Intangible capital, non-rivalry, and growth

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Abstract

We provide an answer to why growth may slow even in the face of technological improvements. Our focus is on the role of intangible assets. Intangible assets are distinct from physical capital in that they are comprised by information that requires a storage medium. A reduction in replication costs for intangible assets enables them to be less rivalrous in use, stimulating growth. However, we show how limits to excludability create a countervailing force. Depending on the strength of property-rights institutions, growth may slow even as technology lowers replication costs for intangibles, enhances their non-rivalry, and creates economies of scale and scope.

Over the past four decades, the importance of intangibles in firms' capital stock has grown substantially (Corrado, Hulten, and Sichel, 2005; Eisfeldt and Papanikolaou, 2014; Crouzet and Eberly, 2019). A key economic feature of intangibles — one that distinguishes them from physical capital — is their *non-rivalry*. When a firm uses an intangible asset for a particular purpose, the asset does not necessarily become unavailable for other uses within the firm. When a firm stamps their brand on a new retail location, the brand remains available to other retail locations. Likewise, when a firm uses one of its patents in a new production facility, the patent remains available to other production facilities. More generally, intangible assets represent information that needs to be stored in order to be used in production (Crouzet, Eberly, Eisfeldt, and Papanikolaou, 2022). Non-rivalry may arise when the technology used to store intangibles makes it easy for the asset to be replicated at low cost within the firm.

Growth theory has long emphasized the fact that the production of non-rival capital goods is a core engine of long-run growth (Romer, 1986, 1990; Jones, 1995). Intuition suggests that the rapid expansion of the intangible capital stock should then have led to a rise in growth overall. In this paper, we argue that this intuition may not be correct.

Our starting point is to note that there are two main differences between ideas, as they are generally treated in endogenous growth models, and intangible assets. First, the intangibles a firm owns may not be *fully* non-rival. The Fourier transform, or the recipe for ketchup (two examples of the type of ideas that are the focus of endogenous growth models) are fully non-rival goods. On the other hand, implementing an inventory management system in a firm's new warehouse requires relocating technical experts there and training employees and management, which might reduce the efficiency with which existing inventory systems are run elsewhere in the company. In other words, given a total stock of intangible capital, increasing the intangibles used in a particular task does not require reducing their use in other tasks one-for-one (as would be the case for physical capital), but it is not entirely costless, either (as would be the case for ideas).

The second difference between ideas, as they are generally treated in endogenous growth models, and intangibles, is that intangibles are partly *excludable*; that is, they generally are not *public* capital goods. In principle, control and cash-flow rights over intangibles can be assigned to particular firms and enforced, just like for physical assets. However, in practice, the degree of excludability is often imperfect. Property rights of certain intangibles, like patents, are well-defined and enforced in many countries, making them effectively excludable capital goods. For other types of intangibles, like managerial methods, organization capital, or even databases and software, excludability is often more difficult to enforce, even in an environment with otherwise well-defined property rights.

The nature of intangibles as information that needs to be stored leads to a unique interplay between technological improvements and institutions that protect property rights. The technological frontier for information storage has evolved from speech to writing to punch cards and finally to electronically-coded software and databases. Speech allowed for intangibles to be alienable from individuals as ideas were communicated to teams. The innovation of writing allowed for intangibles to be stored independently of individuals, increasing their scope and the opportunities for non-rivalrous use. Modern-day electronic data storage, software, and communication allows the same information and processes to be used world-wide at the same time and at large scale. At the same time, the ease with which intangibles can be replicated at high quality and low cost also opens the door to imitation and expropriation. Property-rights institutions can enforce excludability and assign ownership of intangibles so that they can become appropriable assets. Excludability and appropriability have the advantage that incentives to innovate are enhanced, but this comes at a potential cost of lower spillovers.

Our main contribution is to explore the implications of partial non-rivalry and partial excludability for growth. We do this by formalizing these key features and then embedding them into an endogenous growth model. We show that, contrary to what existing intuition from endogenous growth theory might suggest, a higher degree of non-rivalry does not necessarily increase long-run growth. More specifically, the relationship between non-rivalry and growth follows an inverse-U shape. When intangibles are close to rival — and thus resemble physical capital — there are no spillovers from the creation of new intangibles, and thus no growth. Introducing non-rivalry creates growth, but the highest rate of growth in the model is generally *not* achieved when intangibles are fully non-rival, as in the Romer (1990) model. Rather, it generally occurs for an intermediate degree of non-rivalry.

The two main forces that shape equilibrium growth in the model can be broadly described as spillovers and incentives. The spillover channel, described above, is the idea that replication of existing intangibles by new entrants can enhance growth. This is the source of growth generally analyzed in work on endogenous growth, in the case of ideas. In the model, spillovers are affected both by non-rivalry and by excludability. When intangibles are fully excludable, there are no spillovers, and therefore no growth. However, if intangibles are not fully excludable, then non-rivalry can *accelerate* spillovers, by allowing new entrants to replicate the intangible stock of incumbents at a faster rate. This raises returns to the creation of new firms, and thus leads to higher growth. Thus the spillover channel creates a positive relationship between growth and non-rivalry.

The second force is the incentive to create new firms. We model a firm as a collection of product streams (and which can be thought of as retail locations, production facilities, or establishments that the firm has control over), and call the number of product streams the span of the firm. When they are more non-rival, intangibles can be deployed simultaneously across more product streams.

Thus, a higher span raises the marginal return to intangibles, and more so when intangibles are more non-rival. However, by increasing its span, the firm also exposes itself to more competitive risk. In the model, we capture this by assuming that a greater span exposes the firm to a higher risk of expropriation of its product streams by a fringe of imitators. Firms optimally choose span to trade off this competitive risk with the higher returns to intangible capital. As non-rivalry increases, the benefits of higher span grow, so the firm is willing to tolerate a higher degree of competitive risk. Ex-ante, however, the greater competitive risk makes the creation of new firms less attractive. This disincentive effect eventually reduces growth.

Equilibrium growth reflects the balance of these two forces. We show that in general, spillover effects will dominate when intangibles are close to being rival — similar to physical capital —, because in these situations the optimal span of new firms is small and competitive risk is limited. When intangibles are closer to being fully non-rival — similar to ideas in endogenous growth theory —, the optimal span of new firms is high, competitive risk is strong, and disincentive effects are dominant. Nevertheless, we also highlight certain conditions under which spillover effects can strictly dominate, so that the relationship between growth and non-rivalry is increasing. These tend to be situations in which excludability is very weak, so that new firms optimally choose a small span to limit the effects of competitive risk.

A broad implication of the comparative statics of the model is that changes in the cost of storing intangibles, which in our model are equivalent to a change in the degree of non-rivalry of intangibles within the firm, have ambiguous effects on equilibrium growth. Thus, the model highlights the potential downside of, say, improvements in information technology that make the storage of data-related intangible assets easier.

Relation to the literature Our work complements the literature on endogenous growth that starts with Romer (1990). Our contribution is to provide a new model of production using intangible capital and explore its implications for long-run growth. Our model explicitly allows for the degree of non-rivalry and the degree of enforcement of intellectual protection to be structural parameters that could vary with the specific characteristics of the intangible and the technology for transmitting ideas and enforcing property rights. As such, it essentially nests the standard Solow (1956) model where capital is fully rival and there is no long-run growth and the Romer (1990) model in which ideas are fully non-rival and long-run growth obtains. Specifically, we show that the degree of non-rivalry of intangibles maps the continuum between the two types of models and that, surprisingly, it has a non-monotonic effect on growth.

We should emphasize, however, our notion of intangibles is distinct, and likely broader, than

Romer (1990) along two key dimensions. First, we model intangibles as a direct input in production with the firm choosing the scale of implementation subject to concerns about expropriation. By contrast, intangibles in Romer (1990) essentially represent the exclusive right to produce a particular good—or a technology vintage that can be used to produce that good as in Aghion and Howitt (1992). As such, our notion encompasses not only the idea of a new product or specific production process, but also concepts such as organization capital that can deployed at a broader scale or scope. Second, our model features an explicit tradeoff between the scope (or scale) of deployment of the intangible and the likelihood of expropriation by imitators. Indeed, the presence of imitators introduces a further wedge between private and social returns in the creation of new intangibles—by contrast, in Romer (1990) spillovers occur only in the creation of new projects which build on the existing frontier. This tradeoff that limits the scale of deplyoment of intangibles is conceptually distinct from monopolistic competition in Romer (1990) or the need for complementary inputs that are in scarce supply (Atkeson and Kehoe, 2005).

The non-monotonic effect of non-rivalry on long-run growth may be reminiscent of the non-monotonic relation between the degree of competition in the product market and the level of innovation and growth emphasized in Aghion, Bloom, Blundell, Griffith, and Howitt (2005). These two mechanisms have some similarities but are conceptually distinct: the main comparative static we focus on is changes in the degree of non-rivalry within the firm, a notion which is absent in their model. That said, our notion of expropriation by imitators has some conceptual similarities to the loss of rents to competitors. What drives the positive relation between the level of competition and growth in Aghion et al. (2005) is that investing in innovation allows firms to lower the risk of losing monopoly rents—what they term the 'escape the competition effect'. By contrast, in our setting, greater investment in creating intangibles, or deploying them at a greater scale, always increases the risk of expropriation.

Recent advances in this literature have explored the importance of certain non-rival types of capital goods, an in particular data (Farboodi and Veldkamp, 2020; Jones and Tonetti, 2020). For instance, Jones and Tonetti (2020) model the increasing returns that can be achieved through the use of customer data to improve productivity: the more people consume a given product the more data the firm has to improve productivity which leads to higher output and hence more data. Our main contribution relative to this work is to study the role of non-rivalry of intangibles more generally, and to allow for the risk of expropriation and the degree of non-rivalry of intangibles to be partial. Further, our work also relates to work that emphasizes the role of technology diffusion and spillovers (see, e.g. Lucas and Moll, 2014; Stokey, 2015, among others). Relative to these papers, our model emphasizes the dark side of spillovers, namely the fact that innovators cannot fully appropriate the value of their creations.

More broadly, a large literature in macroeconomics and finance has highlighted the macroeconomic implications of the rise in intangible capital for various topics, including investment and asset prices, labor markets, and growth (McGrattan and Prescott, 2010a,b; Haskel and Westlake, 2017; Crouzet and Eberly, 2019, 2021). Our key distinction relative to these papers lies again in how we model the production process using intangibles. Existing work models intangible capital as a factor of production that is qualitatively similar to physical capital—intangible capital is yet another factor of production that can be accumulated subject to adjustment costs. The main difference between tangible and intangible capital in these models is simply due to parameter choices—intangibles are fully excludable and rival capital inputs, but may differ in their rate of depreciation, price, or riskiness. Our contribution to this literature is to revisit the same issues using a different model of production that allows for the key economic differences between physical and intangible capital: their degree of non-rivalry, and their degree of excludability.

Our work also highlights the challenges in measuring intangible assets in the data. Existing work has focused on measuring specific types of intangible capital, including patented innovations (Hall, Jaffe, and Trajtenberg, 2005; Kogan, Papanikolaou, Seru, and Stoffman, 2017; Kelly, Papanikolaou, Seru, and Taddy, 2021); management practices (Bloom and Van Reenen, 2007); software and data-related assets (Bresnahan, Brynjolfsson, and Hitt, 2002); brands and customer capital (Gourio and Rudanko, 2014); or organization capital (Eisfeldt and Papanikolaou, 2013; Bhandari and McGrattan, 2021). Our model shows that the private value of an intangible asset, its acquisition cost, its contribution to aggregate output, and its social value are in general distinct objects that need not coincide.

1 Economic environment

This section describes the model. We start in Section 1.1 by outlining its production side. We then describe labor markets and equilibrium in Section 1.2. We conclude with a discussion of the key assumptions in the model in Section 1.3. For brevity, all proofs and derivations are relegated to Appendix A.1.1.

1.1 Production

Time is continuous. Production takes place within projects, which are indexed by i. We use the notation $\tau(i)$ to denote the date at which project i is created, so that $t - \tau(i)$ is the age of the project. We refer to the owner and creator of the project as the 'entrepreneur', though this term could encompass any stakeholder in the project that has contributed to the creation of its intangible capital, including founders, key employees, or managers. The supply of entrepreneurs is not fixed, but rather there is free entry: households are indifferent between working in production and becoming entrepreneurs and creating new projects.

Existing projects

Each project consists of a collection of product streams, indexed by $s \in [0, x_{i,t}]$, where $x_{i,t}$ is the total number of streams of project i at date t. We will refer to $x_{i,t}$ as the span of the project. Within each project stream, intangible capital, $N_{i,t}(s)$, and labor, $L_{i,t}(s)$, are combined according to a Cobb-Douglas production function with labor share ζ. We start by describing the allocation of labor and intangible capital to different streams. We then discuss how the stock of intangible capital and project span evolve over time.

Static allocation decision At each point in time, the entrepreneur makes two choices: how much total production labor to hire; and how to allocate production labor and intangible capital across product streams. Specifically, they solve:

$$\Pi(x_{i,t}, N_{i,t}; W_t, \rho_{\tau(i)}) = \max_{\{N_{i,t}(s), L_{i,t}(s)\}_{s\in\left[0,x_{i,t}\right]}, L_{i,t}} \int_0^{x_{i,t}} N_{i,t}(s)^{1-\zeta} L_{i,t}(s)^{\zeta} ds - W_t\, L_{i,t} \quad (1)$$

$$\text{s.t.} \quad \int_0^{x_{i,t}} L_{i,t}(s)\, ds \leq L_{i,t} \quad (2)$$

$$\left(\int_0^{x_{i,t}} N_{i,t}(s)^{\frac{1}{1-\rho_{\tau(i)}}}\, ds\right)^{1-\rho_{\tau(i)}} \leq N_{i,t} \quad (3)$$

Here, $N_{i,t}$ denotes the total intangible capital stock of the project, and W_t the wage rate. We make the following central assumption about the parameter $\rho_{\tau(i)}$.

