Intangible Capital, Firm Scope, and Growth*

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Abstract

Intangible assets represent information that needs to be embodied, or stored, in order to be used in production. This information can be replicated and stored in multiple instances, even if imperfectly. Replicability implies that a specific intangible asset can be deployed simultaneously in multiple uses by a single firm, allowing it to expand its scope. At the same time, however, replicability implies a risk that a firm's intangibles will be copied or appropriated by competitors. We embed these properties into an otherwise standard endogenous growth model, and show how improvements in the technology for replicating intangibles can lead to larger firms, an increase in concentration, valuation ratios, and the profit share, but lower growth.

Keywords: Investment, Intangible capital, Technological change, Long-run growth.

JEL codes: G32, G33, O31, O32.

*We are grateful to Philippe Aghion, Andy Atkeson, Anmol Bhandari, Maarten De Ridder, Teresa Fort, Chad Jones, Pete Klenow, Huiyu Li, Ellen McGrattan, Ben Moll, Jane Olmsted-Rumsey, Juliana Salomao, Vincent Stark, Alwyn Young, along with seminar participants at Brown, UT Austin, UT Dallas, Universite Paris Dauphine, Dartmouth Tuck, HEC, University of Houston, INSEAD, ITAM, Kellogg, LBS Economics, LSE Economics, LSE Finance, McGill, Minneapolis Fed, UCL, SAIF, SED, the ICI conference, and the Conference on the Economics of Innovation in Memory of Zvi Griliches.

1 Introduction

What is the difference between ideas, machines, and intangible assets? Intangible assets are ideas that need to be embodied or stored (say, in people) in order for them to be useful in production (Crouzet, Eberly, Eisfeldt, and Papanikolaou, 2022). This has two important implications. First, intangible assets can be replicated, in the sense the same productive idea can be embodied in multiple people or locations within the firm. This replicability need however not be costless or perfect. That is, even if the underlying idea is non-rival in the language of endogenous growth models (Romer, 1986, 1990; Jones, 1995), frictions in the transmission of ideas within firms can limit the replicability of their intangibles.[1] Second, institutions in principle allow intangible assets to be *excludable*: they can assign and enforce cash-flow rights to specific firms—just like physical capital. However, unlike physical assets, enforcing excludability is harder for intangible assets. That is, it is significantly harder to prevent the theft of an idea than the theft of a machine.[2]

Importantly, the degree of replicability is a function of the technology used to store and replicate intangibles: language, writing, the printing press, magnetic tapes, and cloud storage, have facilitated greater replicability of intangible assets over time. How do improvements in this replication technology affect economic growth? To study this question, we build an endogenous growth model with intangible capital. We find that improvements in replication technology may increase growth in the short run, but can lead to lower growth in the long run, even as they lead to higher concentration, valuation ratios, and profit shares.

We begin by formalizing these two properties of partial replicability and excludability in a simple model of production. A given intangible asset can potentially be embodied (stored) in multiple locations. The locations can represent different physical locations or simply different tasks in the production of the final good. The measure of locations that the intangible is stored denotes its scope or scale of implementation. Crucially, the intangible asset is partly replicable, in that increasing the quantity of intangibles stored in a particular location does not require reducing their use in other locations one-for-one (as would be the case for physical capital). Rather, this elasticity of substitution depends on the current state of replication technology: if replication were perfect or costless, increasing the quantity of intangibles stored in one location would not affect the quantity of intangibles stored elsewhere. Crucially, this replicability parameter determines the returns to the scope of implementing intangible

[1] For example, a firm can deploy an inventory management system across all its warehouses. However, the firm may not be able to perfectly replicate the performance of this system in all locations: implementing this system in a new warehouse will require training employees and management, to adapt the system to the location's unique requirements. This training process need not always produced the desired outcome.

[2] Property rights of certain intangibles, like patents, are well-defined and enforced in many countries, making them effectively excludable capital goods. For other types of intangibles, like managerial methods, organization capital, or even databases and software, excludability is often more difficult to enforce, even in an environment with otherwise well-defined property rights.

assets.

What limits the scope of implementation of intangible assets? Our model has no physical costs of implementation, thus in the absence of any other frictions the scope would be infinite: everyone in the world would use the same cutting edge production method. However, a key feature of intangible assets is that property rights cannot be perfectly enforced. Our assumption is that increasing the scale of implementation exposes the original creator of the intangible (the entrepreneur) to the possibility of expropriation. For example, increasing the scale of implementation requires the entrepreneur to share trade secrets with key employees, who in turn can expropriate the entrepreneur by creating their own firms. Thus, by increasing the scale of implementation of intangibles, the entrepreneur can appropriate a smaller share of a larger pie. This tradeoff depends not only on the degree of replicability but also on a second parameter that governs the degree of intellectual property protection. The entrepreneur optimally chooses the scope of implementation to maximize her appropriable value of the new intangible.

We then embed these mechanisms into an otherwise standard endogenous growth model where a fixed measure of workers can be employed either in production or in the creation of new intangibles (entrepreneurs). Our first main result is that, increasing the degree of replicability of intangibles within the firm leads to lower long-run growth through lower levels of entrepreneurship (new entry). In particular, increasing the level of replicability of intangible within the firm results in a larger scale of implementation, which increases the return to entrepreneurship, and therefore growth, in the short run. However, this increase is temporary: the eventual increase in firm scale implies that equilibrium wages in production, and therefore the opportunity cost of entrepreneurship, increases. This increased opportunity cost of entrepreneurship, combined with the fact that entrepreneurs do not appropriate the full value of new firms, implies that in the long run the rate of entrepreneurship and new firm creation falls, which leads to lower levels of growth for all parameters.

Our model has implications for the measurement of intangible capital, which in turn has implications for measuring the Solow residual. Specifically, common approaches to measuring the stock of intangible assets (for instance, the approach adopted by the BEA), construct indices of intangible capital by capitalizing expenditures on the creation of new intangibles. However, these intangible indices crucially miss the scope of implementation: a given R&D expense will generate a larger stock of intangible capital if the results are implemented at a higher scope than in a lesser scope. As a result, improvements in replication technology, which lead to higher scope, will lead to a higher Solow residual.

The negative relation between the replicability technology and long run growth rests on a specific assumption: improvements in replicability technology only affect the within-firm deployment of intangibles. If the scope of implementation of intangibles also affects the degree to which intangibles diffuse in the economy—that is, they lead to greater spillovers to future entrepreneurs—then the relation between replicability and growth becomes non-monotone (inverse U-shaped). Specifically, the higher scope of implementation increases the rate of new knowledge spillovers, which can offset the decline in growth through lower levels of entrepreneurship.

We next calibrate the model to the data. To do so, we use data on firm scope for a panel of public non-financial US firms from Hoberg and Phillips (2024). The cross-sectional relation between firm scope and firm revenue identifies the replicability parameter in our model. Our results indicate that the replicability parameter increases from approximately 0.18 to 0.40 from 1990 to 2019. To identify the extent to which entrepreneurs retain the full value of their intangibles, we use new hand-collected data on equity ownership to quantify the share of equity retained by the founding team (pre-IPO). Importantly, this share declined over the 1996-2016 period, from approximately 65% in 1996 to approximately 45% by the end of 2016. Both of these patterns are consistent with our model tradeoff between higher scope and lower entrepreneurial share.

Our calibrated model is largely consistent with the data. In the long-run, in response to the improvement in replicability, the model generates a lower level of economic growth, higher levels of concentration, lower rates of firm creation, and a higher profit share and Tobin's Q, in many cases by magnitudes that are comparable to the data. However, the transition path is non-monotonic. On impact, economic growth increases, as does measured total factor productivity, mirroring qualitatively and, to an extent, quantitatively the boom experienced by the US economy in the mid-to-late 1990s. In the model, this boom is due to the fact that new firms entering after the technological change can do so at a larger scope. Over the long-run, this advantage is offset by rising wages, which increases the opportunity cost of entrepreneurship, and by the fact that larger scope implies a higher risk of exprorpriation. Both lead to lower entry incentives. But in the short run, wages do not adjust immediately, leading to temporarily high returns to entry, rates of entry, and ultimately, growth.

Relation to the literature Our work complements the literature on endogenous growth that starts with Romer (1990). Our contribution is to provide a new model of production using intangible capital and explore its implications for long-run growth. Our model explicitly allows for the degree of replicability and the degree of enforcement of intellectual protection to be structural parameters that could vary with the specific characteristics of the intangible and the technology for transmitting ideas and enforcing property rights.

That said, there are some key differences between our notion of intangibles and the notion of ideas in existing models of endogenous growth. First, ideas in the most endogenous growth models are fully non-rival. By contrast, we model intangibles as essentially a *partially non-rival* input in production within the firm, with the firm choosing the scope of implementation subject to concerns about expropriation. Second, our notion of intangibles is likely broader. That is, intangibles in Romer (1990) essentially represent the exclusive right to produce a particular good—or a technology vintage that can be used to produce an existing good at lower marginal cost as in Aghion and Howitt (1992); Klette and Kortum (2004); De Ridder (2022); Aghion, Bergeaud, Boppart, Klenow, and Li (2022b). By contrast, our notion encompasses not only the idea of a new product or specific production process, but also concepts such as organization capital that can deployed at a broader scale or scope. Third, our model features an explicit tradeoff between the scope (or scale) of deployment of the intangible and the likelihood of expropriation by other parties. The possibility of expropriation introduces a further wedge between private and social returns in the creation of new intangibles—by contrast, in Romer (1990) spillovers occur only in the creation of new projects which build on the existing frontier. This tradeoff that limits the scale of deployment of intangibles is conceptually distinct from monopolistic competition in Romer (1990) or the need for complementary inputs that are in scarce supply (Atkeson and Kehoe, 2005).

Our modeling of intangible assets is perhaps most similar to McGrattan and Prescott (2009, 2010a), in that the same intangible can be used (in our language, replicated) in multiple locations (countries in their setting). A key difference with their work is that, rather than restricting the number of possible locations, we introduce a cost of expanding scope: the possibility of expropriation. This cost of expanding scope is not a physical cost, which differentiates our model from recent work in which resource costs (either labor scarcity or final output costs) limit firms' scope of operations (Aghion, Bergeaud, Boppart, Klenow, and Li, 2022a, 2023). Along these lines our work also complements the literature that models intangibles as a factor of production that allows firms to reduce the cost of entering new markets (Argente, Moreira, Oberfield, and Venkateswaran, 2020; Hsieh and Rossi-Hansberg, 2022). Relative to these papers, we allow for the replicability of intangibles to be imperfect; we allow for concerns about expropriation to endogenously limit the deployment of intangibles and focus on the implications of the model for long-term growth.

This possibility of expropriation arises because enforcement of cashflow rights over intangibles is imperfect and contracts are potentially incomplete (Grossman and Hart, 1986; Hart and Moore, 1988, 1994; Aghion and Bolton, 1992; Hart and Holmstrom, 2010). The risk of expropriation is intimately linked to increasing the scope of implementation of the intangible assets. Specifically, implementing an idea at a higher scope (replicating the idea) will typically require that the creator of the intangible communicates the idea to a third party—her key employees or by the providers of financial capital. However, ideas, once communicated, can be stolen (the Arrow information paradox, Arrow, 1962). This risk is salient: Bhide (1999) reports that 71% of the founders of firms in the Inc 500 list of fast growing technology firms report that they replicated or modified ideas encountered through previous employment. Financiers can appropriate significant rents by diluting the innovator's stake in the venture, which is often due to contractual features of the VC arrangement (see, e.g. Kaplan and Stromberg, 2004).[3]

[3] For instance, as described in Atanasov, Ivanov, and Litvak (2012), the founder of Pogo.com, an e-gaming company, sued the VCs on the board for issuing complicated derivative securities, effectively reducing his stake from 13% to 0.1%, and then refusing to redeem his stock in violation of prior agreement.

Alternatively, financiers can also transfer the firm's knowledge to other firms that they have a financial stake in (Cabezon and Hoberg, 2024).

Our work also complements the literature on the diffusion of technology through imitation and spillovers (Lucas and Moll, 2014; Perla and Tonetti, 2014; Stokey, 2015; Perla, Tonetti, and Waugh, 2021; Akcigit and Ates, 2019). Similar to Lucas and Moll (2014), we model knowledge as partly a rival input (alternatively, replicable).[4] Relative to these papers, we model intangibles as a replicable capital input, analyze the impact of the degree of replicability on growth, and emphasize the dark side of knowledge spillovers, namely the fact that innovators cannot fully appropriate the value of their creations.

The non-monotonic effect of replicability on long-run growth may be reminiscent of the non-monotonic relation between the degree of competition in the product market and the level of innovation and growth emphasized in Aghion, Bloom, Blundell, Griffith, and Howitt (2005). These two mechanisms have some similarities but are conceptually distinct: the main comparative static we focus on is changes in the degree of replicability within the firm, a notion which is absent in their model. That said, our notion of expropriation by imitators has some conceptual similarities to the loss of rents to competitors. What drives the positive relation between the level of competition and growth in Aghion et al. (2005) is that investing in innovation allows firms to lower the risk of losing monopoly rents—what they term the 'escape the competition effect'. By contrast, in our setting, greater investment in creating intangibles, or deploying them at a greater scale, always increases the risk of expropriation.

Recent advances in this literature have explored the importance of certain non-rival types of capital goods, an in particular data (Farboodi and Veldkamp, 2020; Jones and Tonetti, 2020). For instance, Jones and Tonetti (2020) model the increasing returns that can be achieved through the use of customer data to improve productivity: the more people consume a given product the more data the firm has to improve productivity which leads to higher output and hence more data. Our main contribution relative to this work is to study the role of replicability of intangibles more generally, and to allow for the risk of expropriation and the degree of replicability of intangibles to be partial.

Our model of production using intangible capital implies a strong association between the size of the firm and its scope of operations. In our model, a higher degree of replicability implies a greater complementarity between the stock of intangibles and firm scope. Consistent with this prediction, Ding, Fort, Redding, and Schott (2022) find that firms with greater in-house production of auxiliary services (a measure of intangibles) grow faster and are more likely to enter new industries than other firms.

More broadly, a large literature in macroeconomics and finance has highlighted the macroeconomic

[4]As Lucas and Moll (2014) note, "knowledge is partially rival: it is 'rival' in the short run because people who want to access better knowledge must exert effort and have the good luck to run into the right people; but knowledge is 'nonrival' in the long run in the sense that it is in no way diminished when it spreads from one person to another.

implications of the rise in intangible capital for various topics, including investment and asset prices, labor markets, and growth (McGrattan and Prescott, 2010a,b; Haskel and Westlake, 2017; Crouzet and Eberly, 2019, 2021). Our key distinction relative to these papers lies again in how we model the production process using intangibles. Existing work models intangible capital as a factor of production that is qualitatively similar to physical capital—intangible capital is yet another factor of production that can be accumulated subject to adjustment costs. The main difference between tangible and intangible capital in these models is simply due to parameter choices—intangibles are fully excludable and rival capital inputs, but may differ in their rate of depreciation, price, or riskiness. Our contribution to this literature is to revisit the same issues using a different model of production that allows for the key economic differences between physical and intangible capital: their degree of replicability, and their degree of excludability.

Our work also highlights the challenges in measuring intangible assets in the data. Existing work has focused on measuring specific types of intangible capital, including patented innovations (Hall, Jaffe, and Trajtenberg, 2005; Kogan, Papanikolaou, Seru, and Stoffman, 2017; Kelly, Papanikolaou, Seru, and Taddy, 2021); management practices (Bloom and Van Reenen, 2007); software and data-related assets (Bresnahan, Brynjolfsson, and Hitt, 2002); brands and customer capital (Gourio and Rudanko, 2014); or organization capital (Eisfeldt and Papanikolaou, 2013; Bhandari and McGrattan, 2021). Our model shows that the private value of an intangible asset, its acquisition cost, its contribution to aggregate output, and its social value are in general distinct objects that need not coincide.

2 Economic environment

Section 2.1 describes the main elements of the model. Section 2.2 discusses the main assumptions. Section 2.3 states some basic equilibrium properties. Proofs are in Appendix A.1.

2.1 Description

There are three groups of agents: the household; the final goods producer; and intermediate goods producers, which we refer to as "firms". Figure 1 provides an overview of how these groups interact.

2.1.1 Household and final good producer

Household The household maximizes:

$$U_t = \int_{s \geq t} e^{-\eta s} \log(C_{t+s}) ds,$$

where η is the time discount factor. They supply one unit of labor inelastically. Labor is freely allocated to either entrepreneurship or production. In equilibrium, it must therefore earn exactly the production wage rate W_t. The household owns all firms and receives any dividends paid out. The household's flow budget constraint is:

$$V_t dS_t = (W_t + D_t S_t - P_{Y,t} C_t)\, dt,$$

where V_t is the price of shares issued by firms, D_t the dividend rate, S_t be the number of shares owned by the household (which will be normalized to $S_t = 1$ in equilibrium), and $P_{Y,t}$ the price of the final good. Defining the rate of return on shares as $r_t \equiv \frac{D_t/P_{Y,t} + d(V_t/P_{Y,t})}{V_t/P_{Y,t}}$, then optimality requires that:

$$\frac{dC_t}{C_t} = (r_t - \eta)dt.$$

Final good producer There is a perfectly competitive final good producer that purchases and aggregates intermediate varieties. Their problem is:

$$\min_{\{y_t(m)\}_{m\in[0,M_t]}} \quad \int_0^{M_t} p_t(m) y_t(m) dm$$
$$\text{s.t.} \quad \left(\int_0^{M_t} y_t(m)^\chi dm\right)^{\frac{1}{\chi}} \geq Y_t \qquad [P_{Y,t}]$$

Here, Y_t is total output of the final good, and M_t is the total number of intermediate varieties available at time t. Moreover, $\chi \in [0, 1]$ governs the elasticity of substitution between these varieties, with $\chi = 1$ corresponding to the case of perfect substitutes and no markups, and $\chi = 0$ corresponding to the Cobb-Douglas case, where total expenditure on each product is fixed. This leads to the demand system:

$$y_t(m) = \left(\frac{p_t(m)}{P_{Y,t}}\right)^{-\frac{1}{1-\chi}} Y_t, \quad P_{Y,t} = \left(\int_0^{M_t} p_t(m)^{-\frac{\chi}{1-\chi}}\right)^{-\frac{1-\chi}{\chi}}. \tag{1}$$

2.1.2 Intermediate goods producers

Let J_t represent the number of firms that have entered up to time t, and let $j \in [0, J_t]$ index these firms. Let $\tau(j)$ denote the date at which firm j enters. Each firm exits with constant intensity δ per unit of time; δ represents obsolescence risk. Between entry and exit, firm j is the monopolistic producer of a measure $x(j)$ of different varieties. We refer to $x(j)$ as the "scope" of firm j. As explained in more detail below, firm scope is fixed at the time of entry. Economy-wide, M_t different varieties are produced at time t. We index them by $m \in [0, M_t]$. By the law of large numbers, we have:

$$M_t = \int_0^{J_t} e^{-\delta(t-\tau(j))} x(j) dj.$$

In what follows, we will denote the producer of variety m by $\hat{\jmath}(m)$.

Figure 2 shows the different stages of each firm's problem. At entry ($t = \tau(j)$), the entrepreneur that creates firm j has a total stock of intangible capital $n(j)$, the determination of which we discuss below. At that point, the entrepreneur makes two irreversible choices: the scope of the firm, or how many varieties to produce, $x(j)$; and how to allocate the firm's total intangible stock, $n(j)$, across the production of these varieties. After these choices have been made $(t > \tau(j))$, for each variety within their scope, the entrepreneur makes all remaining production and pricing decisions, including hiring production labor. In what follows, we describe this problem backwards. We first focus on the production and sales stage $(t > \tau(j))$, taking scope $x(j)$ and the allocation of intangible capital across varieties as given. We then discuss the choice of scope and the allocation of intangible capital, and finally explain how the initial total stock of intangible capital is determined, all of which occur at time $t = \tau(j)$.

Production and sales after entry The production technology for each variety is:

$$\forall m \text{ s.t. } \hat{\jmath}(m) = j, \quad y_t(m) = n_t(m)^{1-\zeta} l_t(m)^{\zeta},$$

where $\zeta \in (0, 1)$ is the Cobb-Douglas elasticity of output with respect to labor, $n(m)$ is the amount of intangible capital that the entrepreneur of firm $j = \hat{\jmath}(m)$ has allocated to variety m at entry, and $l_t(m)$ is production labor. The demand and cost functions for variety m at time t thus take the form:

$$p_t(y) = d_t y^{-(1-\chi)}, \quad c_{m,t}(y) = e_t(m) y^{\frac{1}{\zeta}},$$

where:

$$\begin{aligned} d_t &= P_{Y,t} Y_t^{1-\chi}, \\ e_t(m) &= W_t n(m)^{-\frac{1-\zeta}{\zeta}}. \end{aligned}$$

The solution to the monopoly problem for variety m at time t,

$$\pi_t(m) = \max_y \quad p_t(y) y - c_{m,t}(y)$$

consists of setting a markup equal to $1/\chi$:

$$p_t(y) = \frac{1}{\chi} c'_{m,t}(y).$$

Capital is fixed, so the decreasing returns to labor imply that marginal cost is not constant. Derivations reported in Appendix A.1.1 show that the monopoly profits from variety m can then be written as:

$$\pi_t(m) \quad = \quad A_t n(m)^{\omega}, \tag{2}$$

where A_t governs the marginal revenue product of capital invested in the production of any variety:

$$A_t \quad \equiv \quad (1-\zeta\chi) P_{Y,t}^{\frac{1}{1-\zeta\chi}} Y_t^{\frac{1-\chi}{1-\zeta\chi}} \left(\frac{\zeta\chi}{W_t}\right)^{\frac{\zeta\chi}{1-\zeta\chi}}.$$

In Equation (2), we have used the notation:

$$\omega \equiv \frac{(1-\zeta)\chi}{1-\zeta\chi} \in [0,1]. \tag{3}$$

The reduced-form parameter ω captures two sources of concavity in returns to the amount of intangible capital, $n(m)$, invested in variety m. First, there are decreasing returns to labor. Second, whenever $\chi < 1$, the monopolistic firm faces a downward-sloping demand curve for each variety.

The allocation of intangible capital and the choice of scope at entry We now focus on the decisions made by the entrepreneur at entry: how many new varieties to produce, $x(j)$, and how to allocate total intangible capital $n(j)$ across these varieties. Given $x(j)$ and the allocation $\{n(m)\}_{m\in[0,x(j)]}$, the enterprise value of the firm created by the entrepreneur is:

$$v_t\left(x(j), \{n(m)\}_{m\in[0,x(j)]}\right) \quad = \quad \int_{s\geq 0} e^{-\int_0^s (r_{t+u}+\delta)du} \left\{\int_0^{x(j)} \; A_{t+s}\; n(m)^{\omega} dm\right\}$$

The term in curly brackets represents the future flow profits from all the varieties produced by the firm. Thus enterprise value can be expressed as:

$$v_t\left(x(j), \{n(m)\}_{m\in[0,x(j)]}\right) = \nu_t\, A_t \left\{\int_0^{x(j)} \; n(m)^{\omega} dm\right\}$$

where:

$$\nu_t = \int_0^{+\infty} e^{-\int_0^s (r_{t+u}+\delta)du} \frac{A_{t+s}}{A_t} ds. \tag{4}$$

Here, ν_t is the price-to-earnings ratio of firms newly created at time t.

The main decision faced by the entrepreneur concerns the allocation (the storage) of the intangible stock $n(j)$ to each location that produces variet m. Given an initial total intangible capital stock, $n(j)$, the entrepreneur chooses scope and intangible allocation in order to solve:

$$v_t^{(e)}(n(j)) \quad = \quad \max_{x(j),\{n(m)\}_{m\in[0,x(j)]}} \quad \Big(1-\gamma_t(x(j);\lambda)\Big)\ v_t\left(x(j),\{n(m)\}_{m\in[0,x(j)]}\right) \tag{5}$$

$$\text{s.t.} \quad \left(\int_0^{x(j)} n(m)^{\frac{1}{1-\rho}} dm\right)^{1-\rho} \leq n(j). \tag{6}$$

There are two separate assumptions embedded in this expression of the entrepreneur's choice problem.

Assumption 1 (Replicability) *The parameter* ρ *satisfies* $0 \leq \rho \leq 1$.

This assumption has to do with constraint (6), which governs the allocation of intangible capital across varieties. We use this constraint to capture the idea that intangible capital can be a non-rival (replicable) input within the firm. To see why, first consider the case $\rho = 0$. Then, the constraint (6) becomes:

$$\int_0^{x(j)} n(m)dm \leq n(j).$$

Given total intangible capital $n(j)$, allocating more capital to the creation of one variety requires reducing one-for-one its allocation to the production of other varieties. In this case, the constraint (6) implies that capital cannot be replicated within the firm.

By contrast, when $\rho = 1$ the constraint (6) becomes

$$\max_{m\in[0,x(j)]} n(m) \leq n(j).$$

In this case, allocating intangibles to the production of a particular variety does not require reducing the allocation to other varieties. The entrepreneur can allocate the entirety of the intangible stock $n(j)$ to each variety; intangible capital is replicable within the firm.

When $\rho \in (0,1)$, we will say that intangible capital is 'imperfectly replicable' within the firm. The marginal rate of technical substitution of capital between varieties is less than 1 (as when $\rho = 0$) but greater than 0 (as when $\rho = 1$). We associate the parameter ρ with the degree of replicability of intangibles within firm. In Section 2.2.1, we discuss in more detail the constraint (6), its connection to existing work, and examples of assets that exhibit imperfect replicability within the firm.

Assumption 2 (Non-exclusivity) *The function* $\gamma_t : [0,+\infty)^2 \to [0,+\infty)$ *is twice differentiable, strictly increasing, and weakly convex in* x *and satisfies* $\gamma_t(0;\lambda) = 0$ *and* $\lim_{t\to+\infty} \gamma_t(x;\lambda) \geq 1$. *Moreover,* *it is weakly increasing in λ, and satisfies:*

$$\gamma_t(x;0) = 0 \quad \textit{and} \quad \lim_{\lambda \to +\infty} \gamma_t(x;\lambda) > 0. \tag{7}$$

To understand the meaning of this assumption, first note that the objective function of the entrepreneur, Equation (5), implies that they only retain a share $1 - \gamma_t(x;\lambda)$ of the total entreprise value that they create at entry. Thus $\gamma_t(x;\lambda)$ represents the share of enterprise value that is pledged by the entrepreneur to other key stakeholders in the firm at entry.

Assumption (2) expresses two ideas. First, if the entrepreneur chooses to create a firm with larger scope, they must also agree to pledge a *larger* fraction of it to other stakeholders. The second part of the assumption (Equation 7) says that the share pledged to other stakeholders is increasing with respect to a structural parameter, $\lambda \in [0, +\infty)$. This parameter is normalized such that $\lambda = 0$ implies that the entrepreneur does not need to pledge any enterprise value to other stakeholders at entry, while when $\lambda \to +\infty$, they must pledge a share of enterprise value that is strictly bounded away from zero.

We interpret the structural parameter λ as capturing how *non-exclusive* cash flow rights to intangible capital might be. When λ is close to 0, the entrepreneur can expand scope without giving up substantial cash flow rights to other key stakeholders. In this case we say that there is strong exclusivity to the entrepreneur's intangibles. On the other hand, when $\lambda \to +\infty$, the entrepreneur must provide a (potentially large) share of cash-flow rights to other key stakeholders in order to be able increase scope. In this case cash flow rights over the entrepreneur's intangibles are non-exclusive.

Who are key stakeholders? In Section 2.2.2, we discuss a precise microfoundation for Assumption (2) in which key stakeholders are managers or high-skilled workers with knowledge of the firm's intangibles. In this microfoundation, we assume that, in order to increase scope, the entrepreneur must designate a set of key stakeholders and share the firm's intangible assets with them. The number of key workers increases with scope. Furthermore, with intensity λ per unit of time, each key worker may receive the option to start their own firm, and compete with the entrepreneur using the intangibles the entrepreneur had shared with them at entry. Competition is Cournot and reduces the monopoly profits that the entrepreneur would otherwise enjoy. We show that the entrepreneur can deter this competition by offering key workers a fraction $\gamma_t(x;\lambda)$ of monopolistic enterprise value. Crucially, γ_t only depends on firm scope, and satisfies the properties of Assumption (2).

