimally transfers assets between the transactions account and the investment portfolio. The sequence $x_{t_j^+}$, $j = 0, 1, 2, \ldots$, is a stochastic process that is confined to a closed interval $[\pi_1, \pi_2]$. If $x_{t_j^+} = \pi_1$ on any observation date, the consumer consumes the entire transactions account over the interval of time until the next observation date t_{j+1} and so arrives at time t_{j+1} with a zero balance in the transactions account. Thus, $x_{t_{j+1}} = 0 < \pi_1$, so at time t_{j+1} the consumer sells assets from the investment portfolio to make $x_{t_{j+1}^+} = \pi_1$, and the process repeats itself. This argument proves the following proposition.

Proposition 4 *If $x_{t_n^+} = \pi_1$, then $x_{t_j^+} = \pi_1$ for all $j > n$.*

Proposition 4 describes the behavior of $x_{t_j^+}$ once it has reached its absorbing value. The following proposition states that with probability one the consumer will get to the long-run situation in which $x_{t_j^+} = \pi_1$ and, moreover, the consumer will reach this situation in finite time.

Proposition 5 *Let t_n be the first time that $x_{t_n^+} = \pi_1$. Then* $\Pr\{t_n < \infty\} = 1$ *and* $E\{t_n\} < \infty$.

4 Euler Equation at Adjustment Times

Standard representative-consumer models of asset pricing without information costs or transactions costs imply an Euler equation that states that the product of the intertemporal marginal rate of substitution and the excess rate of return on a frictionlessly traded asset has conditional expectation equal to zero. A similar sort of equation holds in the presence of information costs and transactions costs, though, of course, care must be taken to define the intertemporal marginal rate of substitution and the gross rate of return properly in the presence of these costs. To derive the appropriate Euler equation, we use the envelope theorem, or equivalently differentiate the Lagrangian in equation (35) with respect to S_{t_j}, to calculate the marginal valuation of the investment portfolio, $V_S\left(X_{t_j}, S_{t_j}\right)$, at any observation date t_j. Specifically,

$$V_S\left(X_{t_j}, S_{t_j}\right) = \frac{\partial \mathcal{L}\left(X_{t_j}, S_{t_j}\right)}{\partial S_{t_j}} = \beta^{\tau_j} E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)(1-\theta) R_{j+1}(\tau_j)\right\}, \tag{42}$$

which can be rewritten as

$$E_{t_j}\left\{\frac{\beta^{\tau_j} V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)}{V_S\left(X_{t_j}, S_{t_j}\right)}(1-\theta) R_{j+1}(\tau_j)\right\} = 1. \tag{43}$$

Equations (37) and (40) imply that

$$E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)\frac{P_{t_{j+1}}}{P_{t_j}}\right\} = E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right) R_{j+1}(\tau_j)\right\} \tag{44}$$

so that equation (43) can be written as

$$E_{t_j}\left\{\left[\frac{\beta^{\tau_j} V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)}{V_S\left(X_{t_j}, S_{t_j}\right)}\right](1-\theta)\frac{P_{t_{j+1}}}{P_{t_j}}\right\} = 1. \tag{45}$$

Applying the law of iterated projections to equation (45) yields

$$E_{t_j}\left\{\left[\beta^{t_k - t_j}\frac{V_S\left(X_{t_k}, S_{t_k}\right)}{V_S\left(X_{t_j}, S_{t_j}\right)}\right](1-\theta)^{k-j}\frac{P_{t_k}}{P_{t_j}}\right\} = 1, \tag{46}$$

where t_j and $t_k > t_j$ are arbitrary observation dates.

We derive a similar equation for the riskless asset in the investment portfolio by combining equations (40) and (42), and rearranging using the fact that $\frac{\xi_j}{S_{t_j^+}}$ is known at time t to obtain

$$E_{t_j}\left\{\left[\frac{\beta^{\tau_j} V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)}{V_S\left(X_{t_j}, S_{t_j}\right)}\right]\frac{(1-\theta) R^f(\tau_j)}{\Upsilon_j}\right\} = 1, \tag{47}$$

where

$$\Upsilon_j \equiv 1 - \frac{\xi_j}{S_{t_j^+} V_S\left(X_{t_j}, S_{t_j}\right)}. \tag{48}$$

Note that if the constraint $\phi_j \leq 1$ is not binding at time t_j, then $\xi_j = 0$ and $\Upsilon_j = 1$. Now apply the law of iterated expectations to equation (47) to obtain

$$E_{t_j}\left\{\left[\beta^{t_k-t_j}\frac{V_S\left(X_{t_k},S_{t_k}\right)}{V_S\left(X_{t_j},S_{t_j}\right)}\right](1-\theta)^{k-j}\,e^{r_f(t_k-t_j)}\prod_{i=j}^{k-1}\Upsilon_i^{-1}\right\}=1. \tag{49}$$

Subtract equation (49) from equation (46) and divide both sides by $\beta^{t_k-t_j}(1-\theta)^{k-j}$ to obtain

$$E_{t_j}\left\{\frac{V_S\left(X_{t_k},S_{t_k}\right)}{V_S\left(X_{t_j},S_{t_j}\right)}\left[\frac{P_{t_k}}{P_{t_j}}-e^{r_f(t_k-t_j)}\prod_{i=j}^{k-1}\Upsilon_i^{-1}\right]\right\}=0. \tag{50}$$

The next step is to obtain an expression for $\frac{V_S\left(X_{t_k},S_{t_k}\right)}{V_S\left(X_{t_j},S_{t_j}\right)}$. Use the envelope theorem, or equivalently differentiate the Lagrangian in equation (35) with respect to X_{t_j}, to calculate the marginal valuation of the transactions account, $V_X\left(X_{t_j},S_{t_j}\right)$, at any observation date t_j. Specifically,

$$V_X\left(X_{t_j},S_{t_j}\right)=\frac{\partial\mathcal{L}\left(X_{t_j},S_{t_j}\right)}{\partial X_{t_j}}=\beta^{\tau_j}E_{t_j}\left\{V_X\left(X_{t_{j+1}},S_{t_{j+1}}\right)\right\}(1-\theta)\,R^L(\tau_j)+\eta_j. \tag{51}$$

Since the right hand sides of equations (51) and (36) are identical, we obtain

$$V_X\left(X_{t_j},S_{t_j}\right)=U'\left(C\left(t_j,\tau_j\right)\right)=c_{t_j^+}^{-\alpha}, \tag{52}$$

where the second equality follows from equation (13). Equation (52) has a straightforward interpretation. If the consumer receives an additional unit of wealth in the transactions account at observation date t_j, the increase in the expected present value of the consumer's lifetime utility is equal to the increase in utility if the consumer simply consumed that additional unit over the time interval from t_j to t_{j+1}, regardless of whether or not the constraint $X_{t_j^+}-C\left(t_j,\tau_j\right)\geq 0$ binds.

Use the definition of the intratemporal marginal rate of substitution between X_{t_j} and S_{t_j}, $m\left(x_{t_j}\right)\equiv\frac{V_S\left(X_{t_j},S_{t_j}\right)}{V_X\left(X_{t_j},S_{t_j}\right)}$, together with equation (52) to obtain

$$V_S\left(X_{t_j},S_{t_j}\right)=m\left(x_{t_j}\right)V_X\left(X_{t_j},S_{t_j}\right)=m\left(x_{t_j}\right)c_{t_j^+}^{-\alpha}. \tag{53}$$

Substitute equation (53), applied at observation dates t_j and $t_k>t_j$, into equation (50) to obtain

$$E_{t_j}\left\{\frac{m\left(x_{t_k}\right)}{m\left(x_{t_j}\right)}\left(\frac{c_{t_k^+}}{c_{t_j^+}}\right)^{-\alpha}\left[\frac{P_{t_k}}{P_{t_j}}-e^{r_f(t_k-t_j)}\prod_{i=j}^{k-1}\Upsilon_i^{-1}\right]\right\}=0. \tag{54}$$

In standard models without observation costs and transactions costs, the Euler equation would be

$$E_t\left\{\left(\frac{c_s}{c_t}\right)^{-\alpha}\left[\frac{P_s}{P_t}-e^{r_f(s-t)}\right]\right\}=0 \tag{55}$$

for arbitrary dates $s>t$. In general, the Euler equation in the presence of observation costs and transactions costs in equation (54) differs from the standard Euler equation in equation (55) in three ways: 1) The intertemporal marginal rate of substitution in equation (54) contains the ratio $\frac{m\left(x_{t_k}\right)}{m\left(x_{t_j}\right)}$, which is absent (or implicitly equal to one) in the standard Euler equation; 2) the Euler equation in equation (54) contains the product $\prod_{i=j}^{k-1}\Upsilon_i^{-1}$; and 3) in the presence of observation costs, the Euler equation holds only between observation dates, whereas the Euler equation in the standard case holds between any arbitrary pair of dates because all dates are observation dates in the standard case. We will show that in the long run in an interesting special case, the first two of these differences disappear. Before showing this result, we present the following lemma.

