# Default, Debt Maturity and Investment Dynamics*

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**Abstract**

This paper studies the optimal maturity structure of debt in a standard investment model. Firms operate long-term assets, and may want to use long-term debt to reduce short-term refinancing risk. However, long-term financing may lead to debt overhang and distort investment. The maturity structure of debt should trade off these two forces. In numerical calibrations of the model, however, debt maturity is much shorter than observed among US firms. Firms shun long-term debt because debt overhang costs are large, and the benefits from long-term financing small. Potential reconciliations of the model with the data include investment irreversibility and debt covenants.

**Keywords:** debt, maturity, investment, default risk, dilution risk, commitment, financial frictions, firm dynamics.

**JEL Classification Numbers:** E22, E23, G21, G33.

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# 1 Introduction

Recent work on understanding the dynamics of firm-level and aggregate investment has focused on long-term debt contracts.[1] Long-term borrowing is a pervasive feature of the real-world financing arrangements of firms; in 2007, the median share of debt maturing in more than 3 years was 56.5% among publicly traded US corporations (Custódio et al. 2013). Moreover, the introduction of long-term contracts in neo-classical investment models with financial frictions goes a long way toward explaining why firms tend to operate with lower leverage, pay higher credit spreads, and default less often in the data than these models would otherwise predict (Hackbarth et al. 2006, Kuehn and Schmid 2014).[2] Long-term debt also affects the business-cycle properties of investment: with long-term debt, firms adjust leverage more slowly, so that aggregate shocks tend to have more protracted effects on firm-level investment (Gomes et al. 2013). Relative to the standard short-term debt framework (Hennessy and Whited 2005), long-term debt contracts thus provide an attractive way of understanding key aspects of investment dynamics.

This paper takes a slightly different — normative — approach to understanding the role of debt maturity in investment dynamics, and asks whether long-term debt issuance is *optimal* in neo-classical investment models. Specifically, it studies the optimal maturity composition of debt in a frictional investment model where firms can issue a combination of short- and long-term debt, and adjust the maturity mix dynamically. This is by contrast with the literature mentioned above, which takes as given the fact that firms issue only long-term debt, and proceeds to study the positive implications of long-term financing for investment.

In neo-classical investment models, it is not immediately obvious whether the optimal maturity structure should be tilted toward short- or long-term debt. There are two main forces at play, each tightly linked to two key determinants of maturity structure analyzed by the theoretical literature on debt maturity: *short-term refinancing (or liquidity) risk* and *agency costs*. On one hand, firms operate long-lived assets. Using short-term debt may lead to a mismatch between the maturity of assets and liabilities, and hence to problems when refinancing short-term debt. Long-term debt issuance — especially the issuance of zero- or low-coupon bonds — can help mitigate this problem.

[1] See, among others, Gourio and Michaux (2012), Gomes et al. (2013) and Kuehn and Schmid (2014).

[2] By "neo-classical" models, I refer to models where, in the tradition of Hayashi (1982), Abel (1983) or Abel and Eberly (1994), firms operate a production function which takes capital as an input and must adjust investment dynamically. In combination with financial frictions, this forms the basis for workhorse models of dynamic investment such as, among others, Cooley and Quadrini (2001) and Hennessy and Whited (2005).

This *liquidity risk* motive for issuing long-term debt was first emphasized by Flannery (1986) and Diamond (1991). On the other hand, in dynamic investment models, firms lack commitment: in particular, they can take future actions, such as issuing further debt, paying out dividends, repurchasing outstanding debt, or underinvesting, which may drive them closer to default and endanger the claims of existing creditors. This lack of commitment is priced into long-term contracts, and makes long-term debt issuance costly. That is, there may be agency costs associated with long-term debt issuance. The optimal maturity structure should result from a trade-off between the need to manage liquidity risk, and the need to mitigate agency costs. Since the acuteness of each problem may vary over the firm's life-cycle, or depending on the firm's distance to default, the optimal maturity structure may also evolve dynamically.

This paper provides a simple dynamic investment model in which to explore this trade-off. Firms are infinitely lived and operate a decreasing returns to scale technology. Investment in the technology can be financed by a combination of internally accumulated equity or debt issuance. The productivity of investment is affected by idiosyncratic shocks; firms and intermediaries decide on financing and investment before the shock's realization, and cannot write contracts contingent on the shock. Debt issuance is frictional because firms have limited liability and may default if the productivity realization is too low. Default is inefficient because it involves deadweight losses as resources are transfered from debtors (the firm) to creditors. Thus, the fundamental frictions that impede efficient investment are similar to Cooley and Quadrini (2001), Hennessy and Whited (2005) and Gomes and Schmid (2010).