Assumption 1 (Non-rivalry)**.** *The parameter $\rho_{\tau(i)}$ satisfies $0 \leq \rho_{\tau(i)} \leq 1$.*

To understand the role of this assumption, first consider how labor is allocated across production streams. Given a total labor input $L_{i,t}$, adding more labor to a stream requires removing it from another. Thus, the resource constraint (2) implies that labor is perfectly rival within project.

Assumption 1 allows intangibles to be partially non-rival within a project. When $\rho_{\tau(i)} = 0$, the resource constraints for labor (2) and intangibles (3) are the same, as they are both rival inputs; given the total stock of intangibles, adding more intangible capital to a product stream requires taking it away from another one. By contrast, in the special case where intangibles are fully non-rival

$(\rho_{\tau(i)} = 1)$ the constraint (3) becomes

$$\max_{s \in [0, x_{i,t}]} N_{i,t}(s) \leq N_{i,t}. \tag{4}$$

In this case, adding intangibles to a particular stream does not require reducing the input in any other stream: intangible capital is fully non-rival within project.

More broadly, when $\rho_{\tau(i)} \in (0,1)$, intangibles are partially non-rival within the project. The marginal rate of technical substitution between streams is less than 1 (as when $\rho_{\tau(i)} = 0$, or for labor) but greater than 0 (as $\rho_{\tau(i)} = 1$, or fully rival intangibles). Thus we associate the parameter $\rho_{\tau(i)}$ with the degree of non-rivalry of intangibles within project i. We allow the parameter $\rho_{\tau(i)}$ to vary across cohorts of projects, but we assume that its value is fixed within project, because it reflects the particular technological features of the of the technology used to store and reallocate intangibles within the project.

The solution to the allocation problem (1) is symmetric across product streams:

$$\begin{aligned} \forall s \in [0, x_{i,t}], \quad N_{i,t}(s) &= x_{i,t}^{-(1-\rho_{\tau(i)})} N_{i,t}, && (5) \\ L_{i,t}(s) &= \left(\frac{\Lambda_t}{1-\zeta}\right)^{\frac{1}{\zeta}} N_{i,t}(s), && (6) \end{aligned}$$

leading to the following expressions for the total level of profits and labor demand for the project

$$\begin{aligned} L_{i,t} &= x_{i,t}^{\rho_{\tau(i)}} \left(\frac{\Lambda_t}{1-\zeta}\right)^{\frac{1}{\zeta}} N_{i,t}, && (7) \\ \Pi_{i,t} &= x_{i,t}^{\rho_{\tau(i)}} \Lambda_t N_{i,t}\,, && (8) \end{aligned}$$

where we defined:

$$\Lambda_t \equiv (1-\zeta)\left(\frac{\zeta}{W_t}\right)^{\frac{\zeta}{1-\zeta}}. \tag{9}$$

Expropriation risk and the evolution of the project over time After the project is created, the entrepreneur faces a risk of expropriation in each of its product streams. If the entrepreneur is expropriated, ownership of the production stream is transferred to an imitator (discussed below) and the entrepreneur looses ownership of intangibles that had been used in that stream. Expropriation entails a transfer of value from the entrepreneur to a fringe of current imitators, and to future entrepreneurs, due to spillovers. We next discuss this process in more detail.

Assumption 2 (Limits to excludability)**.** *Within each stream, expropriation follows a Poisson process with intensity $\delta(x_{i,\tau(i)})$, where $x_{i,\tau(i)}$ is the initial span of the project. Moreover, expropriation* *is independent across streams. Finally, the function* $\delta(.)$ *satisfies:*

$$\delta(0) \geq 0, \quad \lim_{x \to +\infty} \delta(x) = +\infty, \quad \delta'(x) > 0, \quad \textit{and } \delta''(x) > 0.$$

We associate the function $\delta(.)$ with the degree of excludability of intangibles. The case $\delta(.) = 0$ would correspond to perfect excludability, as no product stream is ever expropriated. On the other hand, $\delta(.) = +\infty$ corresponds to no excludability: immediately after the project is created, all product streams are expropriated.

Assumption 2 implies that at time $t \geq \tau(i)$, the entrepreneur retains control over a stock of intangibles of:

$$N_{i,t} = e^{-\delta(x_{i,\tau(i)})(1-\rho_{\tau(i)})(t-\tau(i))} N_{i,\tau(i)}. \tag{10}$$

Equation (10) highlights the interaction between excludability and non-rivalry. In the case of perfect non-rivalry ($\rho_{\tau(i)} = 1$), expropriation does not affect the stock of intangibles that the entrepreneur owns at all, since $N_{i,t} = N_{i,\tau(i)}$. In this case, the entrepreneur uses the total stock of intangibles of the project in *each* stream. Even if the entrepreneur has been expropriated from one its product streams, expropriation doesn't reduce the stock of intangibles available for the remaining streams. By contrast, in the case in which intangibles are fully rival ($\rho_{\tau(i)} = 0$), the stock of intangibles shrinks at the rate $\delta(x_{i,\tau(i)})$, which is increasing in the number of product streams. In this case expropriation reduces one-for-one the stock of intangibles available for the remaining streams.

Similarly, the fact that the entrepreneur is expropriated implies that the span x under the entrepreneur's control follows

$$x_{i,t} = e^{-\delta(x_{i,\tau(i)})(t-\tau(i))} x_{i,\tau(i)}. \tag{11}$$

As a direct result of equations (10) and (11), the cashflows that the entrepreneur retains rights at $t \geq \tau(i)$ are given by

$$\begin{aligned} \Pi_{i,t} &= e^{-\delta(x_{i,\tau(i)})(t-\tau(i))} \, \Lambda_t \, x_{i,\tau(i)}^{\rho_{\tau(i)}} \, N_{i,\tau(i)} \\ &= e^{-\delta(x_{i,\tau(i)})(t-\tau(i))} \, \frac{\Lambda_t}{\Lambda_{\tau(i)}} \Pi_{i,\tau(i)} \end{aligned} \tag{12}$$

Examining equation (12), we note that the cashflows to the entrepreneur decrease over time for two reasons. First, the market price of labor changes as newly created projects compete for workers with existing projects; this effect is captured by the evolution of the term Λ_t. Existing projects do not depreciate, thus over time as the measure of projects that compete for workers increases, wages increase, which implies that Λ_t has a negative drift. Second, the entrepreneur progressively loses control of the project to imitators; from her perspective, this is equivalent to the cashflows depreciating at a rate $\delta(x_{i,\tau(i)})$.

Spillovers *Spillover intangibles* from project i determine the stock of knowledge that can be freely built on to create new intangibles. We define these spillovers $S_{i,t}$ associated with project i implicitly:

$$N_{i,\tau(i)} \equiv \left(S_{i,t}^{\frac{1}{1-\rho_{\tau(i)}}} + N_{i,t}^{\frac{1}{1-\rho_{\tau(i)}}} \right)^{1-\rho_{\tau(i)}}. \tag{13}$$

Equation (13) defines the stock of spillover intangibles $S_{i,t}$ so that the total intangibles associated with project i (those still used by the entrepreneur, and those that have been expropriated), cannot exceed the initial quantity that the entrepreneur created, $N_{i,\tau(i)}$. Thus the total intangibles initially invested in the project are conserved over time, though their ownership is reallocated over time from the entrepreneur to the rest of the economy (imitators and future entrepreneurs).

Using the law of motion for $N_{i,t}$, we obtain:

$$S_{i,t} = \left(1 - e^{-\delta(x_{i,\tau(i)})(t-\tau(i))}\right)^{1-\rho_{\tau(i)}} N_{i,\tau(i)}. \tag{14}$$

Note that the degree of non-rivalry ρ affects the *speed* at which spillovers from the project are generated. Specifically:

$$S_{i,t} = \begin{cases} N_{i,\tau(i)} - N_{i,t} & \text{if} \quad \rho_{\tau(i)} = 0 \\ N_{i,\tau(i)} & \text{if} \quad \rho_{\tau(i)} = 1 \end{cases}$$

When $\rho_{\tau(i)} = 1$, a new project immediately generates the maximum level of spillovers, since $S_{i,t} = N_{i,\tau(i)}$, regardless of the degree of excludability δ. Recall that when $\rho_{\tau(i)} = 1$, each stream contains the entire intangible stock of the project. Expropriation of any stream immediately makes the whole intangible stock of the project available to outsiders. By contrast, when $\rho_{\tau(i)} = 0$, as the project's stock of intangibles decreases through expropriation, spillover intangibles increase one-for-one. In this case, spillovers happen gradually, at rate $\delta(x_{i,\tau(i)})$.

Last, we define the aggregate stock of spillover intangibles S_t as:

$$S_t = \int_{\tau(i)\leq t} S_{i,t} di = \int_{\tau(i)\leq t} N_{i,\tau(i)}(1 - e^{-\delta(t-\tau(i))})^{1-\rho_{\tau(i)}} di. \tag{15}$$

As we will discuss next, the total amount of spillover intangibles in the economy (15) will be a key input in the creation of new intangibles.

Entrepreneurs and new projects

The first group of agents that benefit from spillovers are the entrepreneurs that create new projects. Entrepreneurs discount future cash flows at rate r. Given an initial span of x and an initial intangible capital input of N, the value of project i at the time of its creation is, $t = \tau(i)$, is:

$$V_t(x, N) = \mathbb{E}_t \left[\int_t^{+\infty} e^{-r(s-t)} \, \Pi_{i,s} \, dt \right] = \Lambda_t \, x^{\rho_t} \, N \, \tilde{v}_t(x) \tag{16}$$

where

$$\tilde{v}_t(x) \equiv \mathbb{E}_t \left[\int_t^{\infty} e^{-(r+\delta(x))(s-t)} \, \frac{\Lambda_s}{\Lambda_t} \, ds \right]. \tag{17}$$

In examining equation (17), we note that the dependence on the initial size of the project x is purely through the resulting risk of expropriation $\delta(x)$.

Starting a new project requires that an entrepreneur spend a unit of labor. The labor spent leads to the same stock of intangibles for all projects that are created at time $t = \tau(i)$, so there is no idiosyncratic risk in the creation of new projects.

Spillover intangibles facilitate the creation of new intangibles. As such, the initial stock of intangibles for a new project created at time t is given by:

$$N_t^e = \nu \, S_t. \tag{18}$$

Here, the parameter ν, which is fixed and exogenous, governs the size of new projects relative to the overall stock of spillover intangibles. The fact that ν is constant helps ensures the existence of a balanced growth path. One can think of this equation as analogous to the equation in a standard neoclassical growth model stating that the investment rate at the intensive margin is constant, since it assumes that the ratio the relationship between the existing capital stock (in this case, the part of the capital stock that generates spillovers, S_t), and new intangibles (in this case, N_t^e) is constant.

The entrepreneur chooses the size of the span of the project x to maximize the total value she can appropriate. Given that all projects newly created at time t are endowed with the same stock of intangibles and face the same wage rate thereafter, the value of new projects (to entrepreneurs) and their optimal span is the same for all new projects created at time t. In particular, the value of a newly-created project at time t is given by

$$V_t = V_t(x, N_t^e) = V_t(x, \nu S_t) = \Lambda_t \, x_t^{\rho_t} \, \nu \, S_t \, \tilde{v}_t(x_t), \tag{19}$$

The optimal choice of project span is determined when the project is created and equals

$$x_t = \arg\max_x \, x^{\rho_t} \, \tilde{v}_t(x). \tag{20}$$

The trade-off between scale and the risk of expropriation will ensure that entrepreneurs choose a finite span. In choosing span in (20), the entrepreneur faces a trade-off between two forces. A higher span raises the marginal revenue product of intangibles (assuming some degree of non-rivalry, $\rho_t > 0$), which is reflected in the term x^{ρ_t} in the equation above. At the same time, a higher span x also increases the likelihood of expropriation within each product stream, $\delta(x)$, thus lowering the marginal revenue product of intangibles. This cost is reflected in the term $\tilde{v}_t(x)$ in the expression above. Since the optimal span is the same for all newly-created projects at time t, $x_{\tau(i),i} = x_t$ for all i such that $\tau(i) = t$, the risk of expropriation is also identical. To simplify notation, in what follows we will denote:

$$\delta_t \equiv \delta(x_t) \quad \text{and} \quad v_t \equiv \tilde{v}_t(x_t). \tag{21}$$

Imitators

The second group of agents that benefit from spillovers are the "imitators". When the product line of a project is expropriated, we assume that it is taken over by a fringe of imitators. These imitators then produce output using the projects' intangibles, but only in the production streams that were expropriated from the entrepreneur. Moreover, they do not face any expropriation risk themselves.

Denote by i^c the imitator associated with each project i. The span of i^c is the complement to the span of the original project, while imitators produce using the stock of spillover intangibles from the project, $S_{i,t}$. Thus their profits are given by:

$$\Pi_{i^c,t} = \Lambda_t \, x_{\tau(i)}^{\rho_{\tau(i)}} \, N_{\tau(i)} - \Pi_{i,t}. \tag{22}$$

Given the above, we can write the total value of the project when it is created as

$$\overline{V}_t = V_t^c + V_t = \Lambda_t \, x_t^{\rho_t} \, \nu \, S_t \, \overline{v}_t \tag{23}$$

where

$$\overline{v}_t \equiv \mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} \, \frac{\Lambda_s}{\Lambda_t} \, ds \right]. \tag{24}$$

Here, the total value of the project equals the sum of the value of the project to the entrepreneur that creates it, and to imitators that eventually benefit from it. Note that this is not the social value of the project, since it does not include the value created by raising the stock of spillover intangibles on which future entrepreneurs will eventually build. Comparing equation (23) to (16), we see that the entrepreneur can appropriate a fraction

$$\theta(x) = \frac{\tilde{v}_t(x)}{\overline{v}_t} \tag{25}$$

of the total value of her project given her choice of span x. Equivalently, the span chosen by the entrepreneur satisfies

$$x_t \;=\; \arg\max_x \; x^{\rho_t}\, \theta_t(x). \tag{26}$$

This re-formulation of the entrepreneur's optimal span choice highlights the fact that by choosing a higher span, and therefore raising expropriation risk and redistributing surplus to imitators, leads the entrepreneur to retain a *smaller* fraction of the project's overall value.