In the particular context of this microfoundation, the key stakeholders are therefore managers or high-skill workers familiar with the intangible capital of the firm. More generally, key stakeholders could also include outside investors with access to the key knowledge assets of the firm, such as venture capital. Key stakeholders are members of the representative household, so that any dividends from their stake in firms gets rebated to the household. Under this interpretation, λ captures degree to which property rights institutions preclude these stakeholders from using the firms' key knowledge assets in outside projects; such institutions could include patents, trademarks, non-competes, or trade secrets.[5]

From (5)-(6), the optimal allocation of intangibles for given scope $x(j)$ is the one that maximizes enterprise value, or equivalently, the one that solves:

$$\begin{aligned} \max_{\{n(m)\}_{m\in[0,x(j)]}} \quad & \int_0^{x(j)} n(m)^{\omega} dm \\ \text{s.t.} \quad & \left(\int_0^{x(j)} n(m)^{\frac{1}{1-\rho}} dm\right)^{1-\rho} \leq n(j). \end{aligned} \tag{8}$$

This allocation problem has a unique symmetric solution:

$$\forall m \;\; \text{s.t.} \;\; \hat{\jmath}(m) = j, \quad n(m) = x(j)^{-(1-\rho)} n(j). \tag{9}$$

In particular, if intangible capital is non-replicable within the firm $(\rho = 0)$, the entrepreneur divides up equally the total capital stock among varieties, $n(m) = n(j)/x(j)$; while if intangible capital is costlessly replicable within the firm $(\rho = 1)$, the entrepreneur is able to invest the entirety of the intangible capital stock $n(j)$ in each of the varieties, $n(m) = n(j)$. Substituting this solution into (5)-(6), the optimal scope of the firm is determined from:

$$x_t \;\; = \;\; \arg\max_{x(j)} \quad \Big(1 - \gamma_t(x(j); \lambda)\Big) \; x(j)^{1-\omega+\rho\omega}. \tag{10}$$

The optimal scope is the same for all firms newly created at time t, so we denote it by x_t. Equation (10) highlights the key trade-off that govern the choice of scope for the firm. On the one hand, greater scope increases enterprise value, and the more so, the higher the degree of replicability ρ (this is captured by the second-term on the right-hand side of Equation 10). This reflects the economies of scope, or the replicability, associated with intangible capital. On the other hand, when $\lambda > 0$, greater scope also reduces the share of enterprise value that the entrepreneur can hope to retain. This reflects the imperfect property rights, or non-exclusivity, associated with intangible capital.[6]

[5] Section 2.2.2 provides a more complete discussion of Assumption (2).

[6] Note that, because enterprise value is proportional to $n(j)^{\omega}$, the choice of scope is independent of the amount of intangible capital $n(j)$ that the entrepreneur has at their disposal to start the firm.

The stock of intangible capital of new firms Let B_t be the stock of public knowledge assets.[7] This stock grows as a result of entrepreneurial effort. Specifically, the law of motion of B_t is:

$$\underbrace{dB_t}_{\text{Public knowledge assets created in } [t,t+dt]} = \underbrace{(L_{E,t}dt)}_{\text{aggregate entrepreneurial effort}} \times \underbrace{f(B_t)}_{\text{entrepreneurial productivity}}, \tag{11}$$

where $L_{E,t}$ is total labor devoted to entrepreneurship, and $f(.)$ is an increasing function. Thus the stock of knowledge assets created per unit of entrepreneurial labor is:

$$n_t = \frac{dB_t}{L_{E,t}dt} = f(B_t).$$

An entrepreneur creating a firm at time t is endowed with an amount of intangible capital equal to n_t:

$$n(j) = n_t \quad \forall j \text{ s.t. } \tau(j) = t. \tag{12}$$

In the baseline version of the model, we will follow Romer (1990) and set:

$$f(B_t) = \xi B_t \quad \Longrightarrow \quad n_t = \xi B_t. \tag{13}$$

Here $\xi > 0$ is a parameter which, in equilibrium, will govern the relative size (in terms of intangible capital) of newly created firms, compared to existing ones. This implies that there can be endogenous growth without population growth. Section 2.2.3 considers an extension in which there are decreasing returns with respect to $L_{E,t}$ in the creation of new public knowledge, as in Jones (1995).

2.1.3 Equilibrium

Definition Labor can be freely allocated to production or entrepreneurship, so we must have:

$$\begin{aligned} L_{E,t} \geq 0 \quad & \text{and} \quad v_t^{(e)}(n_t) = W_t, \quad \text{or} \\ L_{E,t} = 0 \quad & \text{and} \quad v_t^{(e)}(n_t) \leq W_t. \end{aligned} \tag{14}$$

In equilibrium, the final good market and the market for production labor must clear:

$$Y_t = C_t, \quad L_{Y,t} + L_{E,t} = 1, \quad \int_0^{M_t} l_t(m)dm = L_{Y,t},$$

[7] B_t is distinct from the aggregate stock of intangible capital N_t deployed by firms, as well as the measured stock of intangibles, K_t, both of which we define and analyze in Section 2.3.

where $l_t(m)$ is labor demand from the firm producing variety m at time t, and $L_{Y,t}$ is total production labor demand.[8] The stock market must clear, $S_t = 1$; the free-entry condition (14) must hold; and each new entrepreneur's problem, the initial stock of intangible capital must be given by Equation (12), the optimal scope of the firm by Equation (10), and the optimal allocation of intangibles across varieties by Equation (8). We use the final good as the numéraire, so $P_{Y,t} = 1$.

2.2 Discussion of key assumptions

2.2.1 Replicability

We view intangible capital as productive knowledge that must be stored in a particular medium in order to be shared within the firm and used in production. The technology for storing and sharing this information makes intangible capital *replicable* within a firm. Assumption 1 captures this idea. The case $\rho = 0$ makes intangible capital exactly analogous to physical capital, while the case $\rho = 1$ makes intangible capital analogous to total factor productivity, or ideas, in endogenous growth models — which are generally treated as a non-rival within firm.

The state of technology for storing and disseminating specific intangible assets corresponds to different values of ρ. As a first example of intangible assets, consider past data on the purchasing behavior of a firm's customers, which are used in the firm's call centers to sell additional products. In the past, customer data was stored using pen and paper. Currently, customer data is stored in computer memory, which makes it significantly easier to replicate the data and make it available across all call centers in the firm. As such, we argue that this corresponds to the case of an intangible asset with relatively high ρ; further, the availability of computers has likely increased ρ for customer data over time as they can be more easily stored and disseminated within the firm.

As a second example, consider a particular method for organizing production, which is used within a firm's production plants. In this case, the intangible is stored in key employees, since it is part of the human capital of assembly-line workers and production managers. Replicating the production method and making it available to other plants within the firm will require supervision and training, which may be imperfect. Even if the firm could hire and train a sufficient number of production managers, something may still be lost in translation. In our view, this corresponds to the case of an intangible asset with lower ρ. Further, advances in the technology for communicating instructions on the new production method to workers and supervising them will likely increase the intangible's ρ.

We view the degree of replicability of an intangible asset within the firm as an intrinsic feature of the storage technology used to disseminate it. Since intangibles are fixed within firm, we assume that ρ

[8] See Equation (A2) in Appendix A.1.1 for the explicit expression of $l_t(m)$.

is also constant within firm. In Section 4, we study the transitional dynamics of the economy when ρ changes; in this case, newly created firms can take advantage of changes in how well technology allows productive knowledge to be faithfully replicated within the firm.

Note that in the model described above, we *do not* assume that imperfect replicability also applies to the stock of public knowledge, B_t. Instead, we allow all entrepreneurs, in a given period, to use the stock of knowledge B_t to start their firm, with no cost to future entrepreneurs. This helps us focus specifically on the *within-firm* implications of the replicability of intangible assets. In Section 2.2.3, we study an extension to the model where there are spillovers *between* (existing) firms and entrepreneurs that depend on how easy it is to replicate knowledge assets, that is, on the parameter ρ.

Finally, we highlight the connection of Assumption (1) to the idea, often encountered in the literature, that intangible assets enjoy particular "scale economies" (Haskel and Westlake, 2017). Specifically, in the model, the enterprise value of a new firm with scope x and total intangibles n is proportional to:

$$v_t(x, n) \propto x^{1-\omega+\rho\omega} n^{\omega}. \tag{15}$$

When $\rho = 0$ (so that capital cannot be replicated within the firm), enterprise value features jointly constant returns to scope x and to intangible capital n. When instead $\rho > 0$, so that capital is replicable within the firm, enterprise value features *increasing* returns to scope and intangible capital (since $\omega \in [0, 1]$). Moreover the marginal return to scope is increasing with respect to intangible capital, and the more so, the higher value of ρ. Either of these properties can be thought of as capturing the notion of economies of scope or scale associated with intangible assets.

2.2.2 Non-exclusivity

We first provide a microfoundation for Assumption (2), and then discuss more general interpretations.

A microfoundation for the function $\gamma_t(x; \lambda)$ As shortly discussed above, in this microfoundation, in order to choose a higher scope at entry, the entrepreneur must designate a subset of production workers as “key workers”, and share the firm’s intangibles with them.

Assumption 3 (Key workers) *The number of key workers needed to reach scope x is given by:*

$$\theta(x) = \max(0, \alpha(x - \bar{x})^{\kappa}), \tag{16}$$

with $\kappa \geq 1$, $\alpha > 0$, and $\bar{x} \geq 0$. Each key worker is assigned to a variety and has access to the intangible assets $n(m)$ invested in that variety. Moreover, with Poisson intensity λ, each key worker may receive the option to *start a new firm with intangible capital $n(m)$, and compete in a measure $\bar{x}$ of the varieties produced by the firm. Finally, competition in each variety is Cournot.*

Here, $\bar{x}$ is the number of varieties for which the entrepreneur *does not* need to designate key workers. The entrepreneur can therefore increase scope up to $\bar{x}$ without having to share the firm's intangibles with key workers. In most of the analysis below, we will normalize $\bar{x} = 1$. Additionally, the assumption that the entrepreneur only shares $n(m)$ with each key worker captures the idea that invested intangibles $n(m)$ are tied to specific uses. For instance, managers attached to a particular production facility may be those whose organization capital is most adapted to that facility.

Result 1 (Cost of deterrence under non-exclusivity) *At entry, the total value of the option to compete to all key workers designated by the entrepreneur is given by:*

$$\bar{v}_t^{(c)} = \gamma_t(x;\lambda) v_t^{(e)},$$

where the function

$$\gamma_t(x;\lambda) = \theta(x)\ \frac{\bar{x}}{x}\ \psi\ \mu_t(\lambda), \tag{17}$$

satisfies the properties of Assumption (2), with

$$\mu_t(\lambda) = \lambda\ \int_0^{+\infty} e^{-\int_0^s (r_{t+u}+\lambda)du} \frac{A_{t+s}\nu_{t+s}}{A_t \nu_t} ds, \tag{18}$$

while the expression for the constant $\psi \in [0,1]$ is given in Appendix A.1.2.

Proof. See Appendix A.1.2. ■

Thus, with this microfoundation, the function $\gamma_t(x;\lambda)$ can be interpreted as the share of enterprise value that the entrepreneur must give to key workers in order to deter them from exercising their option to compete with the entrepreneur, should that option arrive.

The intuition for the expression of $\gamma_t(x;\lambda)$ in Equation (17) is as follows. $\theta(x)$ is the total number of key workers the entrepreneur hires to reach scope x. The ratio $\bar{x}/x$ represents the fraction of all varieties in which each key worker would compete with the entrepreneur. The constant ψ is the ratio of Cournot profits to the key worker, to monopoly profits to the entrepreneur, in each variety. Finally, the term $\mu_t(\lambda)$ is a discount factor that captures the fact that competition will be delayed, as it is contingent on the Poisson shock materializing, which occurs with intensity λ. This delay term satisfies $\mu_t(0) = 0$ and $\lim_{\lambda\to+\infty} \mu_t(\lambda) = 1$. Importantly, the ratio ψ is independent of aggregate states and of the firm's scope and intangible capital, which ensures that the cost of deterrence can be written as a simple function of the number of key workers.

A key feature of Result (1) is that there is no physical resource cost underlying the share $\gamma_t(x;\lambda)$. More specifically, the underlying assumption is that each production worker is endowed with skill that the entrepreneur may call upon if they designate the production worker as a key worker. This skill is not a substitute for labor, and has no alternative uses, so that it need not earn the wage rate W_t. Instead, each worker is endowed with a sufficiently large supply of it, and does not use it if the entrepreneur does not designate them as a key worker. This feature distinguishes our model from environments in which resource costs (either labor scarcity or final output costs) are the key force limiting firm scope. (see, for instance Aghion et al., 2022a, 2023)

By contrast, what limits the entrepreneur's incentives to choose a higher scope is the limited appropriability of the rents associated with intangible capital. That is, the entrepreneur pledges a fraction $\gamma_t(x;\lambda)$ of enterprise value in order to provide adequate incentives to key workers. In turn, the need to provide incentives comes from imperfect property rights, or non-exclusivity, over the firm's intangible assets — that is, the fact that $\lambda > 0$. In particular, we *do not* assume that the supply of key workers is finite, or that key workers must devote a fraction of their time increasing scope.

In Appendix A.1.2 we show that, while it is always *possible* for the entrepreneur to deter entry from key workers, there may be situations where deterrence is not *optimal* for the entrepreneur ex-post, that is, once the option to compete arrives. These are situations in which the underlying parameters of the Cournot game, (ζ,χ), make profits under Cournot competition not too small *relative to* monopoly profits, so that collusion is difficult to sustain. In those cases deterrence might not be time-consistent for the entrepreneur. However, Appendix A.1.2 shows that if key workers bear a flow cost ε for competing with the entrepreneur in each variety, then there always exists a value $\underline{\varepsilon}$ such that if $\varepsilon \geq \underline{\varepsilon}$, deterrence is time-consistent.

More general interpretations Under the microfoundation just described, enterprise value is shared between entrepreneurs and key workers, whom the entrepreneur needs to rely on in order to increase scope. The key force determining how entry surplus is shared is the need for the entrepreneur to maintain incentives, and avoid the possibility that key workers will start competing firms using the intangibles the entrepreneur shares with them. One literal interpretation of λ in this case is that it represents non-compete clauses (for intangible assets such as production processes, managerial methods, or customer and supplier relations), or patents and trademarks (for intellectual property assets). In either case, exclusivity rights have positive but finite lives, so that $\lambda < +\infty$, but $\lambda > 0$.

More generally, Assumption (2) centers on the idea that, because of imperfect exclusivity, the entrepreneur must share some of the surplus associated with the creation of new knowledge assets with key stakeholders. Aside from key workers, these stakeholders could represent directors, board members, or early investors in the firm (such as venture capital), on whose expertise the entrepreneur must rely in order to deploy their assets at greater scale. The key assumption is that, for this to happen, the entrepreneur must share details of its core knowledge assets with these outsiders, but that the institutions and contracts that assign control and cash flow rights to these assets may be imperfect.[9] This creates an incentive problem, as these outsiders may decide to use the firm's knowledge assets for their own purposes. Additionally, in assumption (2), the share to be pledged increases with scope x. Intuitively, this captures the idea that if an entrepreneur deploys their intangibles at large scale (higher x), they face a higher risk that other key stakeholders will successfully replicate their key intangibles, and implement them in competing firms.[10]

2.2.3 Other assumptions

We have assumed that intangible capital is fixed within a firm, and in particular, cannot be accumulated after date 0. Aggregate intangible capital grows over time, as a result of new cohorts of firms being created, so that the model can still speak to the behavior of aggregate investment. The assumption that all new investment happens in the extensive margin is in line with models of putty-clay technology or models with vintage capital (see, e.g., Kogan, Papanikolaou, and Stoffman, 2020) We view this as a natural assumption in which the technology for storing intangibles is specific within a firm. Drawing the analogy with models with vintage-specific technical change, an increase in ρ in our model has a similar effect: improvements in the technology for storing intangibles displaces existing firms in favor of newly-created and future firms. That said, the model also lends itself to a different interpretation, in which each entrepreneur creates a new project with a particular scope and stock of intangible capital, but where projects may be aggregated within larger firms. We consider such an extension in Section .

Finally, by using Equation (11) to model the creation of new intangible assets, we have assumed that there are direct no spillovers from existing firms to future firms. An implication is that the degree of replicability ρ and the degree of non-exclusivity λ do not directly influence the evolution of the public stock of knowledge, B_t. In Section , we relax this assumption and consider instead a model where the creation of intangibles is influenced by the scope or the stock of intangibles of existing firms.

[9] Incomplete or unenforceable contracts may also affect ownership of physical capital; non-exclusivity is not necessarily specific to intangibles. The model still allows for imperfect property rights when $\rho = 0$.

[10] For example, in the late 80s, software maker Peoplesoft and its founder David Duffield were sued by Integral Systems, which claimed that its software was based on computer code that was stolen from the company while Mr. Duffield worked there. More broadly, Bhide (1999) argues that 71% of the founders of firms in the Inc 500 list of fast growing technology firms report that they replicated or modified ideas encountered through previous employment.

2.3 Basic equilibrium properties

We conclude the section by stating some key results on aggregation and characterization of equilibrium that will be useful in the remainder of the paper.

Aggregation Appendix A.1.4 shows that aggregate demand for production labor can be written as:

$$L_{Y,t} = \chi \frac{\zeta Y_t}{W_t}. \tag{19}$$

With monopolistic competition, the markup $1/\chi$ distorts labor demand downward. Moreover, aggregate output in this economy has the aggregation production function representation:

$$Y_t = N_t^{1-\zeta} L_{Y,t}^{\zeta}, \tag{20}$$

where the quantity index for intangible capital N_t is given by:

$$N_t \equiv \left(\int_0^{M_t} n(m)^{\omega} dm \right)^{\frac{1}{\omega}}. \tag{21}$$

Except when $\omega = 1$ *and* $\rho = 0$, this expression differs from firms' total stock of intangible capital:

$$K_t \equiv \int_0^{J_t} e^{-\delta(t-\tau(j))} n_{\tau(j)} dj. \tag{22}$$

Both markups ($\chi < 1$, which implies $\omega = 1$) and the highly replicable nature of intangible capital $(\rho > 0)$ affect the wedge between N_t and K_t, and thus affect the interpretation of measured quantities such as the Solow residual or returns to capital, a point we come back to in more detail in Section 4.

Equilibrium characterization Finally, we sketch out briefly the characterization of global equilibrium dynamics. Using the aggregation relations above, Appendix A.1.4 shows that the value of a firm at entry relative to the wage rate is:

$$\frac{v_t^{(e)}}{W_t} = \frac{1-\zeta\chi}{\zeta\chi} \left(1 - \gamma_t(x_t;\lambda)\right) x_t^{1-(1-\rho)\omega} \; \nu_t \; L_{Y,t} \left(\frac{\xi B_t}{N_t} \right)^{\omega}. \tag{23}$$

Define:

$$c_t \equiv \left(\frac{\xi B_t}{N_t} \right)^{\omega}. \tag{24}$$

This ratio depends on the size of new firms $(n_t = \xi B_t)$ relative to the total quantity of intangible capital deployed by existing firms, N_t. Using this expression, along with the labor market clearing condition, the free-entry condition (14) can be used to express entrepreneurial effort as:

$$L_{E,t} = \min\Big(1\,,\,\max\Big(0\,,\,1 - \Xi(x_t, c_t, \nu_t)^{-1}\Big)\Big)\,. \tag{25}$$

where:

$$\Xi(x_t, c_t, \nu_t) \equiv \frac{1-\zeta\chi}{\zeta\chi}\;(1-\gamma_t(x_t;\lambda))x_t^{1-(1-\rho)\omega}\;\nu_t\;c_t.$$

Additionally, given a functional form for the share of enterprise value accruing to key stakeholders, γ_t, Equation (10) pins down the optimal scope x_t. as a function of structural parameters and of the function γ_t. Finally, in Appendix A.1.4, we show that the price-to-earnings ratio ν_t satisfies $\nu_t = \nu(c_t)$, where $\nu(.)$ is the solution to a simple first-order ordinary differential equation (ODE). In conjunction with Equations (10) and (25), this ODE characterizes the global dynamics of the economy starting from any value of c_t.[11] The ratio c_t is therefore a sufficient state for this economy.[12]

3 Balanced growth

This section discusses the comparative statics of the model across balanced growth paths (BGPs). We focus on the effects of replicability (ρ) and non-exclusivity (λ).

3.1 Definition

We define a balanced growth as an equilibrium of the model in which the ratio c_t of Equation (24) is constant, and in which the quantity index for aggregate intangible capital N_t is growing at a constant rate g_N. Appendix (A.2) then establishes the following result.

Result 2 (Balanced growth path) *Consider the microfoundation described in Result (2.2.2). For any values of $(\rho,\lambda) \in [0,1] \times \mathbb{R}_+$, there exists a unique balanced growth path. Entrepreneurial effort L_E is constant, the optimal scope of new firms x is constant, the share of enterprise value retained by each entrepreneur, $\gamma(x;\lambda)$, is constant, the price-to-earnings ratio ν is constant, and the size of new firms grows at rate g_N.*

Proof. See Appendix A.2. ■ Appendix A.2 furthermore shows that so long as ξ, the parameter governing the relative size of new firms, is sufficiently larger, the unique balanced growth path features strictly positive entry: $L_E > 0$.

[11] In the microfoundation described in Section 2, optimal scope x_t also depends on the delay term μ_t. However, in Appendix A.1.4, we show that $\mu_t = \mu(c_t)$, where again the function $\mu(.)$ satisfies a first-order ODE; thus in this case the global dynamics of the economy are characterized by a system of two coupled ODEs alongside Equations (10) and (25).

[12] Appendix A.1.4 provides details on numerical computation of global dynamics, using finite-difference methods.

3.2 Comparative statics

Figure 3 shows the comparative statics of the model across balanced growth paths (BGPs). Panel (a) of this figure illustrates two main results.[13] First, the BGP rate of output growth declines when ρ increases, that is, as intangible capital becomes easier to replicate within firm. Second, for a given value of ρ, output growth declines as λ falls, that is, as exclusivity over intangible assets improves.

To clarify both results, first recall that aggregate output growth can always be written as $Y_t = N_t^{1-\zeta} L_{Y,t}^{\zeta}$, where $L_{Y,t}$ is total labor employed in production and N_t is the aggregate intangible capital index defined in Equation (21). Second, recall that in balanced growth this index N_t and the stock of public knowledge B_t must grow at the same rate, since their ratio is constant. Finally, note that because labor supply is fixed, $L_{Y,t}$ and $L_{E,t}$ must both be constant. Using the law of motion for the growth of the public knowledge stock, B_t, output growth is therefore given by:

$$g = (1-\zeta)\ g_N = (1-\zeta)\ \xi\ L_E.$$

Along a BGP the growth rate of output and the rate of entry are proportional, so that understanding how growth varies across BGPs requires understanding how entry varies.

Panels (b) through (d) of Figure 3 illustrate the mechanics underlying the negative relationship between growth and replicability ρ. As ρ increases, optimal scope x increases (panel b). Intuitively, with higher ρ, the creation of intangible assets can benefit from greater economies of scope within each new firm, so firms produce more varieties. The higher enterprise value increases the incentive to enter.

The fact that new firms have greater scope as ρ increases has two additional, but separate effects. The first is that new firms deploy more intangible capital in production, which raises returns to labor and therefore equilibrium wages.[14] Second, creating firms with greater scope also requires pledging a higher fraction of total enterprise value to key stakeholders (panel c), because of non-exclusivity.[15] Both of these effects reduce the incentive to enter.

In isolation, higher wages resulting from increased firm scope would exactly offset the increase in enterprise value that greater firm scope also creates, because of the free-entry condition.[16] As a result, the entry and growth rates would remain fixed across BGPs as ρ increases. A change in ρ would only have an effect on the *level* of output. However, as scope increases, entrepreneur also need to pledge a

[13] These results are proved analytically in Appendix A.2.

[14] Note that this effect would occur even if scope where assumed to be, say, an exogenously increasing function of ρ, instead of being chosen by entrepreneurs.

[15] Creating a firm with larger scope is optimal for entrepreneurs to do *conditional on entry*; loosely speaking, with higher ρ entrepreneurs choose to create a large pie (that is, a higher x) and keep a smaller share of it (that is, a lower value of $(1-\gamma)$).

[16] For instance, this would be the case in a model where scope is an exogenously increasing function of ρ, and the share γ to be pledged to key stakeholders by the entrepreneur is fixed.

larger fraction of enterprise value to key stakeholders, because of non-exclusivity. This creates a further disincentive to entry. On net, higher ρ therefore leads to lower growth.

Intuitively, higher ρ in this model leads to new firms that are larger and deploy their intangible assets at greater scale. However, greater firm scope also bids up the price of production labor, and makes incentive problems with key stakeholders more acute, requiring the entrepreneur to pledge a greater share of enterprise value to that group. Both effects depress entry incentives. Workers that might have chosen entrepreneurship now prefer to become workers and/or key stakeholders in the new, larger businesses.

The second result highlighted in Figure 3 is that, as λ declines (that is, as exclusivity over intangible capital strengthens), growth *declines* in the model. Perhaps counter-intuitively, this result does not reflect weaker spillovers from existing to new businesses, or related business stealing effects, since in the baseline version of the model, spillovers across cohorts of firms are independent of ρ or λ (as per Equation 11). Instead, this result again reflects the combination of general equilibrium wage effects with the need to provide adequate incentives to key stakeholders. Stronger exclusivity encourages entrepreneurs to create firms operating at larger scale, all else equal. This raises both wages and the share of enterprise value that entrepreneurs must pledge to key stakeholders, in turn weakening entry incentives and growth.

The key message is that the replicability of intangible assets need not enhance the long-run growth potential of an economy. Fundamentally this is because replicability interacts with the non-exclusive nature of intangible assets. To reap benefits from the scope economies associated with intangible assets, entrepreneurs must increase their reliance on key stakeholders, and share the firm's core intangibles with them. This creates an incentive problem, which entrepreneurs resolve by retaining a smaller share of enterprise value. Ultimately, though, entry incentives are depressed. The interplay between replicability and non-exclusivity also shapes the dynamics after a change in ρ. So far, however, we have been focusing in the replicability of intangible assets within the firm. In the next section, we provide an extension of the model in which the degree of replicability, ρ, has *positive* effects on the intensity of spillovers across cohorts of firms, and discuss the extent to which this mechanism can overtake the results highlighted in Figure 3.

3.3 Knowledge spillovers increasing in project scope

In our baseline model, improvements in the replicability technology do not directly affect the speed at which new blueprint are created. That is, the growth in new blueprints in equation (13), do not depend on the replicability parameter ρ. However, it may be natural to think that, if a given intangible is implemented at a higher scope, it has a greater potential to generate knowledge spillovers.

To this end, we next consider an extension of our baseline setup, where now the entrepreneurial productivity in generating also depends on the scope of implementation,

$$f(B_t) = \xi B_t \qquad \Longrightarrow \qquad n_t = \xi B_t \, x^{\beta}, \tag{26}$$

where $\beta > 0$ is a parameter governing the strength of this relation.

This extension implies that the relation between ρ and the growth rate of the economy in the balanced growth path is now ambiguous, as we see in the top left panel of Figure 4. In particular, the increase in firm scope x as ρ increases tends to increase long-run growth, as now the rate of blueprint creation is increasing in x.

4 Key economic quantities in the model

Here, we discuss the implications of our model for key economic quantities . In particular, we first show in Section 4.1 that standard empirical measures of capital stock need not adequately capture the input of intangible capital services in production. In Section 4.2, we draw implications for biases in total factor productivity measures based on Solow residuals. In Section 4.3 we derive the implications of our model for (sales) concentration. To conserve space, we relegate all details to Appendix A.1.4.3.

4.1 The measurement of capital

As discussed in Section 2, value added produced in this economy has the following aggregate production representation:

$$Y_t = N_t^{1-\zeta} L_{Y,t}^{\zeta}. \tag{27}$$

Here, $L_{Y,t} = 1 - L_{E,t}$ is the amount of labor devoted to production, and N_t is an index of intangible capital services used in production. The index N_t is given by:

$$N_t \equiv \left(\int_0^{M_t} n(m)^{\omega} dm \right)^{\frac{1}{\omega}}, \tag{28}$$

where M_t is the number of varieties produced by firms at time t.