Lemma 6 *If $x_{t_j}\leq\pi_1$, then (i) $\phi_j<1$ if $\alpha>\frac{\mu-r_f}{\sigma^2}$ and (ii) $\phi_j=1$ if $\alpha\leq\frac{\mu-r_f}{\sigma^2}$.*

Now confine attention to the long run, which we define as the situation in which the sequence x_{t_j} has reached its absorbing value of zero so that $m\left(x_{t_j}\right)=m\left(x_{t_k}\right)=1-\psi_s$. Therefore, in the long run, the ratio $\frac{m\left(x_{t_k}\right)}{m\left(x_{t_j}\right)}$ equals one and the first of the three differences between the Euler equations disappears. If we assume that the consumer is sufficiently risk averse, specifically $\alpha>\frac{\mu-r_f}{\sigma^2}$, then the constraint $\phi_j\leq 1$ does not bind, which implies $\xi_i=0$ for $i=j,j+1...$, so that $\prod_{i=j}^{k-1}\Upsilon_i^{-1}=1$ and the second difference disappears. Therefore, we have proved the following proposition.

Proposition 6 *If $\alpha>\frac{\mu-r_f}{\sigma^2}$, then for arbitrary observation dates $t_k>t_j$ in the long run we have $E_{t_j}\left\{\left(\frac{c_{t_k^+}}{c_{t_j^+}}\right)^{-\alpha}\left[\frac{P_{t_k}}{P_{t_j}}-e^{r_f(t_k-t_j)}\right]\right\}=0$.*

Proposition 6 states that for sufficiently risk-averse consumers, the relevant Euler equation in the long run is the same as the standard Euler equation, provided that attention is confined to observation dates, t_j and t_k. It is worth noting that "sufficiently risk-averse" need not require a very high value of α. For instance, in the baseline set of parameter values in Table 1 later in the paper, $\mu - r_f = 0.04$ and $\sigma = 0.16$, so any value of α greater than 1.5625 will be sufficiently risk averse.

5 Inattention Intervals in the Long Run

We have shown that in the long run, the transactions balance will be zero on all observation dates, and the consumer will sell assets from the investment portfolio to increase the ratio of the transactions account balance to the value of the investment portfolio to π_1. In this section, we focus on this long-run situation and derive an expression for the ratio π_1 and characterize the optimal interval of time between successive observations of the stock market.

It will be convenient to define

$$J(\tau) \equiv \beta^{\tau} E_{t_j} \left\{ \left[(1-\theta) R_{j+1}(\tau) \right]^{1-\alpha} \right\}, \tag{56}$$

where t_j is an observation date, τ is an arbitrary length of time until the next observation date, and ϕ_j has been chosen to maximize $\frac{J(\tau)}{1-\alpha}$. The definition of $J(\tau)$ holds only in Region I. Recall that once the consumer reaches Region I on an observation date, the consumer will be in Region I at all future observation dates, and will choose the same allocation of the investment portfolio and the same interval of time until the next observation at all future observation dates. Thus, the distribution of $R_{j+1}(\tau)$, and hence $J(\tau)$, is invariant to t_j after the consumer has reached Region I. We assume that $J(\tau) < 1$, so that the value function is finite.[9] We use the function $J(\tau)$ to calculate the trigger value π_1 and the optimal interval of time between consecutive observations in the long run, which we denote as τ^*.

Proposition 7 $\pi_1 = (1-\psi_s)\left([J(\tau^*)]^{-\frac{1}{\alpha}} - 1 \right)$.

Proposition 7 expresses π_1 in terms of $J(\tau^*)$, but we still need to determine τ^*. The following proposition provides a nonlinear equation that τ^* must satisfy.

Proposition 8 *If the transactions balance is zero on an observation date, the optimal time until the next observation, τ^*, satisfies* $N(\tau^*) \equiv \left[1 - \frac{1}{\alpha} \frac{h(\tau^*)}{h'(\tau^*)} \frac{J'(\tau^*)}{J(\tau^*)} \right] [J(\tau^*)]^{\frac{1}{\alpha}} = 1$.

[9] For any finite value of $E_{t_j} \left\{ \left[(1-\theta) R_{j+1}(\tau) \right]^{1-\alpha} \right\}$, a small enough value of β will make $J(\tau) < 1$.

Corollary 7 *If the transactions balance is zero on an observation date, the optimal time until the next observation, τ^*, is invariant to the transactions cost parameters ψ_s and ψ_b.*

Corollary 8 *Suppose that the transactions balance is zero on observation date t_j. Let $\{\widetilde{c}_t\}_{t=t_j^+}^{\infty}$ be the path of optimal future consumption if $\psi_s = \psi_b = 0$. Then for arbitrary ψ_s and ψ_b, the path of optimal future consumption is $\{(1-\psi_s)\,\widetilde{c}_t\}_{t=t_j^+}^{\infty}$.*

Corollaries 7 and 8 show that once the consumer has reached a zero transactions balance on an observation date, the transactions cost parameter ψ_s can be viewed as pure consumption tax that does not affect the timing of observations nor the amount of the investment portfolio that is sold on each observation date. However, an increase in ψ_s reduces the amount of liquid assets obtained from any given sale of assets from the investment portfolio and similarly reduces consumption.

Proposition 8 provides an equation that characterizes τ^*, the optimal length of time between consecutive observation dates once the consumer has reached Region I. Table 1 shows the value of τ^* for various values of the parameters. In the baseline calculation, the observation cost is $\theta = 0.0001$, the coefficient of relative risk aversion is $\alpha = 4$, the rate of time preference is $\rho = 0.01$ per year, the rate of return on the liquid asset in the transactions account is $r_L = 0.01$ per year, the rate of return on the riskless bond in the investment portfolio is $r_f = 0.02$ per year, and the rate of return on risky equity has a mean of $\mu = 0.06$ per year and a standard deviation of $\sigma = 0.16$ per year. As shown in the first row of Table 1, when the parameter values take on their baseline values, the optimal value of τ is 0.643 years, or almost 8 months. In each of the subsequent rows of the table, we change one parameter value at a time, keeping the remaining 6 parameters equal to their baseline values. In the second row, we increase θ to 0.001 from its baseline value of 0.0001, and, as a result, the optimal value of τ increases to slightly more than 2 years. That is, a 10-fold increase in θ leads to about a 3-fold increase in τ^*. We will explore this relationship between the observation cost and τ^* further in the next subsection. For the remaining rows in the table, the optimal value of τ is generally between 6 months and 8 months.

5.1 Quadratic Approximation

Table 1 illustrates that even with a tiny observation cost, the optimal interval of time between consecutive observations, τ^*, can be substantial. We will examine the behavior of τ^* for tiny values of θ using a Taylor expansion of the optimality condition in Proposition 8 around

	τ^*
Baseline	0.643
$\theta = 0.001$	2.065
$\rho = 0.02$	0.616
$\alpha = 1.5$	0.568
$r_L = 0$	0.529
$r_f = 0.03$	0.520
$\mu = 0.07$	0.548
$\sigma = 0.20$	0.688

TABLE 1: Optimal τ. Baseline parameter values: $\theta = 0.0001$, $\alpha = 4$, $\rho = 0.01$, $r_L = 0.01$, $r_f = 0.02$, $\mu = 0.06$, and $\sigma = 0.16$.

$\theta = 0$ and $\tau = 0$. The first-order term in τ vanishes at $\tau = 0$, so we use a second-order approximation.

Proposition 9 *If θ is sufficiently small, then τ^* is approximately proportional to $\sqrt{\theta}$, that is $\lim_{\theta \to 0} \frac{\tau^*}{\sqrt{\theta}} \to \kappa$, for an appropriate constant κ.*

Proposition 9 implies that for small θ, $\frac{\tau^*}{\theta}$ is proportional to $\sqrt{\frac{1}{\theta}}$, which becomes arbitrarily large as θ approaches zero. Thus, even though $\tau^* = 0$ when $\theta = 0$, τ^* grows more than proportionally with θ for small positive θ. In words, even when the observation cost is tiny, the optimal interval of time between consecutive observation dates can still be substantial.

As an illustration of fact that τ^* is proportional to $\sqrt{\theta}$, note in Table 1 that when θ increases by a factor of 10 (from the baseline value of 0.0001 to a new value of 0.001), τ^* increases by a factor of 3.21, which is very close to $\sqrt{10}$.

If the consumer is not very risk averse, more precisely, if $\alpha \leq \frac{\mu - r_f}{\sigma^2}$, then we can obtain a simple expression for the proportionality factor κ. Lemma 6 implies that if $\alpha \leq \frac{\mu - r_f}{\sigma^2}$, then in the long run the optimal value of $\phi_j = 1$. In this case, $R_{j+1}(\tau_j) = \frac{P_{t_{j+1}}}{P_{t_j}}$, which implies that $R_{j+1}(\tau_j)$ is lognormal. In the Appendix, we use the lognormality of $R_{j+1}(\tau_j)$, which simplifies the calculation of the derivatives in the Taylor expansion, to prove the following corollary.