1.2 Labor markets and equilibrium

Here, we discuss the labor market and the equilibrium of the model.

Labor can be allocated to two activities: creating new projects, and producing output. We denote by $L_{e,t}$ the amount of labor allocated to the creation of projects. We assume that labor is perfectly substitutable across the two activities, and we normalize total labor supply to 1 throughout, so that $1 - L_{e,t}$ is the amount of labor allocated to production.

There is free entry in the creation of new projects. Since the creation of each project requires spending one unit of labor, free-entry implies that

$$W_t = \Lambda_t\, x_t^{\rho_t}\, \nu\, S_t\, v_t. \tag{27}$$

Total demand for production labor is the sum of demand for labor from each project:

$$1 - L_{e,t} = \left(\frac{\Lambda_t}{1-\zeta}\right)^{\frac{1}{\zeta}} \nu \int_{\tau \le t} x_\tau^{\rho_\tau}\, S_\tau\, L_{e,\tau}\, d\tau. \tag{28}$$

In deriving (28), we exploit the fact that newly-created projects at time t start with the same stock of intangibles $\nu\, S_t$; that each entrepreneurial unit of labor creates one project; and that both entrepreneurs and imitators produce, and therefore have demand for production labor.

Finally, note that given that new projects in a cohort are identical, the law of motion for the stock of spillover intangibles can be rewritten as:

$$S_t = \nu \int_{\tau \le t} S_\tau (1 - e^{-\delta_\tau (t-\tau)})^{1-\rho_\tau} \, L_{e,\tau} \, d\tau. \tag{29}$$

In examining (29), it is useful to characterize the evolution of the stock of spillover intangibles. The following lemma characterizes the growth rate of S_t. Appendix A.1.1 contains the proof.

Lemma 1 (Evolution of spillovers). *Assume that $\rho_t < 1$ for all t. Then the rate of change in the stock of spillover intangibles, g_t, is given by:*

$$\frac{dS_t}{S_t} = g_t \, dt, \tag{30}$$

where:

$$g_t \equiv \nu \int_{\tau \le t} L_{e,\tau} \, \frac{S_\tau}{S_t} (1 - \rho_\tau) \delta_\tau \left(1 - e^{-\delta_\tau \, (t-\tau)}\right)^{-\rho_\tau} e^{-\,\delta_\tau (t-\tau)} d\tau. \tag{31}$$

The lemma has two implications which will be useful when analyzing the balanced growth path.

First, regardless of the process governing ρ_t, but so long as $\rho_t < 1$, then the change in the stock of spillovers is of order dt. The contribution of *existing* projects (those created before time t) to spillovers is of order dt because $\rho_{\tau(i)}$ is fixed within project and expropriation follows a Poisson process. Additionally, when $\rho_t < 1$, new projects (those created at time t) do not immediately contribute to aggregate spillovers; some time must pass before expropriation starts.[1] By contrast, if $\rho_t = 1$, S_t need not have continuous sample paths but can potentially jump, because newly created projects immediately generated spillovers.

Second, equation (31) implies that the drift is a weighted average, across projects from different cohorts τ, of the rate of change in the stock of spillovers created by each cohort. The weights in the average depend on $L_{e,\tau}$, which is the number of projects in that cohort.

We define equilibrium in a standard way, with Appendix A.1.2 providing further details.

Definition 1 (Equilibrium). *Given an exogenous stochastic process for the degree of non-rivalry, $\{\rho_t\}$, an equilibrium of this model is a set of six stochastic processes $\{L_{e,t}, \Lambda_t, W_t, v_t, x_t, S_t\}$ that satisfy Equation (9), the definition of the marginal revenue product of intangibles within each stream; Equations (20) and (21), the value of a new project to entrepreneurs; Equation (27), the free-entry condition; Equation (28), the aggregate demand for production labor; and Equation (29), the law of motion for spillover intangibles.*

Note that we have left the consumption side of the model unspecified. Given the assumption that

[1]Formally, using the law of motion for spillovers (14), we can see that $S_{i,t} \to 0$ as $t \to \tau(i)$ so long as $\rho_{\tau(i)} < 1$.

the interest rate r is fixed, the model can be interpreted as representing an economy with an infinite elasticity of intertemporal substitution and a time discount rate of r. It would be straightforward to extend the model to allow for a finite intertemporal elasticity of substitution.

1.3 Discussion of key assumptions

Here, we discuss the key assumptions of the model.

Non-rivalry We view intangible capital as information that must be stored in a particular medium in order to be used in production. The technology for storing this information makes intangible capital *partially* non-rival within a firm. Our assumption of partial non-rivalry (Assumption 1) captures this idea. The two edge cases $\rho = 0$ and $\rho = 1$ correspond, respectively, to completely rival and completely non-rival intangibles. The case $\rho = 0$ makes intangible capital exactly analogous to physical capital, while the case $\rho = 1$ makes intangible capital analogous to total factor productivity, or ideas in endogenous growth model — which are generally treated as a non-rival within firm.

The current state of storage technology for a specific intangible asset determines ρ. As a first example of intangible assets, consider past data on the purchasing behavior of a firm's customers, which are used in the firm's call centers to sell additional products. In the past, customer data was stored using pen and paper. Currently, customer data is stored in computer memory, which makes it significantly easier to replicate the data and make it available across all call centers in the firm. As such, we argue that this corresponds to the case of an intangible asset with relatively high ρ; further, the availability of computers has likely increased ρ for customer data over time as they can be more easily stored and disseminated.

As a second example, consider a particular method for organizing production, which is used within a firm's production plants. In this case, the intangible is stored in key employees, since it is part of the human capital of assembly-line workers and production managers. Replicating the production method and making it available to other plants within the firm will require supervision and training, which may be costly or imperfect. Even if the firm could hire and train a sufficient number of production managers, something may still be lost in translation. In our view, this corresponds to the case of an intangible asset with lower ρ. Further, advances in the technology for communicating instructions on the new production method to workers and supervising them will likely increase the intangible's ρ.

In general, we view the degree of non-rivalry of the intangible asset as an intrinsic feature of the storage technology used to create it. Since intangibles are fixed within project, we assume that ρ is also constant within a project. However, we allow it to potentially change over time as the technology for storing intangibles improves.

Last, we have implicitly assumed in equation (29) that the total amount of spillovers in the economy, the variable S_t, is the sum of spillovers across existing projects—or stated differently, spillover intangibles are substitutes across projects. We make this assumption because ρ may differ across projects; that is, the storage technology used to create the intangible assets may change. As a result, spillovers from later cohorts may not necessarily enhance those of prior cohorts.

Limits to excludability Our second key assumption (Assumption 2) is that the institutions and contracts that assign control and cash flow rights to intangible assets may be imperfect.[2] Importantly, the effects of contractual incompleteness can be exacerbated by non-rivalry.

To see this interaction, let us revisit the expression for the spillovers generated by an individual project (14). Consider two hypothetical projects, $i = 1, 2$, created at the same date that face the same degree of excludability δ but have different levels of non-rivalry $\rho_{\tau(2)} = \rho_2 > \rho_1 = \rho_{\tau(1)}$. The ratio of the two spillovers generated from these projects equals

$$\frac{S_{2,t}}{S_{1,t}} = \left(\frac{1}{1 - e^{-\delta t}}\right)^{\rho_2 - \rho_1} > 1. \tag{32}$$

Equation (32) indicates that, for a given level of δ, expropriation risk, spillovers build up more rapidly when non-rivalry is higher.[3] Put differently, for any degree of excludability, δ, the ease in which intangibles can be stored, ρ, determines the *speed* at which spillovers occurs. When ρ is close to 1, spillovers happen very quickly, while they happen more slowly when ρ is closer to zero.

Our formulation for $\delta(x)$ assumes that expropriation risk is constant within a project and increases with the initial span of the project. Our assumption that $\delta'(x) > 0$ captures the idea that there are increasing returns to imitation: a competitor might find it easier to copy a project's intangibles if it observes more of the product streams at once, rather than a few product streams individually. The assumption that expropriation risk within project is constant is primarily for tractability; time-dependence in expropriation risk within a project would complicate the analysis but deliver no further insights. That said, since ρ_t may vary over time, different cohorts of projects will face different expropriation risk.

Last, an important assumption regarding limited excludability is that the intangibles that create

[2] Incomplete or unenforceable contracts may also affect ownership of physical capital; imperfect excludability is not specific to intangibles. In fact, the model allows for imperfect property rights over physical capital, which would correspond to a case in which the function δ is strictly positive, but $\rho = 0$. In this case, the evolution of $N_{i,t}$ within a project would represent the progressive expropriation of entrepreneurs from their physical assets.

[3] This example is only meant to illustrate the interaction between the function δ and the parameter ρ. In the full model, the rate of expropriation $\delta_{\tau(i)}$ is a function of the optimal span of the project, which depends on the parameter ρ, so that the comparative static described in this example is more complex. But the basic insight from the example—that for a given level of expropriation risk, the rate at which spillovers build up increases with ρ—would remain.

spillovers for new entrants are *only* those that have already been expropriated, as opposed to the *entire* stock of intangibles, including intangibles used by original project streams that have not yet been expropriated. This assumption sharply distinguishes our model from typical models of endogenous growth, where the entire stock of ideas is a public capital input. Clearly, this may be viewed as an extreme assumption; however, it is meant to capture the idea that there is a downside to strong intellectual property rights enforcement in that it may stifle innovation. In Section 2, we discuss a variant of the model in which the entire stock of intangibles is a public capital input — as in endogenous growth models —, but where entrepreneurs only retain ownership over a fraction of the total value of the project they create. Non-rivalry then tends to depress equilibrium growth. Assuming that the entire stock of intangibles creates spillovers for new projects, but spillover intangibles help more (perhaps because they are in the public domain) would deliver similar insights at the cost of less analytical tractability.

Other assumptions We have made several additional assumptions, two of which are worth discussing in detail. First, we have introduced a fringe of imitators. The imitator fringe does not add to the stock of intangibles, so that they do not have a direct effect on growth. However, the imitators take over the expropriated product streams, and these streams continue generating output rather than being destroyed. Given that these streams employ labor, the existence of these imitators has an indirect (negative) effect on growth: they reduce the amount of labor available for the creation of new projects. The assumption that these projects continue to generate output distinguishes expropriation from capital depreciation and implies that the entrepreneur captures less than the full value of her project. That said, a model in which these production streams were destroyed would deliver similar comparative statics—the fact that imitators and entrepreneurs compete for workers is not a key driver of our results.

Second, we have assumed that intangible capital is fixed within a project, and in particular, cannot be accumulated after date 0, and does not depreciate after date 0. Aggregate intangible capital grows over time, as a result of new cohorts of projects being created, so that the model can still speak to the behavior of aggregate investment. The assumption of no exogenous depreciation is without loss of generality; introducing depreciation complicates the notation, without adding substantial insight. The assumption that all new investment happens in the extensive margin is in line with models of putty-clay technology or models with vintage capital (see, e.g. Kogan, Papanikolaou, and Stoffman, 2020, for an example of a model with a similar structure). We view this as a natural assumption in which the technology for storing intangibles is allowed to vary over time but is specific within a project. Drawing the analogy with models with vintage-specific technical change, an increase in ρ in our model has a similar effect: improvements in the technology for storing intangibles displaces existing projects in favor of newly-created and future projects. That said, we should emphasize the distinction between projects and firms. The model as described has no firms per se. Firms can be arbitrary collections of projects; with additional assumptions on the boundary of the firm we could define firms in the model, but this need not affect the aggregate implications of the model.

2 Balanced growth path

In this section, we discuss the properties of the balanced growth path of the model. We assume that the parameter ρ that governs the storage technology for intangibles is constant,

$$\forall t, \quad \rho_t = \rho, \quad \rho \in [0,1].$$

To keep the exposition concise, we relegate all proofs to Appendix A.2.1.

2.1 Equilibrium growth

We start by clarifying the forces that determine equilibrium growth, and discuss their dependence on the degree of non-rivalry (ρ) and the limits to excludability (the shape of the function $\delta(x)$).

Lemma 2 (Balanced growth path)**.** *A balanced growth path (BGP) is an equilibrium in which $L_{e,t}, x_t$, and v_t are constant while Λ_t, W_t, and S_t are growing at constant rates. For all $\rho \in [0,1]$, if $\nu \geq \frac{\zeta}{1-\zeta}\left(r + \underline{\delta}(\rho)\right)$, then there exists a unique BGP.*

The proof of this lemma is reported in Appendix A.2.1, which also gives the expression for the reduced-form parameter $\underline{\delta}(\rho)$ as a function of other structural parameters.[4] In what follows we give a heuristic description of the forces that help determine growth on the BGP.

