An exact solution for the investment and value of a firm facing uncertainty, adjustment costs, and irreversibility

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Abstract

This paper derives closed-form solutions for the investment and value of a competitive firm with a constant-returns-to-scale production function and convex costs of adjustment. Solutions are derived for the case of irreversible investment as well as for reversible investment. Optimal investment is a non-decreasing function of $q$, the shadow value of capital. Relative to the case of reversible investment, the introduction of irreversibility does not affect $q$, but it reduces the fundamental value of the firm.

Keywords: Investment; Irreversibility

JEL Classification: E22

1. Introduction

Most theoretical analyses of capital investment decisions by firms under uncertainty have focused either on irreversibility of investment or on convex
costs of adjustment. Recently, Abel and Eberly (1994) have shown that an appropriately specified investment cost function can incorporate convex costs of adjustment as well as irreversibility. In this framework, investment is a non-decreasing function of the shadow price of capital, denoted by $q$. In the irreversible investment case, investment is a strictly increasing function of $q$ for values of $q$ above a certain threshold value; for values of $q$ below this threshold value, investment equals zero, and negative investment is never optimal.

In this paper, we present a parametric example of a firm facing convex costs of adjustment and irreversibility, and we provide closed-form solutions for the investment and value of the firm. To our knowledge, the existing literature does not contain any closed-form solutions to problems of this type. Specifically, we examine a continuous-time stochastic model of an infinite-horizon, competitive firm with a constant-returns-to-scale production function. In this case, the value of the firm is a linear function of the firm's capital stock. The slope, $q$, of the value function with respect to capital is the shadow of capital which governs investment decisions. The constant term in the value function is the expected present value of rents to the adjustment technology.

We proceed by first analyzing the investment and value of a competitive firm that faces convex adjustment costs and has the possibility of undertaking negative gross investment. Our motivation for starting with the case of reversible investment is based on substantive as well as expositional considerations. First, the model of reversible investment that we analyze is richer than existing models that have yielded closed-form solutions. Specifically, the models in Abel (1983, 1985) specify convex costs of adjustment but do not include a cost of purchasing capital goods. By not including a cost of purchasing capital and by specifying the marginal adjustment cost to be zero at zero investment, those models are set up so that a positive rate of investment is always optimal. However, once we include the realistic assumption that there is a positive purchase price of capital, there will be situations in which it is optimal for the rate of investment to be zero or negative. Caballero (1991) specifies the cost of investment to include a positive purchase price of capital as well as convex costs

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1 Eisner and Strotz (1963), Lucas (1967), Gould (1968), and Treadway (1969) examined investment under costs of adjustment in the case of certainty. Mussa (1977) and Hayashi (1982) discussed the role of adjustment costs in Tobin's (1969) $q$ theory of investment under certainty, and Abel (1983, 1985) discussed this role under uncertainty. Investment under an irreversibility constraint was introduced by Arrow (1968) in the case of certainty and was later studied under uncertainty by Bernanke (1983), McDonald and Siegel (1986), Bertola (1987), Dixit (1989, 1991), and Pindyck (1988). See Pindyck (1991) for a review of the irreversibility literature, and Dixit and Pindyck (1994) for an extended instructive treatment. In addition, Lucas and Prescott (1971) examined investment under uncertainty with both costs of adjustment and irreversibility, though irreversibility was not a focus of their paper. Indeed, they did not even comment on the assumption of irreversibility in their model.
of adjustment, but does not provide a closed-form solution for investment and the value of the firm as we do.

Second, much of the analytic apparatus needed for the case of irreversible investment is the same as for the case of reversible investment. Because the case of reversible investment is simpler, it provides a useful opportunity for presenting the model, its manipulation, and its basic results.

Third, the case of reversible investment provides a benchmark against which to compare the effects of irreversibility on the investment and fundamental value of the firm. We will show that in our case, with constant returns to scale and perfect competition, the value of $q$ is unaffected by the presence or absence of irreversibility. For values of $q$ high enough to lead to positive investment in the reversible case, the optimal rate of investment is unaffected by the presence of irreversibility. For values of $q$ low enough to lead to negative investment in the reversible case, optimal investment equals zero in the irreversible case. The invariance of $q$ to the presence or absence of irreversibility arises in this case because the value of the firm is linear in the capital stock; the marginal value of an additional unit of capital is independent of the stock of capital, and hence independent of restrictions on the accumulation or decumulation of capital. Although irreversibility does not affect the value of $q$, it does reduce the value of the firm.

Section 2 presents the optimization problem of the competitive firm. The optimal rate of investment and the value of the firm in the case of reversible investment are derived in Section 3. Irreversibility is introduced and analyzed in Section 4. Concluding remarks are presented in Section 5.

2. The optimization problem of the competitive firm

2.1. The price process

We consider a continuous-time model of a competitive firm that sells its output at time $t$ at an exogenously given price $p_t$. The price $p_t$ evolves according to geometric Brownian motion

$$\frac{dp_t}{p_t} = \mu \, dt + \sigma \, dz_t, \quad p_0 > 0, \quad (1)$$

where $\mu$ is the instantaneous drift, $\sigma$ is the instantaneous standard deviation, and $dz_t$ is an increment to a standard Wiener process.

Later in our analysis it will be convenient to have expressions for the expected present value of $p_t^2$ for various values of $\lambda$. Under the geometric Brownian motion in Eq. (1), $E_t\{p_{t+\lambda}^2\}$ grows at a constant rate $\lambda \mu + \frac{1}{2} \sigma^2 \lambda (\lambda - 1)$ as
s increases for a given \( t \). Thus, the present value of \( E_t\{p^i_{t+s}\} \) discounted to time \( t \) at the rate \( R \) is

\[
e^{-Rs} E_t\{p^i_{t+s}\} = p^i_t e^{-Rs} e^{(\lambda_\mu + (1/2)\sigma^2 \lambda (\lambda - 1))s} = p^i_t e^{-f(\lambda; R)s},
\]

(2)

where

\[
f(\lambda; R) \equiv R - \lambda_\mu - \frac{1}{2} \sigma^2 \lambda (\lambda - 1)
\]

(3)

is the growth-rate-adjusted discount rate, equal to the discount rate \( R \) minus the growth rate of \( E_t\{p^i_{t+s}\}, \lambda_\mu + \frac{1}{2} \sigma^2 \lambda (\lambda - 1) \). The growth-rate-adjusted discount rate \( f(\lambda; R) \) is a (concave) quadratic function of \( \lambda \). When \( R > 0 \), the equation \( f(\lambda; R) = 0 \) has two distinct roots, one positive and one negative.