Corollary 9 *If $\alpha < \frac{\mu - r^f}{\sigma^2}$, then $\kappa = \sqrt{\frac{2}{\chi\left(\mu - \alpha\frac{\sigma^2}{2} - r_L\right)}}$, where*[10] $\chi \equiv \frac{\rho - (1-\alpha)\left(\mu - \alpha\frac{\sigma^2}{2}\right)}{\alpha} > 0$.

[10] We assume that $\chi > 0$ so that $J(\tau) < 1$, and the value function is bounded, in a neighborhood of $\tau = 0$. The proof of this corollary shows that when $\phi_j = 1$, $K(\tau) = e^{-\alpha\chi\tau}$. Therefore, the definition of $J(\tau)$ in

We can use this corollary to illustrate the accuracy of the quadratic approximation. In the fourth row of Table 1, the coefficient of relative risk aversion is $\alpha = 1.5$, which is smaller than $\frac{\mu - r^f}{\sigma^2} = \frac{0.06-0.02}{0.16^2} = 1.5625$. Therefore, the condition in the corollary is satisfied. Corollary 9 implies that $\kappa = 56.604$, so that Proposition 9 implies that when $\theta = 0.0001$, $\tau^* \cong \kappa\sqrt{\theta} = (56.604)(0.01) = 0.56604$, which is very close to the value of τ^* (0.568) reported in the fourth row of Table 1.

6 Conclusion

Rules governing infrequent adjustment are typically categorized as time-dependent or state-dependent. Time-dependent rules depend only on calendar time and can optimally result from costs of gathering and processing information. State-dependent rules depend on the value of some state variable, typically reaching some trigger threshold, and can be the optimal response to a transactions cost. Our model shows that these behaviors need not be mutually exclusive, since a consumer may face both costly information and costly transactions. In the general case, the consumer chooses a future date to gather information and reoptimize, but that future date may be state-dependent. Moreover, conditional on the information observed at that future date, the agent's action (or lack thereof) may also be state-dependent. The general model thus retains elements of both state- and time-dependent rules.

In the long run, however, the model converges to a rule that is purely time-dependent. Once the agent arrives at an observation date with a completely depleted transactions account, he will optimally choose to arrive at all subsequent observation dates with zero liquid assets in the transactions account; he would never hold funds in the transactions account in excess of what is needed to finance consumption, and would never return funds to the investment portfolio. In our model, this behavior results from the facts that (a) the consumer can save on costs by synchronizing observation and transactions dates and (b) the consumer would prefer to hold as little as possible of his wealth in the liquid asset because the return on the transactions account is dominated by the return on the investment account.

A version of the consumption Euler equation also holds from one observation date to the next in the long run, if agents are sufficiently risk averse. Therefore, a discrete-time version of the Euler equation would hold for an individual consumer *if* the econometrician's observation

equation (56) implies that $\ln J(\tau) = (1-\alpha)\ln(1-\theta) - \alpha\chi\tau$, so that in a neigborhood of $(\theta, \tau) = (0, 0)$ Proposition 9 implies $\ln J(\tau) \cong (1-\alpha)\ln(1-\theta) - \alpha\chi\kappa\sqrt{\theta} = \left[(1-\alpha)\frac{\ln(1-\theta)}{\sqrt{\theta}} - \alpha\chi\kappa\right]\sqrt{\theta} < 0$ for small θ for $\kappa > 0$ and $\chi > 0$.

dates coincide with the consumer's observation dates. However, for aggregate data, unless consumers are perfectly synchronized, the Euler equation would not hold, since consumption data at any point would include both inattentive consumers and those who observe and reoptimize their portfolios. While they do not test exactly our model, Jagannathan and Wang (2007) show empirically that the Euler equation holds more closely in periods, such as tax dates, when agents are more likely to observe and optimize their consumption and portfolio holdings.

Despite the inexorable tendency toward pure time-dependence in our model, this behavior might not arise if we extend the model to include the arrival of labor income in the transactions account or the observation of attention-grabbing events that occur when the agent is not at an observation date.[11] With these extensions, the consumer might sometimes observe the value of his portfolio when he has a positive holding of liquid assets, thus opening up the possibility of a hybrid of time-dependent and state-dependent behavior. These extensions would allow us to investigate whether the cost savings that result from synchronizing observation and transaction dates would imply that the consumer's long-run behavior is well approximated by a time-dependent rule. The properties of these hybrid state- and time-dependent rules may shed light on not only consumption and portfolio choice, as we have explored here, but also on other settings with infrequent adjustment, such as investment and price-setting by firms.

[11] Recent work by Yuan (2007) has documented that investors appear to react to news that the stock market has reached a new peak.

A Appendix

Optimal consumption between observation dates: The optimal values of consumption during the interval of time from t_j^+ to t_{j+1} satisfy the condition that the product of the intertemporal marginal rate of substitution between times t_j^+ and s, $\left(\frac{c_s}{c_{t_j^+}}\right)^{-\alpha} e^{-\rho(s-t_j)}$, and the gross rate of return between those times, $e^{r_L(s-t_j)}$, equals one, so that

$$c_s = e^{-\frac{\rho - r_L}{\alpha}(s-t_j)} c_{t_j^+}, \text{ for } t_j^+ \leq s \leq t_{j+1}. \tag{A.1}$$

Substituting c_s from equation (A.1) into equation (10) in the text yields

$$C(t_j, \tau_j) = h(\tau_j) c_{t_j^+}, \tag{A.2}$$

where $h(\tau_j)$ is defined in equation (14) in the text. Equations (A.1) and (A.2) imply that

$$c_s = [h(\tau_j)]^{-1} e^{-\frac{\rho - r_L}{\alpha}(s-t_j)} C(t_j, \tau_j), \qquad \text{for } t_j^+ \leq s \leq t_{j+1}. \tag{A.3}$$

Substituting equation (A.3) into equation (11), and using the definition of $h(\tau_j)$ in equation (14) yields $U(C(t_j, \tau_j)) = \frac{1}{1-\alpha}[h(\tau_j)]^{\alpha}[C(t_j, \tau_j)]^{1-\alpha}$, which, along with equation (A.2), implies that $U'(C(t_j, \tau_j)) = c_{t_j^+}^{-\alpha}$.

Proof of Proposition 1. The optimal value of $S_{t_j^+}$ is homogeneous of degree one in X_{t_j} and S_{t_j} so that the optimal value can be written as $S_{t_j^+} = X_{t_j} g\left(\frac{S_{t_j}}{X_{t_j}}\right)$. As a first step toward proving Proposition 1, we state and prove the following two Lemmas.

Lemma 7 *Lemma* $g(0) \geq 0$.

Proof of Lemma 7. (by contradiction). Suppose that $g(0) < 0$. Therefore, if $S_{t_j} = 0$ on an observation date, the consumer sells (short) assets from the investment portfolio to acquire additional liquid assets in the transactions account. However, Lemma 3 implies that if the consumer sells assets from the investment portfolio at time t_j^+, then $C(t_j, \tau_j) = X_{t_j^+}$, so that at the next observation date, t_{j+1}, the consumer will have $X_{t_{j+1}} = 0$ and $S_{t_{j+1}} = (1-\theta) R_{j+1}(\tau_j) S_{t_j^+}$ where $S_{t_j^+} < 0$. Thus, there is a positive probability that the consumer's total wealth will be negative at time t_{j+1}, which is clearly not optimal because $u'(0) = \infty$ along with the standard transversality condition. ■

Lemma 8 *Lemma:* $g(0) \neq 0$.

Proof of Lemma 8. Suppose that $g(0) = 0$. Thus, if $\left(X_{t_j}, S_{t_j}\right) = \left(X_{t_j}, 0\right)$, then $S_{t_j^+} = 0$, which implies that when the consumer arrives at the next observation date the value of the investment portfolio, $S_{t_{j+1}}$, equals zero. By induction, $S_t = 0$ at all future dates. There is no need for the consumer to incur any future observation costs or transaction costs. Therefore, the consumer will optimally consume over the infinite future out of the wealth in the transactions account. For notational simplicity, suppose $t_j = 0$. The optimal path of consumption satisfies the Euler condition $c_t = e^{-\frac{\rho - r_L}{\alpha}t} c_0$ and the budget constraint $X_0 = \int_0^\infty c_t e^{-r_L t} dt$; the transactions account evolves according to $\frac{dX_t}{dt} = r_L X_t - c_t$. Solving this system of equations implies that $X_t = X_0 \exp\left(\frac{r_L - \rho}{\alpha} t\right)$ and $c_t = \omega X_t = \omega X_0 \exp\left(\frac{r_L - \rho}{\alpha} t\right)$ where $\omega \equiv \frac{\rho - (1-\alpha) r_L}{\alpha} > 0$.

Now consider the following alternative plan: The consumer uses $(1 + \psi_b)\gamma X_0$ dollars of the transactions account at time 0^+ to purchase γX_0 dollars of riskless bonds in the investment portfolio and maintains the same path of consumption as above until date T when the transactions account reaches a zero balance. At date T, the consumer sells the entire investment portfolio and transfers the proceeds, net of transactions costs, to the transactions account. We will show that for sufficiently small $\gamma > 0$ this policy will lead to a higher value of the transaction account balance at time T^+ than the original policy of leaving all of the assets in the transactions account at time 0^+.