Key mechanisms First, let $g_t = g$ denote the growth rate of the stock of spillover intangibles. Using Equation (31) from Lemma 1, and the fact that in a BGP entrepreneurial labor $L_{e,t}$ is constant, we see that, given a value for L_e, g is the (unique) solution to the following equation

$$g = \underbrace{\nu\,(1-\rho)\,\mathcal{B}\left(1+\frac{g}{\delta}, 1-\rho\right)}_{n(g;\rho,\delta)} L_e, \tag{33}$$

[4]In short, $\underline{\delta}$ is the expropriation risk corresponding to zero growth, $\delta(\underline{x}(\rho))$, where $\underline{x}(\rho)\delta'(\underline{x}(\rho)) - \rho\delta(\underline{x}(\rho)) = \rho r$.

where, with some abuse of notation, we used $\delta = \delta(x)$ as the short-hand for the constant expropriation risk on the BGP, and where $\mathcal{B}$ is the Beta function. Here, the function $n(g; \rho, \delta)$ captures the return to entrepreneurial labor. Alternatively, since entrepreneurial labor builds on spillovers from existing projects, we can also think of the function n as capturing the intensity of spillovers on the BGP. For a given level of spillover intensity, n, Equation (33) can also be thought of as a demand curve for entrepreneurial labor: a higher rate of spillovers n reduces entrepreneurial labor needed to achieve a particular growth rate,

$$L_e^{(d)}(g; \underset{-}{n}) = \frac{g}{n}. \tag{34}$$

Combining the free-entry condition (27) with the definition of the marginal revenue product of intangibles within stream (9), we see that in a BGP:

$$d\Lambda_t = -\zeta \, g \, \Lambda_t \, dt. \tag{35}$$

Intuitively, for each project, the marginal revenue product of intangibles in each stream is falling because wages are rising — as newer projects with larger intangible inputs are created the demand for labor increases while supply is constant. In turn, this implies that normalized value of a project is given by:

$$\overline{v}_t = \frac{1}{r + \zeta \, g}. \tag{36}$$

Similarly, the normalized value of a project of size x to the entrepreneur equals

$$\tilde{v}(x) \equiv \frac{1}{r + \delta(x) + \zeta \, g}. \tag{37}$$

The entrepreneur chooses the optimal span to maximize (20), which implies the following first-order condition on the BGP,

$$x \, \delta'(x) \tilde{v}(x) = \rho. \tag{38}$$

The left-hand side of this expression captures the costs of higher span — more expropriation risk — while the right-hand side captures the benefits — a higher marginal revenue product of intangible capital, which increases with ρ.

The share of project value retained by the entrepreneur on the BGP equals

$$\theta = \frac{r + \zeta \, g}{r + \delta(x) + \zeta \, g} = 1 - \rho \left(\frac{x \, \delta'(x)}{\delta(x)} \right)^{-1}. \tag{39}$$

This expression helps illustrate the effects of ρ and δ on the optimal share of project retained by the entrepreneur. Keeping the elasticity of $\delta(x)$ fixed, a higher degree of non-rivalry, ρ will tend to reduce the share of total project value accruing to the entrepreneur. This is because when ρ is higher, the entrepreneur generally chooses to operate at a larger span — which increases the total project value — but to keep a smaller share of that larger pie. This is optimal from the standpoint of a committed entrepreneur, but it changes the *relative* returns to entrepreneurial vs. production labor, and thus lowers the supply of entrepreneurial labor. Conversely, any factor that raises the elasticity of δ with respect to x will tend to increase the optimal share of the project retained by the entrepreneur. With a higher elasticity of δ with respect to x, the entrepreneur generally chooses a lower span, in order to avoid expropriation risk. The reduced expropriation risk means that, while the entrepreneur will create an initially smaller project, they will also retain a larger fraction of that smaller pie.

The supply of entrepreneurial labor is determined from the combination of the free-entry condition, equation (27) — which pins down the opportunity cost of using labor in production — with the total labor used in production, equation (28). Together, these two conditions can be written as:

$$L_e^{(s)}(g; \underset{+}{n}, \underset{+}{\theta}) = 1 - \frac{\zeta}{1-\zeta} \int_{\tau \leq t} \frac{S_\tau}{v\, S_t} d\tau = 1 - \frac{\zeta}{1-\zeta} \frac{r + \zeta g}{n\, \theta}. \tag{40}$$

This equation captures the two forces determining the supply of entrepreneurial labor, specifically the ratio of the total stock of intangibles used in production (which determines its opportunity cost) and the average size of new projects (which determines the marginal return to entrepreneurship). The spillover intensity n increases the supply of entrepreneurial labor because, by making the stock of intangibles in the newest (marginal) project higher than in existing projects, it raises the relative returns to entrepreneurial labor.

Equation (40) describes the amount of entrepreneurial labor that is supplied at any given growth rate g. This labor supply schedule declines with growth rates, because higher growth rates imply higher wage growth, which in turn depresses the returns to projects. It increases with spillover intensity n, which raises the productivity of new projects relative to existing ones, and therefore the productivity of entrepreneurial relative to production labor. Finally, and importantly, the labor supply schedule depends positively on the share of project value that is retained by the entrepreneur. Intuitively, when the entrepreneur is able to retain a larger share of overall project value at inception, there is a stronger incentive to create new projects.

Figure 1 illustrates the determination of the equilibrium growth rate $\hat{g}$ as the unique point at which the two schedules (34) and (40) intersect. The graph highlights that two main economic forces drive equilibrium growth in the model: the intensity of spillovers, n, and the fraction of overall project value that each entrepreneur retains, θ. Each has an opposing effect on equilibrium growth. Holding θ constant, higher spillovers reduce entrepreneurial labor demand and increase labor supply,

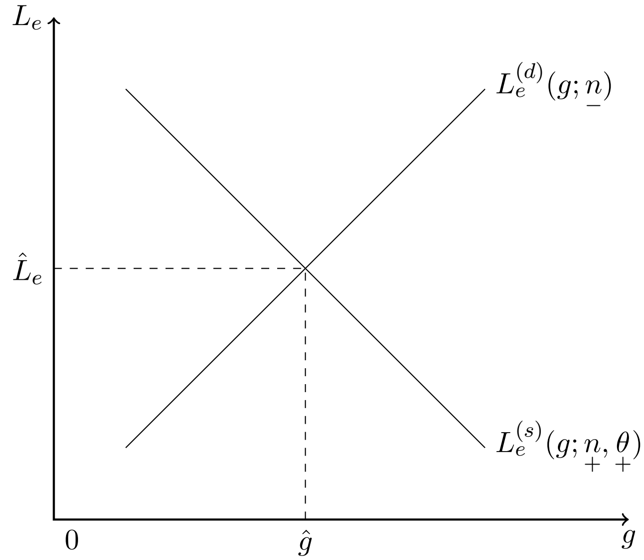

Figure 1: Determination of equilibrium growth rates. The entrepreneurial labor supply curve is described in Equation (40), and the labor demand curve is described in Equation (34).

unambiguously leading to higher equilibrium growth. By contrast, a lower share of project value retained by the entrepreneur θ reduces the supply of entrepreneurial labor, unambiguously leading to lower equilibrium growth rates.[5]

Both the intensity of spillovers n and the fraction of project value retained by the entrepreneur θ are related to the degree of non-rivalry ρ and the limits of excludability, as captured by the function $\delta(x)$. To illustrate the interaction between ρ and δ and how they shape we growth, we next consider two simple variants of the model. In each variant of the model only one of the two key mechanisms—spillover intensity n and the share of project value retained θ—depend on ρ and δ.

A version with $\theta = 1$. Consider first a model in which free-riding by the imitator fringe is eliminated. Assume that in order to enter, the imitator fringe must also spend a fixed amount of labor. Let the cost of entry in labor units be $1-\gamma$ for imitators, and γ for entrepreneurs (they sum to 1 so as to keep this variant comparable to our baseline model). Eliminating the asymmetry between entrepreneurs and imitators leads to two respective free-entry conditions,

$$\begin{aligned}(1-\gamma)\,W_t &= \Lambda_t\, x^{\rho}\,\nu\, S_t\,(\overline{v}-v),\\ \gamma\, W_t &= \Lambda_t\, x^{\rho}\,\nu\, S_t\, v,\end{aligned}$$

[5]Note that this description is somewhat heuristic, since both n and θ are functions of the equilibrium growth rate, g. However, inspecting Equations (34) and (40), we see that so long as n and $n\hat{\theta}$ are weakly decreasing functions of g, then the demand and supply schedules will be monotonic. The proof in Appendix A.2.1 establishes this.

which combined imply the following labor supply schedule for total labor devoted to new projects (by both imitators and entrepreneurs)

$$L_n^{(s)}(g; \underset{+}{n}) = 1 - \frac{\zeta}{1-\zeta}\frac{r+\zeta g}{n}. \tag{41}$$

This is identical to the supply equation (40), except that $\theta = 1$. By contrast, the labor demand equation is unchanged from (34). Thus equilibrium growth would be the same as in the baseline model if θ where constrained to be 1, that is, if entrepreneurs retained all project value.

Appendix A.2.1 shows that the function $n(g; \delta, \rho)$ satisfies:

$$\frac{\partial n}{\partial \rho} \geq 0 \quad \text{and} \quad \frac{\partial n}{\partial \delta} \geq 0. \tag{42}$$

Intuitively, as non-rivalry increases (higher ρ), spillovers accelerate; upon expropriation, imitators capture more of the stock of intangibles of existing projects. Likewise, when limits to excludability increase (higher δ), expropriation risk increases, so that spillover intensity rises. Thus, in this version of the model, higher ρ and higher δ are unambiguously positive for growth as they accelerate spillovers from old to new projects.

A version with constant spillover intensity Next, consider a variant of the model in which the stock of intangible capital of new projects is instead given by:

$$N_t^e = \nu \int_{\tau \leq t} N_\tau \, L_{e,\tau} \, d\tau. \tag{43}$$

In this model, new projects, when they are born, have a size equal to a fraction ν of the *total* intangible capital invested in existing projects, and currently used by both imitators and entrepreneurs. Thus in this version of the model, intangibles are made freely available to new projects, so that non-rivalry and limits to excludability, while they continue to impact the relationship between existing entrepreneurs and imitators, have no bearing on spillovers to new entrepreneurs. In a balanced growth path, letting g now denote the growth rate of N_t^e, we have

$$n = \nu. \tag{44}$$

Thus in this variant of the model, spillover intensity n is independent of ρ and δ. Their only impact on equilibrium growth is through the effect on the distribution of project value between entrepreneurs and imitators, θ. Recalling the discussion around Figure 1, we can now conclude that any change in ρ or δ that raises θ will increase equilibrium growth, by making the incentive for entrepreneurs to enter stronger; conversely, any change in ρ or δ that lower θ will depress growth, by weaking the incentive for entrepreneurs to enter.

2.2 Comparative statics

The discussion above reveals that the comparative statics of spillover intensity n (which is increasing in ρ), and those of the share of project value retained by the entrepreneur (which is decreasing in ρ) suggest potentially opposing effects of non-rivalry ρ and degree of excludability δ on equilibrium growth. To assess the relative strength of these effects, we next turn to explicit comparative statics of the model with respect to ρ and the shape of the function governing limits to excludability δ.

We begin by assuming a specific functional form for the limits to excludability,

$$\delta(x) = \frac{(x-1)^{1+\delta_1}}{1+\delta_1}\delta_0 \qquad \text{where} \quad \delta_0, \delta_1 > 0. \tag{45}$$

Given the above, the first-order condition from the entrepreneur's optimal choice of span (38) implies that on the BGP,

$$\theta \;=\; 1 - \frac{\rho}{1+\delta_1}\frac{\hat{x}-1}{\hat{x}}. \tag{46}$$

The two equations above, together with (34) and (40) fully characterize the BGP.

The degree of non-rivalry ρ

We first consider the comparative statics of the model with respect to the degree of non-rivalry, ρ. It is useful to start with two extreme cases: $\rho = 0$ and $\rho = 1$. When $\rho = 0$, the model features no growth and $g = 0$. To see this, note that if $\rho = 0$ then there is no benefit to increasing span. Thus, entrepreneurs always choose the smallest possible value of $x = 1$, so as to retain as much of project value as possible. Given (45), entrepreneurs choose to retain all of the project: $\theta = 1$. In this case, $\hat{\delta} = 0$, and so there are no spillovers at all. As a result, new projects do not enter, and all labor is allocated to production. The BGP with no growth resembles the steady-state of a model with exogenous (and constant) total factor productivity, such as a real-business cycle model.

In the case of $\rho = 1$, the model closely resembles the Romer (1990) growth model. Any intangibles created by entrepreneurs immediately become available to imitators and future entrepreneurs, so that intangible capital is effectively a public capital good. The spillover intensity becomes constant, $n(g; 1, \delta) = \nu$, regardless of the value of δ. Thus the model with $\rho = 1$ is similar to the second variant of the model discussed in Section 2.1, where the size of new projects is equal to a constant fraction ν of the total existing stock of intangibles. As a result, all of the comparative statics with the enforceability of exclusivity are similar: higher exclusivity incentives entry by entrepreneurs and hence leads to higher growth.

These comparative statics are reported in Figure 2. We use values for the remaining structural parameters, $(r, \zeta, \nu, \delta_0, \delta_1)$ that are set as follows. The values of $r = 0.07$ for the discount rate and $\zeta = 0.7$ for the labor share are standard. The value of $\nu = 0.7$ is chosen so that the growth rate in the version of the model with fully non-rival intangible, $\rho = 1$, is approximately 5%. Finally, we choose $\delta_1 = 0.30$ and $\delta_0 = 0.03$, but explore the impact of the parameters governing the shape of $\delta(x)$ below.[6]

In between the two extremes of $\rho = 0$ and $\rho = 1$, the top left panel of Figure 2 shows that the relationship between growth and non-rivalry ρ can be non-monotonic. The BGP growth rate peaks at degree of non-rivalry of approximately $\rho = 0.7$. Recall that:

$$g = nL_e. \tag{47}$$

The top middle and left panel report the comparative statics for entrepreneurial labor L_e and spillover intensity n. While the latter is increasing strictly with ρ, the former is non-monotonic.

Intuitively, higher ρ exacerbates the effects of expropriation risk, and thus increases spillover intensity. Recall that in equilibrium, spillover intangibles from a particular project are given by:

$$S_{i,t} = \left(1 - e^{-\delta(t-\tau(i))}\right)^{1-\rho} (\nu S_{\tau(i)}).$$

All else equal, spillovers are higher for larger values of ρ. When non-rivalry is higher, at any given rate of expropriation, imitators appropriate more of the overall stock of intangible of the project each time they expropriate a product stream, because more of the project's total intangibles are invested in each product stream. When spillover intensity is higher, the *relative* appeal of choosing entrepreneurship rises, because new projects will have a larger size relative to existing ones. Thus, a higher spillover intensity encourages prospective entrepreneurs to supply more labor.