Define \( PV_t[p^i; R] \) to be the present value (discounted at rate \( R \)) of expected \( p^i \) from time \( t \) onward. Formally, we have

\[
PV_t[p^i; R] \equiv \int_0^\infty E_t\{p^i_{t+s}\} e^{-Rs} ds = p^i_t \int_0^\infty e^{-f(\lambda; R)s} ds = \frac{p^i_t}{f(\lambda; R)},
\]

(4)

where the second equality in Eq. (4) follows from Eq. (2).

2.2. The operating profit and investment cost functions

The firm uses capital \( K_t \) and labor \( L_t \) to produce output \( Y_t \) according to a Cobb–Douglas production function \( Y_t = L_t^\alpha K_t^{1-\alpha} \), where the labor share \( \alpha \) satisfies \( 0 < \alpha < 1 \). The firm pays a fixed wage \( w \) so that its operating profit at time \( t \), which equals revenue minus wages, is \( p_t L_t^\alpha K_t^{1-\alpha} - w L_t \). Because labor can be costlessly and instantaneously adjusted, the firm chooses \( L_t \) to maximize the instantaneous operating profit at time \( t \). The resulting maximized instantaneous operating profit \( \pi(K_t, p_t) \) is

\[
\pi(K_t, p_t) \equiv h p_t^\theta K_t,
\]

(5)

where

\[
\theta \equiv \frac{1}{1 - \alpha} > 1
\]

and

\[
h \equiv \theta^{-\theta}(\theta - 1)^{\theta - 1} w^{1-\theta} > 0.
\]

Notice that \( h p_t^\theta \) is the marginal revenue product of capital at time \( t \).
The firm undertakes gross investment $I_t$ and incurs depreciation at a constant rate $\delta \geq 0$. Thus, the change in the capital stock is

$$dK_t = (I_t - \delta K_t)dt.$$  \hspace{1cm} (6)

Let $c(I_t)$ denote the total cost of investing at rate $I_t$ and assume that $c(I_t)$ is strictly convex.

2.3. The Bellman equation

We assume that the firm operates in complete markets, discounts expected future cash flows at a constant rate $r > 0$, and maximizes the expected present value of its cash flow. The fundamental value of the firm at time $t$ is $V(K_t, p_t)$ where

$$V(K_t, p_t) = \max_{\{I_t \}, \{p_t \}} \left\{ \int_0^\infty \left[ \int hP_{t+s}K_{t+s} - c(I_{t+s}) \right] e^{-rs} ds \right\}. \hspace{1cm} (7)$$

Throughout this paper we will focus on the fundamental value of the firm, thereby ignoring any bubbles on the market value of the firm. The fundamental value of the firm satisfies the following Bellman equation (from this point on, we will suppress time subscripts unless they are needed for clarity):

$$rV(K, p) = \max_I \left[ hp^K - c(I) + \frac{E\{dV\}}{dt} \right]. \hspace{1cm} (8)$$

The right-hand side of Eq. (8) contains the two components of the expected return on the firm over a short interval of time: the instantaneous net cash flow $hp^K - c(I)$, and the expected capital gain $E\{dV\}/dt$. Eq. (8) requires that the sum of these components equals the required return $rV(K, p)$.

The expected capital gain is calculated using Ito's lemma and Eqs. (6) and (1), which describe the evolution of $K$ and $p$, to obtain

$$\frac{E\{dV\}}{dt} = (I - \delta K)V_K + \mu p V_p + \frac{1}{2} \sigma^2 p^2 V_{pp}. \hspace{1cm} (9)$$

It will turn out that investment depends on $V_K$, the marginal valuation of a unit of installed capital. Anticipating this result and emphasizing the relation to the $q$ theory of investment, we define $q \equiv V_K$, which is the shadow value of installed capital, and is non-negative. Substituting $q$ for $V_K$ in Eq. (9), and then
substituting Eq. (9) into Eq. (8) yields

$$rV(K, p) = \max_{I} [h\phi K - c(I) + (I - \delta K)q + \mu pV_p + \frac{1}{2} \sigma^2 p^2 V_{pp}]. \quad (10)$$

We can rewrite Eq. (10) by ‘maximizing out’ the rate of investment to obtain

$$rV(K, p) = h\phi K + \phi - \delta Kq + \mu pV_p + \frac{1}{2} \sigma^2 p^2 V_{pp}, \quad (11)$$

where

$$\phi = \max_{I} [Iq - c(I)]. \quad (12)$$

Note that $\phi$ is the maximized value of rents accruing to the adjustment technology from undertaking investment at rate $I$. When the firm invests at rate $I$ over an interval $dt$ of time, it acquires $I\, dt$ units of capital. Because $q$ is the shadow price of this capital, the firm acquires capital worth $qI\, dt$, but pays $c(I)\, dt$ to increase its capital stock by $I\, dt$. Thus, $qI - c(I)$, the excess of additional value over costs, is the value of the rents accruing per unit time to the firm for undertaking investment at rate $I$.

### 3. Reversible investment

In this section we focus on the case of reversible investment. We begin in Section 3.1 by specifying an investment cost function for which the optimal level of investment can be negative. Also in Section 3.1, we express the optimal rate of investment as a function of $q$. After deriving the differential equation describing the fundamental value of the firm in Section 3.2, we then obtain $q$ as a function of $p$ in Section 3.3. The value to the firm of the adjustment technology is derived in Section 3.4.

#### 3.1. The investment cost function and the optimal rate of investment

Specify the total cost of investing at time $t$, $c(I_t)$ as

$$c(I_t) = bI_t + \gamma I_t^{n(n-1)}, \quad (13)$$

where $b \geq 0$, $\gamma > 0$, and $n \in \{2, 4, 6, \ldots \}$. The cost of undertaking investment $c(I_t)$ has two components: (1) $bI_t$ is the cost of purchasing new capital at a fixed
price of $b$ per unit;\textsuperscript{2} for negative gross investment, $bI_t \leq 0$ represents the proceeds to the firm of selling capital at a price of $b$ per unit. (2) $\gamma I_t^{n/(n-1)}$ is a convex cost of adjustment. When $n = 2$, the cost of adjustment is $\gamma I_t^2$ which is quadratic. In the quadratic case, optimal investment is a linear function of $q$ and the price of capital $b$; however, when $n$ differs from 2, as we allow here, investment may be a nonlinear function of its fundamental determinants. The assumption that $n$ is an even positive integer insures that the adjustment cost function $\gamma I_t^{n/(n-1)}$ is real-valued and convex for negative $I$ as well as for positive $I$. To insure a finite fundamental value of the firm, we assume that $f(n0, r) > 0$.