The date T at which the transactions account reaches zero is such the present value of consumption from time 0^+ to time T equals value of the transactions account at time 0, $[1 - (1 + \psi_b)\gamma] X_0$. Thus, T satisfies $[1 - (1 + \psi_b)\gamma] X_0 = \int_0^T \omega X_0 \exp\left(\frac{r_L - \rho}{\alpha} t\right) \exp(-r_L t)\, dt = \int_0^T \omega X_0 \exp(-\omega t)\, dt$, which implies that T satisfies $(1 + \psi_b)\gamma = \exp(-\omega T)$. At date T the investment portfolio will be worth $S_T = \gamma X_0 \exp(r_f T)$. The consumer can pay the observation cost $\theta(X_T + S_T) = \theta S_T$ and the transactions cost $\psi_s S_T$ to acquire liquid assets in the amount of $(1 - \psi_s - \theta) S_T = (1 - \psi_s - \theta)\gamma X_0 \exp(r_f T)$ in the transactions account at time T^+. Define $D \equiv (1 - \psi_s - \theta)\gamma X_0 \times \exp(r_f T) - X_0 \exp\left(\frac{r_L - \rho}{\alpha} T\right)$, where the first term is the transactions balance at time T^+ under the alternate plan and the second term is the transactions balance at time T^+ under the original plan. If $D > 0$, then the original plan is not optimal. D can be rewritten as $D = [(1 - \psi_s - \theta)\gamma \exp[(r_f - r_L) T] \; - \exp(-\omega T)] \times \exp(r_L T) X_0$. Now use the fact that $(1 + \psi_b)\gamma = \exp(-\omega T)$ to obtain $D = \left[\exp[(r_f - r_L) T] - \frac{1+\psi_b}{1-\psi_s-\theta}\right] \times \gamma(1 - \psi_s - \theta)\exp(r_L T) X_0$, which can be rewritten as $D = \left[((1 + \psi_b)\gamma)^{-\frac{r_f - r_L}{\omega}} - \frac{1+\psi_b}{1-\psi_s-\theta}\right] \times \gamma(1 - \psi_s - \theta)\exp(r_L T) X_0$. Note that $\gamma(1 - \psi_s - \theta)\exp(r_L T) X_0 > 0$. Since $r_f - r_L > 0$ and $\omega > 0$, the term in square brackets is positive for sufficiently small $\gamma > 0$, which implies $D > 0$. Thus, the policy $g(0) = 0$ is not optimal. ■

Proof of Proposition 1 continued: To prove that $\pi_1 > 0$, suppose that $x = 0$ on an observation date. In order to avoid zero consumption, the consumer must choose $\Delta S^{\text{sell}} + \Delta S^{\text{buy}} < 0$. The theorem of the maximum (Stokey and Lucas (1989), p. 62) implies that there is small $\underline{x} > 0$ such that the optimal value of $\Delta S^{\text{sell}} + \Delta S^{\text{buy}}$ is also strictly negative for all x in $[0, \underline{x}]$. Therefore, $\widetilde{v}(x) \neq v(x)$ for all x in $[0, \underline{x}]$, which implies $\pi_1 \geq \underline{x} > 0$.

By definition, $\pi_2 \geq \pi_1$. To prove that π_2 is finite, suppose that $S = 0$ on an observation date, note from Lemmas 7 and 8 that the optimal value of $\Delta S^{\text{sell}} + \Delta S^{\text{buy}} > 0$. Since $S = 0$, we have $x^{-1} = 0$. The theorem of the maximum implies that there is a finite $\overline{x} > 0$ such that the optimal value of $\Delta S^{\text{sell}} + \Delta S^{\text{buy}}$ is also strictly positive for all non-negative $x^{-1} \leq \overline{x}^{-1}$, i.e., for all $x \geq \overline{x}$. Therefore, $\widetilde{v}(x) \neq v(x)$ for all $x \geq \overline{x}$, which implies $\pi_2 \leq \overline{x} < \infty$. Thus we have proved statement 1.

The definition of π_2 implies that for any $x > \pi_2$, $\widetilde{v}(x) \neq v(x)$, so $\Delta S^{\text{sell}} + \Delta S^{\text{buy}} \neq 0$. We have already shown that for $x > \overline{x}$ that $\Delta S^{\text{sell}} + \Delta S^{\text{buy}} > 0$. To show that $\Delta S^{\text{sell}} + \Delta S^{\text{buy}} > 0$ for all $x > \pi_2$, suppose not. That is, suppose that for some $x > \pi_2$, $\Delta S^{\text{sell}} + \Delta S^{\text{buy}} \leq 0$. If $\Delta S^{\text{sell}} + \Delta S^{\text{buy}} = 0$, then $\widetilde{v}(x) = v(x)$, which contradicts the supposition that $x > \pi_2$. If $\Delta S^{\text{sell}} + \Delta S^{\text{buy}} < 0$, then since the optimal value of $\Delta S^{\text{sell}} + \Delta S^{\text{buy}}$ is continuous in x, there is some $x^* > x$ for which $\Delta S^{\text{sell}} + \Delta S^{\text{buy}} = 0$ and $\widetilde{v}(x) = v(x)$, which contradicts the supposition that $x > \pi_2$. Therefore, $\Delta S^{\text{sell}} + \Delta S^{\text{buy}} > 0$. Since $\Delta S^{\text{sell}} \leq 0$ by definition, $\Delta S^{\text{buy}} > 0$, which proves statement 2.

Statement 2 implies that if $x > \pi_2$, the consumer will use liquid assets in the transactions account to buy assets in the investment portfolio until the new value of the transactions account, $X_{new} = X - (1 + \psi_b)\Delta S^{\text{buy}}$, is π_2 times as large as new value of the investment portfolio, $S_{new} = S + \Delta S^{\text{buy}}$. Therefore, $\Delta S^{\text{buy}} = \frac{X - \pi_2 S}{1+\psi_b+\pi_2}$, so $X_{new} = \pi_2 \frac{X+(1+\psi_b)S}{1+\psi_b+\pi_2}$ and $S_{new} = \frac{X+(1+\psi_b)S}{1+\psi_b+\pi_2}$. Since the consumer can move instantaneously from (X, S) to (X_{new}, S_{new}), $V(X, S) \geq V(X_{new}, S_{new})$, and since the consumer chooses to move from (X, S) to (X_{new}, S_{new}), $V(X, S) = V(X_{new}, S_{new}) = \frac{1}{1-\alpha} S_{new}^{1-\alpha} v(\pi_2) = \frac{1}{1-\alpha}\left[\frac{X+(1+\psi_b)S}{1+\psi_b+\pi_2}\right]^{1-\alpha} v(\pi_2)$.

The definition of π_1 implies that for any non-negative $x < \pi_1$, $\widetilde{v}(x) \neq v(x)$, so $\Delta S^{\text{sell}} + \Delta S^{\text{buy}} \neq 0$. We have already shown that for non-negative x in a neighborhood of $x = 0$ that $\Delta S^{\text{sell}} + \Delta S^{\text{buy}} < 0$. To show that $\Delta S^{\text{sell}} + \Delta S^{\text{buy}} < 0$ for all non-negative $x < \pi_1$, suppose not. That is, suppose that for some non-negative $x < \pi_1$, $\Delta S^{\text{sell}} + \Delta S^{\text{buy}} \geq 0$. If $\Delta S^{\text{sell}} + \Delta S^{\text{buy}} = 0$, then $\widetilde{v}(x) = v(x)$, which contradicts the supposition that $x < \pi_1$. If $\Delta S^{\text{sell}} + \Delta S^{\text{buy}} > 0$, then since the optimal value of $\Delta S^{\text{sell}} + \Delta S^{\text{buy}}$ is continuous in x, there is some non-negative $x^* < x$ for which $\Delta S^{\text{sell}} + \Delta S^{\text{buy}} = 0$ and $\widetilde{v}(x) = v(x)$, which contradicts the supposition that $x < \pi_1$. Therefore, $\Delta S^{\text{sell}} + \Delta S^{\text{buy}} < 0$. Since $\Delta S^{\text{buy}} \geq 0$ by definition, $\Delta S^{\text{sell}} < 0$, which proves statement 4.