The middle panels shows the opposing force. As non-rivalry increases, the incentive for entrepreneurs to choose a high span, x, increases. But, while they operate at a larger scale, expropriation risk δ also rises, so they also retain a small share of overall project value, θ. This is optimal from their perspective, but it lowers the *relative* returns to entrepreneurial labor, compared to production labor. Thus it lower the incentive for entrepreneurs to enter.

[6] The functional form that we chose for $\delta(x)$ implicitly normalizes the minimum span of a project to be $x = 1$, which will be chosen in equilibrium only if $\rho = 0$. A more general expropriation risk function with arbitrary minimum span is $(x - x_m)^{\delta_1}$ with $x_m > 0$; x_m will then be the span chosen by projects when $\rho = 0$.

The bottom right panel summarizes the combination of the two effects — the stronger spillovers, and the weaker share of project value retained. The panel reports the product $nv = n\theta v_{tot}$. This summarizes the total *relative* value of entry for a prospective entrepreneur. The value is non-monotonic, peaking at intermediate values of ρ.

Limits to excludability

We now turn to the comparative statics of the model with respect to δ_0. The parameter δ_0 captures the risk of expropriation faced by the project; when δ_0 is higher, the probability of expropriation for the project rises, all else equal.[7]

Figure 3 reports the details of these comparative statics, for a specific degree of non-rivalry of intangibles of $\rho = 0.7$. In order to facilitate the discussion, the results are reported with δ_0 ordered in decreasing values from the left to the right of the horizontal axis. Thus, from left to right, the degree of excludability of intangibles and product streams in existing projects improves.

As for non-rivalry, the relationship between excludability and growth is non-monotonic: higher excludability initial increases BGP growth, but eventually reduces it. However, the mechanisms behind these comparative statics are somewhat different. In particular, the supply of entrepreneurial labor is strictly declining with the degree of excludability, where as it was non-monotonic with respect to non-rivalry.

It may seem counterintuitive that a higher degree of excludability would *reduce* entrepreneurial labor supply (and increase spillover intensity, as shown on the top right panel of Figure 3). The main driver of this result is the optimal choice of span by the entrepreneur. The three middle panels of Figure 3 show that, as excludability improves, entrepreneurs respond by choosing a larger firm span, but retaining a smaller overall share of the project. The smaller relative share of the project retained by committed entrepreneurs lowers the *relative* attractiveness of entrepreneurship, and thus leads to lower entrepreneurial labor supply. This effect dominates when excludability is high.

As excludability improves and entrepreneurs increase span, expropriation risk δ also increases, as highlighted in the middle right panel of Figure 3. This is a consequence of our assumption that δ is increasing with x. As a result of higher expropriation risk, though, spillover intensity increases. This is the effect that dominates when excludability is low.

Note that this non-monotonicity may not hold for all values of $\rho \in]0, 1[$. For instance, as $\rho \to 1$, spillover intensity becomes independent of expropriation risk, δ. In this case, the negative effects of higher excludability on entrepreneurial labor supply, and therefore growth, would dominate. On the other hand, as $\rho \to 0$, the choice of span becomes insensitive to δ_1 — there are no benefits to

[7] By contrast, δ_1 captures the incremental expropriation risk associated with adding an additional product stream.

increasing span, so the entrepreneur seeks to minimize it, and would end up retaining a large share of the project's value. Thus any negative incentive effects would be small, and higher excludability would likely increase equilibrium growth.

When is there an inverse-U shaped relationship between non-rivalry and growth?

Finally, we discuss whether the non-monotonic relationship between non-rivalry ρ and growth is robust to changes in parameter values. We focus on two specific parameters: ν, which captures the strength of the spillover channel; and δ_0, which acts as a shifter in the degree of expropriation risk.

In Appendix A.2.1, we establish that, for any values of δ_0 and ν such that the BGP exists,

$$\frac{\partial g}{\partial \rho}|_{\rho=0} > 0.$$

Thus a sufficient condition for non-rivalry ρ and growth to be non-monotonically related is:

$$\frac{\partial g}{\partial \rho}|_{\rho=1} < 0.$$

In Appendix A.2.1, we show that this condition is also necessary, and moreover, characterize the values of δ_0 and ν for which it holds.

Figure 5 describes the corresponding partition of the parameter (ν, δ_0) into three regions. The first region, in dark grey, are parameter values for which the BGP does not exist at $\rho = 1$.[8] The second region, in a lighter shade of grey, shows the parameter region in which a BGP exists for all values of $\rho \in [0, 1]$, and the relationship between growth and non-rivalry is non-monotonic with respect to ρ. This corresponds to the baseline case discussed above. Finally, the lightest grey region corresponds to parameter values such that a BGP exists for all values of $\rho \in [0, 1]$, and BGP growth increases with ρ.

There are two main take-aways from this graph. First, there is a region of values for δ_0 such that growth is *always* non-monotonically related to non-rivalry, regardless of the value of the spillover parameter ν. In this region, negative incentive effects always dominate spillover effects when $\rho = 1$. This corresponds to low values of δ_0. When δ_0 is low, average expropriation risk is low. In this case entrepreneurs choose a larger project span, x, at the expense of a higher expropriation risk. This choice is optimal conditional on becoming an entrepreneur, but since it also implies that the entrepreneur retains a lower overall share of the project, it makes entrepreneurship less attractive. When δ_0 is sufficiently low, this effect strictly dominates.

The second take-away is that if δ_0 is sufficiently large, the relationship between growth and

[8] Relative to Lemma 2, these are parameter values for which the condition $\nu \geq (\zeta/(1-\zeta))(r + \underline{\delta}(1))$ is violated.

non-rivalry can become strictly increasing. The intuition for this is the converse of the one described above. When δ_0 is sufficiently high, the entrepreneur optimally retains a larger share of the project (while operating with a lower span), making entrepeneurship more attractive, and thus raising entrepreneurial labor supply relative to production labor. However, the figure also shows that this effect dominates only when spillovers are sufficiently weak. Above a certain threshold for ν (specifically, the highest border between the lighter and lightest gray area in the figure, for a given value of δ_0), spillovers become sufficiently strong to make entrepreneurship relatively unattractive at high values of ρ.

Thus, a key message from this figure is that the non-monotonic relationship between growth and non-rivalry holds in a fairly wide range of parameter values for spillover intensity (ν) and limits to excludability δ_0, and in particular when excludability is high (δ_0 is low) or when spillover intensity is high (ν is high).

3 Implications For Macro Trends

This section discusses the implications of the model for the measurement of productivity growth, factor income shares and valuations, and concentration.

3.1 Growth in Measured Productivity

We first ask what the implications of the model are for how to interpret the Solow residual, the most commonly used measure of total factor productivity. Since labor and intangibles are the only two factors of production in the model, we define the Solow residual as:

$$d\log(Z_t) \quad \equiv \quad d\log(\overline{Y}_t) - s_{L,t} d\log(\overline{L}_t) - (1 - s_{L,t}) d\log(K_{N,t}),$$

where $\overline{Y}_t$ is aggregate output, $\overline{L}_t$ is aggregate labor used in production, $s_{L,t} \equiv W_t \overline{L}_t / \overline{Y}_t$ is the labor share, and $K_{N,t}$ is a measure of the productive intangible stock, to be defined below. Note total labor is fixed, $L_{tot,t} = 1$, so:

$$d\log(Z_t) \quad = \quad d\log(\overline{Y}_t) - (1 - s_{L,t}) d\log(K_{N,t}).$$

Appendix A.1.2 shows that on the BGP, total output and the total wage bill are given by:

$$\overline{Y}_t = \frac{\Lambda_t}{1-\zeta} \int_{\tau \leq t} L_{e,\tau} x_\tau^{\rho_\tau} (\nu S_\tau) d\tau = \frac{\nu L_e x^\rho}{g(1-\zeta)} \Lambda_t S_t, \quad W_t \overline{L}_t = \zeta \overline{Y}_t.$$

The labor share is therefore $s_{L,t} = \zeta$. Moreover, on the BGP, $\overline{Y}_t$ grows at rate $(1-\zeta)g$.

Next, we define a replacement cost estimate of the productive intangible capital stock that is analog to measures used in national accounts. On the BGP, aggregate expenditure on intangibles, expressed in output units, are given by:

$$I_{N,t} = W_t L_e.$$

$I_{N,t}$, like W_t, grows at rate $(1-\zeta)g$ on the BGP. We define $K_{N,t}$ as the perpetual inventory estimate of the capital stock associated with these expenditures. Let $\delta_N \geq 0$ be the rate used to capitalize these expenditures, and $\tilde{p}_{N,t}$ be the price index used to deflate them. We assume that on the BGP, $\tilde{p}_{N,t}$ is growing at a constant rate:

$$\frac{d\tilde{p}_{N,t}}{\tilde{p}_{N,t}} = g_{\tilde{P}}.$$

The law of motion for the estimate $K_{N,t}$ is:

$$dK_{N,t} = (1 - \delta_N dt)K_{N,t} + \frac{I_{N,t}}{\tilde{p}_{N,t}}dt.$$

Let the aggregate gross investment rate and the net growth rate of $K_{N,t}$ be defined as:

$$\iota_{N,t} \equiv \frac{I_{N,t}}{\tilde{p}_{N,t}K_{N,t}}, \quad g_{K,t} \equiv \frac{I_{N,t}}{\tilde{p}_{N,t}K_{N,t}}$$

Additionally, assume that $\iota_{N,t}$ and $g_{K,t}$ are constant on the BGP. Since $I_{N,t}$ grows at rate $(1-\zeta)g$, we have:

$$(1-\zeta)g = g_{\tilde{P}} + g_K \qquad \Longrightarrow g_K = (1-\zeta)g - g_{\tilde{P}}.$$

From the law of motion for $K_{N,t}$, the aggregate gross investment rate is then given by $\iota_N = (1-\zeta)g - g_{\tilde{P}} + \delta_N$. Moreover, the Solow residual is given by:

$$\begin{aligned} d\log(Z_t) &\equiv d\log(\overline{Y}_t) - (1 - s_{L,t})d\log(K_{N,t}) \\ &= (1-\zeta)g - (1-\zeta)((1-\zeta)g - g_{\tilde{P}}) \\ &= (1-\zeta)(\zeta g + g_{\tilde{P}}). \end{aligned}$$

Within the model, the correct measure of the price of intangibles is its replacement cost in output units, which is given by:

$$p_{N,t} = \frac{W_t}{\nu S_t} = \Lambda_t x^\rho v.$$

The first equality says that an entrepreneur must spend 1 unit of labor to create νS_t units of intangibles. The second equality follows from the free-entry condition. Since Λ_t is growing at rate $-\zeta g$, we have:

$$\tilde{p}_{N,t} = p_{N,t} \quad \Longrightarrow \quad g_{\tilde{P}} = -\zeta g \quad \Longrightarrow \quad d\log(Z_t) = 0.$$

Thus if $\tilde{p}_{N,t} = p_{N,t}$, the Solow residual will be exactly zero on the BGP.

On the other hand, if the measured price of intangibles, $\tilde{p}_{N,t}$, grows at a different rate from $p_{N,t}$, the Solow residual will not be zero. In this case, any change in structural parameters that affect trends growth, g, will pass through to the Solow residual. For instance, an increase in the degree of non-rivalry of intangibles would impact the Solow residual. The impact may be non-monotonic, with the Solow residual declining for sufficiently high degrees of non-rivalry, as in Section 2.

3.2 Tobin's Q for Intangibles and the profit share

The total value of existing projects (to both entrepreneurs and imitators) is given by:

$$\overline{V}_{tot,t} \quad = \quad \Lambda_t x^\rho v_{tot} \int_{\tau \le t} L_{e,\tau}(\nu S_\tau) d\tau = \frac{p_{N,t}}{\theta} \frac{\nu L_e}{g} S_t.$$

Moreover, the total stock of intangibles is given by:

$$\overline{N}_{tot,t} \quad = \quad \int_{\tau \le t} L_{e,\tau}(\nu S_\tau) d\tau = \frac{\nu L_e}{g} S_t.$$

Thus, aggregate Q is given by:

$$\overline{Q}_t \equiv \frac{\overline{V}_{tot,t}}{p_{N,t}\overline{N}_{tot,t}} \quad = \quad \frac{1}{\theta}$$

Aggregate Q is above 1 because imitators earns rents from expropriation. Note that, from the entrepreneurs' free-entry condition, we have:

$$Q_{e,t} \equiv \frac{\overline{V}_t}{p_{N,t}\overline{N}_{tot,t}} = 1,$$

where $\overline{V}_t$ is the value of existing projects retained by entrepreneurs:

$$\overline{V}_t \quad = \quad \Lambda_t x^\rho v \int_{\tau \le t} L_{e,\tau}(e^{-\delta(t-\tau)} \nu S_\tau) d\tau = p_{N,t} \frac{\nu L_e}{g} S_t.$$

For entrepreneurs only, Q_e is one; all rents from entry flow to imitators.

This can also be seen by looking at factor income shares in the model, rather than valuation ratios. Define the user cost of intangible capital as:

$$R_{N,t} = r - \dot{p}_{N,t} = r + \zeta g = \frac{\theta}{v}.$$

(Note that this definition is consistent with the fact that intangibles do not depreciate, in this model.) Then on the BGP, we have:

$$\begin{aligned} \overline{Y}_t - W_t \overline{L}_t - R_{N,t}(p_{N,t} N_t) &= \frac{\nu L_e x^\rho}{(1-\zeta)g} \Lambda_t S_t - \frac{\nu L_e x^\rho \zeta}{(1-\zeta)g} \Lambda_t S_t - \frac{\theta}{v} (\Lambda_t x^\rho v) \frac{\nu L_e}{g} S_t \\ &= \frac{\nu L_e x^\rho}{g} \Lambda_t S_t \left(1 - \theta\right). \end{aligned}$$

In other words, the pure profit share (that is, the share of total value added earned by owners of projects after competitive payments to capital) is exactly equal to $(1-\zeta)(1-\theta)$, and thus varies one-for-one with Total Q for intangibles, for a given value of ζ. Figure 2 shows the comparative statics of Total Q with respect to ρ. As non-rivalry increases, entrepreneurs choose to retain a smaller fraction of the overall project, increasing the share of rents per unit of capital. Finally, Figure 2 reports imitators' share of total operating revenue, which is given by:

$$s_c = \frac{\int_{\tau \le t} L_{e,\tau} \Pi_{i^c,\tau} d\tau}{\int_{\tau \le t} L_{e,\tau} \Pi_{i,tot,\tau} d\tau} = \frac{1}{1 + \frac{g}{\delta}}.$$

Consistent with the fact that ρ exacerbates the effects of limits to excludability, imitators' share of total operating revenue is increasing with the degree of non-rivalry of intangibles.