Using the parametric specification of the investment cost function in Eq. (13), we can obtain closed-form solutions for investment and the value of the firm. With the investment cost function in Eq. (13) we rewrite Eq. (12) as

$$
\phi = \max_I [(q - b)I - \gamma I_t^{n/(n-1)}].
$$

(14)

The optimal rate of investment $\hat{I}$ is determined by differentiating the term in brackets on the right-hand side of Eq. (14) with respect to $I$, and setting the derivative equal to zero to obtain

$$
\hat{I} = \left[ \frac{n - 1}{n \gamma} \right]^{n-1} (q - b)^{n-1}.
$$

(15)

Eq. (15) indicates that investment is an increasing function of $q$. When the shadow price of capital $q$ is greater than the purchase price of capital $b$, gross investment is positive. When the shadow price $q$ is less than the sale price of capital $b$, the firm sells capital, and gross investment $I$ is negative.\textsuperscript{3} In the special case of quadratic adjustment costs, $n = 2$ and the optimal rate of investment is a linear function of $q$ and $b$.

To determine the value of $\phi$, substitute Eq. (15) into Eq. (14) to obtain

$$
\phi = (q - b)^n \Gamma,
$$

(16)

where

$$
\Gamma \equiv (n - 1)^{n-1} n^{-n} \gamma^{(1-n)} > 0.
$$

\textsuperscript{2}In Abel (1983, 1985) the purchase price of capital $b$ is zero, so the optimal rate of investment is always positive. Allowing the purchase price of capital to be positive as we do here makes the model more realistic and introduces the possibility that the optimal rate of investment is negative.

\textsuperscript{3}Recall that with $n \geq 2$ even, $n - 1$ is odd so that $(q - b)^{n-1}$ has the same sign as $q - b$ and is an increasing function of $q - b$. 

Because the investment cost function is convex, the firm earns rents on inframarginal units of investment when investment is nonzero, that is, when $q \neq b$.

The rents represented by $\phi$ are illustrated in Fig. 1. The curve in each part of Fig. 1, which represents the cost function $c(I)$ specified in Eq. (13), is strictly convex, passes through the origin, and has a slope equal to $b$ at the origin. In Fig. 1a, the marginal value of capital $q$ is greater than the purchase price of capital $b$, so the straight line representing $ql$ is steeper than $c(I)$ at the origin. In this case, $ql$ exceeds $c(I)$ for some positive values of $I$, and the optimal value of investment $I^*$ is the value that maximizes $ql - c(I)$. The rents $\phi$ are shown as the vertical distance between the straight line and the curve at $Z = I$. In Fig. 1b, the marginal value of capital $q$ is less than $b$, so the straight line representing $qZ$ is less steep than $c(I)$ at the origin. In this case, $qZ$ exceeds $c(I)$ for some negative values of $I$. The optimal value of investment $I^*$ is negative in this case, and the value of $\phi$ is again shown as the vertical distance between $qZ$ and $c(I)$ at $I = I^*$.

3.2. The fundamental value of the firm

We have derived the optimal rate of investment as a function of $q$, the marginal value of installed capital. Our next step is to determine $q$ as a function of the price of output $p$, and then to determine the fundamental value of the firm $V(K, p)$. We proceed by hypothesizing that the solution is a linear function of the capital stock. Thus,

$$V(K, p) = q(p)K + G(p),$$

where $q(p)$ and $G(p)$ are functions to be determined. To determine these functions, substitute Eq. (17) into Eq. (11) and use the expression for $\phi$ in Eq. (16) to obtain

$$rqK + rG = hp^qK + (q - b)p\Gamma - \delta Kq + \mu pq pK + \mu p G_p + \frac{1}{2}\sigma^2 p^2 q_{pp}K + \frac{1}{2}\sigma^2 p^2 G_{pp}.$$
This differential equation must hold for all values of $K$. Therefore, the term multiplying $K$ on the left-hand side must equal the sum of the terms multiplying $K$ on the right-hand side. In addition, the term not involving $K$ on the left-hand side must equal the sum of the terms not involving $K$ on the right-hand side. These equalities yield

$$rq = h p^\theta - \delta q + \mu p q_p + \frac{1}{2} \sigma^2 p^2 q_{pp}$$  \hspace{1cm} (19)$$

and

$$rG = (q - b) \gamma + \mu p G_p + \frac{1}{2} \sigma^2 p^2 G_{pp}.$$  \hspace{1cm} (20)$$

These equations have a recursive structure. The differential equation for $q(p)$ in Eq. (19) does not depend on $G(p)$, but the differential equation for $G(p)$ in Eq. (20) depends on $q(p)$. Thus, we will solve Eq. (19) for $q(p)$ and then proceed to solve Eq. (20) for $G(p)$.

3.3. The marginal value of installed capital, $q$

The marginal value of installed capital is obtained by solving the differential equation (19). It can be easily verified by direct substitution that a general solution to this differential equation is

$$q(p) = B p^\theta + A_1 p^{\eta_1} + A_2 p^{\eta_2},$$  \hspace{1cm} (21)$$

where

$$B \equiv \frac{h}{f(\theta; r + \delta)} > 0$$

and $\eta_1 > \eta_2$ are the roots of the quadratic equation $f(\eta; r + \delta) = 0$. These roots satisfy $^4 \eta_1 > n\theta > 2\theta > 0 > \eta_2$.

The particular solution in Eq. (21) $B p^\theta$ equals the present value of expected marginal revenue products of capital $h p^\theta$ accruing to the undepreciated portion of a unit of currently installed capital. $^5$ The terms $A_1 p^{\eta_1}$ and $A_2 p^{\eta_2}$ are solutions

$^4$ For $\xi \geq 0$, $f(\eta; r + \xi) > 0$ for all $\eta \in [0, n\theta]$ because (1) $f(0; r + \xi) = r + \xi > 0$; (2) $f(n\theta; r + \xi) \geq f(n\theta; r) > 0$; and (3) $f(\eta; r + \xi)$ is concave in $\eta$. Therefore $\eta_1 > n\theta$ and $\eta_2 < 0$.