In national income and product accounts, capital indexes are traditionally measured by deflating, then capitalizing past expenditures on capital goods by firms (Herman, Katz, Loebach, and McCulla, 2003; Katz, 2015). Within the context of the model, the corresponding perpetual inventory method would yield an aggregate capital stock K_t that evolves as:

$$\frac{dK_t}{K_t} = \left(\frac{I_t/P_{I,t}}{K_t} - \delta\right) dt, \tag{29}$$

where I_t is spending on the creation of intangibles, in output units, and $P_{K,t}$ is the price (or replacement cost) of intangibles, again expressed in output units. Within the model, these are given by:

$$\begin{aligned} I_t &= W_t L_{E,t}, \\ P_{I,t} &= \frac{W_t}{n_t} = \frac{W_t}{\xi B_t x_t^\beta}. \end{aligned}$$

In particular, the latter expression follows from noting that each unit of entrepreneurial labor generates $n_t = \xi B_t x_t^\beta$ new units of intangible capital. Thus,

$$\frac{dK_t}{K_t} = \left(\frac{\xi B_t x_t^\beta}{K_t} L_{E,t} - \delta\right) dt. \tag{30}$$

Integrating backward, this expression can be written as:

$$K_t = \int_{s \le t} e^{-\delta(t-s)} L_{E,t-s} \underbrace{(\xi B_{t-s} x_{t-s})}_{\equiv n_{t-s}} ds = \int_0^{J_t} e^{-\delta(t-\tau(j))} n_{\tau(j)} dj, \tag{31}$$

where recall that J_t represents the total number of firms that have entered up to time t, and $\tau(j)$ represents the date of entry of firm j. Comparing Equation (31) to Equation (28), we see that there is no a priori reason why the index of capital services N_t, and the measured capital stock K_t should coincide.

4.2 Implied biases for the Solow residual

We define the Solow residual, or measured total factor productivity (TFP), in the model, as:

$$z_t \equiv \frac{Y_t}{K_t^{1-\zeta} L_{Y,t}^{\zeta}} = \left(\frac{N_t}{K_t}\right)^{1-\zeta}. \tag{32}$$

This definition assumes that the correct aggregate capital index N_t cannot be measured directly, but that the perpetual inventory estimate of the capital stock, K_t, can. Because N_t and K_t differ, this will generally lead to biased measures of TFP, which should be constant and equal to 1 in this model.

In balanced growth, the scope of firms is constant: $x_t = x$, and their allocation of intangible capital across varieties is also constant and symmetric: $n(m) = x^{-(1-\rho)} n_{\tau(j)}$, where j is the index of the firm producing good m at time t. Thus we can rewrite the index of aggregate capital services N_t as:

$$\begin{aligned} N_t &= \left(\int_0^{M_t} n(m)^{\omega} dm\right)^{\frac{1}{\omega}} \\ &= \left(\int_0^{J_t} e^{-\delta(t-\tau(j))} x \left(x^{-(1-\rho)} n_{\tau(j)}\right)^{\omega} dj\right)^{\frac{1}{\omega}} \\ &= x^{\frac{1}{\omega}-(1-\rho)} \left(\int_0^{J_t} e^{-\delta(t-\tau(j))} n^{\omega}_{\tau(j)} dj\right)^{\frac{1}{\omega}} \end{aligned}$$

Comparing this expression to Equation (31), the definition of K_t, highlights the fact that two parameters shape the differences between K_t and N_t, and thus biases in measured TFP. The first is the parameter ω, which captures curvature in the revenue function arising from market power. When $\omega = 1$ and firms are perfectly competitive, it can be easily seen that:

$$N_t = x^{\rho} K_t \quad \Longrightarrow \quad z_t = z = x^{\rho(1-\zeta)}. \tag{33}$$

In this case, measured TFP only reflects the fact that firms can deploy intangible assets at greater scale when it is more easily replicable, that is, when ρ is higher. In particular, when $\rho = 0$, $z_t = 1$ and there is no bias in measured TFP.

More generally, when $\omega < 1$ and $\rho > 0$, measured TFP reflects both firms' ability to deploy intangible capital across multiple product lines (which raises measured TFP), and their ability to charge markups (which lowers measured TFP). To see this, using Equation (30), we note that in balanced growth, the Solow residual must be constant, $z_t = z$. Thus it must be that K_t and N_t grow at the same rate, so that:

$$c^{\frac{1}{\omega}} z^{\frac{1}{1-\zeta}} L_E - \delta = \xi x^{\beta} L_E. \tag{34}$$

Using the balanced growth conditions derived in Section 3, this condition can be rewritten as:

$$z = \overbrace{x^{(1-\zeta)\rho}}^{\text{Effect of replicability}} \underbrace{\left(x^{\frac{1-\omega}{\omega}} \frac{\xi x^{\beta} + \frac{\delta}{L_E}}{\left(\omega \xi x^{\beta} + \frac{\delta}{L_E}\right)^{\frac{1}{\omega}}} \right)^{1-\zeta}}_{\text{Effect of markups}}. \tag{35}$$

This expression highlights the two sources of distortions in measured TFP. The first one is the direct effect of replicability, as discussed before; when $\omega = 1$, it is the only source of distortion. On the other hand, the second term represents the distortions in measured TFP created by markups; these exist even when $\rho = 0$, that is, even when intangible capital is rival.

Panels (a) and (b) of Figure 5 report two examples of the comparative statics of z, the bias in the level of the Solow residual, as a function of ρ, the degree of replicability. The top panel uses the same calibration as in the comparative statics reported in Figure 3; in particular, markups are set to $1/\chi = 1.10$, so that $\omega = 0.77$. The bottom panel uses lower markups of $1/\chi = 1.01$, so that $\omega = 0.97$. In both cases, the bias in the level of the Solow residual is positive (that is, $z > 1$). Additionally, the bias increases with the degree of replicability, driven by the term $x^{(1-\eta)\rho}$ in Equation (35). Finally, the bias is larger for lower values of the markup $1/\chi$, highlighting that the second term in Expression (35), the markup effect, tends to bias measured TFP downward.

These comparative statics indicate that replicability can create a large upward bias in the *level* of the Solow residual, relative to measured TFP. In line with this, in Section 5, we will show that during a transition from a BGP with a low value of ρ to a BGP with a high value of ρ, the Solow residual of this economy can temporarily increase, even though underlying total factor productivity is unchanged.

4.3 Concentration

We can explicitly solve for the sales concentration in our economy. In particular, the total revenue share of firm j is given by

$$h_{i,t} \quad = \quad x_{\tau(j)}^{1-(1-\rho_{\tau(j)})\omega} \left(\frac{n_{\tau(j)}}{N_t}\right)^{\omega}. \tag{36}$$

When examining equation (36), recall that our model does not have any heterogeneity within a cohort of firms. As a result, the scope of firm j is purely a function of the time $\tau(j)$ that the firm is created: its revenue share is purely a function of the quality n and optimal scope x of intangibles of firms in cohort $\tau(j)$ relative to the aggregate intangible capital stock N.

The Herfindhal index of concentration can then be written as:

$$H_t \quad \equiv \quad \int_{j=0}^{J_t} h_{i,t}^2 dj = \frac{c_t^2}{u_t}. \tag{37}$$

where

$$u_t \equiv \frac{\left(\xi N_t^{(e)}\right)^{2\omega}}{\int_{s\leq t} e^{-\delta(t-s)} x_s^{2(1-(1-\rho_s)\omega)} L_{E,s} \left(\xi N_s^{(e)}\right)^{2\omega} dj}. \tag{38}$$

Concentration in our model is increasing in the replicability of intangible capital ρ, as we see in Panel (c) of Figure 5. To see why this is the case, note that an increase in the replicability of intangible capital $\rho_{\tau(j)}$ implies that new firms created at $\tau(j)$ have higher scope. Since scope is complementary to the quality of intangibles n, this increase in scope magnifies the impact of the across-cohort differences in n across firms, which leads to higher concentration.

4.4 Valuation Ratios

We next derive the implications of our model for firm valuation ratios. Let V_t denote the enterprise value of all firms currently operating. The total enterprise value in the economy is split between entrepreneurs and the key shareholders; each party appropriates a share $1-\gamma_t$ and γ_t respectively. Dividing enterprise value by the replacement cost of capital, we can write aggregate Q in the economy as

$$Q_t = Q_t^e + \gamma_t \, Q_t \Rightarrow Q_t = \frac{1}{1-\gamma_t} \, Q_t^e,$$

where Q_t^e is the enterprise value that accrues to the entrepreneur divided by the replacement cost of capital $p_{I,t} K_t$. Given our definition of the measured capital stock of intangibles K_t in 4.1, we can write the average Tobin's Q in the economy as

$$Q_t = \underbrace{\frac{1}{1-\gamma_t}}_{\text{Effect of limited excludability}} \quad \underbrace{\frac{c_t^{\frac{1}{\omega}-1} z_t^{\frac{1}{1-\zeta}}}{x_t^{1-(1-\rho_t)\omega}}}_{\text{Effect of markups}}, \tag{39}$$

where z_t is the solow residual defined in (32), which in our model purely reflects the mis-measurement of the intangible capital stock.

As we see in Panel (d) of Figure 5, the value of Tobin's Q along the balanced growth path is increasing in ρ. To understand why this is the case, note that equation (39) is the product of two terms, both which are increasing in ρ.. The first term depends on γ_t and represents the fact that outsiders or key stakeholders earn pure rents as a result of the incentive problem which their ability to copy the firm's core knowledge assets creates. Under full excludability $(\lambda \to 0)$, we have that $\gamma_t \to 0$ and this term approaches one. The second term corresponds to the Tobin's Q faced by entrepreneurs. In the absence of markups $(\omega = 1)$, free entry into entrepreneurship implies that $Q_t^e = 1$. If there are markups, then this term is greater than one, reflecting rents along the BGP that accrue to the entrepreneur.

5 Calibration and transitional dynamics

This section describes the transitional dynamics of the model after a change in ρ. The analysis is quantitative, so Section 5.1 starts by describing the data sources and moments that we use to discipline the model. Section 5.2 then describes the calibration procedure and Section 5.3 discusses the results.

5.1 Data sources and targeted moments

Here we focus on the firm-level data used to construct the moments targeted in our analysis. In the calibration, we also use public, aggregate or semi-aggregate time series covering the population of US businesses; Appendix A.4.1.4 describes these other data sources in more detail.

5.1.1 Firm scope

In order to calibrate the model, we first use data on firm scope for the panel of public non-financial US firms. These data primarily help us calibrate values for the parameter ρ in the quantitative exercises to follow. The criteria we apply to select the panel of firms from Compustat are standard and described in more detail in Appendix A.4.1.1.[17]

The data on firm scope which we use is drawn from Hoberg and Phillips (2024). These data build Part I, item 1 of the 10-K statements of public firms, in which firms provide a description of their business operations. The authors apply textual analysis methods in order to form approximately 300 clusters of terms, each of which characterizes a particular business line or product market that public firms operate in. The authors then construct measures of pairwise similarity between individual 10-K statements at the firm-year level and each cluster. Finally, the firm is flagged as operating in a particular business line or product market if the pairwise similarity measure is sufficiently elevated. The resulting measure, which we refer to as $x_{j,t}$, is a positive number that measures the cardinality of the business lines or product markets in which each firm is active in each year.[18] For the remainder of our analysis we will interpret it as an empirical counterpart to the scope of firms in the model of Section 2.

We use two main moments from these data. First, Figure 6 shows the evolution over time of average scope of firms in our sample, both in the pooled sample (solid red line) and splitting the sample by broad industry groups, based on SIC-1 codes (dark lines). As documented in Hoberg and Phillips (2024), under this measure, the average firm in our sample has experienced an increase in scope over the 1988-2021 period. In the early 1990s, the average firm operated approximately 6 business lines; by the end of the sample, this number had risen to approximately 10. Moreover, this increase was a common phenomenon across broad industry groups. In our subsequent analysis, we will use, as a target for the initial calibration of the model to the early 1990s, the average value of $x_{j,t}$ in the pooled 1988-1992 sample, which is equal to $x = 6.3$.

The second moment we use from these data is the covariance between firm revenue and firm scope. Recall, from Equation (15), that total revenue for a firm with intangible capital n and scope x are

[17] We restrict the sample to 1988-2021 both because this is our period of focus and because the data on firm scope are available only for these years.

[18] Appendix A.4.1.2 provides more details on this measure, and specific examples.

proportional to:

$$s_t(x_{j,t}, n_{j,t}) \propto x_{j,t}^{1-\omega+\rho\omega} n_{j,t}^{\omega}, \tag{40}$$

where the proportionality coefficient only depends on aggregate variables.[19] Thus, the covariance between log revenue of a firm, and the log of its scope, should be informative about the parameter ρ.

In Table 1, we implement this idea by estimating panel regressions of the form:

$$\log(s_{j,t}) = \alpha_{l(j),t} + \beta_x \log(x_{j,t}) + \Gamma_j' X_{j,t} + \varepsilon_{j,t} \tag{41}$$

where j indexes firms, t indexes years, $l(j)$ is the industry of the firm, $s_{j,t}$ is total revenue of the firm, and $X_{j,t}$ contains additional firm-level controls.[20]. The coefficient of interest is β_x, which these specifications effectively estimate as the cross-sectional, conditional elasticity of sales with respect to scope within industry-year cells.

Table 1 first reports the point estimates for β_x in specification (41) for the period 1988-1992. Column (1) does not include any controls and equal-weights the observations, and column (2) additionally weights the observations by the number of firm in each industry-year, so as to obtain a point estimate of β_x that represent an equal-weighted average across industries (since our model does not take a stance on relative industry shares of output). Finally, in column (3), consistent with Equation (40), we add as a control a proxy for the stock of intangible capital, using the capitalized value of R&D expenditures for firm j up to year t.[21] We use this last specification in the remainder of our analysis.

All point estimates are positive and statistically significant. To interpret the economic magnitude of the coefficient β_x, note that Equation (40) implies the following mapping between β_x and ρ:

$$\rho = \frac{\beta_x - (1-\omega)}{\omega}. \tag{42}$$

In the calibration (section 5.2), we will calibrate externally the values of χ and ζ to $\chi = 1/1.05$ and $\zeta = 0.70$. This implies a value of $\omega = 0.86$. In turn, the point estimate of β_x implies a value of $\rho = 0.18$, and the null that $\rho = 0$ can be reject at the 1% level, consistent with our assumption, discussed in Section 2, that intangible assets may be partly replicable within firm.

In Figure 7, we extend this evidence by estimating specification (41) on rolling windows of five years centered around each year in our sample from 1990 to 2019. Figure 7 reports the point estimates from ρ, using Equation (42) and the calibrated value of $\omega = 0.86$, along with 90% confidence intervals constructed the delta method. The figure shows that the point estimates are statistically different from 0 at the 5% level in all years except 1993 and 1994. Most importantly, it shows that the point

[19]Equation (15) refers to total enterprise value, but this is proportional to total flow revenue, as shown in Appendix A.1.
[20]We use the SIC-4 code to classify firms into industry groups
[21]We use log of the capitalized values reported by Peters and Taylor (2017); more details are provided in Appendix A.4.1.1.

estimate increases from 1995 to 2000, and stabilizes after that, around a value of $\rho = 0.40$. In analyzing transitional dynamics, we use this empirical estimate of the increase in ρ as the exogenous force driving structural change in the model.

We conclude by noting that in the model, firms of a given cohort all enter with the same amount of intangible capital and the same scope, so that, strictly speaking, there is no cross-sectional covariance between sales and scope *within a cohort at entry*. As result, in the analysis of transitional dynamics we will careful to compare *average* values of outcomes such as scope or Tobin's Q to their data counterparts. However, we also note that extending the model to allow for idiosyncratic shocks, uncorrelated shocks affecting both firm scope and intangible capital at entry would generate cross-sectional co-movement between sales and scope *within a cohort* that would be sufficient to identify ρ using specification (41), at the expense of further complicating the model.[22]

5.1.2 Firm ownership

In order to calibrate the model, we also use data on the distribution of equity ownership of US public firms. These data provides us with a target for the share $1 - \gamma_t(x_t, \lambda)$ of equity retained by entrepreneurs at the time the firm is created, thus helping us indirectly identify the parameter λ.

The specific information we use is the fraction of equity shares held by the firm's founders, executive directors, or officers, for US public non-financial firms. Because the model primarily speaks to the distribution of enterprise value at the time of entry, we measure this share at the earliest possible date. Specifically, we use initial public offering filings (IPO) to collect data on the distribution of equity ownership between that group and other shareholders prior to IPO. Appendix A.4 reports more detail on data sources, data collection, and sample selection. Thus our precise empirical proxy of $1 - \gamma_t(x_t, \lambda)$ is the pre-IPO share of equity held by public firms' founding team. Four comments are in order.

First, this measure takes a stance on who the "entrepreneur" is. We choose to focus on the founding team (including directors and executive officers) because in many cases, there are several founders who also play executive or director roles at the time of IPO.[23] Second, we focus on equity, not enterprise value.[24] Third, we focus on the information from pre-IPO firms because, for the group of firms on which our calibration is based (US public firms), this is the earliest publicly available data on the distribution of control and cash-flow rights after entry.

Fourth, and most importantly, our empirical proxy implicitly interprets any shareholder in the

[22]Equation (40) only requires that the entrepreneur optimally allocate intangible capital for given $n_{j,t}$ and $x_{j,t}$.

[23]However, in Appendix A.4, we report results from an analysis that identifies specifically the founders using textual analysis of other items of the IPO prospectus, and show that the main results are quantitatively unchanged.

[24]Appendix A.4 shows that while leverage is positive for most IPO firms in our sample, it has not substantially increased over time, so that treating leverage as part of the enterprise value pledged by the entrepreneur to key stakeholders would not qualitatively change the findings reported below.

firm outside of the founding team as a "key stakeholder". In our data, there aretwo main groups of shareholders outside of the founding team: high-skilled employees; and private investment funds (venture capital or private equity). Thus, with our measurement we are taking the stance that, in order to increase scope, the founding team has to rely on these two groups of stakeholders, and must share some of the firm's key knowledge assets with them, at the risk of these assets eventually being used to create projects outside of the firm. In what follows, we refer as these two groups as "outsiders", by contrast with the "insiders" (the founding team) discussed above.

Figure 8 reports the resulting data on the ownership share of the founding team. In this figure, each point represents the average fraction of *pre-IPO* equity shares held by the founding team in a given year. This average is computed across all firms that went through IPO in that year. Thus it represents the distribution of cash flow and control rights for the newest firms in our sample in that year.

There are two main facts to note. First, founding teams generally do not hold the entirety of the firm's equity shares prior to IPO, consistent with our assumption that founders pledge enterprise value to key stakeholders at entry. Second, and most importantly, this share declined over the 1996-2016 period, from approximately 65% in 1996 to approximately 45% by the end of 2016.

In our calibration exercise, when constructing the initial balanced growth path of the model, we will treat the share reported in Figure 8 as of 1996 (which is equal to 64%) as the proxy for $1 - \gamma(x_t, \lambda)$. Moreover, we will highlight that in response to a permanent increase in ρ, the model matches the decline in the ownership share of the founding team both qualitatively and quantitatively.

5.2 Calibration

Next, we describe the baseline calibration from which we start in order to study the effects of an increase in ρ. In this calibration, we match moments of the data in the 5-year window centered in 1990. The model has nine parameters:

$$\{\eta, \zeta, \chi, \rho, \lambda, \alpha, \kappa, \xi, \delta\}. \tag{43}$$

We externally calibrate the first three, and internally calibrate the remaining six.[25] The calibration is reported in Table 2.

The time discount rate, $\eta = 0.02$, reflects the annnual calibration and yields an approximately 5% real rate of return in the BGP. For the Cobb-Douglas elasticity of output with respect to labor ζ and the inverse markup χ, we use the estimates of these parameter values reported in Crouzet and Eberly

[25] By "externally calibrate" we mean that we set these parameters to values used in other work. By "internally calibrate", we mean that we use data moments to assign values to these parameters. Numerically, we do this by minimizing the sum of the squared percentage deviation between data moments and model-implied moments; the procedure is described in more detail in Appendix A.4.2.

(2023) and computed using data for non-financial corporate business sector.[26] Importantly, we will keep the value of the inverse markup in this economy constant in our numerical exercises, so as to isolate specifically the effect of the change in the degree of replicability.

As explained above, our empirical target for ρ is the appropriately scaled covariance between revenue and scope, where our measure of scope is derived from Hoberg and Phillips (2024). Though this empirical target relies on external data, for given values of (χ, ζ), it directly translates to a value for the structural parameter ρ, so that the value for this parameter is set directly in our numerical moment targeting procedure. Likewise, the obsolescence rate in the model, δ, directly maps to the gross exit rate in the data, which we measure using the Business Dynamics Statistics (BDS) dataset.[27]

The four remaining parameters, $\{\lambda, \alpha, \kappa, \xi\}$, are calibrated to minimize the distance between BGP values of four moments and their empirical counterparts for the period 1988-1992. Heuristically, the identification of these moments from the parameters is as follows. Jointly, the degree of non-exclusivity, λ, the parameter α, which scales the number of key stakeholders $\theta(x)$ needed to reach a particular scope x, and the parameter κ, which governs the curvature of the function $\theta(x)$, as per Equation (16), are identified using the share of enterprise value accruing to outsiders, $\gamma(x; \lambda)$, the average scope of firms, x, and aggregate Tobin's Q.[28] A higher degree of non-exclusivity λ increases the share $\gamma(x; \lambda)$ of enterprise value that entrepreneurs must pledge to outsiders to deter competition, while a higher value for the shifter α, all else equal, will induce entrepreneurs to choose a lower scope x. Finally, the curvature of the function $\theta(x)$ is identified using the value of aggregate Tobin's Q. In the model, this value is computed as the ratio of total enterprise value, to the capitalized value of past investment expenditures by entrepreneurs, that is, the measured capital stock K_t, as defined in Equation (22). Appendix A.2.2 shows that in the BGP, aggregate Tobin's Q can be written as:

$$Q = \frac{1}{1-\gamma} \frac{R_K - \eta}{\omega(R_K - \eta) + (1-\omega)\delta}, \tag{44}$$

where R_N is an appropriately defined competitive user cost for intangible capital, the expression of which is reported in Appendix A.2.2. Since the parameter κ influences the steady-state value of the share of enterprise value γ pledged to outsiders, as well a the competitive user cost R_N, matching the value of Tobin's Q helps pin down a value for κ. Finally, recall that in the BGP, the trend growth rate of

[26] Specifically, we use the values in Figure 35 of the Appendix of the paper. These figures show the implied elasticity of output with respect to labor, and the implied markup, when assuming constant returns to scale in production, and when matching simultaneously the labor share of value added and other moments analyzed in Crouzet and Eberly (2023); among these moments, the ratio of measured returns to capital to user costs is the primary source of identification of the size of markups.

[27] Details on these data and the computation of the gross exit rate are reported in Appendix A.4.1.4, and the corresponding time series is reported in Appendix Figure A-1; consistent with the rest of the calibration we use the five-year centered moving average for 1990.

[28] In the quantitative exercises of this section, we normalize the mass of varieties for which entrepreneurs do not need to designate key stakeholders to $\bar{x} = 1$.

the economy is:

$$g = (1 - \zeta)\xi L_E, \tag{45}$$

where L_E is the (endogenous) share of labor allocated to the creation of new firms. Thus all else given, a higher value for ξ yields a higher trend growth rate of output, so that targeting the growth rate of output helps pin down ξ.[29]

5.3 Results

We conduct two main exercises. First, we discuss the changes across two BGPs, our baseline 1988-1992 calibration, and a version meant to match estimates of ρ for 2016-2020, and which can be thought of as the long-run state of the economy after a change in ρ. Second, we discuss the transitional dynamics of the model, and compare them to data on the 1988-2020 of the non-financial corporate business sector.

5.3.1 Comparison of balanced growth paths

Table 3 shows two BGPs of the model side-by-side. The first is the same as our baseline calibration. The second BGP is identical to the first one in all respects, except that we change the value of ρ from $\rho = 0.18$ to $\rho = 0.34$. The latter value is obtained from the same procedure as described in Section 5.1.1, focusing on data for the five-year window centered around 2018, which also corresponds to the value of ρ reported in Figure 7. Thus the only change between the two BGPs is that the degree of replicability of intangible assets within firm has increased in a way that is consistent with the changing cross-section co-movements between revenue and scope.

Two main results are worth highlighting in Table 3. First, with the calibrated change in ρ, the model is able to reproduce qualitatively and, to a large extent, quantitatively, changes in the share of enterprise value pledged to outsiders by entrepreneurs (Figure 8), the average scope of firms (Figure 6), and Tobin's Q (Appendix Figure A-1). Firm scope increases somewhat more than in the data, while the increase in Tobin's Q is only about one-third of what the data indicate. The latter shortcoming of the model is related to the fact that we kept markups constant in this exercise, whereas recent evidence suggests that pure profit shares of US corporations have increased (Barkai, 2020; De Loecker, Eeckhout, and Unger, 2020).

The second result to highlight is the change in the BGP output growth rate. As discussed in Section 3, an increase in ρ will generically reduce trend growth in the baseline version of our model. Quantitatively, the decline generated by this mechanism is about 60bps per annum, whereas in the data, output growth

[29] We measure output growth using BEA data on the growth rate of real value added for the non-financial corporate business sector, as described in Appendix A.4.1.4.

in the NFCB sector in the 2016-2020 window was 160bps lower than in the 1988-1992 window. Thus in isolation, the mechanism we study in this paper can account for approximately one-third of the observed decline in output growth in that sector between the early 1990s and the late 2020s. Of course, additional structural changes (including, but not limited to, an increase in markups), would improve the ability of the model to account for this decline in trend growth. Nonetheless, this exercise highlights that the specific form of technological change on which this paper is focused — a change in firms ability to replicate and deploy their core knowledge assets — can have quantitatively large effects on growth.

5.3.2 Dynamics during the transition

Next, we study the transitional dynamics of the economy between the two BGPs just discussed. In order to construct this transition path, we start the economy in 1990 at the first BGP, described in Table 2. We then simulate the economy until 1995, when we assume that the level of ρ unexpectedly jumps, from its initial value of $\rho = 0.18$, to a new long-run value of $\rho = 0.34$, corresponding to our point estimate for the 2016-2020 window. This jump is depicted in the solid red line in Figure 7. We then compute the transitional dynamics of the economy given the new value of ρ.[30] Note that this change only affects firms created after 1995; other firms still operate using intangible assets that have a lower degree of replicability, so that the transition to a new, higher average value of ρ across firms takes time.

Results are in Figure 9. Panel A reports output growth along the transition path. As discussed in Section 5.3.1, in the long run, the change in ρ induces a decline in output growth, coming from reduced entry. However, Panel A of Figure 9 shows that the transition to this lower-growth BGP is non-monotonic. Specifically, the economy experiences a burst of growth that lasts for about a decade, lining up with the high growth rates of value added in the non-financial corporate sector in the late 1990s and early 2000s. Growth after 2005 subsides in both the model and the data.

As ρ jumps upward, newly created firms now benefit from higher economies of scope, and will therefore start at a higher scope than existing firms. But early on in the transition, most firms still operate using intangible assets associated with the pre-shock value of ρ, and at a lower scope. Thus the mass of new, high-ρ firms is initially too small to put significant upward pressure on wages. Thus the increase in enterprise value associated with economies of scope is not immediately offset by an increase in wages (as will eventually become the case over the longer-run). In other words, the early stages of the transition are favorable to firm creation: entrepreneurs' outside option is relatively unattractive, and wages of production workers remain low. As a result entry is temporarily elevated (as panel F of Figure 9 shows), contributing to a burst in growth.

Panel B of Figure 9 show the *average* scope of firms in the economy along the transition path.[31]

[30] The numerical procedure is described in Appendix A.3.

[31] The computation of the dynamics of average scope, the average outsider share, and concentration along the transition

This average is computed across all firms — those created before the change in ρ, and those created afterwards. The increase in average scope is gradual, reflecting the progressive obsolescence of low-ρ firms. Panel H of Figure 9 shows the transitional dynamics for the Herfindahl index concentration defined in equation (37). As a result of the higher scope of new firms, concentration increases rapidly after the transition, and remains higher in the long-run, reflecting the fact that in the new BGP, newly created firms now have larger scope, which magnifies the impact of the cross-cohort differences in the quality of intangibles on firm revenue.