The model departs from this body of work in only one dimension: firms can issue both short-term debt contracts — the payment of which is due one period after it has been issued — and long-term debt contracts — the repayment of which occurs over a longer period of time. Short-term debt contracts are promises to repay a given dollar amount (or face value) by the end of the period, conditional on the firm not defaulting. Long-term debt contracts, on the other hand, are promises to repay a given face value at some random future date. Each period, the contract may either mature (with a fixed probability), in which case the face value is due, or continue, in which case a coupon payment is due. The advantage of long-term debt is that, because coupons payments are smaller than the face value, expected payments in the current period are smaller than on for short-term contract. On the other hand, the continuation value of debt implicitly depends on actions of the firm in future periods. But the firm cannot commit to any particular path of actions. In particular, actions leading to a decline in the rate of investment will increase the firms' default likelihood, and thus reduce the value of the claims of existing creditors. As a result, equilibrium long-term debt prices typically feature a discount relative to short-term debt; this discount captures the implicit agency costs associated with long-term debt issuance. This model is described in section 2.

In order to quantify the role of endogenous debt maturity choices, section 3 studies key financial and real moments of a numerical calibration of the model. Because the endogenous maturity model nests both the short-term and long-term debt models as particular constrained cases, one can compare the predictions of these three models under a common calibration of structural parameters in order to pinpoint the role of endogenous debt maturity. This calibration is chosen so as to maximize the fit of the long-term model to key moments of investment and profitability of US non-financial firms in Compustat between 1976 and 2014. The long-term debt model's predictions can then be compared to those of the short-term and endogenous maturity models. The key result of this exercise is that the equilibrium maturity structure of debt is short when maturity is endogenous. In particular, it is much shorter than in the data. For instance, with the calibration strategy described above, the mean share of short-term debt is 85% in the model; the data counterpart is 30%. This result is robust to a number of variations of the model, in particular the maturity of the long-term debt contract, the payment of coupons, and the recovery rate of creditors in default. As a result, the endogenous maturity model inherits most of the properties of the short-term debt model, in particular its high leverage ratios and low credit spreads and default rates.

Section 4 then explores the mechanisms behind this result. The key intuitions are as follows. Firms differ along two dimensions: their net worth (or accumulated retained earnings); and their level of debt outstanding. For firms with either high or low net worth, new debt issuances tend to be of short maturity, albeit for different reasons. Firms with high net worth relative to their stock of debt oustanding face limited liquidity risk, so that issuing long-term debt has little advantage for them. As a result, they let any long-term debt roll off their balance sheets, and any net issuances tend to be short-term. Firms with low net worth relative to their stock of long-term debt also tend to shorten debt maturity. These firms are in a quandary: they face both a large debt overhang problem (which should induce them to shorten debt maturity) and high liquidity risk (which should induce them to lengthen it). They resolve this by lowering overall leverage while at the same time shortening the maturity mix. The only firms to lengthen debt maturity are those with net worth close to their oustanding debt stock. These firms issue long-term debt mostly as a way to push their borrowing capacity beyond what is allowed by short-term creditors. The maturity of new issuances eventually starts shorterning when the firm has accumulated enough net worth, has a more limited need for external financing, and faces lower liquidity risk. The equilibrium distribution of maturity thus features mostly firms that are actively reducing the maturity of their debt outstanding, and as a result the typical debt maturity profile is short.

Despite its shortcomings regarding the typical maturity mixed used by firms, some of the qualitative predictions regarding the relationship between leverage, debt maturity, and credit risk (as proxied, in the model, by a combination of net worth and debt outstanding) are borne out in the relatively limited empirical literature on the determinants of debt maturity. Both Barclay and Smith (1995) (for the stock of debt oustanding) and Guedes and Opler (1996) (for new issuances) study the relationship between debt maturity and credit quality in the cross-section, and find that low- and high-credit quality firms are tilted toward short-term debt, while intermediate credit quality firms are tilted toward long-term debt. The most closely related, and most recent evidence is Johnson (2003), which studies jointly the determinants of leverage and maturity. The key finding is that even though unrated firms may choose shorter debt maturity in order to mitigate agency costs, they are still too risky to issue a substantial amount of short-term debt, and thus tend to operate with lower leverage. Rated firms, on the other hand, can both shorten debt maturity in order to reduce agency costs, while using high leverage, as they face little liquidity risk. This paralells closely the model's predictions at the two ends of the spectrum of credit risk: high credit risk firms choosing a short debt maturity and low leverage, and low credit risk firms choosing a short debt maturity and higher leverage. Consistent with this, in the sample of Compustat firms used for the calibration of the model, the correlation of debt maturity with leverage, profitability and investment rates is positive, as it is in the model.