Statement 4 implies that if $x < \pi_1$, the consumer will increase the amount of liquid assets in the transactions account by selling assets in the investment portfolio until the new value of the transactions account, $X_{new} = X - (1 - \psi_s)\Delta S^{\text{sell}}$, is π_1 times as large as new value of the investment portfolio, $S_{new} = S + \Delta S^{\text{sell}}$. Therefore, $\Delta S^{\text{sell}} = \frac{X - \pi_1 S}{1-\psi_s+\pi_1}$, so $X_{new} = \pi_1 \frac{X+(1-\psi_s)S}{1-\psi_s+\pi_1}$ and $S_{new} = \frac{X+(1-\psi_s)S}{1-\psi_s+\pi_1}$. Since the consumer can move instantaneously from (X, S) to (X_{new}, S_{new}), $V(X, S) \geq V(X_{new}, S_{new})$, and since the consumer chooses to move from (X, S) to (X_{new}, S_{new}),

$V(X,S) = V(X_{new}, S_{new}) = \frac{1}{1-\alpha} S_{new}^{1-\alpha} v(\pi_1) = \frac{1}{1-\alpha}\left[\frac{X+(1-\psi_s)S}{1-\psi_s+\pi_1}\right]^{1-\alpha} v(\pi_1)$. ■

Proof of Lemma 1. Subtract equation (38) from equation (39) and use equation (36) to obtain $\lambda_j^{\text{sell}} + \lambda_j^{\text{buy}} = (\psi_b + \psi_s) U'(C(t_j, \tau_j))$. Thus, that as long as at least one of the transactions cost parameters ψ_s and ψ_b is positive, the Lagrange multipliers λ_j^{sell} and λ_j^{buy} must sum to a positive number, so that at least one of the constraints $\Delta S_j^{\text{sell}} \leq 0$ or $\Delta S_j^{\text{buy}} \geq 0$ must bind. ■

Proof of Lemma 2. Multiply and divide the first term on the left hand side of equation (38) by $1-\psi_s$, use equation (40) to replace $E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)(1-\theta) R_{j+1}(\tau_j)\right\}$ by $E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)(1-\theta) R^f(\tau_j)\right\} + \frac{\xi_j}{\beta^{\tau_j} S_{t_j}^+}$ and use $V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right) \geq (1-\psi_s) V_X\left(X_{t_{j+1}}, S_{t_{j+1}}\right)$ from equation (30) to obtain $(1+\psi_b)\eta_j \geq \beta^{\tau_j} E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)\right\} \times (1-\theta)\left[R^f(\tau_j) - \frac{1+\psi_b}{1-\psi_s} R^L(\tau_j)\right] + \frac{\xi_j}{S_{t_j}^+} + \lambda_j^{\text{buy}}$. Now use the definition of $G(\tau_j)$ in equation (41) to rewrite this equation as $\eta_j \geq \frac{1}{1-\psi_s}\beta^{\tau_j} E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)\right\} G(\tau_j) + \frac{\xi_j}{(1+\psi_b) S_{t_j}^+} + \frac{\lambda_j^{\text{buy}}}{1+\psi_b}$. Since $\psi_s < 1$, $\psi_b \geq 0$, $\beta^{\tau_j} E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)\right\} > 0$, $\frac{\xi_j}{(1+\psi_b) S_{t_j}^+} \geq 0$ and $\lambda_j^{\text{buy}} \geq 0$, the assumption that $G(\tau_j) > 0$ implies $\eta_j > 0$, which implies $C = X_{t_j}^+$, and hence $X_{t_{j+1}} = 0$. ■

Proof of Lemma 3. Since $\Delta S_j^{\text{sell}} < 0$, $\lambda_j^{\text{sell}} = 0$, so equation (39) implies $(1-\psi_s)\eta_j = \beta^{\tau_j} E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)(1-\theta) R_{j+1}(\tau_j)\right\} - \beta^{\tau_j} E_{t_j}\left\{V_X\left(X_{t_{j+1}}, S_{t_{j+1}}\right)\right\}(1-\psi_s)(1-\theta) R^L(\tau_j)$. Use equation (40) to replace $E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)(1-\theta) R_{j+1}(\tau_j)\right\}$ by $E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)\right\}(1-\theta) R^f(\tau_j) + \frac{\xi_j}{\beta^{\tau_j} S_{t_j}^+}$ and use $V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right) \geq (1-\psi_s) V_X\left(X_{t_{j+1}}, S_{t_{j+1}}\right)$ from equation (30) to obtain

$$\eta_j \geq \beta^{\tau_j} E_{t_j}\left\{V_X\left(X_{t_{j+1}}, S_{t_{j+1}}\right)\right\}(1-\theta)\left[R^f(\tau_j) - R^L(\tau_j)\right] + \frac{1}{1-\psi_s}\frac{\xi_j}{S_{t_j}^+}.$$

Since $\beta^{\tau_j} E_{t_j}\left\{V_X\left(X_{t_{j+1}}, S_{t_{j+1}}\right)\right\} > 0$, $R^f(\tau_j) > R^L(\tau_j)$, $1-\psi_s > 0$, $1-\theta > 0$, and $\frac{\xi_j}{S_{t_j}^+} \geq 0$, we have $\eta_j > 0$. Since η_j is the Lagrange multiplier on the constraint $X_{t_j^+} - C(t_j, \tau_j) \geq 0$, $\eta_j > 0$ implies $C(t_j, \tau_j) = X_{t_j^+}$, and hence $X_{t_{j+1}} = 0$. ■

Proof of Lemma 4. We will use proof by contradiction. Suppose that $C(t_j, \tau_j) = X_{t_j^+}$, so $X_{t_{j+1}} = 0$, which implies $x_{t_{j+1}} = 0 < \pi_1$. Since Statement 2 in Corollary 1 implies that $V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right) = (1-\psi_s) V_X\left(X_{t_{j+1}}, S_{t_{j+1}}\right)$, substitute $\frac{1}{1-\psi_s} V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)$ for $V_X\left(X_{t_{j+1}}, S_{t_{j+1}}\right)$ in equation (38) to obtain $\eta_j = \frac{1}{1-\psi_s}\beta^{\tau_j} E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)(1-\theta)\left[\frac{1-\psi_S}{1+\psi_b} R_{j+1}(\tau_j) - R^L(\tau_j)\right]\right\} + \frac{\lambda_j^{\text{buy}}}{1+\psi_b}$. Since $x_{t_j} > \pi_2$, the consumer uses some of the liquid asset in the transactions account to buy some assets for the investment portfolio. Therefore, $\Delta S_j^{\text{buy}} > 0$ which implies $\lambda_j^{\text{buy}} = 0$. Since, $1+\psi_b > 0$, $1-\psi_s > 0$, $\lambda_j^{\text{buy}} = 0$ and, by assumption, $E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)(1-\theta)\left[\frac{1-\psi_S}{1+\psi_b} R_{j+1}(\tau_j) - R^L(\tau_j)\right]\right\} < 0$, we have $\eta_j < 0$, which contradicts the fact that the Lagrange multiplier η_j is non-negative. ■

Proof of Lemma 5. If $\phi_j \leq 1$ is not binding, then $\xi_j = 0$, so equation (40) implies that $E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right) R_{j+1}(\tau_j)\right\} = E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right) R^f(\tau_j)\right\}$. The rest of the proof follows from the definition of $G(\tau_j)$ and $E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)\right\} > 0$. ■

Proof of Proposition 3. First, apply the envelope theorem to equation (35) to obtain $V_S\left(X_{t_j}, S_{t_j}\right) = e^{-\rho\tau_j} E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)(1-\theta) R_{j+1}(\tau_j)\right\}$. Lemma 4 along with our assumptions that $x_{t_j} \geq \pi_2$ and $E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)(1-\theta)\left[\frac{1-\psi_s}{1+\psi_b} R_{j+1}(\tau_j) - R^L(\tau_j)\right]\right\} < 0$ imply $C(t_j, \tau_j) < X_{t_j^+}$, which means that $\eta_j = 0$. Accordingly, the envelope theorem implies $V_X\left(X_{t_j}, S_{t_j}\right) = e^{-\rho\tau_j} E_{t_j}\left\{V_X\left(X_{t_{j+1}}, S_{t_{j+1}}\right)\right\}(1-\theta) R^L(\tau_j)$. Now use the definition of $m\left(x_{t_j}\right)$ to obtain

$$m\left(x_{t_j}\right) = \frac{V_S\left(X_{t_j}, S_{t_j}\right)}{V_X\left(X_{t_j}, S_{t_j}\right)} = \frac{E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right) R_{j+1}(\tau_j)\right\}}{E_{t_j}\left\{V_X\left(X_{t_{j+1}}, S_{t_{j+1}}\right)\right\} R^L(\tau_j)}. \tag{A.4}$$

Since $x_{t_j} \geq \pi_2$, we have $m\left(x_{t_j}\right) = 1 + \psi_b$, which allows us to rewrite equation (A.4) as

$$(1+\psi_b) E_{t_j}\left\{V_X\left(X_{t_{j+1}}, S_{t_{j+1}}\right)\right\} R^L(\tau_j) = E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right) R_{j+1}(\tau_j)\right\}. \tag{A.5}$$

Now use the fact that $V_X\left(X_{t_{j+1}}, S_{t_{j+1}}\right) = V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)/m\left(x_{t_{j+1}}\right)$ to rewrite equation (A.5) as

$$E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)\left[R_{j+1}(\tau_j) - \frac{1+\psi_b}{m\left(x_{t_{j+1}}\right)} R^L(\tau_j)\right]\right\} = 0. \tag{A.6}$$