Panel C of Figure 9 reports the average share of enterprise value pledged by entrepreneurs to outsiders. As a result of the higher scope chosen by new firms, the outsider share also gradually rises, as it does in the data. The temporary increase in measured TFP, in panel G of Figure 9, has to do with the higher scope of newly created firms. Recalling the discussion in Section 4.2, measured TFP in the model reflects the discrepancy between K_t, which is the capitalized value of past investment expenditures on the creation of intangible assets, and N_t, which is the intangible capital index defined in equation (21). Aside from markups (which are constant along the transition path), the key difference between these two measures of the capital stock is that only the latter, N_t, takes into account the fact that intangible capital can be deployed across the production of multiple varieties within the firm—captured by firm's optimal scope x. As ρ increases, the wedge between N_t and K_t therefore increases, so that measured TFP growth experiences a burst. This burst reflects the greater scope economies of newly created firms, which the index K_t does not account for.

Panels D, E, and G of Figure 9 show, respectively, the trajectory of Aggregate Tobin's Q, the aggregate profit share, and measured TFP growth. Intuitively, the long-term increase in both Tobin's Q and in the pure profit share is mainly attributable to the increase in the share of total enterprise value that is pledge to key stakeholders, as a resolution of the incentive problem created by non-exclusivity. However, notably, both Tobin's Q and the pure profit share fall in the short-run. This short-run decline in valuation ratios and profits illustrates the similarity between our model and models with vintage-specific technical change (Kogan et al., 2020). Specifically, firms compete for a scarce resource: labor. An increase in the scope of new firms leads to higher equilibrium wages, which depresses the valuation ratios and profitability of existing firms whose scope is not impacted by the increase in ρ.

This pattern is reflected in the increase in the price of intangible capital, which rises on impact. There are two effects shaping the response of $p_{K,t}$. After ρ increases, more entrepreneurial effort is devoted to the creation of new firms, as indicated in Panel F of Figure 9. This leads to an increase in B_t, and therefore to an increase in the stock of intangibles n_t that new firms start their operations with. But at the same time, the wage rate increases after the jump in ρ, as new, larger firms start bidding up the price of production labor. On net, the second effect, the increase in wages, dominates the first

path are all described in more detail in Appendix 9.

effect, the increase in the productivity of entrepreneurs, leading to an increase in the replacement cost of capital. (Over the long-run, the first effect dominates, and the replacement cost of capital eventually declines.) In turn, a higher replacement cost of capital in the short-run depresses the denominator of Tobin's Q, and also increases the competitive cost of capital, leading to a lower pure profit share.

Last, Panel H plots the concentration ratios in the model and in the data. Examining the plot, we see that the model is able to match the 20 log-point increase in the HHI index between the mid-1990s to 2018. As we discussed in Section 4.3, the level of industry concentration in the model is increasing with ρ as scope increases. Since scope is complementary to the quality of intangibles n, this increase in scope magnifies the impact of the across-cohort differences in n, which leads to higher concentration.

6 Conclusion

Intangibles are particular types of capital inputs that differ from physical assets in one important way: they can be replicated, even if imperfectly. This property implies that intangibles can be greatly scaled up within the firm, which can lead to very large (superstar) firms. At the same times, however, the fact that intangibles can be replicated also implies the risk that firms' intangibles will be copied or appropriated by competitor. We embed these properties into an equilibrium model with endogenous growth, and show how increases in the technology for replicating intangibles can lead to an increase in concentration and the profit share and lower growth in the long run.

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	1988-1992			1988-2021		
	(1)	(2)	(3)	(4)	(5)	(6)
$\log(x_{j,t})$	0.43***	0.44***	0.30***	0.63***	0.59***	0.43***
	(8.1)	(8.9)	(6.8)	(10.2)	(12.7)	(10.3)
obs.	18473	18473	18473	135675	135675	135675
s.e. clustering	yr. × ind.	yr. × ind.	yr. × ind.	yr. × ind.	yr. × ind.	yr. × ind.
fixed effects	yr. × ind.	yr. × ind.	yr. × ind.	yr. × ind.	yr. × ind.	yr. × ind.
weighted regression	✗	✓	✓	✗	✓	✓
controls	✗	✗	✓	✗	✗	✓

Table 1: The relationship between firm revenue and firm scope. In all regressions the dependent variable is firm-level revenue, while the independent variables include the log of firm scope, $\log(x_{j,t})$. Specifications (1) and (4) do not contain other controls and equal-weights the observations. Specifications (2) and (5) do not contain other controls but weight observations by the number of firms in the corresponding industry-year cell, where industry is defined using the firm's 4-digit SIC code. Specifications (3) and (6) additionally for a proxy for a firm's intangible capital, the capitalized value of past R&D expenditures, as in Peters and Taylor (2017). Specifications (1) through (3) restrict the sample to the period 1988-1992, while specifications (4) through (6) use the full 1988-2021 panel. The data on firm revenue and estimates of the intangible capital stock are drawn from the panel of Compustat firms for 1988-2021, restricted to non-financial firms, as described in Appendix A.4.1, and the data on firm scope are drawn from Hoberg and Phillips (2024). Appendix A.4.1 reports details on data sources and data construction.

Panel A. Externally calibrated parameters

Parameter	Description	Value	Source
η	Time discount rate	0.02	Annual calibration
ζ	Cobb-Douglas labor elasticity	0.70	Crouzet and Eberly (2023)
$1/\chi$	Markup	1.05	Crouzet and Eberly (2023)

Panel B. Internally calibrated parameters

Parameter	Description	Value	Targeted moment	Model counterpart	Data value	Model value
ρ	Replicability	0.18	Covariance btw. revenue and scope	ρ	0.18	0.18
λ	Non-exclusivity	0.32	Outsider share at IPO	γ	0.36	0.34
α	Level of $\theta(x)$	0.49	Average scope	x	6.3	6.3
κ	Slope of $\theta(x)$	1.18	Tobin's Q	Q	1.6	1.6
ξ	Entrepreneurial productivity	0.34	Output growth	g	0.027	0.027
δ	Obsolescence rate	0.11	Gross exit rate	δ	0.10	0.11

Table 2: Baseline model calibration, 1988-1992. The model has 9 parameters, $\{\eta, \zeta, \rho, \lambda, \alpha, \kappa, \xi, \delta\}$. Panel A reports values for the externally calibrated parameters, $\{\eta, \zeta, \chi\}$. Panel B reports values for the internally calibrated parameters, $\{\rho, \lambda, \alpha, \kappa, \xi, \delta\}$, along with the empirical targets used to calibrate these parameters. Section 5.1 and Appendix A.4.1 discuss data sources and the computation of the targeted moments using these data. Section 5.2 discusses their model counterparts.

	1988-1992 $\rho = 0.18$		2016-2020 $\rho = 0.34$	
Moment	**Data**	**Model**	**Data**	**Model**
Outsider share at IPO	0.36	0.34	0.52	0.51
Average scope	6.3	6.3	10.1	13.6
Tobin's Q	1.6	1.6	3.1	2.1
Output growth	0.027	0.027	0.011	0.021
Gross exit rate	0.10	0.11	0.10	0.08

Table 3: Comparison across balanced growth paths (BGPs), 1988-1992 and 2016-2020. The 1988-1992 BGP is the one calibrated in Table 2. To construct the 2016-2020 BGP, we keep all parameters in the model identical to their 1988-1992 BGP values, except for ρ. We set $\rho = 0.34$ in the 2016-2020 BGP, consistent with the five-year rolling-window estimate for that period of time documented in Figure 7. Section 5.1 and Appendix A.4.1 discuss data sources and the computation of the targeted moments using these data. Section 5.2 discusses their model counterparts.

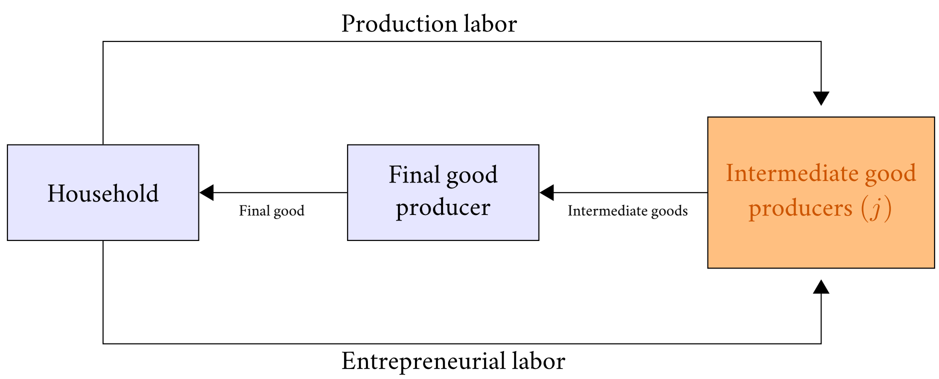

Figure 1: Overview of the model.

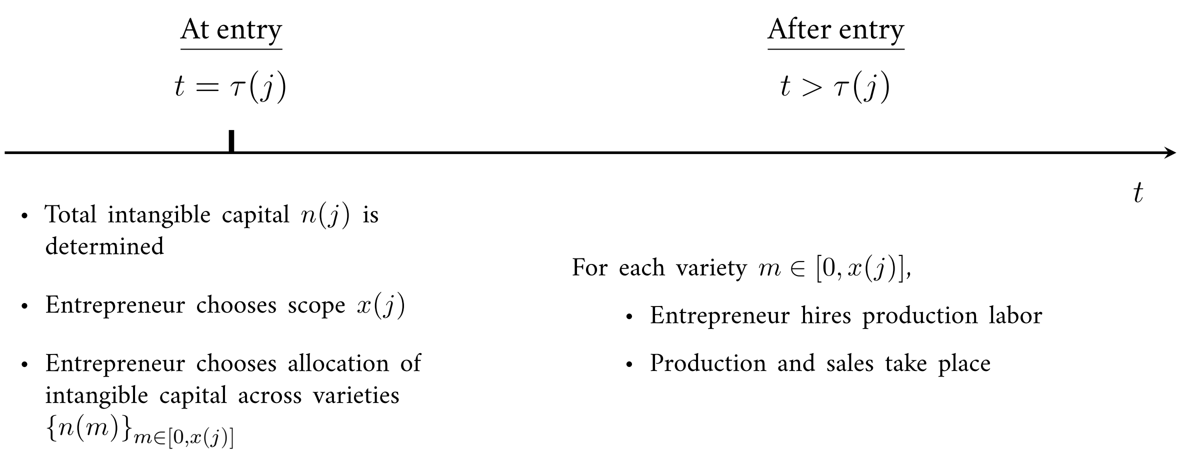

Figure 2: The timing of intermediate producer's j problem.

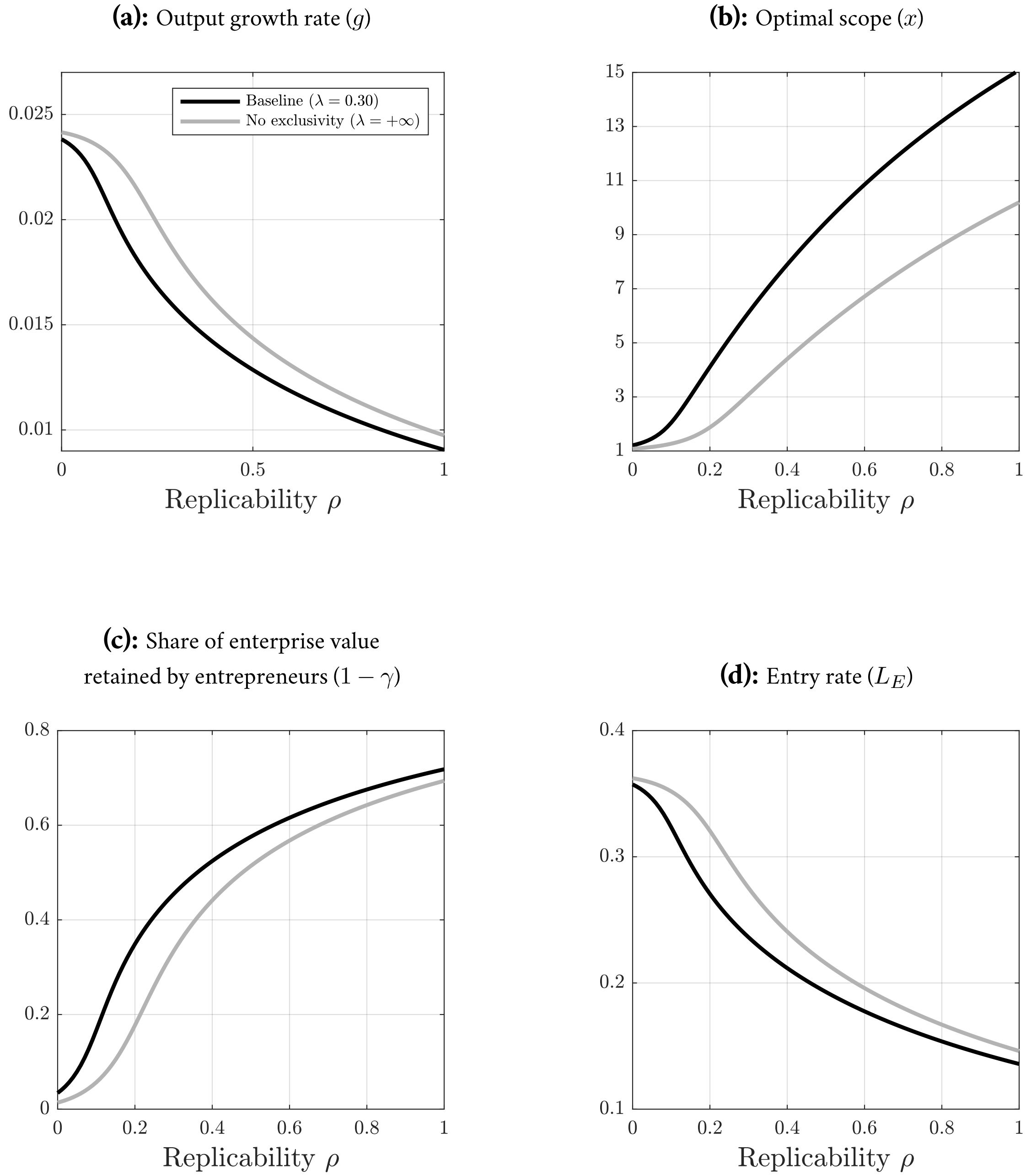

Figure 3: Comparative statics across balanced growth paths in the model of Section 2. The horizontal axis corresponds to different values of the parameter ρ, which governs the degree of non-rivalry of intangible capital within firm. The black line corresponds to a baseline case in which there is partial exclusivity over intangible capital $(\lambda = 0.30)$. The grey line corresponds to a case in which there is no exclusivity over intangible capital ($\lambda = +\infty$). The other parameter values used across all comparative statics are $\eta = 0.02$, $\zeta = 2/3$, $\chi = 1/1.10$, $\xi = 0.2$, $\delta = 0.10$, $\bar{x} = 1$, $\alpha = 2.6$ and $\kappa = 1.3$.

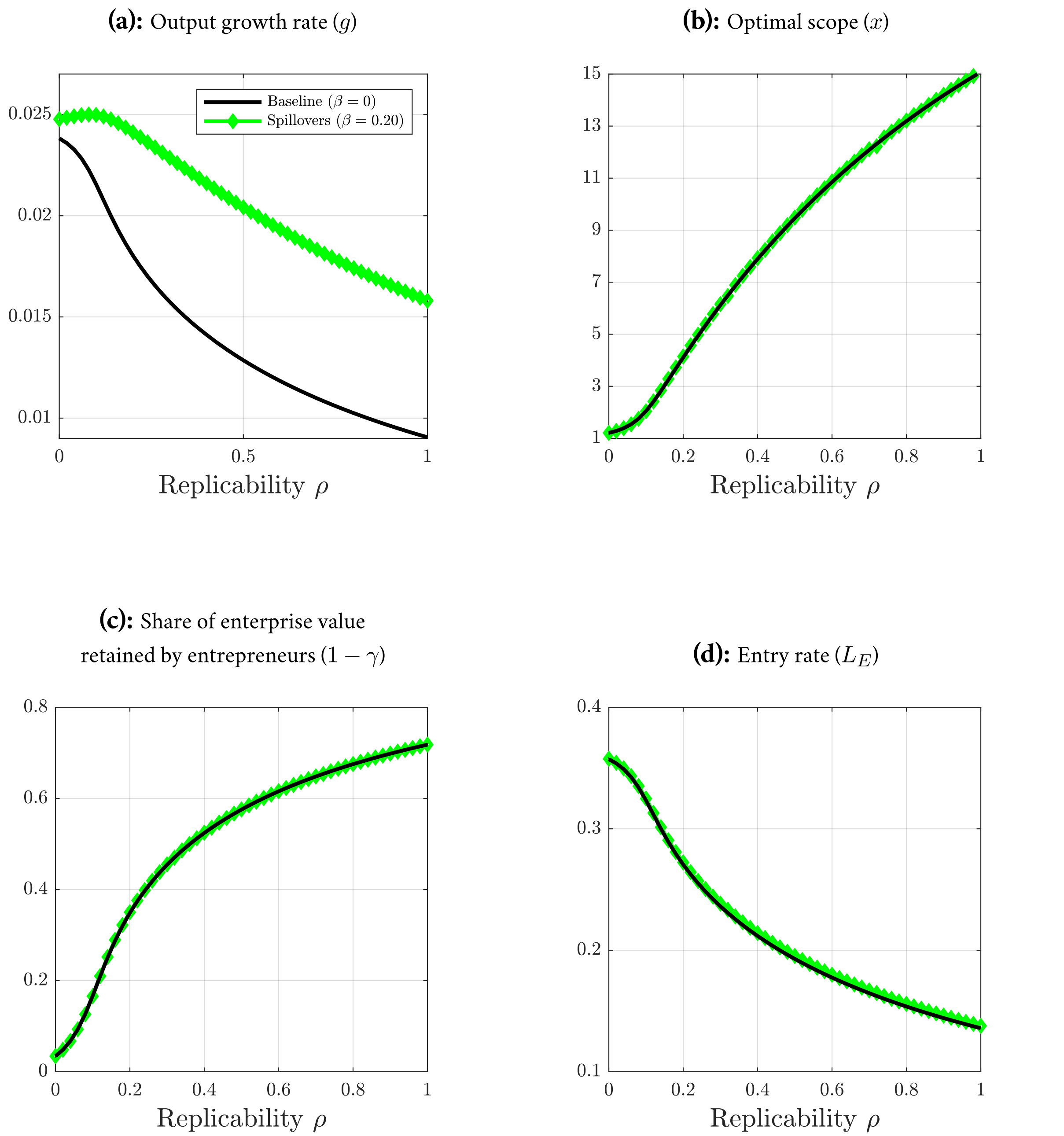

Figure 4: The effects of spillovers on the comparative statics of long-term growth with respect to the degree of non-rivalry, ρ. The horizontal axis corresponds to different values of the parameter ρ. The black line corresponds to a baseline case in which there are no spillovers from firm scope onto the creation of new intangible assets $(\beta = 0)$. The green circled line corresponds to a case in which there are spillovers $(\beta = 0.20)$. The other parameter values used across all comparative statics are $\eta = 0.02$, $\zeta = 2/3$, $\chi = 1/1.10$, $\xi = 0.2$, $\delta = 0.10$, $\bar{x} = 1$, $\alpha = 2.6$ and $\kappa = 1.3$, as in Figure 3.

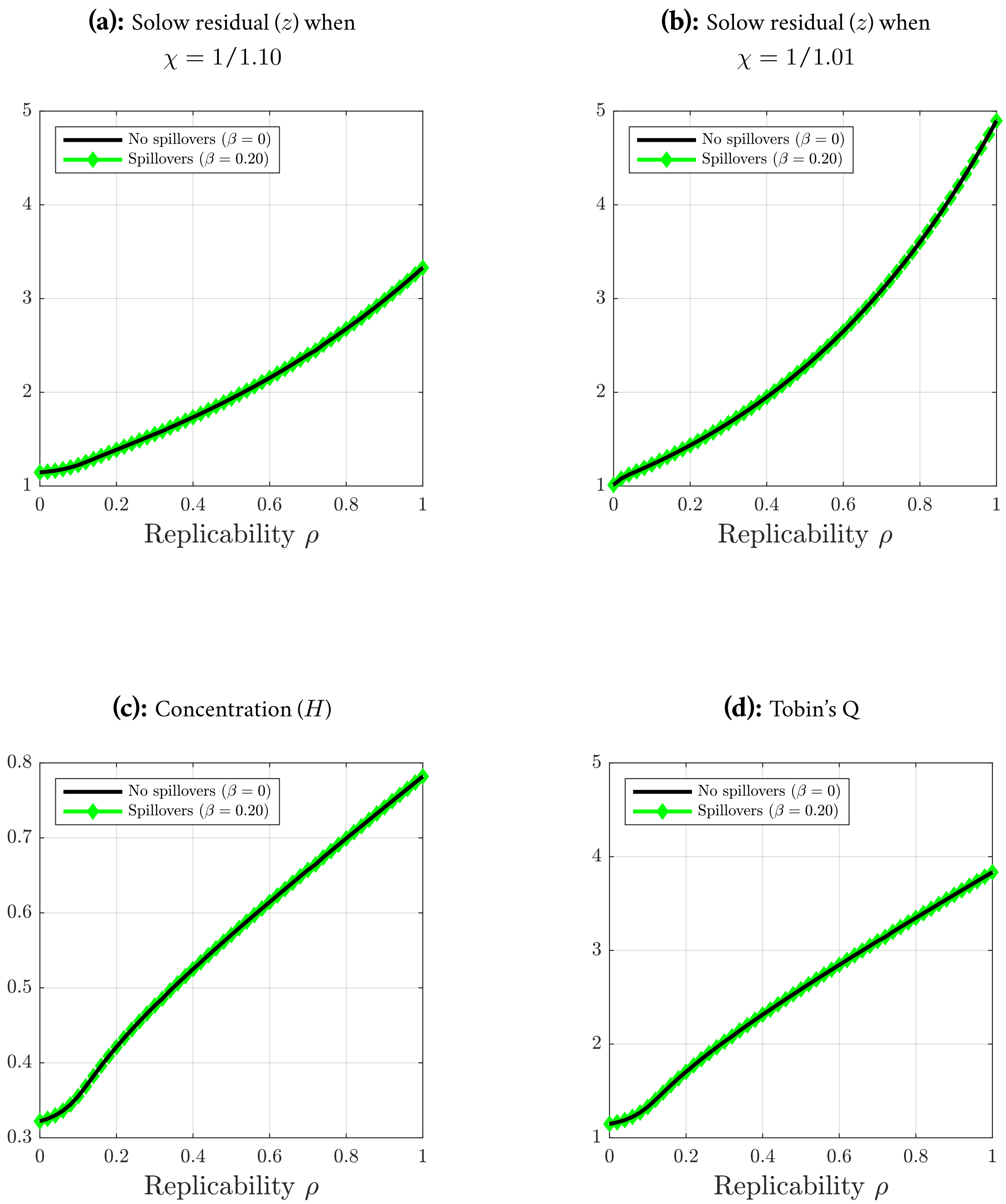

Figure 5: The Solow residual, concentration, and Tobin's Q as a function of the degree of replicability, ρ. The horizontal axis corresponds to different values of the parameter ρ. In all panels, the black line corresponds to a baseline case in which there are no spillovers from firm scope onto the creation of new intangible assets $(\beta = 0)$. The green circled line corresponds to a case in which there are spillovers $(\beta = 0.20)$. Except for the bottom left panel, the parameter values used are $\eta = 0.02$, $\zeta = 2/3$, $\chi = 1/1.10$, $\xi = 0.2$, $\delta = 0.10$, $\bar{x} = 1$, $\alpha = 2.6$ and $\kappa = 1.3$. In the bottom panel, we set $\chi = 1/1.01$, corresponding to markups of 1% instead of 10%.

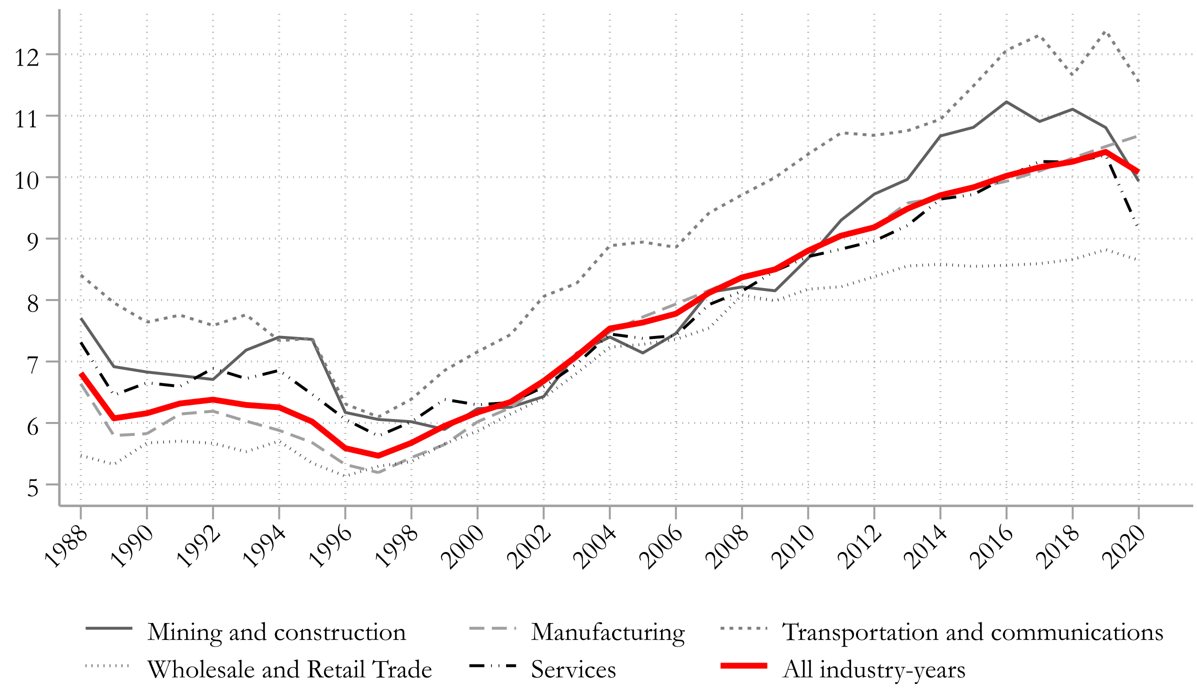

Figure 6: Average scope among US public firms, 1988-2021. On the vertical axis, the units represent the average number of business lines that firms operate in as reported in their 10-K statement, either at the industry level (dark lines) or in the pooled sample (solid red line). The industry classification is based on the SIC-1 code. The sample includes Compustat non-financial firms for 1988-2021, as described in Appendix A.4.1. The data on firm scope are drawn from Hoberg and Phillips (2024). Appendix A.4.1 reports details on data sources and data construction.