The failure of the model to account for the typical maturity mix used by firms can be interpreted as suggesting that the agency costs associated with long-term debt issuance are particularly large, and the debt overhang problem particularly severe. A sign of the severity of these issues is that firms sometimes reduce the maturity mix by repurchasing outstanding long-term debt, as opposed to issuing more short-term debt. These debt repurchases are carried out using the firms' retained earnings, so that they come at the expense of investment in fixed capital. Thus, when firms are over-indebted and devote resources to debt repurchases, they ulterior growth is slow. The endogeneity of investment, and thus the default threshold, is key to this result. Indeed, it is by drastically reducing investment rates that firms are able to lower the outstanding value of long-term debt contracts sufficiently to repurchase them.

Two potential extensions to the model could help mitgate agency costs, or equivalently create better incentives for long-term debt issuance. The first is to allow for debt covenants to be written, in particular covenants on early debt repurchases; these convenants would precisely constrain firms' ability to repurchase debt at fire-sale prices; in turn, they would improve the prices of long-term debt. The second possibility would be to assume that investment is partially irreversible; this also limits firm's ability to use internal financing towards debt repurchases and maturity shortening.

An additional contribution of the paper is to address issues associated with equilibrium multiplicity in debt markets that may arise with defaultable debt contracts. The intuition behind the existence of multiple equilibria in this class of models is the following. For any level of debt issuance, if debt prices are high, the firm will be able to invest substantially; all other things equal, this will reduce its likelihood of default, and in turn, justify the high debt price. The converse is true if debt prices are low. In the context of a mixed-maturity model, I show that this mutliplicity problem is amplified. This is because even conditional on not defaulting on the current period, the firm may default at some future point in time. Different levels of *expected* default probabilities, and thus *current* long-term debt prices, may then be consistent with lenders' zero-profit conditions. The paper proposes an algorithm to select equilibria that provide firms with the highest possible level of capital, for any given level of long-term debt issuance, and it studies the solutions to the model under this assumption.

The rest of this paper is organized as follows. Section 2 describes the model, and discusses equilibrium multiplicity. Section 3 compares numerical simulations from the model to its short- and long-term versions, and establishes the result that optimal debt maturity tends to be short. Section 4 then provides some analytical results on the optimality of short-term debt, and draws a link between these results and the optimal firm policies in the numerical solution of the model. Section 5 concludes.

**Related literature** This paper is related to three strands of literature. First, it contributes to the corporate finance literature on the optimal maturity structure of debt, and in particular a subset of papers which study how the debt overhang problem of Myers (1977) interacts with long-term debt issuance in dynamic environments. These models typically build on Leland (1994a) and Leland (1994b). Moyen (2007) contrasts the investment policy of firms that issue short-term debt and continuously adjust leverage, to that of a firm issuing long-term debt but not readjusting it; she finds that underinvestment as a result of debt overhang can occur in both models, but it is more dramatic in long-term debt models. Titman and Tsyplakov (2007) study also study a dynamic leverage model in the style of Leland (1994b), and find that shorter-term debt tends to mitigate underinvestment, but also lead to more frequent default. Recent work by Diamond and He (2014) emphasizes that short-term debt can lead to "future debt overhang": if debt matures *before* future investment decisions are made, it may lead to lower future investment ("ex-post"); if, in turn, initial and future investment decisions are related, this may lead to under-investment "ex-ante". Relative to this body of work, this paper departs in that the firm can continuously readjust both its stock of capital and the oustanding maturity of its debt dynamically . This opens up the possibility for the large debt repurchases and the under-investment spells, thus magnifying the debt overhang problem associated with long-term debt issuance.