Multiply both sides of equation (A.6) by $\frac{1-\psi_s}{1+\psi_b}$ to obtain

$$E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)\left[\frac{1-\psi_s}{1+\psi_b} R_{j+1}(\tau_j) - \frac{1-\psi_s}{m\left(x_{t_{j+1}}\right)} R^L(\tau_j)\right]\right\} = 0. \tag{A.7}$$

From this point on, the proof proceeds by contradiction. Suppose – counterfactually – that $\Pr\left\{x_{t_{j+1}} \leq \pi_1\right\} = 1$, which implies that $m\left(x_{t_{j+1}}\right) = 1 - \psi_s$. Therefore, equation (A.7) implies $E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)\left[\frac{1-\psi_s}{1+\psi_b} R_{j+1}(\tau_j) - R^L(\tau_j)\right]\right\} = 0$, which contradicts the assumption in the statement of the Proposition. ■

Proof of Proposition 5. Equation (3) implies $S_{t_{j+1}} = (1-\theta)\left[\phi_j \frac{P_{t_{j+1}}}{P_{t_j}} + \left(1-\phi_j\right) R^f(\tau_j)\right] S_{t_j^+}$. Since $\mu > r_f$, the optimal value of ϕ_j is nonzero.[12] Also, $1-\theta > 0$ and $S_{t_j^+} > 0$. $\frac{P_{t_{j+1}}}{P_{t_j}} > 0$ is

[12]To see that $\phi_j \neq 0$, suppose otherwise. Since $\phi_j = 0$, both $S_{t_{j+1}}$ and $X_{t_{j+1}}$ are known at time t_j so we have

$$E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)\left[\frac{P_{t_{j+1}}}{P_{t_j}} - \left(R^f\right)^{\tau_j}\right]\right\} = V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right) E_{t_j}\left\{\frac{P_{t_{j+1}}}{P_{t_j}} - \left(R^f\right)^{\tau_j}\right\} > 0,$$

where the inequality follows from $\mu > r_f$. If $\phi_j = 0$, then $\xi_j = 0$ so equation (37) implies $V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right) E_{t_j}\left\{\frac{P_{t_{j+1}}}{P_{t_j}} - \left(R^f\right)^{\tau_j}\right\} = 0$, which contradicts the inequality above.

conditionally lognormal with support $(0,\infty)$, so $(1-\theta)\,\phi_j \frac{P_{t_{j+1}}}{P_{t_j}} S_{t_j^+}$ is also conditionally lognormal with support $(0,\infty)$. Conditional on ϕ_j and $S_{t_j^+}$, $\left(1-\phi_j\right)(1-\theta)\, R^f(\tau_j)\, S_{t_j^+}$ is a constant, so that, conditional on S_{t_j}, $S_{t_{j+1}}$ is distributed according to a nontrivial translated lognormal distribution. Therefore, $\delta(x) \equiv \Pr\left\{x_{t_{j+1}} \leq \pi_1 | x_{t_j^+} = x\right\} > 0$ for any x in $[\pi_1, \pi_2]$. Since ϕ_j^* is continuous in x_j, this probability is continuous in x for $x \in [\pi_1, \pi_2]$.[13] Since $[\pi_1, \pi_2]$ is closed and bounded (hence compact), there is a $\delta^* > 0$ such that $\delta(x) \geq \delta^*$ for $\pi_1 \leq x \leq \pi_2$. Suppose that $x_{t_0^+} = x > \pi_1$. We will show that:

$$\Pr(t_n > t_k) \leq (1-\delta^*)^k \text{ for } k = 1, 2, 3... \tag{A.8}$$

To show (A.8) it is easiest to employ an induction argument. First, note that (A.8) holds for $k=1$, since $\Pr\{t_n > t_1\} = \Pr\left\{x_{t_1} > \pi_1 | x_{t_0^+} = x\right\} < 1-\delta^*$. Second, as we show next, if (A.8) holds for k, then it must also hold for $k+1$. To see this, observe that

$$\Pr(t_n > t_{k+1}) = \Pr(t_n > t_{k+1} \text{ and } t_n > t_k), \tag{A.9}$$

because t_n can only be larger than t_{k+1} if it is also larger than t_k. Using (A.9) we have:

$$\begin{aligned} \Pr(t_n > t_{k+1}) &= \frac{\Pr(t_n > t_{k+1} \text{ and } t_n > t_k)}{\Pr(t_n > t_k)} \Pr(t_n > t_k) \\ &= \Pr\left(x_{t_{k+1}} > \pi_1 | x_{t_k^+} > \pi_1\right) \Pr(t_n > t_k) \\ &\leq (1-\delta^*)(1-\delta^*)^k = (1-\delta^*)^{k+1} \end{aligned} \tag{A.10}$$

where the second line follows from Bayes Rule, while the third line follows from the construction of δ^* and the inductive assumption (A.8). In light of (A.10), equation (A.8) follows by induction. Letting $k \to \infty$ in (A.8) we obtain $\Pr\{t_n < \infty\} = 1$. Finally

$$\begin{aligned} E(t_n) &= \sum_{k=1}^{\infty} k \Pr(t_n = t_k) \\ &= 1 + \sum_{k=1}^{\infty} \Pr(t_n > t_k) \\ &\leq 1 + \sum_{k=1}^{\infty} (1-\delta^*)^k = \frac{1}{\delta^*} < \infty. \end{aligned}$$

The second line follows by applying summation by parts[14] and the last line by using (A.8). ■

[13] The theorem of the maximum (See e.g. Stokey and Lucas (1989)) implies that the portfolio share ϕ_j^* is continuous in x_{t_j}. This implies that $\delta\left(x_{t_j}\right)$ is continuous in x_{t_j}.

[14] $\sum_{k=1}^{\infty} k \Pr(t_n = t_k) = \sum_{k=1}^{\infty} \sum_{j=1}^{k} \Pr(t_n = t_k) = \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} \Pr(t_n = t_k) = \sum_{j=1}^{\infty} \left(\Pr(t_n = t_j) + \sum_{k=j+1}^{\infty} \Pr(t_n = t_k)\right) = \sum_{j=1}^{\infty} (\Pr(t_n = t_j) + \Pr(t_n > t_j)) = 1 + \sum_{j=1}^{\infty} \Pr(t_n > t_j)$.

Proof of Lemma 6. If $x_{t_j} \leq \pi_1$, which implies $x_{t_{j+1}} = 0$, the optimal value of $\phi_j \leq 1$ maximizes $\varphi\left(\phi_j;\alpha\right) \equiv \frac{1}{1-\alpha} E_{t_j}\left\{\left((1-\theta)\left[\phi_j \frac{P_{t_j+\tau_j}}{P_{t_j}} + \left(1-\phi_j\right) e^{r_f \tau_j}\right]\right)^{1-\alpha}\right\}$. Define α^* such that $\arg\max_{\phi_j} \varphi\left(\phi_j;\alpha^*\right) = 1$ and note that $\varphi'\left(1;\alpha^*\right) = 0$.

Differentiating the definition of $\varphi\left(\phi_j;\alpha\right)$ with respect to ϕ_j and setting $\phi_j = 1$ yields

$$\varphi'\left(1;\alpha\right) = (1-\theta)^{1-\alpha}\left(E_{t_j}\left\{\left(\frac{P_{t_j+\tau_j}}{P_{t_j}}\right)^{1-\alpha}\right\} - e^{r_f\tau_j} E_{t_j}\left\{\left(\frac{P_{t_j+\tau_j}}{P_{t_j}}\right)^{-\alpha}\right\}\right).$$

Use the fact that $\frac{P_{t_j+\tau_j}}{P_{t_j}}$ is lognormal to obtain

$$\varphi'\left(1;\alpha\right) = (1-\theta)^{1-\alpha}\left(\exp\left[(1-\alpha)\left(\mu - \frac{1}{2}\alpha\sigma^2\right)\tau_j\right] - e^{r_f\tau_j}\exp\left[-\alpha\left(\mu + \frac{1}{2}(-\alpha-1)\sigma^2\right)\tau_j\right]\right).$$

Further rearrangement yields

$$\varphi'\left(1;\alpha\right) = (1-\theta)^{1-\alpha}\exp\left[\left(-\alpha\mu + r_f - \frac{1}{2}\alpha(1-\alpha)\sigma^2\right)\tau_j\right] \times \left[\exp\left(\left(\mu - r_f\right)\tau_j\right) - \exp\left(\alpha\sigma^2\tau_j\right)\right],$$

which implies that

$$\varphi'\left(1;\alpha\right) \lesseqqgtr 0 \text{ as } \alpha \gtreqqless \alpha^* \equiv \left(\mu - r_f\right)/\sigma^2.$$