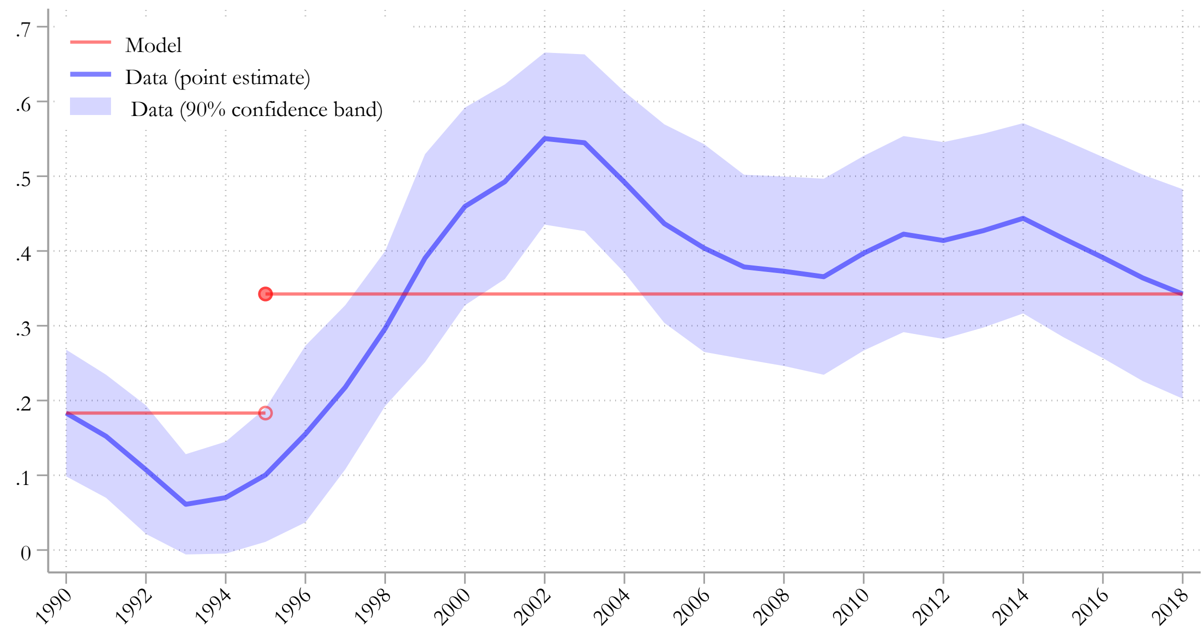

Figure 7: Rolling window estimate of ρ among US public firms, 1990-2018. In each year, the solid line reports the point estimate for the elasticity of firm revenue to firm scope, estimated from the same specification as in Column (3) of Table 1. The point estimate β_x has been translated to a point estimate for ρ using $\rho = (\beta_x - (1-\omega))/\omega$. We use a value of $\omega = 0.86$, which corresponds to the externally calibrated values for χ and ζ used in Section 5.2, $\chi = 1/1.05$ and $\zeta = 0.70$. The solide blue line represents the point estimate for ρ, and the shaded area represents the 90% confidence interval around the point estimate for ρ, constructed using the delta method. The red line represents the shock to ρ that we use to construct the transitional dynamics between 1990 and 2018 in the quantitative exercise of Section 5.3.2. The data on firm revenue and estimates of the intangible capital stock are drawn from the panel of Compustat firms for 1988-2021, restricted to non-financial firms, as described in Appendix A.4.1, and the data on firm scope are drawn from Hoberg and Phillips (2024). Appendix A.4.1 reports details on data sources and data construction.

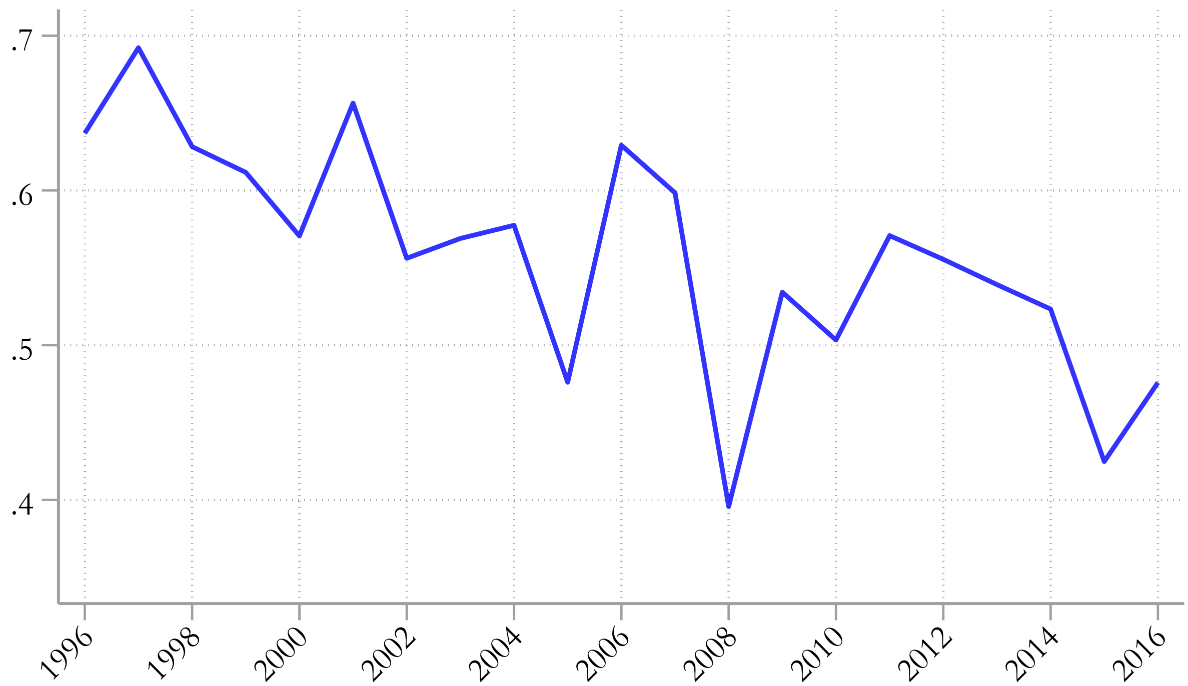

Figure 8: Share of firm equity held by founders prior to initial public offering (IPO). The share is measured using IPO prospectuses, and each point on the line reports the average share for firms that IPO in that year. The sample consists of all non-financial firms that went through an IPO from 1996 to 2016, excluding IPOs related to spin-offs, leveraged buy-outs, or mergers. Section 5.1.2 and Appendix A.4.1.3 reports details on data sources and data construction.

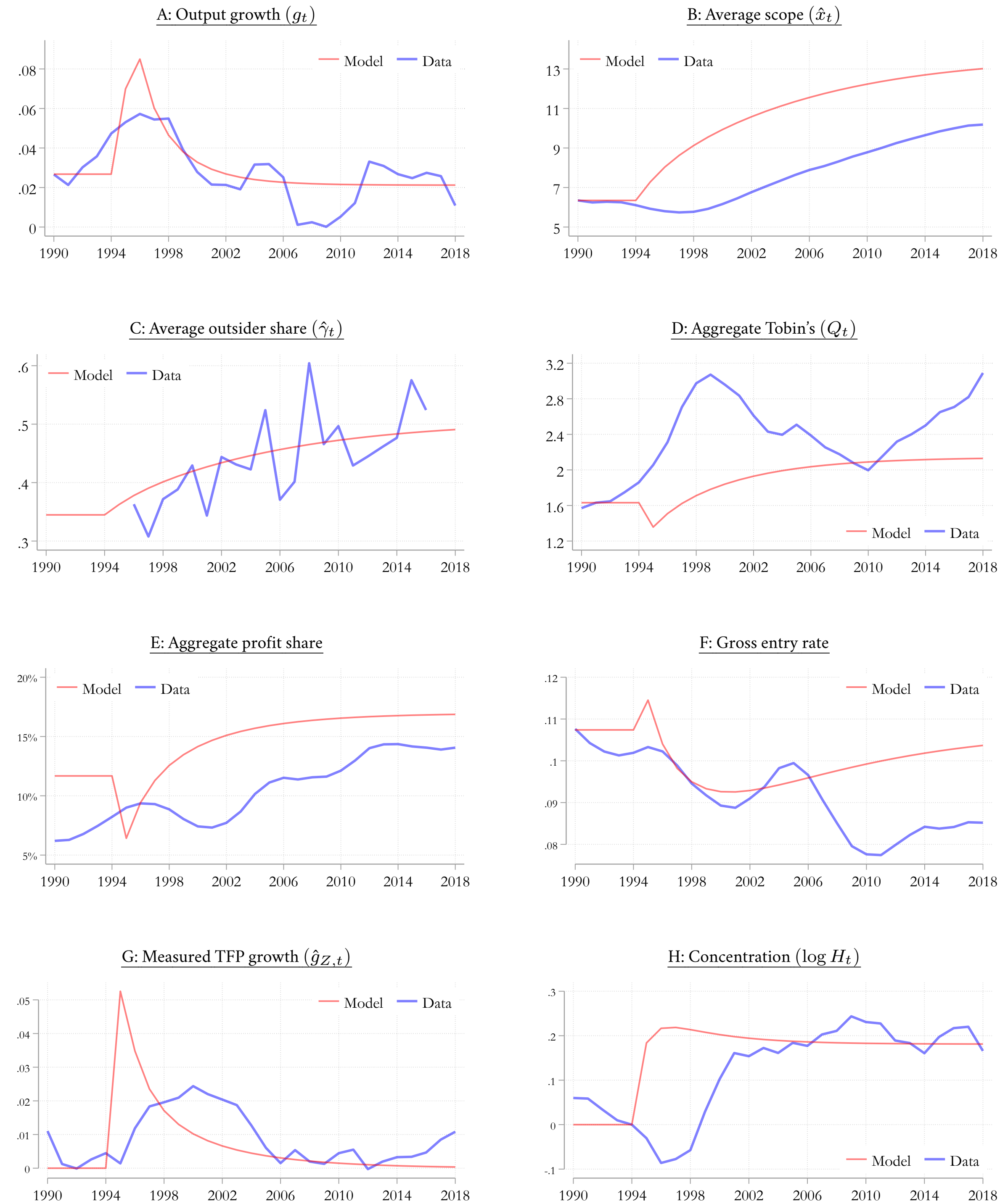

Figure 9: Transitional dynamics after an increase in ρ. The initial balanced growth path (BGP) is described in Table 2. From 1990 to 1995, the economy is assumed to evolve along that balanced growth path. The value of ρ, the degree of replicability, increases from $\rho = 0.18$ to $\rho = 0.34$ in 1995, as depicted in Figure 7. Each panel then reports the transitional dynamics for a particular endogenous variable in the model (red line), along with its data counterpart (blue line). Section 5.1 and Appendix A.4.1 discuss data sources and the computation of the empirical counterparts to model moments, and Section 5.3.2 and Appendix A.3 reports details on the computation of transitional dynamics in the model and of the different variables reported in this graph.

Internet Appendix for "Intangible Capital, Firm Scope, and Growth"

Nicolas Crouzet, Janice Eberly, Andrea Eisfeldt, and Dimitris Papanikolaou

June 2024

A.1 Model

This Appendix contains details on the model and the results of Section 2.

A.1.1 Monopoly problem for variety m

Let:

$$\begin{aligned} d_t &= P_{Y,t} Y_t^{1-\chi} \\ e_t(m) &= W_t n(m)^{-\frac{1-\zeta}{\zeta}} \quad \forall m \ \text{ s.t. } \ \hat{\jmath}(m) = j. \end{aligned}$$

The optimality condition for the monopoly problem for variety m at time t,

$$\pi_t(m) = \max_y \quad p_t(y) y - c_{m,t}(y)$$

is:

$$p_t(y) = \frac{1}{\chi} c'_{m,t}(y).$$

Solving for output y, we obtain the following expressions:

$$\begin{aligned} p_t(m) &= d_t^{\frac{1-\zeta}{1-\zeta\chi}} e_t(m)^{\frac{(1-\chi)\zeta}{1-\zeta\chi}} (\zeta\chi)^{-\frac{(1-\chi)\zeta}{1-\zeta\chi}} \\ y_t(m) &= d_t^{\frac{\zeta}{1-\zeta\chi}} e_t(m)^{-\frac{\zeta}{1-\zeta\chi}} (\zeta\chi)^{\frac{\zeta}{1-\zeta\chi}} \\ c_t(m) &= d_t^{\frac{1}{1-\zeta\chi}} e_t(m)^{-\frac{\zeta\chi}{1-\zeta\chi}} (\zeta\chi)^{\frac{1}{1-\zeta\chi}} \\ \pi_t(m) &= d_t^{\frac{1}{1-\zeta\chi}} e_t(m)^{-\frac{\zeta\chi}{1-\zeta\chi}} (\zeta\chi)^{\frac{\zeta\chi}{1-\zeta\chi}} (1-\zeta\chi) \end{aligned}$$

Using the expressions for d_t and $e_t(m)$, flow profits from variety m for firm j at time t are:

$$\begin{aligned} \forall m \quad \text{s.t.} \quad \hat{\jmath}(m) = j, \quad \pi_t(m) &= A_t n(m)^{\frac{(1-\zeta)\chi}{1-\zeta\chi}}, \\ A_t &= P_{Y,t}^{\frac{1}{1-\zeta\chi}} Y_t^{\frac{1-\chi}{1-\zeta\chi}} W_t^{-\frac{\zeta\chi}{1-\zeta\chi}} (1-\zeta\chi) (\zeta\chi)^{\frac{\zeta\chi}{1-\zeta\chi}}. \end{aligned} \tag{A1}$$

This is the expression given in the main text, substituting:

$$\omega \equiv \frac{(1-\zeta)\chi}{1-\zeta\chi}$$

where possible. Moreover, production labor demand for variety m is given by:

$$\forall m \quad \text{s.t.} \quad \hat{\jmath}(m) = j, \quad l_t(m) = P_{Y,t}^{\frac{1}{1-\zeta\chi}} Y_t^{\frac{1-\chi}{1-\zeta\chi}} W_t^{-\frac{1}{1-\zeta\chi}} (\zeta\chi)^{\frac{1}{1-\zeta\chi}} n(m)^{\frac{(1-\zeta)\chi}{1-\zeta\chi}}$$
$$= \frac{\zeta\chi}{1-\zeta\chi} \frac{A_t}{W_t} n(m)^{\omega}. \tag{A2}$$

A.1.2 A microfoundation for the function $\gamma_t(x;\lambda)$

Here we provide a derivation of Result 1. To do this, we start by setting up the Cournot problem for a particular variety m. With this we characterize the ratio ψ between profits under Cournot competition for the key worker, to profits under monopoly for the entrepreneur. We then turn to characterizing the value of the option to compete for each key worker at the date that the firm is created. Combining this, we finally establish Result 1.

A.1.2.1 Cournot competition for variety m

We drop time subscript where they are not necessary. Throughout, we use the notation:

$$N(m) \equiv \sum_j n(j,m)$$
$$s_n(j,m) \equiv n(j,m)/N(m)$$
$$l(m) \equiv \sum_j l(j,m)$$
$$y(m) \equiv \sum_j y(j,m)$$
$$s_x(j,m) \equiv x(j,m)/x(m), \quad x \in \{l,y\}$$

Additionally, we define:

$$l^*(m) = \frac{\zeta P_Y Y^{1-\chi}}{W}.$$

Without markups $(\chi = 0)$, this is total labor demand from firms producing variety m.

Solution for an arbitrary distribution of markups Consider the general case of a producer whose first-order condition is:

$$p(m) = \mu(j,m)\frac{W}{\zeta}\left(\frac{y(j,m)}{n(j,m)}\right)^{\frac{1-\zeta}{\zeta}}, \tag{A3}$$

where for now we take the markups $\{\mu(j,m)\}_{j=1}^{J}$ as given. (Recall that the distribution of intangibles across firms is also given.) Then, first, we have that:

$$y(j,m) \propto n(j,m)\mu(j,m)^{-\frac{\zeta}{1-\zeta}}. \tag{A4}$$

Therefore,

$$s_y(j,m) = \frac{s_n(j,m)\mu(j,m)^{-\frac{\zeta}{1-\zeta}}}{\sum_j s_n(j,m)\mu(j,m)^{-\frac{\zeta}{1-\zeta}}}. \tag{A5}$$

Writing the production function as:

$$l(j,m) = n(j,m)\left(\frac{y(j,m)}{n(j,m)}\right)^{\frac{1}{\zeta}} = n(j,m)\mu(j,m)^{-\frac{1}{1-\zeta}}, \tag{A6}$$

we also see that:

$$s_l(j,m) = \frac{s_n(j,m)\mu(j,m)^{-\frac{1}{1-\zeta}}}{\sum_j s_n(j,m)\mu(j,m)^{-\frac{1}{1-\zeta}}}. \tag{A7}$$

Since:

$$s_y(j,m)y(m) = y(j,m) = l(j,m)^{\zeta}n(j,m)^{1-\zeta} = \left(s_l(j,m)l(m)\right)^{\zeta} n(j,m)^{1-\zeta}, \tag{A8}$$

we can write:

$$y(m) = l(m)^{\zeta}n(m)^{1-\zeta}, \tag{A9}$$

where:

$$n(m) \equiv \frac{N(m)}{\Delta_n(m)} \tag{A10}$$

and where we defined:

$$\Delta_n(m) \equiv \frac{\left(\sum_{j=1}^{J} s_n(j,m)\mu(j,m)^{-\frac{1}{1-\zeta}}\right)^{\frac{\zeta}{1-\zeta}}}{\left(\sum_{j=1}^{J} s_n(j,m)\mu(j,m)^{-\frac{\zeta}{1-\zeta}}\right)^{\frac{1}{1-\zeta}}}. \tag{A11}$$

Finally, using the production function, note that:

$$\frac{y(j,m)}{n(j,m)} = \frac{y(j,m)}{(y(j,m)/l(j,m)^{\zeta})^{\frac{1}{1-\zeta}}} = \left(\frac{l(j,m)}{y(j,m)}\right)^{\frac{\zeta}{1-\zeta}} \tag{A12}$$

We can therefore rewrite the optimality condition for the producer as:

$$P_Y \left(\frac{Y}{y(m)}\right)^{1-\chi} = \frac{W}{\zeta}\mu(j,m)\frac{s_l(j,m)}{s_y(j,m)}\frac{l(m)}{y(m)}. \tag{A13}$$

Since, for all (j,m):

$$\mu(j,m)\frac{s_l(j,m)}{s_y(j,m)} = \frac{\sum_{j=1}^{J} s_n(j,m)\mu(j,m)^{-\frac{\zeta}{1-\zeta}}}{\sum_{j=1}^{J} s_n(j,m)\mu(j,m)^{-\frac{1}{1-\zeta}}} \equiv \Delta_l(m) \geq 1, \tag{A14}$$

we can rewrite this expression as:

$$l(m) = \frac{l^*(m)}{\Delta_l(m)}\ y(m)^{\chi}. \tag{A15}$$

Note that:

$$(\Delta_l(m))^{\zeta}(\Delta_n(m))^{1-\zeta} \quad = \quad \left(\sum_{j=1}^{J} s_n(j,m)\mu(j,m)^{-\frac{\zeta}{1-\zeta}}\right)^{-(1-\zeta)} \equiv \Delta(m), \tag{A16}$$

so we can write output in market m as:

$$y(m) = \frac{1}{\Delta(m)} l^*(m)^{\zeta} N(m)^{1-\zeta}\ y(m)^{\chi\zeta}. \tag{A17}$$

Note that $\Delta(m)$ only depends on the markups set by firms and on their output shares. Taking $\Delta(m)$ as given, we can then use this expression to solve for $y(m)$:

$$y(m) = \left(\frac{1}{\Delta(m)} l^*(m)^{\zeta} N(m)^{1-\zeta}\right)^{\frac{1}{1-\zeta\chi}} \tag{A18}$$

The wedge $\Delta(m)$ represents distortions arising because of markups. When all firms have the same markup, $\Delta(m) = \mu(m)^{\zeta} > 1$. If, on top of that, firms charge different markups, $\Delta(m)$ is even higher.

Finally, we can compute the profits of each firm. We have:

$$\begin{aligned}
p(m)y(m) &= P_Y Y^{1-\chi} y(m)^{\chi} \\
&= \frac{W}{\zeta} l^*(m) y(m)^{\chi} \\
&= \frac{W}{\zeta} \Delta_l(m) l(m) \\
p(m)y(j,m) &= \frac{W}{\zeta} \Delta_l(m) l(m) s_y(j,m) \\
W l(j,m) &= W s_l(j,m) l(m)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\pi(j,m) &= \left(1-\zeta \frac{s_l(j,m)}{s_y(j,m)\Delta_l(m)}\right) s_y(j,m) \frac{W}{\zeta} \Delta_l(m) l(m) \\
&= \left(1-\frac{\zeta}{\mu(j,m)}\right) s_y(j,m) \frac{W}{\zeta} \Delta_l(m) l(m) \\
&= \left(1-\frac{\zeta}{\mu(j,m)}\right) \frac{W}{\zeta} l^*(m) y(m)^{\chi} s_y(j,m) \\
\pi(j,m) &= \left(1-\frac{\zeta}{\mu(j,m)}\right) \left(\frac{\zeta}{W}\right)^{\frac{\zeta\chi}{1-\zeta\chi}} \left(P_Y^{\frac{1}{1-\chi}} Y\right)^{\frac{1-\chi}{1-\zeta\chi}} \left(\frac{1}{\Delta(m)}\right)^{\frac{\chi}{1-\zeta\chi}} N(m)^{\frac{(1-\zeta)\chi}{1-\zeta\chi}} s_y(j,m)
\end{aligned}$$

Finally, for use later, note that for an arbitrary distribution of markups, the following relationship holds between the market share, the markup, the distortion term $\Delta(m)$ and the stock of intangibles:

$$s_y(j,m) = \Delta(m)^{\frac{1}{1-\zeta}} s_n(j,m) \mu(j,m)^{-\frac{\zeta}{1-\zeta}} \tag{A19}$$

This follows just from looking at the expressions given above for $s_y(j,m)$ and $\Delta(m)$.

Solution under Cournot competition with $J > 2$ firms Suppose there are J firms producing variety m, and without loss of generality, rank the firms by the amount of intangible capital they have invested in the market, so that:

$$n(1,m) \geq n(2,m) \geq \ldots \geq n(J,m) > 0.$$

Denote by:

$$\phi(j,m) \equiv \frac{n(j,m)}{n(1,m)} \leq 1.$$

Then we have that the shares $s_n(j,m)$ are given by:

$$s_n(j,m) = \frac{\phi(j,m)}{\phi(m)} \quad , \quad \phi(m) \equiv \sum_{j=1}^{J} \phi(j,m).$$

Under Cournot competition, the first-order condition for each producer leads to:

$$\mu(j,m) = \frac{1}{1-(1-\chi)s_y(j,m)}.$$

Using Equation (A19), it must then be that:

$$s_y(j,m) = \Delta(m)^{\frac{1}{1-\zeta}} s_n(j,m) \left(1-(1-\chi)s_y(j,m)\right)^{\frac{\zeta}{1-\zeta}}.$$

Therefore each output share must satisfy:

$$h(s_y(j,m)) = s_n(j,m)^{1-\zeta}\Delta(m)$$

where the function h is defined as:

$$h(z) \equiv \frac{z^{1-\zeta}}{(1-(1-\chi)z)^{\zeta}} \tag{A20}$$

The function h is a strictly increasing bijection of $[0,1]$ to $\left[0, \chi^{-\zeta}\right]$, so we can write:

$$s_y(j,m) = h^{-1}\left(s_n(j,m)^{\zeta}\Delta(m)\right).$$

Finally, the output shares have to add up to 1. Therefore, $\Delta(m)$ must be the unique solution to:

$$1 = \sum_{j=1}^{J} h^{-1}\left(s_n(j,m)^{1-\zeta}\Delta(m)\right). \tag{A21}$$

Note that this solution satisfies:

$$\forall j, \quad \Delta(m)s_n(j,m)^{1-\zeta} \leq \chi^{-\zeta} \quad \implies \Delta(m) \leq (J\chi)^{-\zeta}.$$

We can rewrite Equation (A21) using the relative weights $\phi(j, m)$:

$$1 = \sum_{j=1}^{J} h^{-1}\left(\left[\frac{\phi(j,m)}{\phi(m)}\right]^{1-\zeta}\Delta(m)\right). \tag{A22}$$

This shows that the distortion $\Delta(m)$ only depends on a χ, ζ, and the relative weights $\{\phi(j, m)\}_{j=1}^{J}$. Profits for each firm will be given by:

$$\begin{aligned}\pi^{(c)}(j,m) \quad &= \quad \left(\frac{\zeta}{W}\right)^{\frac{\zeta\chi}{1-\zeta\chi}}\left(P_Y^{\frac{1}{1-\chi}}Y\right)^{\frac{1-\chi}{1-\zeta\chi}}\left(\frac{1}{\Delta(m)}\right)^{\frac{\chi}{1-\zeta\chi}} N^{(c)}(m)^{\frac{(1-\zeta)\chi}{1-\zeta\chi}} \\ &\times \left[1-\zeta\left(1-(1-\chi)h^{-1}\left(\left[\frac{\phi(j,m)}{\phi(m)}\right]^{1-\zeta}\Delta(m)\right)\right)\right] h^{-1}\left(\left[\frac{\phi(j,m)}{\phi(m)}\right]^{1-\zeta}\Delta(m)\right)\end{aligned}$$

Here the superscript (c) refers to the Cournot outcome.

This expression can be related to the monopoly case. Suppose that firm $j = 1$ is the monopolist, so that we can rewrite the monopoly profits, Equation (2)

$$\pi(m) = (1-\zeta\chi)\left(\frac{\zeta\chi}{W}\right)^{\frac{\zeta\chi}{1-\zeta\chi}}\left(P_Y^{\frac{1}{1-\chi}}Y\right)^{\frac{1-\chi}{1-\zeta\chi}} n(1,m)^{\frac{(1-\zeta)\chi}{1-\zeta\chi}}.$$

Moreover note that we can rewrite total intangible capital in market m as:

$$N^{(c)}(m) = \phi(m)n(1,m).$$

Then we see that:

$$\begin{aligned}\frac{\pi^{(c)}(j,m)}{\pi(m)} \quad &= \quad \left(\frac{\chi^{-\zeta}}{\Delta(m)}\right)^{\frac{\chi}{1-\zeta\chi}}(\phi(m))^{\frac{(1-\zeta)\chi}{1-\zeta\chi}} \\ &\times \frac{1-\zeta\left(1-(1-\chi)h^{-1}\left(\left[\frac{\phi(j,m)}{\phi(m)}\right]^{\zeta}\Delta(m)\right)\right)}{1-\zeta\chi} h^{-1}\left(\left[\frac{\phi(j,m)}{\phi(m)}\right]^{1-\zeta}\Delta(m)\right)\end{aligned} \tag{A23}$$

This expression again only depends on χ, ζ, and the relative weights $\{\phi(j, m)\}_{j=1}^{J}$.

Solution under Cournot competition with $J = 2$ firms Everything above applies to the particular case of $J = 2$. In that case the profits of the potential competitor with capital $n^{(c)}(2, m) = \phi n(1, m)$, relative to the profits of firm 1 if they remain a monopolist, can be written as:

$$\tilde{s}^{(c)}(\phi) \equiv \frac{\pi^{(c)}(m)}{\pi(m)} = \left(\frac{\chi^{-\zeta}}{\Delta}\right)^{\frac{\chi}{1-\zeta\chi}} (1+\phi)^{\frac{(1-\zeta)\chi}{1-\zeta\chi}}$$
$$\times \frac{1-\zeta\left(1-(1-\chi)h^{-1}\left(\left[\frac{\phi}{1+\phi}\right]^{1-\zeta}\Delta\right)\right)}{1-\zeta\chi} h^{-1}\left(\left[\frac{\phi}{1+\phi}\right]^{1-\zeta}\Delta\right)$$

where the function h is given as in Equation (A20), and Δ is now implicitly given by:

$$1 = h^{-1}\left(\left[\frac{1}{1+\phi}\right]^{1-\zeta}\Delta\right) + h^{-1}\left(\left[\frac{\phi}{1+\phi}\right]^{1-\zeta}\Delta\right) \tag{A24}$$

Relation to the constant ψ in Result (1) Each key worker specializes in one variety, so that the entrepreneur only shares the corresponding amount of intangibles with them. This amount is given by:

$$n(m_0) = x^{-(1-\rho)}n,$$

where m_0 is the variety in which the key worker specializes, and n is the total amount of intangible capital created by the entrepreneur, and where we have omitted the j subscripts, so that $n(m)$ will represent the amount of intangible capital referred to as $n(1,m)$ in the derivations above. A key worker that has started a firm will choose to invest their intangibles symmetrically across the $\bar{x}$ varieties in which they can compete with the entrepreneur. Thus they will invest:

$$n(2,m) = \bar{x}^{-(1-\rho)}n(m_0)$$

in each of the varieties that they compete with the entrepreneur in. Therefore the ratio of capital stocks for any particular variety where there is competition is:

$$\frac{n(2,m)}{n(m)} = \frac{\bar{x}^{-(1-\rho)}n(m_0)}{n(m)} = \bar{x}^{-(1-\rho)}.$$

Moreover, we assume that, for each variety in which the key worker competes, they must incur competition costs equal to a constant fraction $\varepsilon \in [0,1]$ of output produced.[32] As a result, the constant ψ in Result (1) is then given by:

$$\psi \equiv (1-\varepsilon)\tilde{s}^{(c)}\left(\bar{x}^{-(1-\rho)}\right).$$

[32]This fraction is a deadweight loss, but it does not affect aggregate resource cosntraints because Cournot competition remains off-equilibrium in the model.