The second strand of literature to which this paper is related are recent models of dynamic capital structure, following Cooley and Quadrini (2001), Hennessy and Whited (2005) and Miao (2005).[3] These models study the implications of costly external financing for the cross-section and dynamic properties of firm-level and aggregate investment, as well as for equity returns (Gomes and Schmid 2010). The contribution of this paper to this line of research is to make maturity choices an integral part of the capital structure decision of firms, and show that long-term debt issuance may not always be optimal. The closest paper to mine in this respect is a recent contribution by Yamarthy (2016). This paper tackles the question of the cyclicality of the long-term debt share. Similar to this paper, the model considers endogenous investment. However, it assumes that all short-term debt is risk-free, and that short-term debt issuance is limited by a simple collateral constraint. This not only allows the model to match, by construction, the observed long-term

[3] A non-exhaustive list of important contributions to this line of research are Miao (2005), Hennessy and Whited (2005), Miao and Wang (2007), Hennessy and Whited (2007), Strebulaev (2007), Gomes and Schmid (2010), Gourio and Michaux (2012), (Kuehn and Schmid 2014) and Sundaresan et al. (2015).

debt share; it also simplifies substantially the solution method, as firms only choose long-term debt issuance. This is useful in order to match business-cycle dynamics of debt maturity, but sidesteps the question of the optimal maturity structure, which is the focus of this paper.

Finally, this paper is closely related to the large and recent international finance literature on the optimal maturity structure of government debt, following the work of Hatchondo and Martinez (2009), Chatterjee and Eyigungor (2012), Arellano and Ramanarayanan (2012) and Aguiar and Amador (2013). In particular, I model the maturity choice using the approach of Arellano and Ramanarayanan (2012), by allowing firms to issue both short- and long-term debt. Relative to this literature, the key difference is that the amount of debt raised by the firm has an impact on investment, output, and default probabilities. By contrast, in this literature, output tends to be exogenous, or at least independent of the current level of debt issuance. An exception is Mendoza and Yue (2012); however, in that framework, producers take the interest rate on working capital loans as given, whereas in the context of the model studied in this paper, firms face menus of prices, and thus internalize the impact of their production decision on their cost of capital. The endogeneity of investment and debt prices opens the door to equilibrium multiplicity, as discussed above.[4] Additionally, it worsens the commitment problem, in the sense that the firm can choose to underinvest in order to repurchase debt outstanding in bad times. This mechanism is similar to the "hedging" advantage of long-term debt in the international finance literature, whereby the value of long-term debt falls after a bad shock. In the context of this model, however, the debt repurchasing policy instead *endogenously* generates a fall in output produced by the firm, and thus a fall in the value of outstanding long-term debt.

## 2 A neo-classical investment model with short and long term debt

This section describes a simple model of investment dynamics in which both short- and long-term financing are available to firms. I start with the key economic features of the environment, and in particular the debt financing options available to firm.

[4] An additional important difference between this model and standard models of sovereign debt, such as Eaton and Gersovitz (1981) and Cole and Kehoe (2000), is in the *timing* of borrowing decisions and productivity realizations. Whether altering the timing of shocks and borrowing, in this model, would moderate the multiplicity problem, is an open question.

### 2.1 Key features of the model

Time is discrete. There are two groups of infinitely-lived, risk-neutral agents: firms and financial intermediaries. Firms are risk-neutral and discount the future at rate $\beta^{-1}$; their objective is to maximize the present value of dividends. Financial intermediaries discount the future at a lower rate than firms, $(1+r) \leq \beta^{-1}$.[5]

**Production technology** The production technology operated by firms takes physical capital $k$ as an input, and has decreasing returns to scale:

$$\mathbf{y}(\phi, k) = \phi k^{\zeta}.$$

Here, $\phi$ denotes firm-level productivity, and $\zeta$ is the degree of returns to scale. $\phi$ is assumed to be i.i.d. across firms and over time, with cumulative distribution function $F$. Capital is long-lived, and depreciates at rate $\delta_k$. Resources available after production, inclusive of depreciated capital, are given by:

$$\boldsymbol{\pi}(k, \phi) = \mathbf{y}\,(k, \phi) + (1-\delta_k)k = \phi k^{\zeta} + (1-\delta_k)k.$$

The support of $F$ is assumed to be $\Phi = \mathbb{R}_+$, so that $\boldsymbol{\pi}(k, \phi)$ is bounded from below by $(1-\delta_k)k$.