3.3 Concentration

Finally, we discuss the implications of the model for concentration. As discussed in Section 1, boundaries of the firm are not endogenously determined in this model. We use projects as our notion of firm boundaries to measure concentration.[9]

On the BGP, sales from a particular project, and aggregate sales, are given by:

$$\begin{aligned} Y_{i,t} &= \frac{\Lambda_t}{1-\zeta} x^\rho \nu S_{\tau(i)} \\ \overline{Y}_t &= \frac{\Lambda_t}{1-\zeta} x^\rho \nu \frac{S_t}{n}. \end{aligned}$$

[9] Additionally, we do not distinguish between sales and value added in our model, so we use the two terms interchangeably in this section.

Thus the sales share of a project is:

$$s_{i,t} \quad = \quad ne^{-g(t-\tau(i))}.$$

From this, the Herfindhal index of projects is:

$$H \quad = \quad \int_{\tau(i)\leq t} s_{i,t}^2 di = n^2 L_e \int_{\tau\leq t} e^{-2g(t-\tau)} d\tau = \frac{n}{2}.$$

Thus in the baseline model, the Herfindhal index of projects is exactly proportional to spillovers. The intuitive reason for this result is that when spillover intensity is higher, projects are born at a bigger size *relative to* previous cohorts. This tends to accentuate concentration.

One concern with this measure of concentration is that entrepreneurs and imitators may not directly compete in the same markets; indeed, imitation effectively expels entrepreneurs from specific product streams, to the benefit of competitors. Similar steps as above show that the Herfindhal index of sales *among entrepreneurs* is given by:

$$H_e \quad = \quad \frac{n}{2}\left(1 + \frac{\delta}{g}\right).$$

The additional term in the Herfindhal index for entrepreneurs captures the fact that the market share will also depend upon the rate at which entrepreneurs loose market share to the imitators. When equilibrium expropriation risk δ is low, entrepreneurs retain large market shares for a longer while, and so concentration within entrepreneurs is lower.

Figure 4 reports the two measures of concentration together, as a function of the degree of non-rivalry. Recall that Section 2 established that spillover intensity is strictly increasing as a function of non-rivalry, ρ; thus concentration among projects must monotonically increase with non-rivalry. Among entrepreneurs, the effect of non-rivalry is potentially ambiguous, since at low levels of non-rivalry, both δ and g are increasing. The example of Figure 4 suggests that the effects of rising expropriation risk (and therefore fewer entrepreneurs left per cohort) tend to dominate, and that concentration increases monotonically with non-rivalry even among entrepreneurs.

Overall, we see how changes in the degree of non-rivalry in the model will produce a non-monotonic relationship between growth and concentration: the two grow in tandem when non-rivalry is low, but eventually diverge as intangibles approach complete non-rivalry.

4 Conclusion

Intangibles are particular types of capital inputs that differ from physical assets in one important way: they may be partly non-rival within the firm. The main contribution of this paper is to show the degree of non-rivalry of intangibles has an ambiguous effect on growth. While non-rivalry increases spillovers from existing to new firms, it also increases the risk that firms' intangibles will be copied or appropriated by competitor. The former effect stimulates entry, but the latter inhibits it. The model thus sheds light on the macroeconomic implications of the technological changes in the way that intangible assets are codified, and which determines their degree of non-rivalry. Concrete examples include innovations in information technology used to store data-related intangibles, or managerial innovations used to create organization capital and communicate it to employees.

The paper leaves at least three questions unanswered. First, we have only compared steady-states, but the model can also speak to transitional dynamics between balanced growth paths, as the technological or legal environment around intangibles changes. The transitional dynamics may be non-monotonic, with spillovers dominating in the short-run but disincentive effects dominating in the long-run. Second, equilibria in this economy are generically inefficient because current entrants do not internalize the effect of their investment decisions on future ones; thus the model could speak to questions of industrial policy. Third, some of the first-order conditions of the model suggest ways of estimating directly the value of ρ, the degree of non-rivalry, and the value of δ, the equilibrium degree of competitive risk, in the data. We leave these to future research.

References

Aghion, P., N. Bloom, R. Blundell, R. Griffith, and P. Howitt (2005, 05). Competition and Innovation: an Inverted-U Relationship*. The Quarterly Journal of Economics 120(2), 701–728.

Aghion, P. and P. Howitt (1992). A model of growth through creative destruction. Econometrica 60(2), 323–351.

Atkeson, A. and P. J. Kehoe (2005). Modeling and measuring organization capital. Journal of political Economy 113(5), 1026–1053.

Bhandari, A. and E. R. McGrattan (2021). Sweat equity in us private business. The Quarterly Journal of Economics 136(2), 727–781.

Bloom, N. and J. Van Reenen (2007). Measuring and explaining management practices across firms and countries. The quarterly journal of Economics 122(4), 1351–1408.

Bresnahan, T. F., E. Brynjolfsson, and L. M. Hitt (2002). Information technology, workplace organization, and the demand for skilled labor: Firm-level evidence. The quarterly journal of economics 117(1), 339–376.

Corrado, C., C. Hulten, and D. Sichel (2005). Measuring capital and technology: an expanded framework. In Measuring capital in the new economy, pp. 11–46. University of Chicago Press.

Crouzet, N. and J. Eberly (2019). Understanding weak capital investment: the role of market power and intangibles. 2018 Jackson Hole Symposium, Federal Reserve Bank of Kansas City.

Crouzet, N. and J. Eberly (2021). Rents and Intangible Capital: A Q+ Framework. The Journal of Finance forthcoming.

Crouzet, N., J. C. Eberly, A. L. Eisfeldt, and D. Papanikolaou (2022, August). The economics of intangible capital. Journal of Economic Perspectives 36(3), 29–52.

Eisfeldt, A. L. and D. Papanikolaou (2013). Organization capital and the cross-section of expected returns. The Journal of Finance 68(4), 1365–1406.

Eisfeldt, A. L. and D. Papanikolaou (2014). The value and ownership of intangible capital. American Economic Review 104(5), 189–94.

Farboodi, M. and L. Veldkamp (2020). Long-run growth of financial data technology. American Economic Review 110(8), 2485–2523.

Gourio, F. and L. Rudanko (2014). Customer capital. Review of Economic Studies 81(3), 1102–1136.

Hall, B. H., A. Jaffe, and M. Trajtenberg (2005). Market value and patent citations. The RAND Journal of Economics 36(1), 16–38.

Haskel, J. and S. Westlake (2017). Capitalism without capital. Princeton University Press.

Jones, C. I. (1995). R & d-based models of economic growth. Journal of political Economy 103(4), 759–784.

Jones, C. I. and C. Tonetti (2020). Nonrivalry and the Economics of Data. The American Economic Review, 2819–58.

Kelly, B., D. Papanikolaou, A. Seru, and M. Taddy (2021). Measuring technological innovation over the long run. American Economic Review: Insights 3(3), 303–20.

Kogan, L., D. Papanikolaou, A. Seru, and N. Stoffman (2017, 03). Technological Innovation, Resource Allocation, and Growth*. The Quarterly Journal of Economics 132(2), 665–712.

Kogan, L., D. Papanikolaou, and N. Stoffman (2020). Left behind: Creative destruction, inequality, and the stock market. Journal of Political Economy 128(3), 855–906.

Lucas, R. E. and B. Moll (2014). Knowledge growth and the allocation of time. Journal of Political Economy 122(1), 1–51.

McGrattan, E. R. and E. C. Prescott (2010a). Technology capital and the us current account. 100(4), 1493–1522.

McGrattan, E. R. and E. C. Prescott (2010b). Unmeasured investment and the puzzling us boom in the 1990s. American Economic Journal: Macroeconomics 2(4), 88–123.

Romer, P. M. (1986). Increasing returns and long-run growth. Journal of political economy 94(5), 1002–1037.

Romer, P. M. (1990). Endogenous technological change. Journal of political Economy 98(5, Part 2), S71–S102.

Solow, R. (1956). A contribution to the theory of economic growth. The Quarterly Journal of Economics 70(1), 65–94.

Stokey, N. L. (2015). Catching up and falling behind. Journal of Economic Growth 20(1), 1–36.

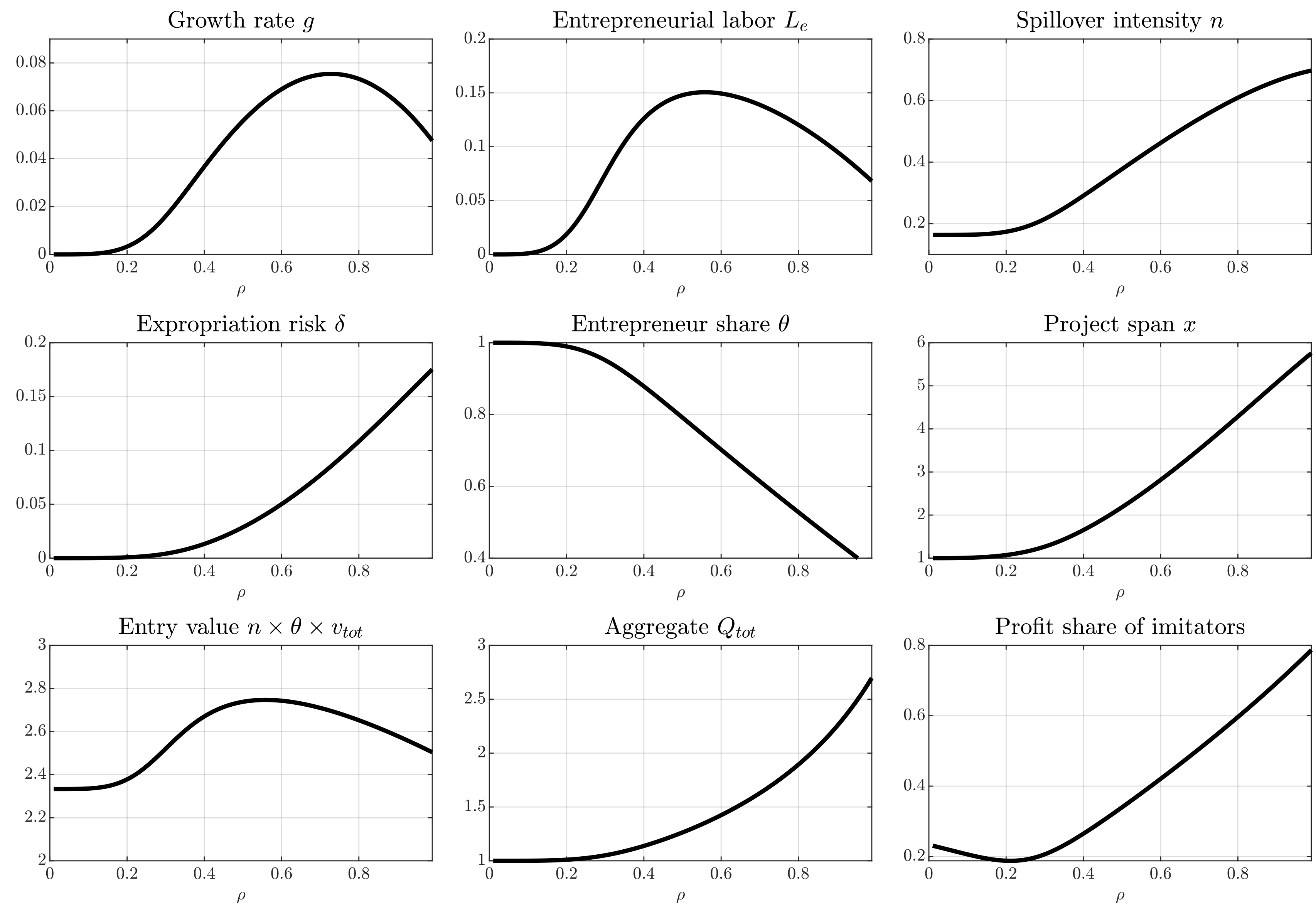

Figure 2: Comparative statics of the balanced growth path with respect to the degree of non-rivalry, ρ. The expropriation risk function is assumed to be $\delta(x) = \frac{(x-1)^{1+\delta_1}}{1+\delta_1}\delta_0$. Parameter values used are $r = 0.07$, $\zeta = 0.7$, $\nu = 0.7$, $\delta_0 = 0.03$, and $\delta_1 = 0.3$.