Summary We summarize this discussion in the following Lemma.

Lemma 3 (Cournot competition between the entrepreneur and a key worker) *The ratio between the flow profits of a key worker competing à la Cournot with the entrepreneur in a particular variety m, and the monopoly profits of the entrepreneur in that variety, is given by:*

$$\psi \equiv (1-\varepsilon)\tilde{s}^{(c)}\left(\bar{x}^{-(1-\rho)}\right), \tag{A25}$$

where:

$$\tilde{s}^{(c)}(\phi) = \left(\frac{\chi^{-\zeta}}{\Delta}\right)^{\frac{\chi}{1-\zeta\chi}}(1+\phi)^{\frac{(1-\zeta)\chi}{1-\zeta\chi}} \tag{A26}$$

$$\times \frac{1-\zeta\left(1-(1-\chi)h^{-1}\left(\left[\frac{\phi}{1+\phi}\right]^{1-\zeta}\Delta\right)\right)}{1-\zeta\chi}h^{-1}\left(\left[\frac{\phi}{1+\phi}\right]^{1-\zeta}\Delta\right)$$

$$1 = h^{-1}\left(\left[\frac{1}{1+\phi}\right]^{1-\zeta}\Delta\right)+h^{-1}\left(\left[\frac{\phi}{1+\phi}\right]^{1-\zeta}\Delta\right) \tag{A27}$$

$$h(z) \equiv \frac{z^{1-\zeta}}{(1-(1-\chi)z)^{\zeta}} \quad \forall z\in[0,1] \tag{A28}$$

$$\phi \equiv \bar{x}^{-(1-\rho)} \tag{A29}$$

In particular, ψ is independent of m, of aggregate states, time, or of the amount of intangible capital invested by the entrepreneur in m. If $\bar{x}=1$, it only depends on (ε,χ,ζ).

We interpret ε as flow costs that the key worker must bear in order to compete in each variety. The reason for introducing ε is the following. In order for the entrepreneur to find it ex-post profitable to deter the outsider from entry, it must be that total profits under Cournot competition (to both the key worker and the entrepreneur) are less than or equal to monopoly profits. Otherwise, collusion and entry deterrence are not possible. This condition can be rewritten as:

$$1 \geq \tilde{s}^{(c)}(\phi)+\tilde{s}^{(e,c)}(\phi), \tag{A30}$$

where $\tilde{s}^{(e,c)}(\phi)$ is equal to the ratio of entrepreneur profits under Cournot competition, to their profits under monopoly. Using the derivations above, this latter ratio is given by:

$$\tilde{s}^{(e,c)}(\phi) = \left(\frac{\chi^{-\zeta}}{\Delta}\right)^{\frac{\chi}{1-\zeta\chi}}(1+\phi)^{\frac{(1-\zeta)\chi}{1-\zeta\chi}} \tag{A31}$$

$$\times \frac{1-\zeta\left(1-(1-\chi)h^{-1}\left(\left[\frac{1}{1+\phi}\right]^{1-\zeta}\Delta\right)\right)}{1-\zeta\chi}h^{-1}\left(\left[\frac{1}{1+\phi}\right]^{1-\zeta}\Delta\right)$$

We cannot prove that condition (A30) holds for all values of (ϕ, ζ, χ). Moreover, numerical examples suggest that this need not be the case, even when $\phi = 1$ (the symmetric case, where the entrepreneur and competitor have the same amount of capital invested in the market). However, we can show that there always exists $\epsilon \in [0, 1]$ such that:

$$1 \geq (1-\varepsilon)\tilde{s}^{(c)}(\phi) + \tilde{s}^{(e,c)}(\phi), \tag{A32}$$

see monopoly profits of the entrepreneur are always greater than their Cournot profits. In this case, there always exists $\underline{\varepsilon}(\zeta, \chi, \phi)$ such that:

$$1 = (1-\underline{\varepsilon}(\zeta, \chi, \phi))\tilde{s}^{(c)}(\phi) + \tilde{s}^{(e,c)}(\phi). \tag{A33}$$

In words, there are flow costs of entry for the potential competitor such that the entrepreneur is ex-post indifferent between deterring entry and not deterring it. In the main model, we assume that if condition (A30) does not hold, ε is such that:

$$\varepsilon \geq \underline{\varepsilon}(\zeta, \chi, \phi).$$

This guarantees the deterrence strategy is time-consistent. In certain calibrations, we will assume that the condition holds with equality, so that the entrepreneur is indifferent between deterring entry of key workers and competing with them à la Cournot.

A.1.2.2 The cost of deterrence

Next, we turn to deriving the value of the option to compete to the key worker.

Lemma 4 (The value of the option to compete for a key worker) *Let t be the date of entry of the entrepreneur. At time t, the value of the option to open a competing firm for each key worker is given by:*

$$v_t^{(c)} = \bar{x}\ \psi\ \frac{v_t^{(e)}}{x}\ \mu_t(\lambda), \tag{A34}$$

where:

$$\mu_t(\lambda) = \lambda \int_0^{+\infty} e^{-\int_0^s (r_{t+u}+\lambda)du} \frac{A_{t+s}\nu_{t+s}}{A_t \nu_t} ds. \tag{A35}$$

Proof. Let π_s denote total flow profits of the entrepreneur under monopoly at some date $s \geq t$ after the firm has been created.

$$\pi_s \equiv \int_0^x \pi_s(m)dm.$$

Suppose that at time $s \geq t$, a key worker has already received and exercised the option to compete. Then using Lemma 3, the *total* profits of a key worker operating a firm that produces $\bar{x}$ varieties are given by:

$$\pi_s^{(c)} = \bar{x}\ \psi\ \frac{\pi_s}{x}.$$

The first term is the number of varieties produced; the second term is the ratio of the profits of the key worker to the monopoly profits of the entrepreneur for each variety; and the third term is the monopoly profit per variety.

In present value terms, at any date $s \geq t$ after the arrival of the option to compete, the value function of a competing key worker is therefore related to the entrepreneur's value function through:

$$v_s^{(c)} = \bar{x}\ \frac{\psi}{x}\ \frac{v_s^{(e)}}{x}.$$

Before the option arrives, the latent value function of each outsider follows:

$$(r_s ds)v_s^{(c)} \quad = \quad dv_s^{(c)} + (\lambda ds)\left(\bar{x}\frac{\psi v_s^{(e)}}{x} - v_s^{(c)}\right)$$

Integrating, we obtain that at entry, the latent value function of each outsider at entry is:

$$\begin{aligned} v_t^{(c)} &= \bar{x}\ \frac{\psi}{x}\ \lambda\ \int_0^{+\infty} e^{-\int_0^s (r_{t+u}+\lambda)du} v_{t+s}^{(e)} ds \\ &= \frac{\bar{x}}{x}\ \psi\ \mu_t(\lambda)\ v_t^{(e)}, \end{aligned}$$

where:

$$\mu_t(\lambda) \equiv \lambda\ \int_0^{+\infty} e^{-\int_0^v (r_{t+u}+\lambda)du} \frac{A_{t+v}\nu_{t+v}}{A_t\nu_t} dv.$$

■

This result shows that the value of the option to compete for each key worker is proportional to the entrepreneur's overall value function. The ratio of the two only depends on the number of varieties in which the key worker would compete, relative to the total number of varieties in which the entrepreneur is the monopolist (the term $\bar{x}/x$), the ratio of profits under Cournot competition to monopoly profits for each variety (the term ψ), and a delay term, $\mu_t(\lambda)$, which captures the fact that the key worker would, on average, have to wait $1/\lambda$ periods before opening their own firm. Importantly, the delay term $\mu_t(\lambda)$ only depends on aggregate variables.[33]

Using this lemma, we now arrive at the expression for $\gamma_t(x;\lambda)$ under this microfoundation.

Result 5 (Cost of deterrence under non-exclusivity) *The total value of the option to compete to all key workers is given by:*

$$\bar{v}_t^{(c)} = \gamma_t(x;\lambda) v_t^{(e)},$$

where:

$$\gamma_t(x;\lambda) \equiv \theta(x)\ \bar{x}\ \frac{\psi}{x}\ \mu_t(\lambda). \tag{A36}$$

Proof. This follows from multiplying the value of the option to compete, Equation (A34), by the total number of key workers, $\theta(x)$. ■

A.1.3 Optimal scope

In this section we study the problem of choosing the optimal scope of the firm. Recall that the optimal scope of the firm must

$$x^* \quad = \quad \arg\max_x \quad \Big(1-\gamma(x)\Big)\ x^{1-(1-\rho)\omega}. \tag{A37}$$

Here, for clarity, we have omitted the time and firm subscripts, as well as the explicit dependence of γ on the parameter λ that governs the degree of non-exclusivity of intangible assets. In the general formulation of the model, the function $\gamma(.)$ only needs to satisfy the conditions outlined in Assumption 2. In the microfoundation described in Section 2.2.2, the function $\gamma(.)$ has a specific analytical expression, given in Result (1).

In what follows we give the conditions under which the solution to problem (A37) exists and is unique. We start by showing that the conditions laid out in Assumption 2 are sufficient.

Result 6 *If Assumption 2 holds, there exists a unique optimal scope x^* that solves problem (A37).*

Proof. The objective $h(x) = (1-\gamma(x))x^{1-(1-\rho)\omega}$ is continuously differentiable, so any interior extrema must satisfy:

$$h'(x) = 0 \quad\iff\quad (1-\gamma(x))(1-(1-\rho)\omega) = \gamma'(x)x. \tag{A38}$$

[33] Add proof that $\mu_t(\lambda)$ satisfies $\mu_t(0) = 0$ and $\lim_{\lambda\to+\infty}\mu_t(\lambda) = 1$.

The left-hand side is monotonically decreasing, from $1-(1-\rho)\omega$ to a value that is weakly negative, as $\lim_{x\to+\infty}\gamma(x)\geq 1$. Moreover, because $\gamma(x)$ is strictly increasing, continuously differentiable and weakly convex, $\lim_{x\to 0^+} < +\infty$, $\gamma'(x)\geq 0$, and $\gamma'(.)$ is increasing. As a result $x\to\gamma'(x)x$ is strictly increasing, from 0 to $\lim_{x\to+\infty} x\gamma'(x)$. Thus equation (A38) has a unique solution, so the objective function in problem (A37) has a unique interior extremum, which must be a maximum because the objective function is concave. Since $h(0)=0$ and $\lim_{x\to+\infty} h(x)<0$, this interior maximum is also the global maximum. ■

While these conditions do not hold in the specific microfoundation outline in Section 2.2.2, we show that new firms still have an optimal, finite scope in that version of the model so long as a simple restriction on the shape of the function $\theta(x)$, which governs the number of key stakeholders required to reach scope x, holds.

Result 7 *Suppose that κ, ρ and ω satisfy:*

$$[xxx]. \tag{A39}$$

Then, there exists a unique optimal scope x^* that solves problem (A37).

Proof. In the microfoundation of Section 2.2.2, the function $\gamma(x)$ is given by:

$$\gamma(x)=\begin{cases}\alpha\psi\mu\dfrac{\bar{x}}{x}\left(x-\bar{x}\right)^{\kappa} & \text{if} \quad x\geq\bar{x}\\[2ex] 0 & \text{if} \quad x<\bar{x}\end{cases} \tag{A40}$$

This function is twice continuously differentiable, strictly increasing for $x\geq\bar{x}$, but not necessarily weakly convex, so that Result (6) does not necessarily apply. We start by rewriting this function as:

$$p(u)=\begin{cases}Au^{-(\kappa-1)}\left(1-u\right)^{\kappa} & \text{if} \quad u\leq 1\\[2ex] 0 & \text{if} \quad u>1\end{cases} \tag{A41}$$

where:

$$A=\alpha\psi\mu\bar{x}^{\kappa} \tag{A42}$$

and:

$$u\equiv\frac{\bar{x}}{x}. \tag{A43}$$

With this parametrization, $p(\bar{x}/x)=\gamma(x)$. Thus we can rewrite the choice of optimal scope as:

$$x^*=\arg\max_u h(u), \tag{A44}$$

where:

$$h(u) = \begin{cases} \left(1 - Au^{-(\kappa-1)}\left(1-u\right)^{\kappa}\right) u^{-(1-(1-\rho)\omega)} & \text{if} \quad u \leq 1 \\ 0 & \text{if} \quad u > 1 \end{cases} \tag{A45}$$

This objective function is continuously differentiable so that any interior extremum must satisfy:

$$h'(u) = 0 \quad \iff \quad (1-(1-\rho)\omega)u^{-1}\left(1 - Au^{-(\kappa-1)}\left(1-u\right)^{\kappa}\right) = Au^{-\kappa}(1-u)^{\kappa-1}\left(\kappa+u-1\right)$$

This condition can be rewrriten as:

$$1-(1-\rho)\omega = Au^{-(\kappa-1)}(1-u)^{\kappa-1}\left(\kappa+u-(1-\rho)\omega\right)$$

The right-hand side of this expression is a function that is equal to $+\infty$ at $u = 0^{+}$ and 0 at $u = 1$, so there is at least one solution to this equation. We are then looking for sufficient conditions for the solution to be unique. For this, a sufficient condition is that the function:

$$q(u) = \left(\frac{1}{u} - 1\right)^{\kappa-1}\left(\kappa+u-(1-\rho)\omega\right) \tag{A46}$$

be strictly decreasing on $(0,1]$. ■

A.1.4 Equilibrium characterization

Here we provide derivations for the basic equilibrium properties of the model summarized in Section 2.3 of the main paper. Throughout, in order to make notation lighter, we omit the dependence of the function $\gamma_t(x_t;\lambda)$ on the structural parameter λ, and simply write:

$$\gamma_t(x_t;\lambda) = \gamma_t(x_t). \tag{A47}$$

A.1.4.1 Aggregation

For each m, from the first-order condition of the producer, we have that:

$$y_t(m) \;\propto\; n(m)^{\frac{1-\zeta}{1-\zeta\chi}}.$$

Therefore, output of each variety can be written as:

$$y_t(m) = \left(\frac{n(m)}{N_t}\right)^{\frac{1-\zeta}{1-\zeta\chi}} Y_t, \tag{A48}$$

where the aggregate capital stock N_t is defined as:

$$N_t \equiv \left(\int_0^{M_t} n(m)^{\frac{(1-\zeta)\chi}{1-\zeta\chi}} dm \right)^{\frac{1-\zeta\chi}{(1-\zeta)\chi}}. \tag{A49}$$

Additionally, using Equation (A2), we have:

$$l_t(m) \propto n(m)^{\frac{(1-\zeta)\chi}{1-\zeta\chi}}.$$

Therefore,

$$l_t(m) = \left(\frac{n(m)}{N_t} \right)^{\omega} L_{Y,t}. \tag{A50}$$

Using this, we obtain the following expression for output of variety m:

$$y_t(m) = l_t(m)^{\zeta} n(m)^{1-\zeta} = L_{Y,t}^{\zeta} \left(\frac{n(m)}{N_t^{\zeta}} \right)^{\frac{1-\zeta}{1-\zeta\chi}}.$$

Comparing this to Equation (A48), we see that total output Y_t is given by:

$$Y_t = N_t^{1-\zeta} L_{Y,t}^{\zeta}. \tag{A51}$$

Moreover, using Equation (A2), we can derive aggregate labor demand as a function of output, capital, and the wage rate:

$$L_{Y,t} = (\zeta\chi)^{\frac{1}{1-\zeta\chi}} N_t^{\frac{(1-\zeta)\chi}{1-\zeta\chi}} Y_t^{\frac{1-\chi}{1-\zeta\chi}} W_t^{-\frac{1}{1-\zeta\chi}}.$$

Using the aggregate production function to eliminate N_t, we arrive at the following expression:

$$L_{Y,t} = \chi \frac{\zeta Y_t}{W_t}. \tag{A52}$$

Finally, using the expression for labor demand as a function of A_t in Equation (A52), and the labor market clearing condition, we can express A_t as:

$$\frac{A_t}{W_t} = \frac{1-\zeta\chi}{\zeta\chi} L_{Y,t} N_t^{-\frac{(1-\zeta)\chi}{1-\zeta\chi}}. \tag{A53}$$

Thus, the value of a firm at entry, with intangible capital $n_t(j) = n_t$, relative to the wage rate, is given by:

$$\frac{v_t^{(e)}}{W_t} = \frac{1-\zeta\chi}{\zeta\chi} (1-\gamma_t(x_t)) x_t^{1-(1-\rho)\omega} \nu_t \; L_{Y,t} \left(\frac{n_t}{N_t} \right)^{\frac{(1-\zeta)\chi}{1-\zeta\chi}}. \tag{A54}$$

Substituting for $n_t = \xi B_t$ from Equation (13) we obtain the expression reported in the main text.

A.1.4.2 Characterization of global dynamics

In what follows we focus on the particular case of the microfoundation described in Section (2.2.2), since that is the case we focus on in the rest of the paper. We establish the following result.

Result 8 (Global dynamics) *Let:*

$$c_t \equiv \left(\frac{\xi B_t}{N_t}\right)^{\omega} \geq 0. \tag{A55}$$

Moreover, define the two functions:

$$\boldsymbol{\gamma}(x,\mu) = \theta(x)\,\bar{x}\,\frac{\psi}{x}\,\mu \tag{A56}$$

$$\boldsymbol{x}(\mu) = \arg\max_x \;(1-\boldsymbol{\gamma}(x,\mu))x^{1-(1-\rho)\omega} \tag{A57}$$

Then the equilibrium values of the variables $(L_{E,t}, x_t, \nu_t, \mu_t)$ *must satisfy:*

$$L_{E,t} = L_E(c_t), \quad x_t = \tilde{x}(c_t), \quad \nu_t = \nu(c_t), \quad \mu_t = \mu(c_t), \tag{A58}$$

where the functions $\{L_E(.), \tilde{x}(.), \nu(.), \mu(.)\}$ *solve the following system of non-linear differential equations:*

$$\begin{aligned} 0 = {} & \left(\eta + \tilde{x}(c)^{1-(1-\rho)\omega} L_E(c)c\right)\nu(c) \\ & -1 - \left(\omega\xi L_E(c) + \delta - \tilde{x}(c)^{1-(1-\rho)\omega} L_E(c)c\right) c\partial_c\nu(c) \end{aligned} \tag{A59}$$

$$\begin{aligned} 0 = {} & \left(\lambda - \delta + \frac{1}{\nu(c)}\right)\mu(c) \\ & -\lambda - \left(\omega\xi L_E(c) + \delta - \tilde{x}(c)^{1-(1-\rho)\omega} L_E(c)c\right) c\partial_c\mu(c) \end{aligned} \tag{A60}$$

$$L_E(c) = \min\left(1\,,\,\max\left(0\,,\,1 - \Xi(c)^{-1}\right)\right) \tag{A61}$$

where we have defined:

$$\tilde{x}(c) = \boldsymbol{x}\left(\mu(c)\right) \tag{A62}$$

$$\tilde{\gamma}(c) = \boldsymbol{\gamma}\left(\tilde{x}(c), \mu(c)\right) \tag{A63}$$

$$\Xi(c) = \frac{1-\zeta\chi}{\zeta\chi}\tilde{x}(c)^{1-(1-\rho)\omega}\left(1-\tilde{\gamma}(c)\right)c\nu(c) \tag{A64}$$

Proof. We proceed by guessing and verifying that the following ratio is a sufficient state:

$$c_t = \left(\frac{n_t}{N_t}\right)^{\omega} = \left(\frac{\xi B_t}{N_t}\right)^{\omega}.$$

Our derivations below verify this guess. First, we write the index for aggregate capital as:

$$N_t^{\omega} = \int_{s \leq t} x_s^{1-(1-\rho)\omega} L_{E,s} e^{-\delta(t-s)} n_s^{\omega} ds.$$

Differentiating with respect to time, we obtain that for all $t \geq 0$,

$$\frac{d(N_t^{\omega})}{N_t^{\omega}} = \left(x_t^{1-(1-\rho)\omega} L_{E,t} c_t - \delta\right) dt. \tag{A65}$$

Second, recall that the law of motion for B_t is:

$$\frac{dB_t}{B_t} = \xi L_{E,t} dt.$$

Since $n_t = \xi B_t$, we have:

$$\frac{d([n_t^{(e)}]^{\omega})}{[n_t^{(e)}]^{\omega}} = \omega \xi L_{E,t} dt.$$

Combining these equations, we obtain the law of motion for c_t:

$$\frac{dc_t}{c_t} = \left(\omega \xi L_{E,t} + \delta - x_t^{1-(1-\rho)\omega} L_{E,t} c_t\right) dt. \tag{A66}$$

Under our guess, ν_t can be expressed as a time-invariant function of c_t only, that is:

$$\nu_t = \nu(c_t).$$

Next, we derive a first-order differential equation characterizing $\nu(.)$. To do this, first note that since:

$$r_t dt = \frac{dC_t}{C_t} + \eta dt = \frac{dY_t}{Y_t} + \eta dt$$

we can rewrite the definition of ν_t as:

$$\nu_t = \int_0^{+\infty} e^{-(\eta+\delta)s} \frac{A_{t+s}/Y_{t+s}}{A_t/Y_t} ds.$$

Second, using the definition of A_t in Equation (A53), we note that:

$$\frac{A_t}{Y_t} = (1-\zeta\chi)N_t^{-\omega}.$$

Consider the integral:

$$h_t = \int_0^{+\infty} e^{-(\delta+\eta)s} N_{t+s}^{-\omega} ds.$$

Then by the Feynman-Kac formula there exists a function $h(N_t^\omega)$ such that $h_t = h(N_t^\omega)$, where h follows:

$$(\delta+\eta)h(N_t^\omega) = N_t^{-\omega} + \left[\frac{1}{dt}\frac{d(N_t^\omega)}{(N_t^\omega)}\right](N_t^\omega)h'((N_t^\omega)). \tag{A67}$$

Under our guess of $\nu_t = \nu(c_t)$, we have $h(N_t^\omega) = (N_t^{-\omega})\nu(c_t)$, so:

$$\left[\frac{1}{dt}\frac{d(N_t^\omega)}{(N_t^\omega)}\right](N_t^\omega)h'(N_t^\omega) = -(N_t^{-\omega})\left[\frac{1}{dt}\frac{d(N_t^\omega)}{(N_t^\omega)}\right]\nu(c_t) + N_t^{-\omega}\left[\frac{1}{dt}\frac{dc_t}{c_t}\right]c_t\partial_c\nu(c_t).$$

Replacing these expressions into Equation (A67), and multiplying by $Z_t > 0$, we obtain:

$$(\delta+\eta)\nu(c_t) = 1 - \left[\frac{1}{dt}\frac{d(N_t^\omega)}{(N_t^\omega)}\right]\nu(c_t) + \left[\frac{1}{dt}\frac{dc_t}{c_t}\right]c_t\partial_c\nu(c_t).$$

Finally, we can use Equations (A65) and (A66) to substitute out the two drift terms, and obtain:

$$\left(\eta + x_t^{1-(1-\rho)\omega}L_{E,t}c_t\right)\nu(c_t) = 1 + \left(\xi\omega L_{E,t} + \delta - x_t^{1-(1-\rho)\omega}L_{E,t}c_t\right)c_t\partial_c\nu(c_t).$$

Likewise, we write:

$$\begin{aligned}
\mu_t &= \lambda\int_0^{+\infty} e^{-\int_0^v (r_{t+u}+\lambda)du}\frac{A_{t+v}\nu_{t+v}}{A_t\nu_t}dv \\
&= \lambda\int_0^{+\infty} e^{-(\eta+\lambda)v}\frac{A_{t+v}\nu_{t+v}}{A_t\nu_t}\frac{Y_t}{Y_{t+v}}dv
\end{aligned}$$

Define:

$$U_t \equiv \frac{(1-\zeta\chi)N_t^\omega}{\nu_t} = \frac{Y_t}{\nu_t A_t} \tag{A68}$$

We have:

$$U_t^{-1}\mu_t = \lambda\int_0^{+\infty} e^{-(\eta+\lambda)v}U_{t+v}^{-1}dv,$$

Defining:

$$g(U_t) \quad = \quad \int_0^{+\infty} e^{-(\eta+\lambda)v} U_{t+v}^{-1} dv,$$

we get that g follows the ODE:

$$(\lambda+\eta)g(U_t) = U_t^{-1} + \left[\frac{1}{dt}\frac{dU_t}{U_t}\right] U_t g'(U_t). \tag{A69}$$

Under the guess that $\mu_t = \mu(c_t)$, we have:

$$\lambda g(U_t) = \left(U_t^{-1}\mu(c_t)\right). \tag{A70}$$

Differentiating this and substituting, we obtain that μ must follow:

$$(\lambda+\eta)\mu(c_t) = \lambda - \left[\frac{1}{dt}\frac{dU_t}{U_t}\right]\mu(c_t) + \left[\frac{1}{dt}\frac{dc_t}{c_t}\right] c_t\partial_c\mu(c_t).$$

Finally, we have that:

$$\left[\frac{1}{dt}\frac{dU_t}{U_t}\right] = \left[\frac{1}{dt}\frac{d(N_t^\omega)}{N_t^\omega}\right] - \left[\frac{1}{dt}\frac{dc_t}{c_t}\frac{c_t\partial_c\nu(c_t)}{\nu(c_t)}\right].$$

Substituting and simplifying:

$$\begin{aligned}(\lambda+\eta)\mu(c_t) \quad &= \quad \lambda - \left(x_t^{1-(1-\rho)\omega}L_{E,t}c_t - \delta\right)\mu(c_t) + \frac{c_t\partial_c\nu(c_t)}{\nu(c_t)}\left(\omega\xi L_{E,t} + \delta - x_t^{1-(1-\rho)\omega}L_{E,t}c_t\right)\mu(c_t) \\ &+ \quad \left(\omega\xi L_{E,t} + \delta - x_t^{1-(1-\rho)\omega}L_{E,t}c_t\right)c_t\partial_c\mu(c_t)\end{aligned}$$

Multiplying by $\nu(c_t)$ we rewrite this as:

$$\begin{aligned}(\lambda+\eta)\mu(c_t)\nu(c_t) \quad &= \quad \nu(c_t)\lambda - \left(x_t^{1-(1-\rho)\omega}L_{E,t}c_t - \delta\right)\mu(c_t)\nu(c_t) \\ &+ \quad \left(\omega\xi L_{E,t} + \delta - x_t^{1-(1-\rho)\omega}L_{E,t}c_t\right)c_t\partial_c\nu(c_t)\mu(c_t) \\ &+ \quad \left(\omega\xi L_{E,t} + \delta - x_t^{1-(1-\rho)\omega}L_{E,t}c_t\right)c_t\partial_c\mu(c_t)\nu(c_t)\end{aligned}$$

Recall that the ODE for $\nu(.)$ is:

$$(\delta+\eta)\nu(c_t) \quad = \quad 1 - \left(x_t^{1-(1-\rho)\omega}L_{E,t}c_t - \delta\right)\nu(c_t) + \left(\omega\xi L_{E,t} + \delta - x_t^{1-(1-\rho)\omega}L_{E,t}c_t\right)c_t\partial_c\nu(c_t)$$

Multiplying by $\mu(c_t)$ we have:

$$\begin{aligned}(\delta+\eta)\mu(c_t)\nu(c_t) &= \mu(c_t)-\left(x_t^{1-(1-\rho)\omega}L_{E,t}c_t-\delta\right)\nu(c_t)\mu(c_t)\\ &+ \left(\omega\xi L_{E,t}+\delta-x_t^{1-(1-\rho)\omega}L_{E,t}c_t\right)c_t\nu'(c_t)\mu(c_t)\end{aligned}$$

Taking the difference with the expression above we obtain:

$$(\lambda-\delta)\mu(c_t)\nu(c_t) = \lambda\nu(c_t)-\mu(c_t)+\left(\omega\xi L_{E,t}+\delta-x_t^{1-(1-\rho)\omega}L_{E,t}c_t\right)c_t\partial_c\nu(c_t)\nu(c_t)$$

Dividing through by $\nu(c_t)$:

$$(\lambda-\delta)\mu(c_t) = \lambda-\frac{\mu(c_t)}{\nu(c_t)}+\left(\omega\xi L_{E,t}+\delta-x_t^{1-(1-\rho)\omega}L_{E,t}c_t\right)c_t\partial_c\mu(c_t)$$

which we can rewrite as:

$$\left(\lambda-\delta+\frac{1}{\nu(c_t)}\right)\mu(c_t)=\lambda+\left(\omega\xi L_E(c_t)+\delta-x_t^{1-(1-\rho)\omega}L_E(c_t)c_t\right)c_t\partial_c\mu(c_t).$$

The other equilibrium equations are the free-entry condition, and the optimal choice of scope. The optimal choice of scope can be simply rewritten as:

$$x_t=\boldsymbol{x}(\mu(c_t)),$$

and the associated share of enterprise value accruing to outsiders as:

$$\gamma_t=\boldsymbol{\gamma}(\boldsymbol{x}(\mu(c_t)),\mu(c_t)).$$

Along with the total labor resource constraint, $L_{E,t}\leq 1$, we can rewrite the free-entry condition as:

$$L_E(c_t)=\min\left(1\,,\,\max\left(0\,,\,1-\Xi(c_t)^{-1}\right)\right)$$

where given $\mu(.)$, the function $\Xi(.)$ is defined as:

$$\Xi(c) \equiv \frac{1-\zeta\chi}{\zeta\chi}\boldsymbol{x}(\mu(c))^{1-(1-\rho)\omega}\left(1-\boldsymbol{\gamma}(\boldsymbol{x}(\mu(c)),\mu(c))\right)c\nu(c)$$

Thus we can summarize the global solution to the model with the following set of equations:

$$\left(\eta+x(\mu(c_t))^{1-(1-\rho)\omega}L_E(c_t)c_t\right)\nu(c_t) = 1+\left(\omega\xi L_E(c_t)+\delta-\boldsymbol{x}(\mu(c_t))^{1-(1-\rho)\omega}L_E(c_t)c_t\right)c_t\nu'(c_t)$$

$$\left(\lambda-\delta+\frac{1}{\nu(c_t)}\right)\mu(c_t) \;=\; \lambda+\left(\omega\xi L_E(c_t)+\delta-\boldsymbol{x}(\mu(c_t))^{1-(1-\rho)\omega}L_E(c_t)c_t\right)c_t\mu'(c_t)$$

$$L_E(c_t) \;=\; \min\left(1\,,\,\max\left(0\,,\,1-\Xi(c_t)^{-1}\right)\right)$$

This is a system of differential equations in the three unknown functions $(\nu(.),\mu(.),L_E(.))$, which must be satisfied by $(\nu_t,\mu_t,L_{E,t})$ in any equilibrium. This verifies our guess that c_t is a sufficient state. ■

A.1.4.3 Other key model variables

Here, we define and derive the dynamics of several other key model variables.