$^5$ The present value of expected marginal revenue products of capital $h p^\theta$ accruing to the undepreciated portion of a unit of currently installed capital is

$$\int_0^\infty \mathbb{E}_\pi(h p^\theta) e^{-r s} ds = \frac{h p^\theta}{f(\theta; r + \delta)},$$

where the equality follows directly from Eq. (4). The coefficient $B \equiv h/f(\theta; r + \delta)$ is positive because $h > 0$ and (see the previous footnote) $f(\theta; r + \delta) > 0$. 

to the homogeneous part of the differential equation (19). The expected growth rates of \( A_1 p^n \) and \( A_2 p^n \) are both equal to \( r + \delta \). We refer to these terms as bubbles because they are unrelated to the underlying fundamentals (cash flows). Restricting attention to the fundamental value of \( q \) (i.e., ruling out bubbles on the shadow price of installed capital) implies that \( A_1 = A_2 = 0 \). Thus, \( q \) can be written as simply

\[
q(p) = Bp^\theta. \tag{22}
\]

The expression for \( q \) does not involve any of the parameters of the adjustment cost function. The value of \( q \) is simply the present value of expected marginal revenue products, and for a competitive firm with constant returns to scale to scale the marginal revenue product of capital is exogenous. Because the path of marginal revenue products does not depend on the firm's investment, \( q \) is independent of the specification of the adjustment cost function. Also because \( p \) evolves according to geometric Brownian motion with \( p_0 > 0 \), \( p^\theta \) cannot be negative. Thus, as mentioned earlier, \( q \) cannot be negative.

The definition of \( B \) in Eq. (21) together with the definition of the growth-rate-adjusted discount rate in Eq. (3) and the fact that \( \theta > 1 \) implies that \( B \) is an increasing function of \( \sigma^2 \). Thus, \( q \) is an increasing function of the instantaneous variance \( \sigma^2 \). Because investment is an increasing function of \( q \), investment in an increasing function of \( \sigma^2 \) for a given value of the output price \( p \). This result is the same as that found by Hartman (1972), Abel (1983), and Caballero (1991).

### 3.4. The value of the adjustment technology

The intercept term \( G(p) \) in the fundamental value of the firm equals the present value of the expected rents, \( \phi \), accruing to the adjustment technology represented by the convex cost of adjustment function as illustrated in Fig. 1. We describe these rents as accruing to the adjustment technology because the adjustment technology is a scarce resource. Each firm has access to only one adjustment technology characterized by \( c(I) \). If a firm had access to two identical adjustment technologies, any rate of investment \( I \) could be achieved at lower cost by installing half of the new investment using one adjustment technology at a cost \( c(I/2) \) and installing the other half of new investment using the other adjustment technology at a cost \( c(I/2) \), because \( c(I) > 2c(I/2) \) for \( I \neq 0 \). Equivalently, because \( G(p) > 0 \), a firm with capital stock \( K \) would have an incentive to divide itself into two plants with capital stock \( K/2 \) because the value of the two plants together would be \( 2G(p) + q(p)K \). In our model, such a division into two plants is prevented by the assumption that each firm has access only to one adjustment technology with investment cost function \( c(I) \). It is this scarcity of the adjustment technology that gives rise to the rents discussed here.
The function $G(p)$ is determined by the differential equation (20). Let $GP(p)$ denote a particular solution to Eq. (20). It can be verified by direct substitution that the following expression is a particular solution to Eq. (20):²

$$GP(p) = PV_t[(q - b)^n r].$$

Since $\phi = (q - b)^n r$, the particular solution is the present value of the expected rents $\phi$.

We can obtain a general solution to Eq. (20) by adding the particular solution in Eq. (23) and the solution to the homogeneous part of the differential equation. The solution to the homogeneous part of Eq. (20) is $C_1 p^{\omega_1} + C_2 p^{\omega_2}$, where $\omega_1 > \omega_2$ are the roots of the quadratic equation $f(\omega; r) = 0$.

The expected growth rates of $C_1 p^{\omega_1}$ and $C_2 p^{\omega_2}$ are both equal to $r$. These terms are bubbles in the sense that they are unrelated to the underlying fundamentals (cash flows) of the firm. Our definition of $V(K_1, p_1)$ in Eq. (7) as the fundamental value of the firm rules out these bubbles. Formally, this assumption implies that $C_1 - C_2 = 0$, so that $G(p) = GP(p) = PV_t[\phi; r]$.

We have shown that the fundamental value of the firm is comprised of two (additive) parts: (1) the value of existing capital $qK$ which is the expected present value of the returns to the existing capital stock; and (2) the value of the adjustment technology $PV_t[\phi; r]$ which equals the present value of the expected rents accruing to current and future investment through the adjustment technology.

4. The case of irreversible investment

Now we consider the case of irreversible investment. Fortunately, much of the mathematics of this case is the same as for the case of reversible investment discussed in earlier sections. We take advantage of this overlap to abbreviate the derivations in this section and to focus on aspects that are specific to irreversibility. We can also use the results of earlier sections as a benchmark for comparison to understand the impact of irreversibility on the firm's decisions and its value.

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²A binomial expansion of $(q - b)^n$ combined with the fact that $q = Bp^\theta$ yields $(q - b)^n = \Gamma \sum_{j=0}^{\infty} \frac{n!}{j!(n-j)!} B^j (-b)^{n-j}$. Using the definition of the present value operator in Eq. (4) with $\lambda = j\theta$ and $R = r$, and recognizing that this operator is linear, yields $PV_t[(q - b)^n r; r] - \Gamma \sum_{j=0}^{\infty} \frac{n!}{j!(n-j)!} B^j (-b)^{n-j}$. Direct substitution verifies that setting $G$ equal to the right-hand side of this equation satisfies Eq. (20).