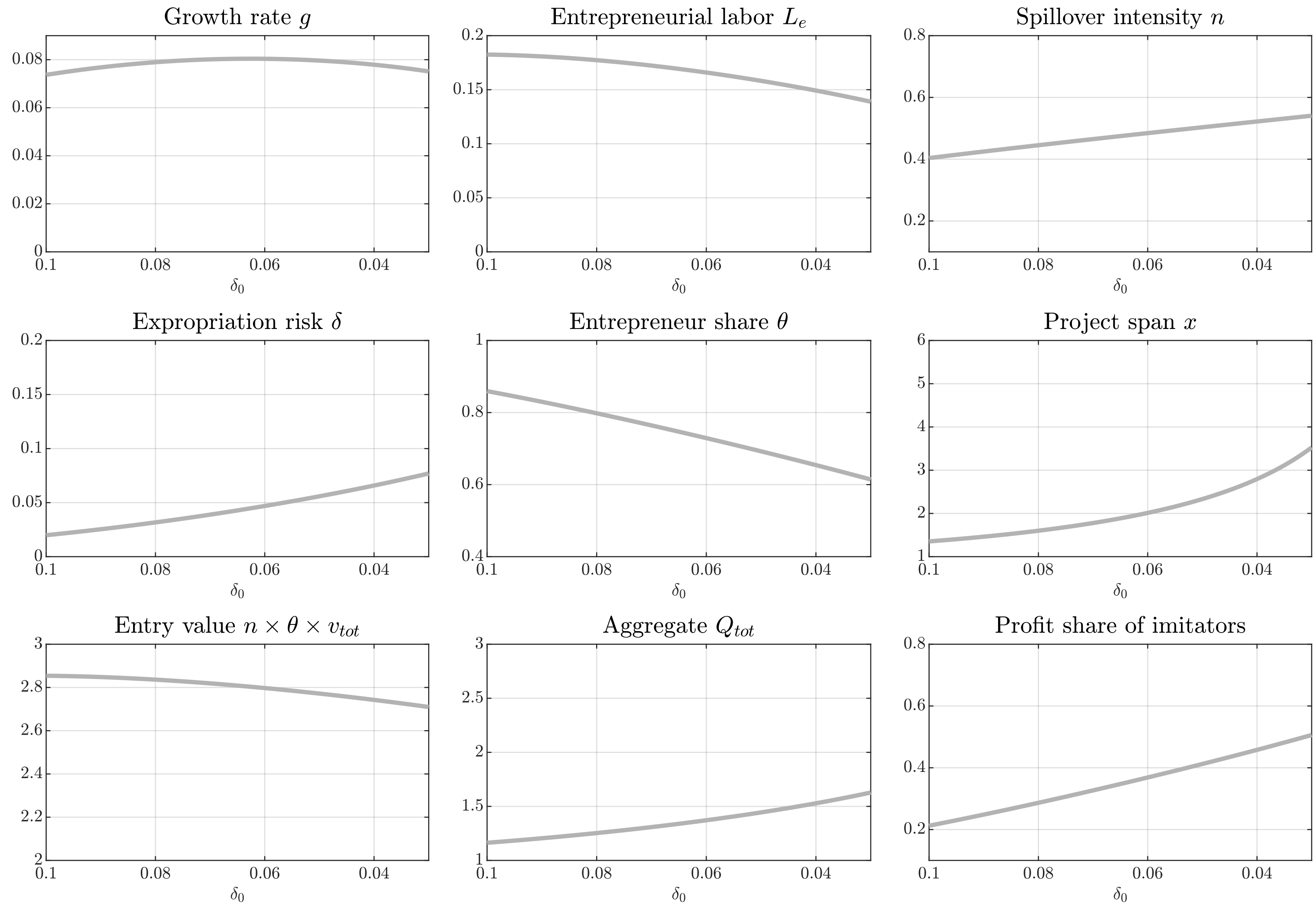

Figure 3: Comparative statics of the balanced growth path with respect to limits to excludability, δ_0. The expropriation risk function is assumed to be $\delta(x) = \frac{(x-1)^{1+\delta_1}}{1+\delta_1}\delta_0$. Parameter values used are $r = 0.07$, $\zeta = 0.7$, $\nu = 0.7$, $\rho = 0.7$, and $\delta_1 = 0.3$. The horizontal axis has decreasing values of δ_0 from left to right, corresponding to an increasing degree of excludability of intangibles and product streams.

Figure 4: Equilibrium concentration of sales as a function of the degree of non-rivalry of intangibles. The expropriation risk function is assumed to be $\delta(x) = \delta_0 \frac{(x-1)^{\delta_1}}{1+\delta_1}$. Parameter values used are $r = 0.07$, $\zeta = 0.7$, $\nu = 0.7$, $\delta_1 = 0.3$ and $\delta_0 = 0.03$. The figure reports the Herfindhal index of sales in the baseline model.

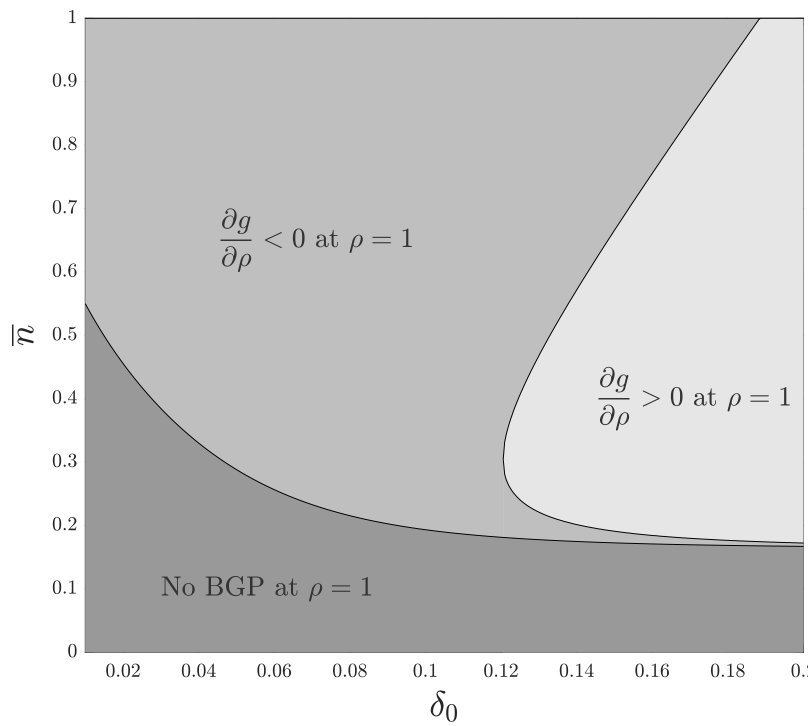

Figure 5: Parameter values for which the relationship between growth and non-rivalry is non-monotonic. The vertical axis has values of δ_0, the limits to excludability, with higher δ_0 corresponding to less excludable intangibles. The horizontal axis has value of ν, the maximum spillover intensity, which is also the spillover intensity when $\rho = 1$. The darkest region corresponds to values of (ν, δ_0) for which the model has no BGP when $\rho = 1$. The lighter gray region corresponds to values of (ν, δ_0) for which the derivative of the equilibrium BGP growth rate with respect to ρ, evaluated at $\rho = 1$, is strictly negative, so that the relationship between growth and non-rivalry is non-monotonic. The lightest gray region corresponds to the values of (ν, δ_0) for which the derivative of the equilibrium BGP growth rate with respect to ρ, evaluated at $\rho = 1$, is strictly positive, so that the relationship between growth and non-rivalry is strictly increasing.

Appendix

A.1 Appendix to Section 1

A.1.1 Proofs

Law of motion for $N_{i,t}$. Given that the probability of expropriation is independent across streams and constant over time, the number of streams follows:

$$x_{i,t} = x_{i,\tau(i)} e^{-\delta(x_{i,\tau(i)})(t-\tau(i))},$$

Aggregating across streams, total intangibles in use in the project follow:

$$\begin{aligned} N_{i,t+dt} &= \left(\int_0^{x_{i,t}+dx_{i,t}} N_{i,t}(s)^{\frac{1}{1-\rho_{\tau(i)}}} di\right)^{1-\rho_{\tau(i)}} \\ &= \left(\int_0^{x_{i,t}+dx_{i,t}} x_{i,t}^{-1} N_{i,t}^{\frac{1}{1-\rho_{\tau(i)}}} di\right)^{1-\rho_{\tau(i)}} \\ &= N_{i,t}\left(1+\frac{dx_{i,t}}{x_{i,t}}\right)^{1-\rho_{\tau(i)}}. \end{aligned}$$

Leting $\hat{N}_{i,t} = N_{i,t}^{\frac{1}{1-\rho}}$, the equation above implies that:

$$\frac{d\hat{N}_{i,t}}{\hat{N}_{i,t}} = \frac{dx_{i,t}}{x_{i,t}}.$$

Thus,

$$\frac{\hat{N}_{i,t}}{\hat{N}_{i,\tau(i)}} = \frac{x_{i,t}}{x_{i,\tau(i)}}$$

Substituting $\hat{N}_{i,t}$ for $N_{i,t}$ and using the law of motion for $x_{i,t}$ above gives the result. ■

Proof of Lemma 1. Assume that $\rho_t < 1$ for all t. Then:

$$\begin{aligned} S_{t+dt} &= \overbrace{0}^{\text{spillovers from projects started at } t+dt} \\ &+ \underbrace{\nu \int_{\tau(i)\le t} S_{\tau(i)} \left(1 - e^{-\delta_{\tau(i)}(t+dt-\tau(i))}\right)^{1-\rho_{\tau(i)}} di.}_{\text{spillovers from existing projects}} \end{aligned}$$

To see why there are no immediate spillovers from new projects, recall that spillovers from an individual project are given by:

$$S_{i,t} = \left(1 - e^{-\delta_{\tau(i)}(t-\tau(i))}\right)^{1-\rho_{\tau(i)}} \nu S_{\tau(i)},$$

which is zero when $t = \tau(i)$, i.e. when the project is just created, except if $\rho_{\tau(i)} = 1$. Using the expression above, we get:

$$\begin{aligned}
dS_t &= \nu \int_{\tau(i)\le t} S_{\tau(i)} \left[\left(1 - e^{-\delta_{\tau(i)}(t+dt-\tau(i))}\right)^{1-\rho_{\tau(i)}} - \left(1 - e^{-\delta_{\tau(i)}(t-\tau(i))}\right)^{1-\rho_{\tau(i)}}\right] di \\
&= \nu \int_{\tau(i)\le t} S_{\tau(i)} \left(1 - e^{-\delta_{\tau(i)}(t-\tau(i))}\right)^{1-\rho_{\tau(i)}} \left[\left(\frac{1 - e^{-\delta_{\tau(i)}(t+dt-\tau(i))}}{1 - e^{-\delta_{\tau(i)}(t-\tau(i))}}\right)^{1-\rho_{\tau(i)}} - 1\right] di \\
&= \nu \int_{\tau(i)\le t} S_{\tau(i)} \left(1 - e^{-\delta_{\tau(i)}(t-\tau(i))}\right)^{1-\rho_{\tau(i)}} \left[\left(1 + \frac{e^{-\delta_{\tau(i)}(t-\tau(i))}}{1 - e^{-\delta_{\tau(i)}(t-\tau(i))}} \delta_{\tau(i)} dt\right)^{1-\rho_{\tau(i)}} - 1\right] di \\
&= (\nu dt) \int_{\tau(i)\le t} S_{\tau(i)} \left(1 - e^{-\delta_{\tau(i)}(t-\tau(i))}\right)^{-\rho_{\tau(i)}} e^{-\delta_{\tau(i)}(t-\tau(i))} (1 - \rho_{\tau(i)}) \delta(\hat{x}_{\tau(i)}) di \\
&= (\nu dt) \int_{\tau\le t} S_\tau L_{e,\tau} (1 - \rho_\tau) \delta_\tau \left(1 - e^{-\delta_\tau (t-\tau)}\right)^{-\rho_\tau} e^{-\delta(\hat{x}_\tau)(t-\tau)} d\tau.
\end{aligned}$$

In the last line, we used the fact that $L_{e,\tau}$ projects get created in each cohort τ. This establishes the result. ■

A.1.2 Equilibrium conditions

The set of equilibrium conditions is:

$$\begin{aligned}
\Xi_t &= (1-\zeta)\left(\frac{\zeta}{W_t}\right)^{\frac{\zeta}{1-\zeta}} \\
x_t &= \arg\max_x x^{\rho_t} \tilde{v}_t(x) \\
v_t &= \tilde{v}_t(x_t) \\
W_t &= \Xi_t x_t^{\rho_t} (\nu S_t)\, v_t \\
1 - L_{e,t} &= \nu \left(\frac{\Xi_t}{1-\zeta}\right)^{\frac{1}{\zeta}} \int_{\tau(i)\le t} x_{\tau(i)}^{\rho_{\tau(i)}} S_{\tau(i)} di \\
S_t &= \nu \int_{\tau(i)\le t} S_{\tau(i)} (1 - e^{-\delta_{\tau(i)}(t-\tau(i))})^{1-\rho_{\tau(i)}} di
\end{aligned}$$

where the function $\tilde{v}_t(x)$ is given by:

$$\tilde{v}_t(x) \equiv \mathbb{E}_t \left[\int_t^{\infty} e^{-(r+\delta(x))(s-t)} \, \frac{\Xi_s}{\Xi_t} \, ds \right].$$