Quantity, price, and user cost cost of intangible capital The total stock of intangible capital of active firms is given by:

$$\begin{aligned} K_t &\equiv \int_0^{J_t} e^{-\delta(t-\tau(j))} n_{\tau(j)} dj \\ &= \int_{s\le t} e^{-\delta(t-s)} L_{e,s} n_s ds. \end{aligned}$$

Consistent with measurement in national accounts, this is the capitalized value of past flow expenditures on intangible capital, which are given by $L_{e,t}n_t$. Since $n_t=\xi B_t$, we have:

$$\begin{aligned} \frac{dK_t}{K_t} &= \left(L_{E,t}\frac{\xi B_t}{K_t}-\delta\right)dt \\ &= \left(L_{E,t}c_t^{\frac{1}{\omega}}z_t^{\frac{1}{1-\zeta}}-\delta\right)dt, \end{aligned}$$

where we have defined:

$$z_t \equiv \left(\frac{N_t}{K_t}\right)^{1-\zeta}. \tag{A71}$$

This variable follows:

$$\frac{dz_t}{z_t} = (1-\zeta)\left(\frac{dN_t}{N_t}-\frac{dK_t}{K_t}\right) = -\frac{1-\zeta}{\omega}\frac{dc_t}{c_t}+(1-\zeta)\left(\xi L_{E,t}+\delta-L_{E,t}c_t^{\frac{1}{\omega}}z_t^{\frac{1}{1-\zeta}}\right)dt \tag{A72}$$

Having characterized the dynamics of $\left\{c_t, L_{E,t}, x_t\right\}$ is thus sufficient to characterize the dynamics of z_t, and therefore of K_t.

In the model, the correct measure of the price of capital is its replacement cost in output units, which is given by:

$$p_{K,t} = \frac{W_t}{n_t} = \frac{W_t}{\xi B_t}.$$

Combining the labor market clearing condition with the aggregate production function gives:

$$W_t = \chi\zeta \left(\frac{N_t}{1 - L_{E,t}}\right)^{1-\zeta} \tag{A73}$$

Combining this with the definition of c_t, we obtain that the price of capital is given by:

$$p_{K,t} = \chi\zeta c_t^{-\frac{1}{\omega}} N_t^{-\zeta} \left(1 - L_{E,t}\right)^{-(1-\zeta)}.$$

Thus, having characterized the dynamics of $\left\{c_t, L_{E,t}, N_t\right\}$ is sufficient to compute the price of capital at any point in time.

Finally, the definition of the user cost, or competitive cost of capital, in the model, is:

$$R_{K,t} \equiv r_t + \delta - \frac{1}{dt}\frac{dp_{K,t}}{p_{K,t}} = \eta + \delta + \frac{1}{dt}\frac{dY_t}{Y_t} - \frac{1}{dt}\frac{dp_{K,t}}{p_{K,t}}. \tag{A74}$$

Thus the user cost can be computed from knowing the dynamics of output (which depend only upon the dynamics of $L_{E,t}$ and N_t), and the dynamics of $p_{K,t}$ (which additionally depend on the sufficient state c_t). With these definitions, the total flow competitive cost of intangible capital per unit of value added are then given by:

$$C_{K,t} \equiv \frac{R_{K,t} p_{K,t} K_t}{Y_t} = R_{K,t}\ z_t^{-\frac{1}{1-\zeta}} c_t^{-\frac{1}{\omega}} \frac{\zeta\chi}{1 - L_{E,t}}. \tag{A75}$$

Income distribution Labor income is always:

$$W_t L_t = \zeta\chi Y_t.$$

The remainder of value added, $(1 - \zeta\chi)Y_t$, is allocated to either competitive costs of capital, or pure rents. Pure rents are defined as the residual capital income after accounting for the competitive cost of capital:

$$Re_t = \left(1 - \zeta\chi - C_{K,t}\right) Y_t.$$

Tobin's Q Let V_t denote the enterprise value of all firms currently operating. It can be written as:

$$\begin{aligned} V_t &= A_t \nu_t N_t^{\omega} \\ &= \frac{1-\zeta\chi}{\zeta\chi}(W_t L_{Y,t})\nu_t \\ &= (1-\zeta\chi) Y_t \nu_t, \end{aligned}$$

where we used the fact that:

$$A_t = \frac{1-\zeta\chi}{\zeta\chi} W_t L_{Y,t} N_t^{-\omega} = (1-\zeta\chi) Y_t N_t^{-\omega}.$$

The free-entry condition can be written as:

$$\begin{aligned} W_t &= \frac{1-\zeta\chi}{\zeta\chi}(1-\gamma_t) x_t^{1-(1-\rho_t)\omega} \nu_t (W_t L_{Y,t}) c_t \\ &= (1-\zeta\chi)(1-\gamma_t) x_t^{1-(1-\rho_t)\omega} c_t \nu_t Y_t \\ &= (1-\gamma_t) x_t^{1-(1-\rho_t)\omega} c_t \; V_t. \end{aligned}$$

Thus:

$$\frac{V_t}{W_t} = \frac{1}{1-\gamma_t} \frac{1}{x_t^{1-(1-\rho_t)\omega} c_t}$$

Tobin's Q is therefore given by:

$$\begin{aligned} Q_t &\equiv \frac{V_t}{p_{K_N,t} K_{N,t}} \\ &= \frac{V_t}{(W_t/(\xi B_t)) K_{N,t}} \\ &= \frac{V_t}{W_t} \frac{\xi B_t}{K_{N,t}} \\ &= \frac{V_t}{W_t} \frac{\xi B_t}{N_t} \frac{N_t}{K_{N,t}} \\ &= \frac{V_t}{W_t} c_t^{\frac{1}{\omega}} z_t^{\frac{1}{1-\zeta}} \end{aligned}$$

$$= \frac{1}{1-\gamma_t} \frac{c_t^{\frac{1}{\omega}} z_t^{\frac{1}{1-\zeta}}}{x_t^{1-(1-\rho_t)\omega} c_t}$$

$$Q_t = \frac{1}{1-\gamma_t} \frac{c_t^{\frac{1}{\omega}-1} z_t^{\frac{1}{1-\zeta}}}{x_t^{1-(1-\rho_t)\omega}}.$$

In the expression above, the first term represents the fact that outsiders or key stakeholders earn pure rents as a result of the incentive problem which their ability to copy the firm's core knowledge assets creates. The second term, which may be larger than 1 even when there are no pure rents to outsiders (that is, when $\gamma_t = 0$), represents the present value of the rents that entrepreneurs earn as a result of markups.

Measured TFP In the model, given the quantity of capital K_t, measured TFP is:

$$\widehat{TFP}_t = \frac{Y_t}{K_t^{1-\zeta} L_{Y,t}^{\zeta}} = \left(\frac{N_t}{K_t}\right)^{1-\zeta} = z_t,$$

where z_t was defined in Equation (A71), and its law of motion was characterized in Equation (A72).

Concentration Sales of each variety are given by:

$$s_t(m) = Y_t^{\frac{1-\chi}{1-\zeta\chi}} W_t^{-\frac{\zeta\chi}{1-\zeta\chi}} (\zeta\chi)^{\frac{\zeta\chi}{1-\zeta\chi}} n_t(m)^{\omega},$$

so that total sales for firm j are:

$$s_{j,t} = Y_t^{\frac{1-\chi}{1-\zeta\chi}} W_t^{-\frac{\zeta\chi}{1-\zeta\chi}} (\zeta\chi)^{\frac{\zeta\chi}{1-\zeta\chi}} x_{\tau(j)}^{1-(1-\rho_{\tau(j)})\omega} n_{\tau(j)}^{\omega}.$$

Aggregate sales are simply:

$$S_t = Y_t.$$

Thus the sales share of firm j is:

$$h_{i,t} = Y_t^{-\omega} W_t^{-\frac{\zeta\chi}{1-\zeta\chi}} (\zeta\chi)^{\frac{\zeta\chi}{1-\zeta\chi}} x_{\tau(j)}^{1-(1-\rho_{\tau(j)})\omega} n_{\tau(j)}^{\omega}$$

$$= x_{\tau(j)}^{1-(1-\rho_{\tau(j)})\omega} \left(\frac{n_{\tau(j)}}{N_t}\right)^{\omega}$$

$$
\begin{aligned}
&= x_{\tau(j)}^{1-(1-\rho_{\tau(j)})\omega} \left(\frac{\xi B_t}{N_t}\right)^{\omega} \left(\frac{\xi N_{\tau(j)}^{(e)}}{\xi N_t^{(e)}}\right)^{\omega} \\
&= c_t \left(\frac{\xi N_{\tau(j)}^{(e)}}{\xi N_t^{(e)}}\right)^{\omega} x_{\tau(j)}^{1-(1-\rho_{\tau(j)})\omega}.
\end{aligned}
$$

The Herfindhal index of sales can then be written as:

$$
\begin{aligned}
H_t &\equiv \int_{j=0}^{J_t} h_{i,t}^2 dj \\
&= c_t^2 \int_{j=0}^{J_t} \left(\frac{\xi N_{\tau(j)}^{(e)}}{\xi N_t^{(e)}}\right)^{2\omega} x_{\tau(j)}^{2(1-(1-\rho_{\tau(j)})\omega)} dj \\
&= c_t^2 (\xi N_t^{(e)})^{-2\omega} \underbrace{\int_{s\leq t} e^{-\delta(t-s)} x_s^{2(1-(1-\rho_s)\omega)} L_{E,s} \left(\xi N_s^{(e)}\right)^{2\omega} dj}_{\equiv G_t}.
\end{aligned}
$$

G_t evolves as:

$$
\begin{aligned}
dG_t &= \left(x_t^{2(1-(1-\rho_s)\omega)} L_{E,t} \left(\xi N_t^{(e)}\right)^{2\omega} - \delta G_t\right) dt \\
&= \left(x_t^{2(1-(1-\rho_s)\omega)} L_{E,t} u_t - \delta\right) G_t dt
\end{aligned}
$$

where we defined:

$$
u_t \equiv \frac{\left(\xi N_t^{(e)}\right)^{2\omega}}{G_t}. \tag{A76}
$$

u_t evolves as:

$$
\begin{aligned}
\frac{du_t}{u_t} &= 2\omega\xi L_{E,t} dt - \frac{dG_t}{G_t} \\
&= 2\omega\xi L_{E,t} dt - \left(x_t^{2(1-(1-\rho)\omega)} L_{E,t} v_t - \delta\right) dt \\
&= \left((2\omega\xi - u_t x_t^{2(1-(1-\rho)\omega)}) L_{E,t} + \delta\right) dt
\end{aligned}
$$

Finally, we can rewrite H_t as:

$$H_t = \frac{c_t^2}{u_t}.$$

Thus having characterized the dynamics of c_t and u_t is sufficient to construct the evolution of C_t.

Other variables Define the average share of outsiders as:

$$\bar{\gamma}_t = J_t^{-1} \underbrace{\int_{s \geq t} e^{-\delta(t-s)} L_{E,s} \gamma_s ds}_{\equiv \Omega_t}$$

Here J_t is the number of firms, which evolves as:

$$dJ_t = (L_{E,t} - \delta J_t)dt.$$

Then Ω_t evolves as:

$$\frac{d\Omega_t}{\Omega_t} = \left(L_{E,t} \frac{\gamma_t}{\Omega_t} - \delta \right) dt$$

In steady-state,

$$\bar{\gamma}_t = \gamma_t = \gamma \quad \Longrightarrow \quad \Omega = J\gamma = \frac{L_E}{\delta}\gamma.$$

We can then use the equation above to compute the evolution of Ω_t, from which $\bar{\gamma}_t$ can then be derived.

A.2 Balanced growth analysis

This Appendix contains details on results of Section 3. We focus again on the microfoundation described in described in Section (2.2.2). As in Appendix A.1.4.2, we will use the two functions:

$$\boldsymbol{\gamma}(x, \mu) = \theta(x)\, \bar{x}\, \frac{\psi}{x}\, \mu \tag{A77}$$

$$\boldsymbol{x}(\mu) = \arg\max_x \; (1 - \boldsymbol{\gamma}(x, \mu)) x^{1-(1-\rho)\omega} \tag{A78}$$

where recall that:

$$\theta(x) = \begin{cases} 0 & \text{if} \quad x \leq \bar{x} \\ \alpha(x - \bar{x})^{\kappa} & \text{if} \quad x \geq \bar{x} \end{cases} \tag{A79}$$

A.2.1 Existence and unicity of balanced growth path

Proof of result 2. Using Result A.1.4.2 from Appendix A.1, a balanced growth path of the economy can be characterized by setting the drift terms in the differential equations characterizing the global solution of the model to zero. This yields:

$$
\begin{aligned}
c &= \boldsymbol{x}(\mu)^{-(1-(1-\rho)\omega)}\left(\omega\xi + \frac{\delta}{L_E}\right) \\
\nu &= \frac{1}{\eta + \delta + \xi\omega L_E} \\
\mu &= \frac{\lambda\nu}{1 + (\lambda - \delta)\nu} \\
L_E &= \min\left(1\,,\ \max\left(0\,,\ 1 - \Xi(c)^{-1}\right)\right),
\end{aligned}
$$

implying in particular that if balanced growth path exists it must feature constant values for ν, μ (and therefore x), and L_E. Thus we need to prove that this system of equations always has a unique solution. [Proof to be finished.] ■

Heuristic derivation The equations derived above and that characterize the balanced growth path are not particularly transparent. So here we also provide a more heuristic derivation of these conditions.

On a balanced growth path, the required return on capital must be constant:

$$r_t = r = g + \eta. \tag{A80}$$

Moreover, from the aggregate labor demand condition, Equation (19), we see that the wage rate must grow at rate g. From the aggregate production function (19), we see that the aggregate intangible capital index N_t must grow at rate:

$$g_N = \frac{1}{1-\zeta} g.$$

This, along with Equation (A53), implies that A_t grows at rate:

$$g_A = g - \frac{(1-\zeta)\chi}{1-\zeta\chi}\frac{g}{1-\zeta} = \frac{1-\zeta\chi-\chi}{1-\zeta\chi} g.$$

Given that A_t is growing at a constant rate and $r = g + \eta$, we then have:

$$\nu_t = \nu = \frac{1}{r + \delta - \frac{1-\zeta\chi-\chi}{1-\zeta\chi} g} = \frac{1}{\eta + \delta + \frac{\chi}{1-\zeta\chi} g}. \tag{A81}$$

Consider the case of a balanced growth path with strictly positive entry: $L_{E,t} = L_E = 1 - L_Y > 0$. Then it must be that:

$$\frac{v_t(\xi B_t)}{W_t} = 1,$$

and therefore, using Equation (23), it must be that:

$$c^{\frac{1}{\omega}} = \frac{\xi B_t}{N_t},$$

for some constant c to be determined. If the ratio of B_t to N_t is constant, then B_t must be growing at rate g_N. Given the law of motion for B_t this means that:

$$g_N = \xi L_E \quad \implies \quad g = (1-\zeta)\xi L_E \tag{A82}$$

(Note that this implies that the growth rate and the entry rate are proportional to each other.)

To arrive at the condition that pins down L_E, we need to solve for the constant c. This is were we "rely" on the balanced growth assumption more strongly. First, we write the aggregate intangible capital index as:

$$\begin{aligned}
N_t &= \left(\int_0^{M_t} n(m)^{\frac{(1-\zeta)\chi}{1-\zeta\chi}} dm\right)^{\frac{1-\zeta\chi}{(1-\zeta)\chi}} \\
&= \left(\int_0^{J_t} \left(\int_{m \text{ s.t. } \hat{\jmath}(m)=j} n(m)^{\frac{(1-\zeta)\chi}{1-\zeta\chi}} dm\right) dj\right)^{\frac{1-\zeta\chi}{(1-\zeta)\chi}} \\
&= \left(\int_0^{J_t} \left(\int_{m \text{ s.t. } \hat{\jmath}(m)=j} \left(x(j)^{-(1-\rho)} n_{\tau(j)}(j)\right)^{\frac{(1-\zeta)\chi}{1-\zeta\chi}} dm\right) dj\right)^{\frac{1-\zeta\chi}{(1-\zeta)\chi}} \\
&= \left(\int_0^{J_t} \left(x(j) \left(x(j)^{-(1-\rho)} n_{\tau(j)}(j)\right)^{\frac{(1-\zeta)\chi}{1-\zeta\chi}}\right) dj\right)^{\frac{1-\zeta\chi}{(1-\zeta)\chi}} \\
&= \left(x^{1-(1-\rho)\frac{(1-\zeta)\chi}{1-\zeta\chi}} \int_0^{J_t} n_{\tau(j)}(j)^{\frac{(1-\zeta)\chi}{1-\zeta\chi}} dj\right)^{\frac{1-\zeta\chi}{(1-\zeta)\chi}} \\
&= x^{\frac{1-\zeta\chi}{(1-\zeta)\chi}-(1-\rho)} \left(\int_0^{J_t} n_{\tau(j)}(j)^{\frac{(1-\zeta)\chi}{1-\zeta\chi}} dj\right)^{\frac{1-\zeta\chi}{(1-\zeta)\chi}}
\end{aligned}$$

where recall that $\tau(j)$ is the date of entry for firm j. We have:

$$n_{\tau(j)}(j) = n_{\tau(j)} = \xi B_{\tau(j)} = \xi e^{-\frac{g}{1-\zeta}(t-\tau(j))} B_t$$

Here, the last expression uses the fact that, along a balanced growth path, B_t must be growing at rate $g_N = \frac{g}{1-\zeta}$. Given that in a balanced growth path, the same number of firms (each with the same stock of intangibles) enter each period, we can change variables in the above expression for n_t:

$$N_t = x^{\frac{1-\zeta\chi}{(1-\zeta)\chi}-(1-\rho)} \left(\int_0^{+\infty} \left(\underbrace{e^{-\frac{g}{1-\zeta}s}\xi B_t}_{\substack{\text{Capital of firms} \\ \text{that entered at time } t-s}} \right)^{\frac{(1-\zeta)\chi}{1-\zeta\chi}} \underbrace{e^{-\delta s} L_E ds}_{\substack{\text{Mass of firms that entered at time } t-s \\ \text{and have not exited by time } t}} \right)^{\frac{1-\zeta\chi}{(1-\zeta)\chi}}$$

We can use this expression to relate N_t and B_t:

$$\begin{aligned} N_t &= x^{\frac{1-\zeta\chi}{(1-\zeta)\chi}-(1-\rho)} \xi B_t \left(L_E \int_0^{+\infty} e^{-\left(\frac{g}{1-\zeta}\frac{(1-\zeta)\chi}{1-\zeta\chi}+\delta\right)s} ds \right)^{\frac{1-\zeta\chi}{(1-\zeta)\chi}} \\ &= x^{\frac{1-\zeta\chi}{(1-\zeta)\chi}-(1-\rho)} \xi B_t \left(L_E^{-1} \left(\frac{\chi g}{1-\zeta\chi} + \delta \right) \right)^{-\frac{1-\zeta\chi}{(1-\zeta)\chi}} \\ &= x^{\frac{1-\zeta\chi}{(1-\zeta)\chi}-(1-\rho)} \xi B_t \left(L_E^{-1} \left(\frac{\chi(1-\zeta)\xi L_E}{1-\zeta\chi} + \delta \right) \right)^{-\frac{1-\zeta\chi}{(1-\zeta)\chi}} \\ &= x^{\frac{1-\zeta\chi}{(1-\zeta)\chi}-(1-\rho)} \xi B_t \left(\frac{(1-\zeta)\chi\xi}{1-\zeta\chi} + \frac{\delta}{L_E} \right)^{-\frac{1-\zeta\chi}{(1-\zeta)\chi}} \end{aligned}$$

(In particular, when $\chi = 1$ and $\delta = 0$, we have $N_t = x^\rho B_t$ on the balanced growth path.) Therefore,

$$c^{\frac{1}{\omega}} = \frac{\xi B_t}{N_t} = x^{(1-\rho)-\frac{1}{\omega}} \left(\omega\xi + \frac{\delta}{L_E} \right)^{\frac{1}{\omega}} \tag{A83}$$

Putting the pieces together, we can write the free-entry condition as:

$$1 = \frac{1-\zeta\chi}{\zeta\chi} \frac{1-L_E}{\delta+\eta+\omega\xi L_E} (1-\gamma(x(\mu),\mu)) \left(\omega\xi + \frac{\delta}{L_E} \right) \tag{A84}$$

Moreover, let g_A be the (constant) growth rate of A in balanced growth. Then we have that:

$$\nu_t = \nu = \frac{1}{r+\delta-g_A}.$$

Therefore,

$$\mu_t \;=\; \frac{\lambda}{r+\lambda-g_A} = \mu$$

Therefore,

$$\frac{\lambda\nu}{1+(\lambda-\delta)\nu} = \frac{\lambda}{r+\delta-g_A+\lambda-\delta} = \mu. \tag{A85}$$

When there is positive entry, equations (A81), (A82), (A83), (A84), and (A85) are the same as those obtained from the global solution to the model.

A.2.2 Expressions for key model variables along the BGP

Here we provide derivations for the analytical expressions of key model variables along the BGP.

Measured TFP In balanced growth:

$$c^{\frac{1}{\omega}} z^{\frac{1}{1-\zeta}} = \xi + \frac{\delta}{L_E} \quad \text{and} \quad x^{(1-(1-\rho)\omega)} c = \omega\xi + \frac{\delta}{L_E},$$

so that measured TFP is given by:

$$z = x^{(1-\zeta)\rho} \left(x^{\frac{1-\omega}{\omega}} \frac{\xi + \frac{\delta}{L_E}}{\left(\omega\xi + \frac{\delta}{L_E}\right)^{\frac{1}{\omega}}} \right)^{1-\zeta}$$

Note that without markups, in balanced growth we simply have:

$$z = x^{(1-\zeta)\rho}.$$

Otherwise, measured TFP reflects both markup-driven misallocation, and the fact that returns to scope drive a wedge between N_t (the aggregate intangible capital index) and K_t (the measured stock of intangibles).

Quantity, price, and user cost cost of intangible capital In balanced growth, the quantity of capital K_t is simply given by:

$$K_t = z^{-\frac{1}{1-\zeta}} N_t, \tag{A86}$$

where the BGP value of z was characterized above. The price of capital is growing at a constant rate, given by:

$$\frac{1}{dt}\frac{dp_{K,t}}{p_{K,t}} = -\frac{\zeta}{1-\zeta}g, \tag{A87}$$

where g is the trend growth rate of output. Therefore, the user cost of capital is:

$$R_K = \eta + \delta + g + \frac{\zeta}{1-\zeta}g = \eta + \delta + \frac{1}{1-\zeta}g. \tag{A88}$$

Finally, competitive capital costs, relative to value added, are given by:

$$C_K = \left(\eta + \delta + \frac{1}{1-\zeta}g\right) z^{-\frac{1}{1-\zeta}} c^{-\frac{1}{\omega}} \frac{\zeta\chi}{1-L_E} = (1-\zeta\chi)(1-\gamma)\frac{\omega\xi + \frac{\delta}{L_E}}{\xi + \frac{\delta}{L_E}}. \tag{A89}$$

Income distribution Labor income is always:

$$W_t L_t = \zeta\chi Y_t.$$

With our definition of the user cost, competitive returns to capital in balanced growth are given by:

$$R_N p_{K_N,t} K_{N,t} = (1-\zeta\chi) Y_t (1-\gamma)\frac{\omega\xi + \frac{\delta}{L_E}}{\xi + \frac{\delta}{L_E}}.$$

This implies the following distribution of income on the balanced growth path:

$$\begin{aligned}
Y_t \; = \; & \zeta\chi Y_t && \text{(labor income)} \\
+ \; & (1-\zeta\chi) Y_t (1-\gamma)\frac{\omega\xi + \frac{\delta}{L_E}}{\xi + \frac{\delta}{L_E}} && \text{(competitive capital income)} \\
+ \; & (1-\zeta\chi) Y_t \left(\underbrace{\gamma}_{\text{to outsiders}} + \overbrace{(1-\gamma)(1-\omega)\frac{\xi}{\xi + \frac{\delta}{L_E}}}^{\text{to entrepreneurs}} \right) && \text{(rents/pure profits)}
\end{aligned}$$

With $\omega < 1$ competitive returns to capital are lower than they would be under perfect competition.

Tobin's Q In a balanced growth path with no markups ($\omega = 1$):

$$Q = \frac{1}{1-\gamma}.$$

In a balanced growth path with markups $(\omega < 1)$,

$$Q = \frac{1}{1-\gamma} \frac{\xi + \frac{\delta}{L_E}}{\omega\xi + \frac{\delta}{L_E}} > \frac{1}{1-\gamma}$$

where the second term captures rents due to markups (and would be there even in the case of no competition from outsiders i.e. $\gamma \to 0$).

Concentration Finally, recall that concentration can be written as:

$$H_t = \frac{c_t^2}{u_t},$$

It will be constant in balanced growth, and so will u_t. Cancelling out the drift in the derivation of the law of motion for u_t gives:

$$ux^{2(1-(1-\rho)\omega)} = 2\omega\xi + \frac{\delta}{L_E},$$

implying that concentration will be:

$$\begin{aligned} H &= \frac{c^2 x^{2(1-(1-\rho)\omega)}}{2\omega\xi + \frac{\delta}{L_E}} \\ &= \frac{\left(\omega\xi + \frac{\delta}{L_E}\right)^2}{2\omega\xi + \frac{\delta}{L_E}}. \end{aligned}$$

A.3 Numerical solution method

Outside of balanced growth, we solve the model globally, using finite difference methods. This appendix describes our method, which consists of approximating a solution to the system of differential equations described in Result (A.1.4.2). We also describe how the transitional dynamics of the economy are then constructed.