**Financial friction** Every period, the purchase of capital goods by firms is financed via a combination of internal resources, or "inside equity", $e$, and debt issuance. Financial intermediaries are perfectly competitive, and make zero profits from lending to firms. Lending is frictional because of a combination of limited liability and bankruptcy costs. Firms and financial intermediaries contract before the level of productivity of the firm is known. Once productivity is realized, the firm may choose or be forced to default. In this case, creditors only retrieve:

$$(1-\chi)\pi(k, \phi) \quad , \quad \chi \in \left]0, 1\right].$$

**Short and long-term debt** Firms can issue either short-term debt ($s$), long-term debt ($l$), or some combination of both. Debt contracts are a promise to repay fixed amounts of output at

[5] This assumption is a transparent way of ensuring that firms will be willing to use debt financing in equilibrium. One could alternatively assume identical discount rates, but a tax shield for interest payments on debt; the model would formally be identical.

future dates. The face value of all debt contracts is normalized to 1. Contracting occurs before the productivity level of the firm, $\phi$, has been realized. A short-term debt contract is a promise to repay one unit of output within a period, conditional on no default.

Long-term debt contracts, on the other hand, have a random maturity date.[6] With probability $\delta \in ]0,1]$, long-term debt does not mature; in this case, conditional on no default, a coupon payment $c \in [0,1]$ is made.[7] With probability $1-\delta$, the debt matures and a payment of 1 unit of output is made. Long-term debt matures independently across contracts and over time. For each outstanding long-term contract, conditional on no default, firms thus expect to repay a total of $\kappa = \delta c + (1-\delta)$ to financial intermediaries. The duration of these contracts is thus:

$$m = \frac{1}{1 - \frac{\delta}{1+r}}.$$

The use of random maturity is useful because it reduces the state-space of the firm's problem. If a firm issues $n_l$ new long-term contracts today, a fraction $(1-\delta)$ will mature, so that tomorrow, only $\delta n_l$ claims will remain. As a result, the total number of outstanding long-term debt contracts at the beginning of the period is a sufficient state variable for summarizing past debt issuance; it is not necessary to keep track of the past timing of issuances. Total outstanding bonds *at the beginning of the period* are denoted by $l_m$. *Net* issuance of long-term debt can then be rewritten as $n_l = l - \delta l_m$, where $l$ denotes the number of outstanding contracts *after issuance* in the current period. Using this notation, we can rewrite the relationship between sources and uses of funds as:

$$k = e + q_L(l - \delta l_m) + q_S s.$$

The stock of outstanding long-term debt after issuance, $l$, can satisfy $l \gtrless \delta l_m$. If $l > \delta l_m$, the firm is issuing new long-term contracts, whereas if $l < \delta l_m$, it is using internal ressources $e$ to repurchase outstanding long-term contracts.

In equilibrium, the price of debt will adjust so as to satisfy the zero profit condition of the firm, conditional on the variables describing the firm: internal ressources, outstanding long-term debt

[6] Random maturity contracts, and their formally close cousins, geometrically declining consols, are the standard tool for modelling long-duration debt contracts both in corporate finance (Leland and Toft 1996) and in international finance (Hatchondo and Martinez 2009).

[7] When $c = 1$, the contract is equivalent to a consol with coupons that decline at rate $\delta$.

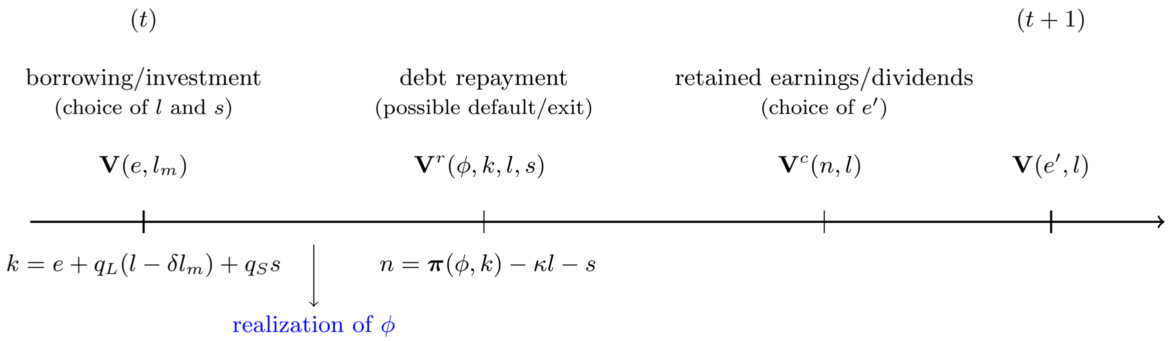


Figure 1: Timing of an individual firm's problem.

contracts, desired new long-term debt, desired short-term debt issuance. Therefore:

$$q_L \quad = \mathbf{q}_L(e, l_m, l, s) \quad , \quad q_S \quad = \mathbf{q}_S(e, l_m, l, s).$$

The determination of these equilibrium price schedules is described below.