Differentiate $\varphi\left(\phi_j;\alpha\right)$ twice with respect to ϕ_j to obtain

$$\varphi''\left(\phi_j;\alpha\right) = -\alpha E_{t_j}\left\{(1-\theta)^{-\alpha+1}\left(\phi_j\frac{P_{t_j+\tau_j}}{P_{t_j}} + (1-\phi)e^{r_f\tau_j}\right)^{-\alpha-1}\left(\frac{P_{t_j+\tau_j}}{P_{t_j}} - e^{r_f\tau_j}\right)^2\right\} < 0,$$

which implies that $\varphi\left(\phi_j;\alpha\right)$ is concave. If $\alpha > \alpha^*$, then $\varphi'\left(1;\alpha\right) < 0$, so the concavity of $\varphi\left(\phi_j;\alpha\right)$ implies that the optimal value of ϕ_j is less than one and the Lagrange multiplier on the constraint $\phi_j \leq 1$ is $\xi_j = 0$. If $\alpha \leq \alpha^*$, then $\varphi'\left(1;\alpha\right) \geq 0$, so the concavity of $\varphi\left(\phi_j;\alpha\right)$ implies that the optimal value of ϕ_j equals one. If $\alpha < \alpha^*$, the Lagrange multiplier on the constraint $\phi_j \leq 1$ is $\xi_j > 0$. ■

Proof of Proposition 7. As a first step, we introduce the conditional value function $\widehat{V}\left(X_{t_j}, S_{t_j};\tau\right)$, which is defined for an arbitrary τ as

$$\widehat{V}\left(X_{t_j}, S_{t_j};\tau\right) = \max_{C(t_j,\tau),\Delta S_{t_j}^{\text{sell}},\Delta S_{t_j}^{\text{buy}},\phi_j} \frac{1}{1-\alpha}\left[h(\tau)\right]^{\alpha}\left[C\left(t_j,\tau\right)\right]^{1-\alpha} + \beta^{\tau} E_{t_j}\left\{V\left(X_{t_j+\tau}, S_{t_j+\tau}\right)\right\}. \tag{A.11}$$

Note that $V\left(X_{t_j}, S_{t_j}\right) = \max_{\tau} \widehat{V}\left(X_{t_j}, S_{t_j};\tau\right)$ and define τ^* as the optimal value of τ, so that $V\left(X_{t_j}, S_{t_j}\right) = \widehat{V}\left(X_{t_j}, S_{t_j};\tau^*\right)$. The conditional value function takes τ as given. Define $\widehat{\pi}_1(\tau)$

as the value of $x_{t_j^+}$ that is attained by the solution to the maximization on the right hand side of equation (A.11) when $X_{t_j} = 0$, and observe that $\pi_1 = \widehat{\pi}_1(\tau^*)$. We first use the definition of $J(\tau)$ and recall the definition of $v(x)$ in equation (21) to obtain an expression for $\widehat{\pi}_1(\tau)$.

Lemma 9 $\widehat{\pi}_1(\tau) = \left(\frac{1-\psi_s}{v(0)}\right)^{\frac{1}{\alpha}} h(\tau)\left[J(\tau)\right]^{-\frac{1}{\alpha}}$.

Proof of Lemma 9. Suppose that time t_j is an observation date and that $\left(X_{t_j}, S_{t_j}\right)$ is in Region I. Therefore, the transactions balance at the next observation date is $X_{t_j+\tau} = 0$. Use the first-order conditions with respect to $C(t_j,\tau)$ and ΔS^{sell} in equations (36) and (39), respectively, to obtain $(1-\psi_s)U'(C(t_j,\tau)) = \beta^\tau E_{t_j}\left\{V_S\left(X_{t_j+\tau}, S_{t_j+\tau}\right)(1-\theta)R_{j+1}(\tau)\right\} - \lambda_j^{\text{sell}}$. Rewrite this equation using (i) $C(t_j,\tau) = X_{t_j^+} = \widehat{\pi}_1(\tau)S_{t_j^+}$ so equation (12) implies $U'(C(t_j,\tau)) = [h(\tau)]^\alpha\left[\widehat{\pi}_1(\tau)S_{t_j^+}\right]^{-\alpha}$; (ii) equation (21) which implies that $V_S\left(X_{t_j+\tau}, S_{t_j+\tau}\right) = V_S\left(0, S_{t_j+\tau}\right) = S_{t_j+\tau}^{-\alpha}v(0) = \left((1-\theta)R_{j+1}(\tau)S_{t_j^+}\right)^{-\alpha}v(0)$; and (iii) in Region I, $\Delta S_j^{\text{sell}} > 0$, so $\lambda_j^{\text{sell}} = 0$ to obtain $(1-\psi_s)[h(\tau)]^\alpha[\widehat{\pi}_1(\tau)]^{-\alpha} = \beta^\tau E_{t_j}\left\{[(1-\theta)R_{j+1}(\tau)]^{1-\alpha}\right\}v(0)$. Use the definition of $J(\tau)$ in equation (56) to obtain $(1-\psi_s)[h(\tau)]^\alpha[\widehat{\pi}_1(\tau)]^{-\alpha} = J(\tau)v(0)$, which implies $\widehat{\pi}_1(\tau) = \left(\frac{1-\psi_s}{v(0)}\right)^{\frac{1}{\alpha}} h(\tau)\left[J(\tau)\right]^{-\frac{1}{\alpha}}$. ■

Lemma 9 is not a complete solution for $\widehat{\pi}_1(\tau)$ because it depends on $v(0)$. Nevertheless, it will prove helpful in solving for τ^*, the optimal value of τ, when the transactions account balance is zero on an observation date, and for the value of $\widehat{\pi}_1(\tau^*)$. As a step toward calculating τ^*, the following lemma presents an expression for the conditional value function evaluated at $x = 0$.

Lemma 10 $\widehat{v}(0;\tau) = \left[1 + \frac{\widehat{\pi}_1(\tau)}{1-\psi_s}\right]^\alpha J(\tau)v(0)$.

Proof of Lemma 10. Suppose that time t_j is an observation date, and at time t_j the transactions account has a zero balance, so that $x_{t_j} = 0$. The consumer will set $\Delta S_j^{\text{buy}} = 0$ and will choose ΔS_j^{sell} so that $X_{t_j^+} = \widehat{\pi}_1(\tau)S_{t_j^+}$. Using equations (8) and (9) along with $X_{t_j} = 0$, $\Delta S_j^{\text{buy}} = 0$, and $X_{t_j^+} = \widehat{\pi}_1(\tau)S_{t_j^+}$ implies $S_{t_j^+} = \frac{1-\psi_s}{1-\psi_s+\widehat{\pi}_1(\tau)}S_{t_j}$. Lemma 3 implies that for $x_{t_j} = 0$, $C(t_j,\tau_j) = X_{t_j^+} = \widehat{\pi}_1(\tau)S_{t_j^+}$. The next observation date is $t_j+\tau$, and on this observation date $X_{t_j+\tau} = 0$ and $S_{t_j+\tau} = (1-\theta)R_{j+1}(\tau_j)S_{t_j^+}$. Therefore, the conditional value function in equation (A.11) can be written as $\widehat{V}\left(0, S_{t_j};\tau\right) = \frac{1}{1-\alpha}[h(\tau)]^\alpha\left(\widehat{\pi}_1\frac{1-\psi_s}{1-\psi_s+\widehat{\pi}_1(\tau)}S_{t_j}\right)^{1-\alpha} + \beta^\tau E_{t_j}\left\{V\left(0,(1-\theta)R_{j+1}(\tau)\frac{1-\psi_s}{1-\psi_s+\widehat{\pi}_1(\tau)}S_{t_j}\right)\right\}$. Like the value function $V\left(X_{t_j}, S_{t_j}\right)$, the conditional value function $\widehat{V}\left(X_{t_j}, S_{t_j};\tau\right)$ is homogeneous of degree $1-\alpha$ in X_{t_j} and S_{t_j}, so $\widehat{V}\left(X_{t_j}, S_{t_j};\tau\right) = \frac{1}{1-\alpha}S_{t_j}^{1-\alpha}\widehat{v}\left(x_{t_j}\right)$. Use this equation and equation (21) for $V\left(X_{t_j}, S_{t_j}\right)$ to rewrite the conditional value function as $\widehat{v}(0;\tau) = \left(\frac{1-\psi_s}{1-\psi_s+\widehat{\pi}_1(\tau)}\right)^{1-\alpha}\left[[h(\tau)]^\alpha[\widehat{\pi}_1(\tau)]^{1-\alpha} + \beta^\tau E_{t_j}\left\{[(1-\theta)R_{j+1}(\tau)]^{1-\alpha}\right\}v(0)\right]$. Use the definition of $J(\tau)$ in equation (56) and use Lemma 9 to rewrite the conditional value function as $\widehat{v}(0;\tau) =$

$\left(\frac{1-\psi_s}{1-\psi_s+\widehat{\pi}_1(\tau)}\right)^{1-\alpha}\left[\frac{1}{1-\psi_s}\widehat{\pi}_1(\tau)\,v(0)\,J(\tau)+J(\tau)\,v(0)\right]=\left(\frac{1-\psi_s}{1-\psi_s+\widehat{\pi}_1(\tau)}\right)^{-\alpha}J(\tau)\,v(0)=\left(1+\frac{\widehat{\pi}_1(\tau)}{1-\psi_s}\right)^{\alpha}J(\tau)\,v(0).$
■