³In the special case in which $b = 0$, Eq. (23) yields $GP(p) = \Gamma q^\theta (n\theta; r)$ which is equivalent to the intercept of the linear value function in Eq. (11a) of Abel (1983).
4.1. The modified investment cost function and the optimal rate of investment

Rather than simply assume that it is impossible for gross investment to be negative, we modify the investment cost function \( c(I) \) for negative values of gross investment so that it is never optimal for the firm to undertake negative gross investment. Specifically, assume that

\[
c(I_t) = \begin{cases} 
bl_t + \gamma l_t^n(n-1) & \text{for } I_t \geq 0, \\
g(I_t) > 0 & \text{for } I_t < 0,
\end{cases}
\] (24)

where we continue to assume that \( b \geq 0, \gamma > 0, \) and \( n \) is an even positive integer. Note that for nonnegative values of \( I_t \), the investment cost function in Eq. (24) is identical to that in Eq. (13). However, we have changed the investment cost function for negative gross investment. Specifically, for all negative values of gross investment, the cost of investment \( c(I_t) \) is positive, which, as we show below, implies that negative gross investment will never be optimal. Compared to the investment cost function in Eq. (13), we have increased the cost of undertaking negative investment.\(^8\)

For values of \( q \geq b \), the optimal rate investment is positive and identical to that in the case of reversible investment. For values of \( q < b \), \( qI - c(I) < 0 \) for all nonzero values of \( I \) as illustrated in Fig. 2. Thus, the maximal value of \( qI - c(I) \), which is zero, is attained by setting \( I = 0 \) when \( q < b \). Since the optimal rate of investment cannot be negative when the cost function is given by Eq. (24), we describe this case as one in which investment is irreversible. We summarize these findings for the case of irreversible investment as

\[
\hat{I} = \max \left\{0, \left[ \frac{n-1}{ny} \right]^{n-1} (q - b)^{n-1} \right\}
\] (25)

and

\[
\phi = (\max[0, q - b])^n. \quad (26)
\]

4.2. The fundamental value of the firm

Eq. (25) gives the optimal rate of investment as a function of the shadow price of capital \( q \). The next step, as in the case of reversible investment in Section 3, is

\(^8\)More precisely, in the reversible case the only rates of negative investment that could be optimal are those for which the cost of investment is negative. For all these rates of negative investment, we have increased the cost of investment.
to determine \( q \) as a function of the price of output \( p \). We proceed by hypothesizing that \( V(K, p) \) is a linear function of the capital stock, and that there are two regimes: regime \( H \) applies for values of \( q \) greater than or equal to \( b \); regime \( L \) applies for values of \( q \) less than or equal to \( b \). Thus,

\[
V^{(i)}(K, p) = q^{(i)}(p)K + G^{(i)}(p), \quad i = \begin{cases} 
L & \text{for } q \leq b, \\
H & \text{for } q \geq b,
\end{cases}
\]

(27)

where \( q^{(i)}(p) \) and \( G^{(i)}(p) \) are functions to be determined. To determine these functions, substitute Eq. (27) into Eq. (11) and use the expression for \( \phi \) in Eq. (26). As in the reversible case in Section 3, the term multiplying \( K \) on the left-hand side of the resulting equation must equal the sum of the terms multiplying \( K \) on the right-hand side. In addition, the term not involving \( K \) on the left-hand side must equal the sum of the terms not involving \( K \) on the right-hand side. These conditions yield

\[
rq^{(i)} = hp^\theta - \delta q^{(i)} + \mu pq_p^{(i)} + \frac{1}{2}\sigma^2 p^2 q_{pp}^{(i)}, \quad i = L, H, \tag{28}
\]

\[
rG^{(i)} = (\max[0, q - b])p^\Gamma + \mu pG_p^{(i)} + \frac{1}{2}\sigma^2 p^2 G_{pp}^{(i)}, \quad i = L, H. \tag{29}
\]

These equations correspond to Eqs. (19) and (20) in the reversible investment case. As in Section 3, we exploit the recursive structure of these equations by solving Eq. (28) for \( q^{(i)}(p) \) and then solving Eq. (29) for \( G^{(i)}(p) \).
4.3. The solution for q and the optimal rate of investment

The solution for \( q^{(i)}(p) \) is determined from Eq. (28), which is identical to Eq. (19). Therefore, a general solution to this equation is

\[
q^{(i)}(p) = Bp^\theta + A_1^{(i)}p^{\eta_1} + A_2^{(i)}p^{\eta_2}, \quad i = L, H,
\]

where \( B, \eta_1 \) and \( \eta_2 \) are identical to their values in Eq. (21) (the reversible investment case). The particular solution \( Bp^\theta \) is identical to that in Eq. (21). The only new aspect of Eq. (30) is that the coefficients \( A_1^{(i)} \) and \( A_2^{(i)} \) in the homogeneous part of the solution can potentially differ across the two regimes \( L \) and \( H \). We now explore this possibility.

The two regimes \( L \) and \( H \) meet when \( q = b \). Let \( p^* \) denote the value(s) of \( p \) for which \( q = b \). The value matching condition requires that \( q^{(L)}(p^*) = q^{(H)}(p^*) \), and what is often called the smooth pasting condition requires that \( q_p^{(L)}(p^*) = q_p^{(H)}(p^*) \). Applying these conditions to the expression for \( q^{(i)}(p) \) in Eq. (30) yields \( A_1^{(L)} = A_1^{(H)} \) and \( A_2^{(L)} = A_2^{(H)} \). Thus, the general solution for \( q^{(i)}(p) \) is the same in both regimes and is identical to Eq. (21). From this point on, the solution method for \( q(p) \) is identical to the reversible case. As in the reversible case, we focus on the fundamental solution so \( A_1^{(i)} = A_2^{(i)} = 0 \), for \( i = L, H \).

To summarize our comparison of the reversible and irreversible investment cases so far, we have shown that the value of \( q \) is identical in both cases. Furthermore, for values of \( q \) greater than or equal to \( b \), investment is the same in the reversible case and in the irreversible case. The only difference in investment behavior occurs when \( q < b \). In this situation, investment is negative in the reversible case and is zero in the irreversible case.

4.4. The value of the adjustment technology

Although the shadow price \( q \) is unaffected by whether or not investment is reversible, the rents to the adjustment technology, represented by the intercept \( G(p) \), depend on whether or not investment is reversible. In the reversible case, the adjustment technology provides the firm with the opportunity to undertake a profitable action – sell capital – when \( q \) is very low. However, this profitable opportunity is not provided by the adjustment technology in the irreversible case, and thus the present value of expected rents to the adjustment technology is smaller under irreversibility than under reversibility.