## 2.2 The problem of an individual firm

The timing of the firm's problem within a period is summarized in figure 1. Each period, there are three stages: first, the firm chooses how much to borrow and invest; second, productivity is realized, and the firm chooses whether to default or repay and continue; third, conditional on not defaulting, the firm chooses how to allocate its resources between retained earnings and dividends.

**Retained earnings and dividend issuance** If the firm has not defaulted on its debt, it allocates it resources $n$ dividends and retained earnings. It solves the following problem:

$$\mathbf{V}^c(n, l) \quad = \quad \max_{0 \leq e' \leq n} \quad n - e' \quad + \quad \beta \mathbf{V}(e', l) \tag{1}$$

where $\mathbf{V}$ is the value of the firm at the beginning of the following period. The dividend issuance/retained earnings problem is constrained by the fact that next period equity, $e'$ cannot exceed cash on hand $n$. This can be interpreted as a restriction on the issuance of outside equity. Additionally, firms cannot start the following period with negative resources ($e' \geq 0$). Implicitly, this prevents firms from using future debt issuance as way to pay out dividends.

**Debt repayment** After the firm's productivity has been realized, resources available for debt repayment, $\boldsymbol{\pi}(\phi, k)$, become know. The net present value of default for the firm is assumed to be fixed, and is given by $V_d$. Since it cannot issue outside equity at the following stage, the firm must default whenever the realization of $\phi$ is such that:

$$\boldsymbol{\pi}(\phi, k) = \phi k^{\zeta} + (1 - \delta_k)k \leq \kappa l + s.$$

Additionally, the firm can default voluntarily, if its continuation value is below its value in default. The value of default for the firm is assumed to be fixed, and given by $V_d$. Thus, the default decision can be summarized as:

$$\mathbf{V}^r(\phi, k, l, s) = \mathbb{1}_{\{n \leq 0\}} \times V_d + \mathbb{1}_{\{n > 0\}} \times \max\Big(V_d, \mathbf{V}^c(n, l)\Big) \tag{2}$$

$$s.t. \quad n = \phi k^{\zeta} + (1 - \delta_k)k - \kappa l - s \tag{3}$$

The default value of the firm $V_d$ can be interpreted as outside options available to the managers, conditional on exiting. The assumption that the default value of a firm is exogenous is not crucial to the results that follow; it could, for example, depend on the resources of the firm $\boldsymbol{\pi}(\phi, k)$. It is however important that the firm may not be able to keep operating after default; otherwise, firms may chose default in order to wipe out existing creditors and keep operating in future periods.[8]

**Complete formulation of firm problem** At the beginning of the period, firms choose borrowing and investment in order to maximize the expected present value of future dividend. Using the previous discussion, the problem of an individual firm can be written as:

$$\mathbf{V}(e, l_m) = \max_{l,s} \int_{\phi \in \Phi} \mathbf{V}^r(\phi, k, l, s) dF(\phi) \tag{4}$$

$$s.t. \quad k = e + q_L(l - \delta l_m) + q_S s \tag{5}$$

$$l \geq 0 \quad , \quad s \geq 0 \quad , \quad k \geq 0 \tag{6}$$

$$q_S = \mathbf{q}_S(e, l_m, l, s) \quad , \quad q_L = \mathbf{q}_L(e, l_m, l, s) \tag{7}$$

[8]Additionally, this definition of the default and continuation set assumes that if it is indifferent between the two options, the firm always chooses default, but this assumption is not important to the results.

where $\mathbf{V}^r(\phi, k, l, s)$ is given by (2)-(3), and $\mathbf{V}^c(n, l)$ is given by (1). The firm's objective is to maximize its expected value at the repayment stage; this value includes the possibility of default, as captured by (2)-(3). The balance sheet constraint (5) captures the sources and uses of funds, while the non-negativity constraints (6) state the firm cannot lend, either on short- or long-term debt markets, and that it must operate with a positive amount of capital. Finally, the firm takes the contingent price schedules of debt contracts as given, as stated by constraint (7).