Project-level variables are given by:

$$\begin{aligned}
x_{i,t} &= e^{-\delta_{\tau(i)}(t-\tau(i))} x_{\tau(i)} \\
N_{i,t} &= e^{-\delta_{\tau(i)}(1-\rho_{\tau(i)})(t-\tau(i))} (\nu S_{\tau(i)}) \\
Y_{i,t} &= \frac{\Xi_t}{1-\zeta} e^{-\delta_{\tau(i)}(t-\tau(i))} \, x_{\tau(i)}^{\rho_{\tau(i)}} (\nu S_{\tau(i)}) \\
L_{i,t} &= \left(\frac{\Xi_t}{1-\zeta} \right)^{\frac{1}{\zeta}} e^{-\delta_{\tau(i)}(t-\tau(i))} \, x_{\tau(i)}^{\rho_{\tau(i)}} (\nu S_{\tau(i)}) \\
\Pi_{i,t} &= \Xi_t e^{-\delta_{\tau(i)}(t-\tau(i))} x_{\tau(i)}^{\rho_{\tau(i)}} (\nu S_{\tau(i)}) \\
V_{i,t} &= \Xi_t x_t^{\rho_t} (\nu S_t) v_t \\
x_{i^c,t} &= \left(1 - e^{-\delta_{\tau(i)}(t-\tau(i))} \right) x_{\tau(i)} \\
N_{i^c,t} &= \left(1 - e^{-\delta_{\tau(i)}(t-\tau(i))} \right)^{1-\rho_{\tau(i)}} (\nu S_{\tau(i)}) \\
Y_{i^c,t} &= \frac{\Xi_t}{1-\zeta} \left(1 - e^{-\delta_{\tau(i)}(t-\tau(i))} \right) x_{\tau(i)}^{\rho_{\tau(i)}} (\nu S_{\tau(i)}) \\
L_{i^c,t} &= \left(\frac{\Xi_t}{1-\zeta} \right)^{\frac{1}{\zeta}} \left(1 - e^{-\delta_{\tau(i)}(t-\tau(i))} \right) x_{\tau(i)}^{\rho_{\tau(i)}} (\nu S_{\tau(i)}) \\
\Pi_{i^c,t} &= \Xi_t \left(1 - e^{-\delta_{\tau(i)}(t-\tau(i))} \right) x_{\tau(i)}^{\rho_{\tau(i)}} (\nu S_{\tau(i)}) \\
V_{i^c,t} &= \Xi_t x_t^{\rho_t} (\nu S_t)(v_{t,tot} - v_t) \\
x_{tot,i,t} &= x_{\tau(i)} \\
N_{tot,i,t} &= \nu S_{\tau(i)} \\
Y_{tot,i,t} &= \frac{\Xi_t}{1-\zeta} x_{\tau(i)}^{\rho_{\tau(i)}} (\nu S_{\tau(i)}) \\
L_{tot,i,t} &= \left(\frac{\Xi_t}{1-\zeta} \right)^{\frac{1}{\zeta}} x_{\tau(i)}^{\rho_{\tau(i)}} (\nu S_{\tau(i)})
\end{aligned}$$

$$\begin{aligned}
\Pi_{tot,i,t} &= \Xi_t x_{\tau(i)}^{\rho_{\tau(i)}}(\nu S_{\tau(i)}) \\
V_{tot,i,t} &= \Xi_t x_t^{\rho_t}(\nu S_t) v_{t,tot} \\
v_{tot,t} &= \mathbb{E}_t\left[\int_t^{\infty} e^{-r(s-t)}\frac{\Xi_s}{\Xi_t}\,ds\right] \\
\overline{N}_t &= \int_{\tau(i)\leq t} \nu S_{\tau(i)} di \\
\overline{Y}_t &= \frac{\Xi_t}{1-\zeta}\int_{\tau(i)\leq t} x_{\tau(i)}^{\rho_{\tau(i)}}(\nu S_{\tau(i)}) di \\
\overline{L}_t &= \left(\frac{\Xi_t}{1-\zeta}\right)^{\frac{1}{\zeta}}\int_{\tau(i)\leq t} x_{\tau(i)}^{\rho_{\tau(i)}}(\nu S_{\tau(i)}) di
\end{aligned}$$

A.2 Appendix to Section 2

A.2.1 Proofs

Existence and unicity of balanced growth path. The two conditions that determine the equilibrium values of (g, x) can be written as:

$$g = n\left(\frac{g}{\delta(x)}, \rho\right) - \frac{\zeta}{1-\zeta}\left(r + \zeta g + \delta(x)\right), \tag{1}$$

$$x\delta'(x) = \rho\left(r + \zeta g + \delta(x)\right), \tag{2}$$

where:

$$\begin{aligned}
h(y,z) &\equiv \nu y \mathcal{B}(y, 2-z), \\
\delta(x) &\equiv \frac{(x-1)^{1+\delta_1}}{1+\delta_1}\delta_0.
\end{aligned}$$

We first note the following properties of the function h:

$$\begin{aligned}
\forall y>0, \quad n(y,1) &= \nu y \mathcal{B}(y,1) = \nu \\
\forall y>0, z\in[0,1], \quad \frac{\partial n}{\partial y}(y,z) &= \nu\left(1 + y\psi(y) - y\psi(y+2-z)\right)\mathcal{B}(y, 2-z) \leq 0 \\
\forall y>0, z\in[0,1], \frac{\partial n}{\partial z}(y,z) &= \nu\left(\psi(y+2-z) - \psi(2-z)\right) y\mathcal{B}(y,2-z) > 0
\end{aligned}$$

Here ψ denotes the Digamma function and $\mathcal{B}$ denotes the Beta function. Throughout we fix the values of $\delta_0 > 0$, $\delta_1 > 0$, $\zeta \in]0,1[$, and $r > 0$. We analyze separately two cases: $\rho \in]0,1]$ and $\rho = 0$.

The case $\rho \in]0,1]$ First, assume that $\rho \in]0,1]$. We can rewrite Equation (2) as:

$$R(x) \equiv x\delta'(x) - \rho\delta(x) = \delta_0(x-1)^{\delta_1}\left(x - \frac{\rho}{1+\delta_1}(x-1)\right) = r + \zeta g.$$

The function $R(x)$ satisfies

$$\begin{aligned} R(1) &= 0 \\ \lim_{x\to+\infty} R(x) &= +\infty \\ R'(x) &= \delta_0(x-1)^{\delta_1-1}\left(\delta_1 x + (1-\rho)(x-1)\right) > 0 \end{aligned}$$

Thus R is a strictly increasing bijection from $[1,+\infty[$ to $[0,\infty[$. Let R^{-1} denotes its inverse, which is also strictly increasing. For any $g \geq 0$, let:

$$x(g) = R^{-1}(\rho(r+\zeta g)). \tag{3}$$

Note that:

$$x(g) \geq \underline{x} \equiv R^{-1}(\rho r) > 1 = R^{-1}(0).$$

where $\underline{x} > 1$ is a lower bound on the equilibrium span chosen by the firm. Correspondingly, define the rate of expropriation corresponding to growth rate g as:

$$\tilde{\delta}(g) = \delta(x(g)), \tag{4}$$

and the lower bound on the equilibrium expropriation risk:

$$\underline{\delta} \equiv \delta(\underline{x}) > 0. \tag{5}$$

With this notation, we can rewrite Equation (1) as:

$$g = n\left(\frac{g}{\tilde{\delta(g)}}, 2-\rho\right) - \frac{\zeta}{1-\zeta}(r + \zeta g + \tilde{\delta}(g)) \equiv M(g). \tag{6}$$

Let $M(g)$ be the function on the right-hand side of this equation. Since $\tilde{\delta(g)} > 0$ for all $g \geq 0$, we have that $\lim_{g \to 0} g/\tilde{\delta(g)} = 0$. Thus,

$$\lim_{g \to 0} M(g) = \nu - \frac{\zeta}{1-\zeta}(r + \underline{\delta}).$$

Moreover, when $\rho = 1$, $h\left(\frac{g}{\tilde{\delta(g)}}, 2-\rho\right) = \nu$ for all $g > 0$, and otherwise, $h\left(\frac{g}{\tilde{\delta(g)}}, 2-\rho\right) \to 0$ as $g \to +\infty$. Either way, we have:

$$\lim_{g \to +\infty} M(g) = -\infty.$$

Thus, if $M(g)$ is strictly decreasing, then a necessary and sufficient condition for the existence of an equilibrium is that:

$$M(0) = \nu - \frac{\zeta}{1-\zeta}(r + \underline{\delta}) \geq 0. \tag{7}$$

In what follows we establish that indeed, $M(g)$ is strictly decreasing with respect to g.

First note that as highlighted above, $h(y, 2-\rho)$ is strictly decreasing with respect to y when $\rho < 1$, and constant when $\rho = 1$. So it is sufficient to establish that the function:

$$m(g) \equiv \frac{\tilde{\delta}(g)}{g}$$

is strictly increasing with respect to g. A necessary and sufficient condition for $m(g)$ to be increasing is that:

$$1 \leq \frac{g\tilde{\delta}'(g)}{\tilde{\delta}(g)} = \frac{gx'(g)}{x(g)} \frac{x(g)\delta'(x(g))}{\delta(x(g))}.$$

We have:

$$\frac{x(g)\delta'(x(g))}{\tilde{\delta}(g)} = (1 + \delta_1)\frac{x(g)}{x(g) - 1}.$$

Furthermore,

$$\begin{aligned}
\frac{gx'(g)}{x(g)} &= \frac{\rho\zeta g}{x(g)R'(x(g))} \\
&= \frac{\rho\zeta g}{x(g)\delta_0(x(g)-1)^{\delta_1}\left(\delta_1 \frac{x(g)}{x(g)-1} + 1 - \rho\right)}
\end{aligned}$$

Thus the inequality becomes:

$$1 \quad \leq \quad \frac{\rho\zeta g}{\delta(x(g))\left(\delta_1 \frac{x(g)}{x(g)-1} + 1 - \rho\right)}.$$

This condition is equivalent to:

$$\delta(x(g)) \geq (x(g)-1)\rho(r + \delta(x(g))).$$

The case $\rho = 0$ In this case, the set of necessary conditions for a BGP collapse to:

$$g \quad = \quad \nu \frac{1}{1+\frac{g}{\delta}} - \frac{\zeta}{1-\zeta}\left(r + \zeta g + \delta(x)\right), \tag{8}$$

$$x\delta'(x) \quad = \quad 0, \tag{9}$$

which we can rewrite as

$$g(g+\delta) \quad = \quad \nu\delta - \frac{\zeta}{1-\zeta}\left(r + \zeta g + \delta(x)\right)(g+\delta), \tag{10}$$

$$x\delta'(x) \quad = \quad 0. \tag{11}$$

The unique solution is $g = 0$ and $x = 1$, so that $\delta(x) = 0$. (Any solution with $\delta > 0$ would also require $\delta'(x) > 0$ and $x > 0$, which would violate the second condition.) Finally, note that the first equilibrium condition then shows that:

$$\lim_{\rho \to 0^+} \frac{g}{\delta} = \frac{1-\zeta}{\zeta}\frac{\bar{n}}{r} - 1.$$

■

Regions of non-monotonicity. Throughout we fix the values of $\delta_1 > 0$, $\zeta \in]0,1[$, and $r > 0$.

First, assume that $\nu > 0$ and $\delta_0 > 0$. From the proof of Lemma 2, we know that when $\rho = 1$, $g/\delta(x)$ is finite. We now differentiate the system of Equations (1)-(2) with respect to ρ, and evaluate the differentials at $\rho = 1$. We obtain:

$$\frac{\partial g}{\partial \rho} \quad = \quad \frac{\partial n}{\partial \rho} - \frac{\zeta}{1-\zeta}\left(r + \zeta\frac{\partial g}{\partial \rho} + \delta'(x)\frac{\partial x}{\partial \rho}\right),$$

$$x\delta''(x)\frac{\partial x}{\partial \rho} \quad = \quad x\delta'(x) + \zeta\frac{\partial g}{\partial \rho}.$$

Here, we have used, in particular, the fact that $\frac{\partial n}{\partial y}(y,1) = 0$ to simplify the first equation. We have also guessed that $\frac{\partial x}{\partial \rho}$ and $\frac{\partial g}{\partial \rho}$ are both finite at $\rho = 1$, but we will verify this guess below. We can rewrite the system as:

$$
\begin{aligned}
\frac{1-\zeta+\zeta^2}{1-\zeta}\frac{\partial g}{\partial \rho} + \frac{\zeta}{1-\zeta}\delta'(x)\frac{\partial x}{\partial \rho} &= \frac{\partial n}{\partial \rho} - \frac{\zeta}{1-\zeta}r \\
-\zeta\frac{\partial g}{\partial \rho} + x\delta''(x)\frac{\partial x}{\partial \rho} &= x\delta'(x).
\end{aligned}
$$

The discriminant is:

$$
\Delta(x) = \frac{1-\zeta+\zeta^2}{1-\zeta}x\delta''(x) + \frac{\zeta^2}{1-\zeta}\delta'(x) > 0
$$

where the latter inequality is because the optimal span is $x > 1$, and δ is convex. The solution is:

$$
\begin{aligned}
\frac{\partial g}{\partial \rho} &= \frac{1}{\Delta(x)}\left(x\delta''(x)\left(\frac{\partial n}{\partial \rho} - \frac{\zeta}{1-\zeta}r\right) - \frac{\zeta}{1-\zeta}x(\delta'(x))^2\right) \\
\frac{\partial x}{\partial \rho} &= \frac{1}{\Delta(x)}\left(\zeta\left(\frac{\partial n}{\partial \rho} - \frac{\zeta}{1-\zeta}r\right) + \frac{1-\zeta+\zeta^2}{1-\zeta}x\delta'(x)\right)
\end{aligned}
$$

We rewrite the solution for $\frac{\partial g}{\partial \rho}$ as:

$$
\begin{aligned}
\frac{\partial g}{\partial \rho} &= \frac{x\delta''(x)}{\Delta(x)}\left(\frac{\partial n}{\partial \rho} - \frac{\zeta}{1-\zeta}\left(r + \frac{(\delta'(x))^2}{\delta''(x)}\right)\right) \\
&= \frac{1-\zeta}{1-\zeta+\zeta^2+\zeta^2\frac{\delta'(x)}{x\delta''(x)}}\left(\frac{\partial n}{\partial \rho} - \frac{\zeta}{1-\zeta}\left(r + \frac{(\delta'(x))^2}{\delta''(x)}\right)\right).
\end{aligned}
$$

We finally substitute the functional form for $\delta(x)$ and note that $\delta_0(x-1)^{1+\delta_1} = (1+\delta_1)\delta(x)$ to get:

$$
\frac{\partial g}{\partial \rho} = \frac{1-\zeta}{1-\zeta+\zeta^2+\frac{\zeta^2}{\delta_1}\frac{x-1}{x}}\left(\frac{\partial h}{\partial \rho} - \frac{\zeta}{1-\zeta}\left(r + \frac{1+\delta_1}{\delta_1}\delta(x)\right)\right).
$$

This expression shows that:

$$
\frac{\partial g}{\partial \rho} < 0 \quad \Longleftrightarrow \quad \frac{\partial n}{\partial \rho} < \frac{\zeta}{1-\zeta}\left(r + \frac{1+\delta_1}{\delta_1}\delta(x)\right).
$$

Figure 5 then reports a partition of the space of admissible values for (ν, δ_0) such that this inequality holds. ■