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9The value matching condition implies that \( A_1^{(L)} - A_1^{(H)} = p^* (\eta_1 - \eta_1) (A_2^{(H)} - A_2^{(L)}) \) and the smooth pasting condition implies that \( A_2^{(L)} - A_2^{(H)} = (\eta_2/\eta_1) p^* (\eta_2 - \eta_1) (A_1^{(H)} - A_1^{(L)}) \). Since \( \eta_2/\eta_1 \) and \( p^* \) are both nonzero and \( \eta_2/\eta_1 \neq 1 \), these conditions imply that \( A_2^{(H)} - A_2^{(L)} = 0 \) and \( A_1^{(L)} - A_1^{(H)} = 0 \).
The function $G(p)$ is determined by the differential equation in Eq. (29). This differential equation contains the term $(\max[0, q - b])^\pi \Gamma$ which equals 0 in regime $I$, but equals $(q - b)\pi \Gamma$ in regime $H$. We need to solve the differential equation separately for each regime.

We begin with the simpler case, which is regime $L$. Since $(\max[0, q - b])^\pi \Gamma = 0$ in this regime, the differential equation (29) is homogeneous. Specifically,

$$rG^{(L)} = \mu pG_p^{(L)} + \frac{1}{2}\sigma^2 p^2 G_{pp}^{(L)}.$$  

(31)

The general solution to this homogeneous differential equation is

$$G^{(L)}(p) = C_1^{(L)} e^{\omega_1 p} + C_2^{(L)} e^{\omega_2 p},$$  

(32)

where $\omega_1 > \omega_2$ are the roots of $f(\omega; r) = 0$ as in the reversible case, and $C_1^{(L)}$ and $C_2^{(L)}$ are constants to be determined. The roots satisfy $\omega_1 > 0$ and $\omega_2 < 0$. Thus, $p^{\omega_2}$ becomes arbitrarily large as $p$ approaches zero. However, the fundamental value of the firm approaches zero as $p$ approaches zero. Since we are focusing on the fundamental value of the firm, thereby ruling out bubbles, $C_2^{(L)}$ must equal zero and $G^{(L)}(p)$ can be written as

$$G^{(L)}(p) = C_1^{(L)} e^{\omega_1 p}.$$  

(33)

Eq. (33) gives the fundamental value of the firm with no capital in regime $L$ where $q < b$. Even though a firm in this situation has no capital and is not currently undertaking gross investment, it will have a positive value because of the prospect that one day $q$ may rise above $b$, and it will become profitable for the firm to invest. In this case, $C_1^{(L)} e^{\omega_1}$ is not a bubble. We will determine the value of $C_1^{(L)}$ after we solve for $G^{(H)}(p)$.

In regime $H$, $q \geq b$ so that $(\max[0, q - b])^\pi \Gamma = (q - b)\pi \Gamma$. Thus, the differential equation (29) becomes

$$rG^{(H)} = (q - b)\pi \Gamma + \mu pG_p^{(H)} + \frac{1}{2}\sigma^2 p^2 G_{pp}^{(H)}.$$  

(34)

This differential equation is identical to the differential equation (20) in the reversible case. Therefore, the particular solution $GP(p)$ in Eq. (23) is also the particular solution of the differential equation in Eq. (34).

We can obtain a general solution to Eq. (34) by adding the particular solution in Eq. (23) and the solution to the homogeneous part of the differential equation. The homogeneous part of the differential equation (34) is identical to the

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These inequalities are obtained by setting $\xi = 0$ in footnote 4.
differential equation (31). Thus, the solution to the homogeneous part of Eq. (34) is $C_1^{(H)} p^{\omega_1} + C_2^{(H)} p^{\omega_2}$. Since $\omega_1 > n\theta$, $p^{\omega_1}$ dominates $GP(p)$ and $p^{\omega_2}$ as $p$ grows without bound. We are focusing on the fundamental value of the firm, thereby ruling out bubbles, so $C_1^{(H)} = 0$.

The coefficients $C_1^{(L)}$ and $C_2^{(H)}$ still remain to be determined. These two coefficients can be determined using the value matching condition $G^{(L)}(p^*) = G^{(H)}(p^*)$ and the smooth pasting condition $G^{(L)}_p(p^*) = G^{(H)}_p(p^*)$. Appendix A shows how these two conditions lead to the following solutions for $G^{(L)}$ and $G^{(H)}$. It is more convenient to write these expressions as functions of $q \in B_{pe}$ rather than of $p$:

\begin{align*}
G^{(L)}(q) &= \left[ (\omega_1 - \omega_2) \prod_{j=0}^{n} (\omega_1 - j\theta) \right]^{-1} \frac{2\Gamma b^{n}\theta n!}{\sigma^2} \left( \frac{q}{b} \right)^{\omega_1/\theta}, \quad (35) \\
G^{(H)}(q) &= PV \left[ (q - b)^n; r \right] \Gamma \\
&\quad + \left[ (\omega_1 - \omega_2) \prod_{j=0}^{n} (\omega_2 - j\theta) \right]^{-1} \frac{2\Gamma b^{n}\theta n!}{\sigma^2} \left( \frac{q}{b} \right)^{\omega_2/\theta}. \quad (36)
\end{align*}

Recall that $\omega_1 > n\theta > 0 > \omega_2$. Therefore, Eq. (35) implies that $G^{(L)}(q) > 0$ because $\omega_1 - \omega_2 > 0$ and $\omega_1 - j\theta > 0$ for $j = 0, 1, \ldots, n$. Thus, as discussed earlier, even when $q < b$ so that it is not currently profitable for the firm to undertake positive gross investment, the prospect that $q$ will eventually exceed $b$ means that the present value of the rents accruing to the adjustment technology is positive.

Eq. (36) allows a direct comparison of the present value of rents to the adjustment technology in the reversible and irreversible cases. Recall that the first term on the right-hand side of Eq. (36), $PV \left[ (q - b)^n; r \right] \Gamma$, equals $G(p)$ in the reversible case. Thus, the difference between $G(p)$ in the irreversible and reversible cases is given by the second term in Eq. (36). Since $\omega_1 - j\theta < 0$ for $j = 0, 1, \ldots, n$, and second term in Eq. (36) contains the product of an odd number $(n + 1)$ of such terms, the second term in Eq. (36) is negative. That is, $G^{(H)}$ in the irreversible case is smaller than $PV \left[ (q - b)^n; r \right] \Gamma$. The reason for this result is clear. In the case of reversible investment the current rents to the adjustment technology are $(q - b)^n$ regardless of whether $q$ is greater than, less than, or equal to $b$. However, in the irreversible case, current rents to the adjustment technology are $0$ (which is less than $(q - b)^n$) whenever $q < b$. Even when $q$ is currently greater than $b$, the prospect that $q$ may eventually fall below $b$ means that the present value of expected rents to the adjustment technology is smaller in the case of irreversible investment than in the reversible case.
5. Conclusions

We have derived closed-form solutions for the optimal investment and fundamental value of a competitive firm under uncertainty. These solutions were obtained under the assumptions that the firm has a constant-returns-to-scale production technology and convex costs of investing. We have solved for investment and fundamental value in both a reversible and an irreversible investment case. Optimal investment is a nondecreasing function of \( q \), the shadow value of capital.