Proof of Proposition 7 continued: Recall that $\tau^*=\arg\max_\tau \widehat{v}(0;\tau)$ so $v(0)=\widehat{v}(0;\tau^*)=\left[1+\frac{\widehat{\pi}_1(\tau^*)}{1-\psi_s}\right]^{\alpha}J(\tau^*)\,v(0)$. Therefore, $1=\left[1+\frac{\widehat{\pi}_1(\tau^*)}{1-\psi_s}\right]^{\alpha}J(\tau^*)$, so $1+\frac{\widehat{\pi}_1(\tau^*)}{1-\psi_s}=[J(\tau^*)]^{-\frac{1}{\alpha}}$, which implies $\widehat{\pi}_1(\tau^*)=(1-\psi_s)\left([J(\tau^*)]^{-\frac{1}{\alpha}}-1\right)$. ■

Proof of Proposition 8. To find the optimal value of τ, first differentiate the conditional value function $\widehat{v}(0;\tau)$ in Lemma 10 with respect to τ and set the derivative equal to zero to obtain $\alpha\frac{\widehat{\pi}_1(\tau)}{1-\psi_s}\frac{\widehat{\pi}_1'(\tau)}{\widehat{\pi}_1(\tau)}=-\left(1+\frac{\widehat{\pi}_1(\tau)}{1-\psi_s}\right)\frac{J'(\tau)}{J(\tau)}$. Differentiate the expression for $\widehat{\pi}_1(\tau)$ in Lemma 9 with respect to τ to obtain $\frac{\widehat{\pi}_1'(\tau)}{\widehat{\pi}_1(\tau)}=\frac{h'(\tau)}{h(\tau)}-\frac{1}{\alpha}\frac{J'(\tau)}{J(\tau)}$ and substitute this expression for $\frac{\widehat{\pi}_1'(\tau)}{\widehat{\pi}_1(\tau)}$ into the preceding expression to obtain $\frac{\widehat{\pi}_1(\tau)}{1-\psi_s}\frac{h'(\tau)}{h(\tau)}=-\frac{1}{\alpha}\frac{J'(\tau)}{J(\tau)}$. Evaluate this equation at $\tau=\tau^*$ and use Proposition 7 to obtain $[J(\tau^*)]^{-\frac{1}{\alpha}}-1=-\frac{1}{\alpha}\frac{J'(\tau^*)}{J(\tau^*)}\frac{h(\tau^*)}{h'(\tau^*)}$, which can be rearranged to obtain $1=\left[1-\frac{1}{\alpha}\frac{J'(\tau^*)}{J(\tau^*)}\frac{h(\tau^*)}{h'(\tau^*)}\right][J(\tau^*)]^{\frac{1}{\alpha}}\equiv N(\tau^*)$. ■

Proof of Proposition 9. Write the function $J(\tau)$ defined in equation (56) as $J(\tau)=(1-\theta)^{1-\alpha}K(\tau)$, where $K(\tau)\equiv\beta^\tau E_{t_j}\left\{\left[\phi_j\frac{P_{t_j+\tau}}{P_{t_j}}+\left(1-\phi_j\right)e^{r_f\tau}\right]^{1-\alpha}\right\}$. Since the optimal portfolio share ϕ_j in Region I is independent of the observation cost θ, $K(\tau)$ is invariant to the observation cost. Thus, $J(\tau)$ is the product of one term that depends on the observation cost θ but is independent of τ, and a second term that depends on τ but is independent of the observation cost θ. Therefore, we can write the optimality condition in Proposition 8 as $M(\tau^*)(1-\theta)^{\frac{1-\alpha}{\alpha}}=1$, where $M(\tau)\equiv\left[1-\frac{1}{\alpha}\frac{h(\tau)}{h'(\tau)}\frac{K'(\tau)}{K(\tau)}\right][K(\tau)]^{\frac{1}{\alpha}}$. We will now examine the value of τ^* in a neighborhood of $\theta=0$. The following functions evaluated at $\tau=0$ will be helpful: $h(0)=0$, $h'(0)=1$, $h''(0)=-\omega$, and $K(0)=1$. Therefore, $M(0)=1$, so that when $\theta=0$, $M(0)(1-\theta)^{\frac{1-\alpha}{\alpha}}=1$, and hence $\tau^*=0$ satisfies the optimality condition $M(\tau^*)(1-\theta)^{\frac{1-\alpha}{\alpha}}=1$.

To examine the behavior of the optimality condition in the neighborhood of $\theta=0$ and $\tau=0$, we calculate the Taylor series expansion of $M(\tau)$ around $\tau=0$. First, differentiate the definition $M(\tau)$ with respect to τ to obtain

$$M'(\tau)=\frac{1}{\alpha}\frac{h(\tau)}{h'(\tau)}\left(\frac{h''(\tau)}{h'(\tau)}-\frac{K''(\tau)}{K'(\tau)}+\left(1-\frac{1}{\alpha}\right)\frac{K'(\tau)}{K(\tau)}\right)\frac{K'(\tau)}{K(\tau)}[K(\tau)]^{\frac{1}{\alpha}}. \tag{A.12}$$

Evaluate equation (A.12) at $\tau=0$ using the fact that $h(0)=0$ to obtain $M'(0)=0$. Therefore, the linear term in the Taylor series expansion of $M(\tau)$ around $\tau=0$ is zero, so we move on to the quadratic term in this expansion. Differentiate equation (A.12) with respect to τ, and evaluate the resulting expression at $\tau=0$ using $h(0)=0$, $h'(0)=1$, $h''(0)=-\omega$, and $K(0)=1$ to obtain

$$M''(\tau)=-\frac{1}{\alpha}\left[\omega K'(\tau)+K''(\tau)+\left(\frac{1}{\alpha}-1\right)\left[K'(\tau)\right]^2\right]. \tag{A.13}$$

The optimal value of τ satisfies $M(\tau)(1-\theta)^{\frac{1-\alpha}{\alpha}} = 1$ or equivalently

$$M(\tau) = (1-\theta)^{\frac{\alpha-1}{\alpha}}. \tag{A.14}$$

Expanding each side of (A.14) in a Taylor fashion around $\tau = 0$, $\theta = 0$ and keeping terms up to the second order gives

$$M(0) + M'(0)\tau + \frac{1}{2}M''(0)\tau^2 = 1 + \frac{1-\alpha}{\alpha}\theta + \frac{1}{2\alpha}\left(\frac{1-\alpha}{\alpha}\right)\theta^2. \tag{A.15}$$

Using the facts that $M(0) = 1$, $M'(0) = 0$ and simplifying equation (A.15) implies that $\frac{1}{2}M''(0)\tau^2 = \frac{1-\alpha}{\alpha}\theta + \frac{1}{2\alpha}\left(\frac{1-\alpha}{\alpha}\right)\theta^2$, or equivalently $\frac{1}{2}M''(0)\frac{\tau^2}{\theta} = \frac{1-\alpha}{\alpha} + \frac{1}{2\alpha}\left(\frac{1-\alpha}{\alpha}\right)\theta$. Taking the limit as $\theta \to 0$ leads to $\lim_{\theta\to 0}\left(\frac{\tau^2}{\theta}\right) = \frac{2}{M''(0)}\frac{1-\alpha}{\alpha}$, or

$$\lim_{\theta\to 0}\frac{\tau}{\sqrt{\theta}} = \kappa \equiv \sqrt{\frac{2}{M''(0)}\frac{1-\alpha}{\alpha}}. \tag{A.16}$$

■

Proof of Corollary 9. In the long run, $x_{t_j} = 0$ on any observation date t_j. Therefore, Lemma 6 implies that $\phi_j = 1$. Since, conditional on P_{t_j}, $\log(P_{t_j+\tau}) \sim N\left(\log(P_{t_j}) + \left(\mu - \frac{\sigma^2}{2}\right)\tau, \sigma^2\tau\right)$, it follows that $(1-\alpha)\log(P_{t_j+\tau}) \sim N\left((1-\alpha)\left[\log(P_{t_j}) + \left(\mu - \frac{\sigma^2}{2}\right)\tau\right], (1-\alpha)^2\sigma^2\tau\right)$. Therefore, since $\phi_j = 1$, the definition of $K(\tau)$ implies that $K(\tau) = e^{-\rho\tau}E_{t_j}\left(\frac{P_{t_j+\tau}}{P_{t_j}}\right)^{1-\alpha} = e^{-\alpha\chi\tau}$, where $\chi \equiv \frac{\rho-(1-\alpha)\left(\mu-\alpha\frac{\sigma^2}{2}\right)}{\alpha} > 0$ to keep the value function finite. Differentiating this expression for $K(\tau)$ twice with respect to τ yields $K'(0) = -\alpha\chi$ and $K''(0) = \alpha^2\chi^2$, so that equation (A.13) implies that $M''(0) = (\omega - \chi)\chi = \frac{\chi}{\alpha}(1-\alpha)\left(\mu - \alpha\frac{\sigma^2}{2} - r_L\right)$. Using this expression for $M''(0)$ in (A.16) implies that for small θ, $\tau \simeq \sqrt{\frac{2\theta}{\chi\left(\mu-\alpha\frac{\sigma^2}{2}-r_L\right)}}$. ■

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