We assume that code has been constructed to evaluate numerically the two functions:

$$\boldsymbol{x}(\mu) = \arg\max_x \; (1 - \boldsymbol{\gamma}(x,\mu))x^{1-(1-\rho)\omega}$$

$$\boldsymbol{\gamma}(x,\mu) \quad = \quad \theta(x)\,\bar{x}\,\frac{\psi}{x}\,\mu$$

for arbitrary values of $\bar{x}$ and $\mu \in [0,1]$. To make notation lighter, for a particular value of the delay term $\mu(c_t,t)$, we define:

$$\tilde{x}(c_t,t) \quad = \quad \boldsymbol{x}\left(\mu(c_t,t)\right) \tag{A90}$$

$$\tilde{\gamma}(c_t,t) \quad = \quad \boldsymbol{\gamma}\left(\tilde{x}(c_t,t),\mu(c_t,t)\right) \tag{A91}$$

Given a value for $\mu(c_t,t)$, one can compute directly $\tilde{x}(c_t,t)$ and $\tilde{\gamma}(c_t,t)$. For computation, we rewrite the three equilibrium conditions including time drifts, as follows:

$$\begin{aligned} 0 \quad = \quad & \left(\eta + \tilde{x}(c_t,t)^{1-(1-\rho)\omega} L_E(c_t,t)c_t\right)\nu(c_t,t) \\ & -1 - \partial_t \nu(c_t,t) \\ & -\left(\omega\xi L_E(c_t,t) + \delta - \tilde{x}(c_t,t)^{1-(1-\rho)\omega} L_E(c_t,t)c_t\right) c_t \partial_c \nu(c_t,t) \end{aligned} \tag{A92}$$

$$\begin{aligned} 0 \quad = \quad & \left(\lambda - \delta + \frac{1}{\nu(c_t,t)}\right)\mu(c_t,t) \\ & -\lambda - \partial_t \mu(c_t,t) \\ & -\left(\omega\xi L_E(c_t,t) + \delta - \tilde{x}(c_t,t)^{1-(1-\rho)\omega} L_E(c_t,t)c_t\right) c_t \partial_c \mu(c_t,t) \end{aligned} \tag{A93}$$

$$L_E(c,t) \quad = \quad \min\left(1\,,\,\max\left(0\,,\,1 - \Xi(c_t,t)^{-1}\right)\right) \tag{A94}$$

where, as before, the function $\Xi(c_t,t)$ is defined as:

$$\Xi(c_t,t) = \frac{1-\zeta\chi}{\zeta\chi}\tilde{x}(c_t,t)^{1-(1-\rho)\omega}\left(1-\tilde{\gamma}(c_t,t)\right)c_t\nu(c_t,t).$$

We added time drifts to the two differential equations because we will solve the system backward in time, as explained below.

We will compute discrete approximations of the functions $(\nu,\mu,\tilde{x},\tilde{\gamma},L_E)$ at $N_c \times N_t$ points of the state-space. Denote these points by (c_i,t_j), where $1 \leq i \leq N_c$ and $1 \leq j \leq N_t$. The points in the c and t dimension will be equally spaced by Δ_c and Δ_t, respectively. We will denote $\nu_{i,j} = \nu(c_i,t_j)$ and $\nu_{\cdot,j}$ for the $N_c \times 1$ column vector $(\nu_{i,j})_{i=1}^{N_c}$ representing the step-j iterate for ν (the notation will be similar for other functions). We will obtain the stationary solution by starting from arbitrary values for $(\nu_{\cdot,N_t},\mu_{\cdot,N_t},\tilde{x}_{\cdot,N_t},\tilde{\gamma}_{\cdot,N_t},L_{E\cdot,N_t})$, and then solving backward iteratively in time. Specifically, at each time step j, we make the following three updates successively:

1. Update $\nu_{\cdot,j}$ using the step-$j+1$ value and policy functions;
2. Update $\mu_{\cdot,j}$ using the step-$j+1$ value and policy functions;
3. Update $\tilde{x}_{\cdot,j}$ and $\tilde{\gamma}_{\cdot,j}$ using $\mu_{\cdot,j}$ and Equations (A90)-(A91);
4. Update $L_{E,\cdot,j}$ using $(\nu_{\cdot,j}, \tilde{x}_{\cdot,j}, \tilde{\gamma}_{\cdot,j})$ and Equation (A94).

Step 1: Computation of $\nu_{\cdot,j}$ We define the forward and backward difference approximations of $\partial_c \nu$ as follows:

$$\partial_c^{(b)} \nu_{i,j} := \frac{\nu_{i,j} - \nu_{i-1,j}}{\Delta_c}, \qquad \partial_c^{(f)} \nu_{i,j} := \frac{\nu_{i+1,j} - \nu_{i,j}}{\Delta_c}$$

The finite difference approximation of $\partial_c \nu(c,t)$ with an upwinding scheme is then:

$$\partial_c \nu(x_i, t_j) := \mathbb{I}_{\{g_{c,i,j+1} < 0\}} \partial_c^{(b)} \nu_{i,j} + \mathbb{I}_{\{g_{c,i,j+1} \geq 0\}} \partial_c^{(f)} \nu_{i,j}$$

Here $g_{c,i,j+1}$ is the drift rate of c_t at (c_i, t_{j+1}):

$$g_{c,i,j+1} = \left(\omega \xi L_{E,i,j+1} + \delta - (\tilde{x}_{i,j+1})^{1-(1-\rho)\omega} L_{E,i,j+1} c_i\right) c_i. \tag{A95}$$

For given $(f_{i,j+1}, L_{E,i,j+1}, \tilde{x}_{i,j+1})_{i=1}^{N_c}$, the discretization of Equation (A92) leads to the following system of N_c equations, here expressed in matrix form:

$$\left(\eta I_{N_c \times N_c} + diag_{N_c \times N_c}(s_{\cdot,j+1})\right) \nu_{\cdot,j} - \frac{\nu_{\cdot,j+1} - \nu_{\cdot,j}}{\Delta_t} - \Pi^{(\nu)}_{\cdot,j+1} - \mathcal{L}_{j+1} \nu_{\cdot,j} = 0. \tag{A96}$$

Here:

$$\begin{aligned} s^{(\nu)}_{\cdot,j+1} &= (\tilde{x}_{\cdot,j+1})^{1-(1-\rho)\omega} \odot L_{E,\cdot,j+1} \odot c, \\ \Pi^{(\nu)}_{\cdot,j+1} &= \mathbf{1}_{N_c \times 1} \end{aligned}$$

with c denoting the $N_c \times 1$ vector of discretized values of for c_t, $\odot$ denoting the element-wise product of two equal-size vectors or matrices, the power function is element-wise, and $\mathbf{1}_{N_c \times 1}$ is a vector of ones. Additionally, $\mathcal{L}_{j+1}$ is the linearized Feynam-Kac operator, an $N_c \times N_c$ matrix. It is a tri-diagonal matrix encoding the dynamic evolution of c_t. Its diagonal elements ($\mathcal{L}_{i,i,j+1}$) and off-diagonal terms

($\mathcal{L}_{i,i-1,j+1}$ and $\mathcal{L}_{i,i+1,j+1}$) are:

$$\mathcal{L}_{i,i,j+1} = -\frac{|g_{c,i,j+1}|}{\Delta_c}$$
$$\mathcal{L}_{i,i-1,j+1} = -\frac{\min(0, g_{c,i,j+1})}{\Delta_c}$$
$$\mathcal{L}_{i,i+1,j+1} = \frac{\max(0, g_{c,i,j+1})}{\Delta_c}$$

We can further simplify Equation (A96) as:

$$B^{(\nu)}_{j+1}\nu_{\cdot,j} = z^{(\mu)}_{j+1},$$

with

$$\begin{aligned} B^{(\nu)}_{j+1} &= \left(\left(\eta + \frac{1}{\Delta_t}\right) I_{N_c \times N_c} + diag_{N_c\times N_c}(s^{(\nu)}_{\cdot,j+1})\right) - L_{j+1} \\ z^{(\nu)}_{j+1} &= \frac{1}{\Delta_t}\nu_{\cdot,j+1} + \Pi^{(\nu)}_{\cdot,j+1}. \end{aligned}$$

In constructing $\mathcal{L}$ at each step, we also impose the boundary conditions:

$$\begin{aligned} \mathcal{L}_{1,1,j+1} &= -\frac{\min(0, g_{c,1,j+1})}{\Delta_c} - \frac{|g_{c,1,j+1}|}{\Delta_c} = -\mathcal{L}_{1,2,j+1} \\ \mathcal{L}_{N_c,N_c,j+1} &= = \frac{\max(0, g_{c,N_c,j+1})}{\Delta_c} - \frac{|g_{c,N_c,j+1}|}{\Delta_c} = -\mathcal{L}_{N_c,N_c-1,j+1} \end{aligned}$$

If there were "shadow" knots at $i = 0$ and $i = N_c + 1$, respectively, these conditions would impose that the stationary solution satisfies:

$$\nu_0 = \nu_1 \quad \text{and} \quad \nu_{N_c+1} = \nu_{N_c},$$

so that:

$$\partial^{(b)}_c \nu_1 = 0 \quad \text{and} \quad \partial^{(\nu)}_c \nu_{N_c} = 0.$$

Thus this is similar to assuming that the two following boundary conditions hold:

$$\partial_c \nu(\underline{c}) = 0 \quad \text{and} \quad \partial_c \nu(\overline{c}) = 0.$$

The normalizations above also guarantee that each line of $\mathcal{L}_{.,j+1}$ add up to 0, ensuring that it represents a Markov intensity matrix. After constructing these matrices, the updated function ν_j will be:

$$\nu_{.,j} = \left(B^{(\nu)}_{j+1}\right)^{-1} z^{(\nu)}_{j+1}.$$

Step 2: Computation of $\mu_{.,j}$ The discretization of Equation (A93) leads to the following system of N_c equations, here expressed in matrix form:

$$\left((\lambda-\delta)I_{N_c\times N_c} + diag_{N_c\times N_c}(s^{(\mu)}_{.,j+1})\right)\mu_{.,j} - \frac{\mu_{.,j+1}-\mu_{i,j}}{\Delta_t} - \Pi^{(\mu)}_{.,j+1} - \mathcal{L}_{j+1}\mu_{.,j} = 0. \quad \text{(A97)}$$

Here:

$$\begin{aligned} s^{(\mu)}_{.,j+1} &= 1./f(.,j+1), \\ \Pi^{(\mu)}_{.,j+1} &= \lambda \mathbf{1}_{N_c\times 1} \end{aligned}$$

Here all the notation is as before, except for ./ which we use to represent the element-wise inverse of a vector. We can further simplify Equation (A97) as:

$$B^{(\mu)}_{j+1}\mu_{.,j} = z^{(\mu)}_{j+1},$$

with

$$\begin{aligned} B^{(\mu)}_{j+1} &= \left(\left(\lambda-\delta+\frac{1}{\Delta_t}\right)I_{N_c\times N_c} + diag_{N_c\times N_c}(s^{(\mu)}_{.,j+1})\right) - L_{j+1} \\ z^{(\mu)}_{j+1} &= \frac{1}{\Delta_t}\mu_{.,j+1} + \Pi^{(\mu)}_{.,j+1}. \end{aligned}$$

Then the updated function $\mu_{.,j}$ will be:

$$\mu_{,.j} = \left(B^{(\mu)}_{j+1}\right)^{-1} z^{(\mu)}_{j+1}.$$

We don't discuss step 3, since it is immediate if numerical evaluation of $\boldsymbol{x}$ and $\boldsymbol{\gamma}$ is feasible.

Step 4: Update of $L_{E,.,j}$ We update $L_{E,.,j}$ using:

$$L_{E,.,j} = \min\left(\, 1\ ,\ \max\left(\, 0\ ,\ 1 - 1./\Xi_{.,j}\right)\,\right)$$

$$\Xi_{.,j} \equiv \frac{1-\zeta\chi}{\zeta\chi} \tilde{x}_{.,j} \odot (1-\tilde{\gamma}_{.,j}) \odot \Big(1./\left(c \odot f_{.,j}\right)\Big)$$

where the min and max operators are element-wise.

Initialization and bounds on state-space We initialize all vectors $(\nu_{.,N_t}, \mu_{.,N_t}, \tilde{x}_{.,N_t}, \tilde{\gamma}_{.,N_t}, L_{E.,N_t})$ to constants (i.e., the entries in each vector are the same), where the entry in each vector corresponds to the value of that variable in the balanced growth path. Additionally, for the bounds on the state space, we use the steady-state values of c for $\rho = 0$ and $\rho = 1$, respectively.

Transitional dynamics In Section 5.3.2, we compute the transitional dynamics of the economy between the two balanced growth paths described in Table 3. In order to do this, we proceed as follows. First, we obtain the global solution for the two calibrations of the model corresponding to the two balanced growth paths described in Table 3. Second, we discretize the time dimension of the model to a monthly frequency, starting with 1990m1 as our initial month. We assume that in 1990m1, the economy is in the first BGP, and we use the policy functions from that BGP to construct all variables of interest from 1990m1 to 1994m12. We then assume that in 1995m1, the value of ρ jumps, from $\rho = 0.18$ to $\rho = 0.34$. We then use the law of motion characterizing the evolution of c_t, the sufficient state, and derived in Appendix (A.1.4.2), and the global solution under this new value of ρ, to construct the joint evolution of $(x_t, L_{E,t}, \mu_t, \nu_t, c_t)$ along the transition path. The rest of the quantities of interest can then be derived from these four variables.

A.4 Data sources and calibration

A.4.1 Data sources

This section contains information on the data sources, data construction, and sample selection for the different data sources used in the empirical exercise of Section 5.

A.4.1.1 Compustat sample

Our main sample is drawn from the Compustat-CRSP merged database. We keep only firms incorporated in the USA (`fic`="USA"); firms with non-missing two-digit SIC code; firms with two-digit SIC codes not equal to 49 and outside of the $60-69$ and $91-99$ range (utilities, financials, and multinationals, respectively). We also drop American Depository Receipts (ADRs). Finally, we drop any firm-year observations with assets (`at`) and sales (`sale`) beow $\$1$m, weakly negative gross property plant and equipment (ppget), and strictly negative capital expenditures (capx), cash (che), debt due in one year or less (dlc), and debt due in more than one year (dltt).

In order to add a firm-level control for the stock of intangible capital, we merge in the data from Peters and Taylor (2017), and used the variable k_int_know, which represents a firm-level capitalized value of past R&D expenditures, as our proxy for the stock of intangible capital of a given firm. We use one plus the log of that stock as an additional control in Specification 41, in order to include in our sample firms for which this estimate of the stock is unavailable; this effectively normalizes the stock of intangible capital to a common industry-year constant for these firm-year observations. Prior to merging with scope data, the 1988-2021 panel we construct contains 173827 observations for 16136 distinct firms.

A.4.1.2 Data on firm scope

The data on firm scope is taken from Hoberg and Phillips (2024); here we provide a brief overview of the methodology underlying the measurement of scope in that paper. The measure is based on textual analysis of 10K data. Per Regulation S-K, firms must report, in their 10Ks, a description of their business. This description includes, among other information, a description of "revenue-generating activities, products and/or services, and any dependence on revenue-generating activities, key products, services, product families or customers, including governmental customers."[34]

Information contained in the 10K data is processed using the doc2vec algorithm (Mikolov, Chen, Corrado, and Dean, 2013). This algorithm first maps each business description to a 300-dimensional vector, with each dimension representing clusters of words that are semantically related. (The model is trained using data from 1997 10-K filings.) Using k-means clustering on the subsample of single-segment firms, vectorized business descriptions are translated into 450 different candidate "industries", each of which is characterized by a specific "dialect", or set of central words used to describe it. This set of 450 industries is then further refined, eliminating industries that are redundant or represent clusters of boilerplate text, to arrive at a final set of 300 "industries". This set of industries represents a rotation of the initial clusters of semantically related components from business descriptions, and provides a text-based alternative to the existing segments data from Compustat. Appendix 3 of Hoberg and Phillips (2024) provides more detail on these classification steps.

Having constructed the 300 industries and their corresponding dialects, Hoberg and Phillips (2024) then score each individual firm-year 10K business description in terms of its similarity to the each of the 300 industry-specific dialects. The scoring is *not* done using cosine similarity, out of the concern that firms operating across many different industries might be penalized because their descriptions of each

[34] See 17 CFR paragraph 229.101 - (Item 101) Description of business, for instance at law.cornell.edu\/cfr\/text\/17\/229.101.

industry would be short (and thus have a low cosine similarity with each industry dialect). Instead, for the pooled set of all words appearing across business descriptions, Hoberg and Phillips (2024) construct an importance weight for each word relative to each industry's dialect. They then compute, for each year/firm by industry pair, a weighted average fraction of the words from the industry's dialect that appear in the year/firm's 10K description, where the weights are proportional to the importance of that word in the industry's dialect. This "importance weighting" is meant to help capture cases where a firm's business description contains few words relating to a particular industry, but those words closely match core words that are highly specific to that industry.[35]

Finally, having obtained this importance-weighted measure of the similarity between each year/firm business description and each industry's dialect, the authors then classify a firm as being operating in that industry if its similarity score is in the top 2% of all similarity scores for that industry in that year. A firm's total scope is then the sum of the number of industries in which the firm is deemed to operate. It thus has an interpretation as a cardinal measure of the set of industries that a firm is deemed to be most likely to operate in within a given year.

A.4.1.3 Data on firm ownership

For collecting the data on firm ownership used to compute the fraction of equity shares held by the founding team, we proceed as follows. We first identify the sample of firms with a public offering between 1996 and 2016, along with their founding date, provided by Loughran and Ritter (2004).[36] Within this sample we then drop any cases where the IPO is related to a spin-off, merger, or leveraged buy-out. Finally, we only keep cases where the listing was on the NYSE or NASDAQ.

We then web-scrape IPO prospectuses from the EDGAR website. We specifically focus on form $S1$ and on the associated details regarding firm ownership reported in form 424B. The main item which we use, in either form, is the capital table reported under the header "Principal and selling stockholders". This capital table gives the number and fraction of total equity shares of different classes held by executive officers and directors (each of whom is individually named), as well as a list of 5% or greater shareholders. This capital table represents the distribution of equity ownership prior to the IPO.

From this table, our primary proxy for $1 - \gamma_t(x_t)$ is the sum of all the fraction of shares held by the first group of shareholders (executive officers and directors), plus the fraction of shares retained by the founder if they are named in the list of 5% or greater shareholders but are not part of the first

[35] The importance weights are defined for each word by industry pair. Within each industry's dialect, the authors first identify a core group of words that appear more frequently. The importance of each word by industry pair is then defined as the product of two terms. The first term reflects how specific to a particular industry the group of core words is (using the HHI of the corresponding group of words across industries), and doesn't depend the specific word in the word-by-industry pair. The second term reflects how close each word is to the core words in the industry.

[36] The data are available on Jay Ritter's website, `https://site.warrington.ufl.edu/ritter/ipo-data/`.

group. In order to identify the founders, we manually search the rest of the EDGAR filing for mentions of "founder" or "founders" and cross-check the names mentioned with the names in the capital table. Additionally, we note that we focus only on class A shares.

Finally, we also collect information on the identify of other shareholders pre-IPO. We do this in two steps. First, using the same capital tables described above, we identify, among the list of 5% or greater shareholders, any venture capital or private equity fund that holds 5% of equity shares as of the time of the S1 report. Second, we collect data on stock options held by key employees prior to IPO.

Add summary statistics on the IPOs we selected, as well as a time series for the number of IPOs per year and the average age of the firm at IPO Add robustness tests.

A.4.1.4 Other data

The data for the real growth rate of value added in the non-financial corporate sector is drawn from the National Income and Product Account (NIPA) Table 1.14; the corresponding FRED series is B455RX1Q027SBEA. The data for profits as $\%$ of value added come from the National Income and Product Account (NIPA) Table 1.15; the corresponding FRED series is A463RD3Q052SBEA. The ratio is defined as corporate profits with inventory valuation plus capital consumption adjustments, to total gross value added. Corporate profits are defined as follows: domestic corporate gross value added, minus compensation of employees, minus taxes on production and imports less subsidies, minus consumption of fixed capital (depreciation), minus net interest payments, minus net business current transfer payments. Conceptually, this measure is thus close to accounting measures of net income, except that it does not net out corporate income taxes (since we do not include these in our model).

The data for the gross business entry and exit rates are drawn from the Census' Business Dynamics Statistics (BDS) data.[37] In order to obtain a set of industries approximately consistent in scope with the rest of the data, we drop the two-digit NAICS sectors corresponding to agriculture, utilities, finance and insurance, real estate rental and leasing, and public administration (11, 22, 52, 53, and 92). We define the gross entry rate as $e_t = (N_t - N_{t-1} + D_t)/N_{t-1}$, where N_t is the number of firms in year t and D_t is the number of firm deaths in year t, and the gross exit rate as $o_t = D_t/N_{t-1}$.

The data for Tobin's Q comes from aggregation of Compustat data. For each year in our sample, we define aggregate entreprise value V_t as the sum, across all firms in our sample, of the market value of equity shares outstanding (using the CRSP variables `prc` and `shrout`), plus the book value of debt outstanding (the sum of Compustat variables `dlc` and `dltt`). We define the stock of productive capital K_t as the sum of the one-year lagged value of net property, plant and equipment (Compustat variable

[37] Specifically, we use the one-way table aggregated at the sectoral level, which is available at https://www2.census.gov/programs-surveys/bds/tables/time-series/2021/bds2021_sec.csv.

ppent), plus the sum of the capitalized value of R&D expenditures (variable k_int_know from Peters and Taylor (2017)), plus the sum of the capitalized value of organizational capital expenditures (variable k_int_org from Peters and Taylor (2017)). The resulting time-series plot for Tobin's Q is obtained as the ratio V_t / K_t.

The data for the Herfindhal index of sales also comes from our Compustat sample. For each year and each NAICS-3D industry, we compute the Herfindahl index of sales as the sum of the square revenue share of firms in that industry. For each year, we then take the weighted average of the HHI across all the sectors in our sample, using the total revenue of the sector as weight. Finally, the data for total factor productivity is drawn from Fernald (2014); we use the utilization-adjusted, annual time series for the business sector.

A.4.2 Calibration procedure

In order to internally calibrate the model to match the moments listed in Table 2, we proceed as follows. Let Θ represent the vector of structural parameters of the model to be internally calibrated:

$$\Theta = (\rho, \lambda, \alpha, \kappa, \xi, \delta)' . \tag{A98}$$

The numerical solution for the BGP provides a mapping $\Xi : \Theta \to \Xi(\Theta)$, where $\Xi(\Theta)$ is a vector containing the moments targeted by our calibration procedure:

$$\Xi(\Theta) = (\rho, \gamma, x, Q, g, \delta)' . \tag{A99}$$

We calibrate Θ by minimizing the following objective function:

$$\min_{\Theta} \left(\left(\Xi(\Theta) - \widehat{\Xi} \right) ./ \widehat{\Xi} \right)' \left(\left(\Xi(\Theta) - \widehat{\Xi} \right) ./ \widehat{\Xi} \right) . \tag{A100}$$

Here, $\widehat{\Theta}$ is the empirical counterpart of the set of targeted moments, which is reported in Table 2, and $./$ is element-wise division. Note that we include (ρ, δ) in the set of internally calibrated parameters, even though these parameters directly map to two of our targeted moments. This is to allow the fit for (ρ, δ) not to be perfect, since they also influence other moments of the model. However, in our internal calibration, the values for (ρ, δ) chosen in the minimization procedure end up being close to the targeted moments.

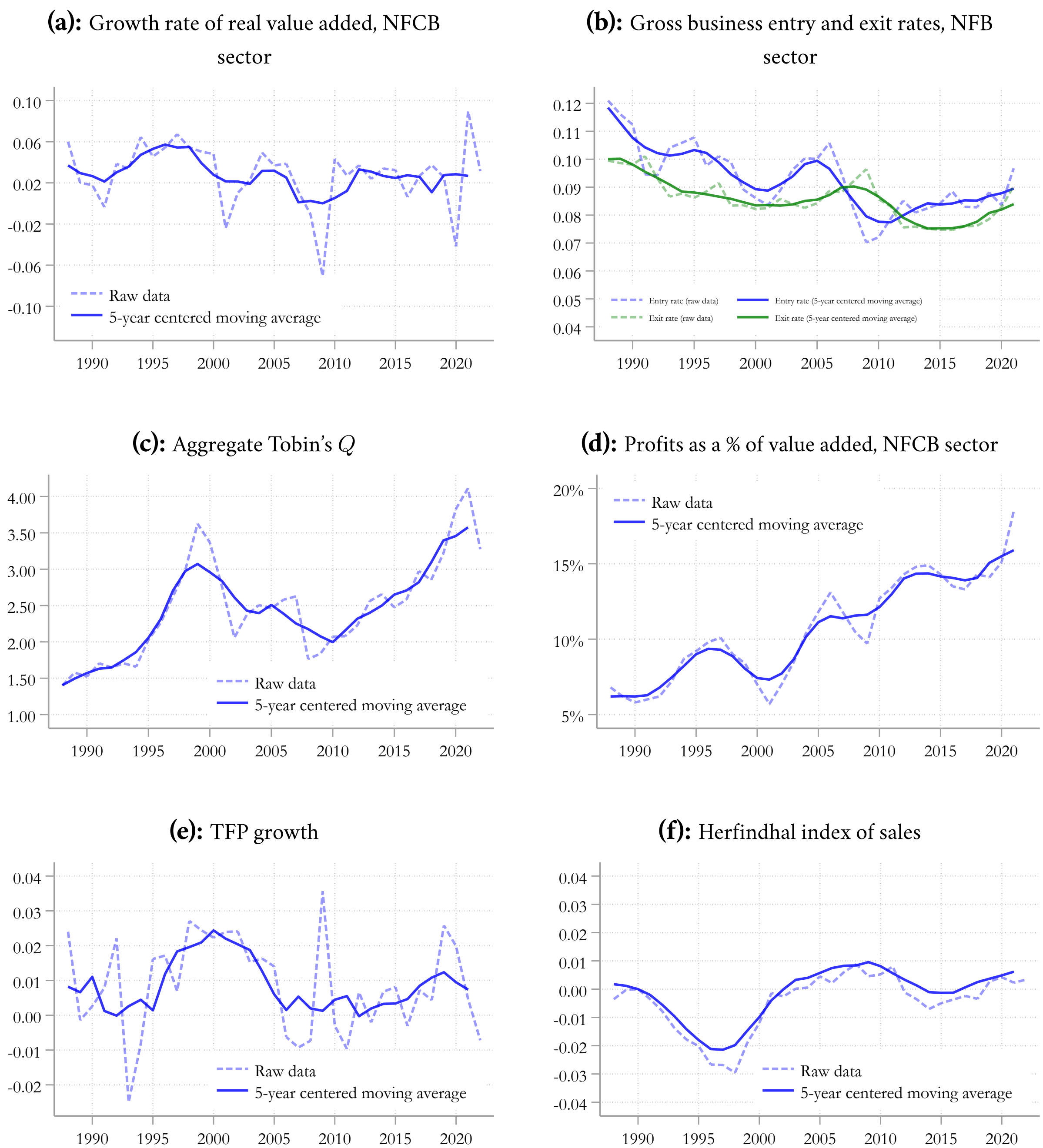

Figure A-1: Other aggregate moments used in the calibration or evaluation of the model. The top left shows the growth rate of real value added for the non-financial corporate business (NFCB) sector. The top right panel shows the gross business entry and exit rates for the non-financial business (NFB) sector. The middle left graph shows aggregate total Tobin's Q for the NFCB sector, with the denominator defined as the sum of estimates of the physical and intangible capital stocks. The middle right graph shows the ratio of profits to value added for the NFCB sector. The bottom left show total factor productivity growth for the business sector, from Fernald (2014), and the bottom right shows the average Herfindahl index of sales across NAICS 3-digit sectors in Compsutat. Data sources are reported in Appendix A.4.1.4.