The shadow value of an additional unit of capital \( q \) is unaffected by irreversibility in our model. The only effect of irreversibility on investment behavior is to set investment to zero when it would otherwise be negative. Irreversibility does, however, affect the valuation of the firm. Irreversibility implies that the firm has a very costly technology for disinvesting. The value of the firm will reflect this cost disadvantage compared to an otherwise identical firm with reversible investment; thus, the value of the firm is reduced by irreversibility. An implication of this finding is that average \( q \), computed as the ratio of the firm’s value to its capital stock \( V(K, p)/(bK) \), and often used as an empirical proxy for \( q \), is reduced by irreversibility despite the fact that (marginal) \( q \) is unaffected. Irreversibility therefore reduces the difference between marginal and average \( q \), since average \( q \) exceeds marginal \( q \) by the ratio of the present value of rents to the capital stock \( G(p)/(bK) \).

The invariance of (marginal) \( q \) to the imposition of irreversibility is in contrast to results reported in the irreversible investment literature, where imposition of a nonnegativity constraint on investment reduces the marginal value of additional capital as a result of what is often called the ‘option value of waiting’. In our model, the value function is linear in capital, so \( q \) does not depend on the capital stock. Consistent with Pindyck’s (1993) argument, the firm does not ‘kill an option’ when it invests, since its investment behavior does not affect the current or future return to capital.

We have confined our attention to a competitive firm with constant returns to scale so that the operating profit function of the firm is linear in capital. Therefore, the marginal operating profit of capital is invariant to the capital stock. However, if the firm has some monopoly power and/or if the production function exhibits decreasing returns to scale, the operating profit function of the firm will be strictly concave in the capital stock, and the marginal operating profit of capital will be strictly decreasing in the capital stock. The case with a strictly concave operating profit function is substantially more difficult, but

\[\text{This is true for a given value of the capital stock } K.\]

\[\text{For example, see McDonald and Siegel (1986), Dixit (1989), Bertola (1987), and Pindyck (1988).}\]
would allow analysis of the 'option value' found in some irreversible investment models.  

Appendix A. Applying the boundary conditions to $G^{(H)}(p)$ and $G^{(L)}(p)$

Recall that we have argued in the text that $C^*_1 = 0$ so that

$$G^{(H)}(p) = GP(p) + C^*_2 p^{\omega_2}. \quad (A.1)$$

Now evaluate the particular solution $GP(p)$ and its derivative at $p = p^*$ using

$$GP(p) = \Gamma \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} \frac{1}{f(j; \theta, r)} B^j p^j (-b)^{n-j}$$

from footnote 6 and the fact that $BP^* = b$:

$$GP(p^*) = \Gamma b^n \sum_{j=0}^{n} (-1)^{n-j} \frac{n!}{j!(n-j)!} f(j; \theta, r), \quad (A.2)$$

$$GP_p(p^*) = \left( \frac{\theta}{p^*} \right) \Gamma b^n \sum_{j=0}^{n} (-1)^{n-j} \frac{n!}{j!(n-j)!} f(j; \theta, r). \quad (A.3)$$

The value-matching condition $G^{(L)}(p^*) = G^{(H)}(p^*)$ implies, using Eqs. (33) and (A.1), that

$$C^*_1 p^{\omega_1} = GP(p^*) + C^*_2 p^{\omega_2}. \quad (A.4)$$

The smooth pasting condition $G_p^{(L)}(p^*) = G_p^{(H)}(p^*)$ implies, using Eqs. (33) and (A.1), that

$$\omega_1 C^*_1 p^{\omega_1} - 1 = GP_p(p^*) + \omega_2 C^*_2 p^{\omega_2} - 1. \quad (A.5)$$

Eqs. (A.4) and (A.5) are two linear equations in the two unknown constants $C^*_1 p^{\omega_1}$ and $C^*_2 p^{\omega_2}$. Solving these two linear equations yields

$$C^*_1 p^{\omega_1} = p^* GP_p(p^*) - \omega_2 GP(p^*) \over \omega_1 - \omega_2 \quad (A.6)$$

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13 Abel et al. (1996) analyze the option value and relate the option value to $q$ for an operating profit function that is concave in $K$. However, their analysis is confined to a two-period discrete-time model.
and
\[ C_2^{(H)} \rho^{*\omega_2} = \frac{p^*GP_p(p^*) - \omega_1 GP(p^*)}{\omega_1 - \omega_2}. \] (A.7)

Now use the fact that \( q/b = (p/p^*)^\theta \) to observe that \( C_1^{(L)} \rho^{\omega_1} = C_1^{(L)} \rho^{*\omega_1}(p/p^*)^{\omega_1} = C_1^{(L)} \rho^{*\omega_1}(q/b)^{\omega_1/\theta} \) so that Eq. (A.6) implies
\[ C_1^{(L)} \rho^{\omega_1} = \left\{ \frac{p^*GP_p(p^*) - \omega_1 GP(p^*)}{\omega_1 - \omega_2} \right\} \left( \frac{q}{b} \right)^{\omega_1/\theta}. \] (A.8)

Similarly, observe that \( C_2^{(H)} \rho^{\omega_2} = C_2^{(H)} \rho^{*\omega_2}(p/p^*)^{\omega_2} = C_2^{(H)} \rho^{*\omega_2}(q/b)^{\omega_2/\theta} \) so that Eq. (A.7) implies
\[ C_2^{(H)} \rho^{\omega_2} = \left\{ \frac{p^*GP_p(p^*) - \omega_1 GP(p^*)}{\omega_1 - \omega_2} \right\} \left( \frac{q}{b} \right)^{\omega_2/\theta}. \] (A.9)

Next use Eqs. (A.2) and (A.3) to calculate
\[ p^*GP_p(p^*) - \omega_1 GP(p^*) = b^n \sum_{j=0}^{n} (j\theta - \omega_1) D_j (-1)^{n-j}, \] (A.10)

where \( D_j = \frac{\Gamma(n)!}{[j!(n-j)!] f(j\theta; r)]. \)

Substituting Eq. (A.10) into Eq. (A.8), and recalling that \( G^{(L)}(p) = C_1^{(L)} \rho^{\omega_1} \), we obtain
\[ G^{(L)}(p) = \left( \frac{q}{b} \right)^{\omega_1/\theta} b^n \sum_{j=0}^{n} \frac{(j\theta - \omega_2) D_j (-1)^{n-j}}{\omega_1 - \omega_2}. \] (A.11)

Substituting Eq. (A.10) into Eq. (A.9), and recalling that \( G^{(H)}(p) = GP(p) + C_2^{(H)} \rho^{\omega_2} \), we obtain
\[ G^{(H)}(p) = GP(p) + \left( \frac{q}{b} \right)^{\omega_2/\theta} b^n \sum_{j=0}^{n} \frac{(j\theta - \omega_1) D_j (-1)^{n-j}}{\omega_1 - \omega_2}. \] (A.12)

We can simplify the expressions in Eqs. (A.11) and (A.12) using the following lemma.
Lemma. Let \( \omega_i \) and \( \omega_k \) be the two real roots of the quadratic equation
\[
f(x; r) \equiv r - \mu x - \frac{1}{2} \sigma^2 x(x - 1) = 0.
\]
Define
\[
S_n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!f(j; r)} (-1)^{n-j}.
\]

Then
\[
S_n = \frac{2n! \theta^n/\sigma^2}{\prod_{j=0}^{n}(\omega_k - j \theta)}.
\]

Proof. Observe that the quadratic function \( f(x; r) \) can be written as
\[
f(x; r) = \frac{1}{\sigma^2} (\omega_k - x)(x - \omega_i),
\]
so that \( f(j; r) = \frac{1}{\sigma^2} (\omega_k - j \theta)(j \theta - \omega_i) \). Therefore, \( (j \theta - \omega_i)/f(j; r) = 1/[\frac{1}{\sigma^2}(\omega_k - j \theta)] \) and we can rewrite \( S_n \) as
\[
S_n = \frac{2}{\sigma^2} \sum_{j=0}^{n} \frac{1}{\omega_k - j \theta} \frac{n!}{j!(n-j)!} (-1)^{n-j}.
\] (A.13)

Applying Eq. (A.13) for \( n - 1 \) yields
\[
S_{n-1} = \frac{2}{\sigma^2} \sum_{j=0}^{n-1} \frac{1}{\omega_k - j \theta} \frac{(n-1)!}{j!(n-1-j)!} (-1)^{n-1-j}.
\] (A.14)

Multiply both sides of Eq. (A.14) by \(-n \theta\) to obtain
\[
-n \theta S_{n-1} = \frac{2}{\sigma^2} \sum_{j=0}^{n-1} \frac{\theta}{\omega_k - j \theta} \frac{n!}{j!(n-1-j)!(n-j)} (n-j)(-1)^{n-j}.
\] (A.15)

Observe that when \( j = n \), the summand on the right-hand side of Eq. (A.15) is zero, so we can increase the upper limit on the summation index \( j \) from \( n - 1 \) to \( n \). Performing this change and rearranging yields
\[
-n \theta S_{n-1} = \frac{2}{\sigma^2} \sum_{j=0}^{n} \frac{\theta(n-j)}{\omega_k - j \theta} \frac{n!}{j!(n-j)!} (n-j)(-1)^{n-j}.
\] (A.16)
Now replace $\theta(n - j)$ by $(\theta n - \omega_k) + (\omega_k - j\theta)$ to obtain

$$- n\theta S_{n-1} = \frac{2}{\sigma^2} \sum_{j=0}^{n} \left[ \frac{\theta n - \omega_k}{\omega_k - j\theta} \right] \left[ \frac{n!}{j!(n-j)!} \right] (-1)^{n-j}$$

$$+ \frac{2}{\sigma^2} \sum_{j=0}^{n} \left[ \frac{\omega_k - j\theta}{\omega_k - j\theta} \right] \left[ \frac{n!}{j!(n-j)!} \right] (-1)^{n-j}. \quad (A.17)$$

The second summation on the right-hand side of Eq. (A.17) is simply the binomial expansion of $(1 - 1)^n$, which is, of course, zero. Using Eq. (A.13) to simplify the first term on the right-hand side of Eq. (A.17) we obtain

$$- n\theta S_{n-1} = (\theta n - \omega_k) S_n. \quad (A.18)$$

Therefore,

$$S_n = \left[ \frac{n\theta}{\omega_k - \theta n} \right] S_{n-1}. \quad (A.19)$$

To solve the difference equation in (A.19), we need a boundary condition. Observe (using Eq. (A.13)) that when $n = 1$, we have

$$S_1 = \frac{2}{\sigma^2} \left[ \left[ \frac{1}{\omega_k} \right] (-1) + \frac{1}{\omega_k - \theta} \right] = \frac{2\theta/\sigma^2}{\omega_k(\omega_k - \theta)}. \quad (A.20)$$

Eqs. (A.19) and (A.20) together imply

$$S_n = \frac{2n! \theta^n/\sigma^2}{[\prod_{j=0}^{n} (\omega_k - j\theta)]}. \quad (A.21)$$

Now use the lemma to rewrite Eqs. (A.11) and (A.12) as

$$G^{(L)}(p) = \left( \frac{q}{b} \right)^{\omega_1/\theta} b^n \frac{2\Gamma n! \theta^n/\sigma^2}{(\omega_1 - \omega_2)\prod_{j=0}^{n} (\omega_1 - j\theta)} \quad (A.22)$$

and

$$G^{(M)}(p) = GP(p) + \left( \frac{q}{b} \right)^{\omega_2/\theta} b^n \frac{2\Gamma n! \theta^n/\sigma^2}{(\omega_1 - \omega_2)\prod_{j=0}^{n} (\omega_2 - j\theta)}, \quad (A.23)$$

which are identical to Eqs. (35) and (36) in the text.
References

Pindyck, R.S., 1988, Irreversible investment, capacity choice, and the value of the firm, American Economic Review 78, 969–985.