## 2.3 Financial intermediation and equilibrium

### 2.3.1 Default and repayment sets

Before discussing the pricing of debt, it is first convenient to express the default decision of an individual firm in terms of a continuation and a default set. Let:

$$\mathcal{C}(k, l, s; \mathbf{V}^c) \equiv \left\{ \quad \phi \in \Phi \quad s.t. \qquad \begin{array}{l} \phi k^\zeta + (1-\delta_k)k - (\kappa l + s) > 0 \\ \mathbf{V}^c\Big(\phi k^\zeta + (1-\delta_k)k - (\kappa l + s), l\Big) > V_d \end{array} \right\}, \tag{8}$$

$$\mathcal{D}(k, l, s; \mathbf{V}^c) \equiv \Phi \setminus \mathcal{C}(k, l, s; \mathbf{V}^c).$$

A firm with capital $k$ and liabilities $(l, s)$ will not default, if and only if its realized productivity satisfies $\phi \in \mathcal{C}(k, l, s; \mathbf{V}^c)$. This formulation of the continuation and default sets emphasizes that debt in this model can be *current-default* risk free, in the sense that current liabilities, $\kappa l + s$, may be small enough that the firm will honor them regardless of the productivity realization. This occurs when:

$$(1-\delta_k)k \geq \kappa l + s \quad \text{and} \quad \mathbf{V}^c\Big((1-\delta_k)k - (\kappa l + s), l\Big) \geq V_d.$$

### 2.3.2 Debt pricing

Financial intermediaries take the value and policy functions of firms, $\{\mathbf{V}, \mathbf{V}^c\}$ and $\{\mathbf{l}, \mathbf{s}, \mathbf{e}'\}$, as given. The debt prices they offer are conditional on the observable characteristics of the firm at the time of borrowing: the desired debt structure $(l, s)$, as well as on the firm's state variables, $(e, l_m)$. For any $(e, l_m, l, s) \in \mathbb{R}^4_+$, prices offered must be such that the net present value of future lending profits is zero. Debt prices must therefore satisfy:

$$\mathbf{q}_S(e, l_m, l, s) = \frac{1}{1+r}\mathbb{E}_\phi\left[\mathbb{1}_{\{\phi\in\mathcal{C}(k,l,s;\mathbf{V}^c)\}}\right] + \frac{1}{1+r}\mathbb{E}_\phi\left[\mathbb{1}_{\{\phi\in\mathcal{D}(k,l,s;\mathbf{V}^c)\}}\frac{(1-\chi)\boldsymbol{\pi}(\phi,k)}{s+\kappa l}\right] \quad (9)$$

$$\mathbf{q}_L(e, l_m, l, s) = \kappa\mathbf{q}_S(e, l_m, l, s) + \frac{\delta}{1+r}\mathbf{q}_C(k, l, s) \quad (10)$$

$$\mathbf{q}_C(k, l, s) = \mathbb{E}_\phi\left[\mathbb{1}_{\{\phi\in\mathcal{C}(k,l,s;\mathbf{V}^c)\}}\hat{\mathbf{q}}_L\left(\,\tilde{\mathbf{e}}(\phi, k, s, l)\ ,\ l\,\right)\right] \quad (11)$$

$$k = e + \mathbf{q}_L(e, l_m, l, s)(l - \delta l_m) + \mathbf{q}_S(e, l_m, l, s)s \quad (12)$$

Equation (9) is the zero-profit condition for short-term lenders. This condition states that the price of debt should be equal to the expected discounted cash flows from the contract. The first terms on the right-hand side of (9) captures the expected principal repayment conditional on no default. The second term is the value of the contract if the firm defaults. In this case, creditors receive a portion $(1-\chi)$ of the output produced by the firm. This value is assumed to be shared according to parri-passu between creditors which had a claim on the firm this period. This includes short-term creditors (the term $s$) and expected payment to long-term creditors (the term $\kappa l$).

Equation (10) is the zero-profit condition for long-term lenders. It states that the price of a long-term debt contract, $\mathbf{q}_L(e, l_m, l, s)$, should be equal to the sum of its expected payoff in the current period, $\kappa\mathbf{q}_S(e, l_m, l, s)$, plus its expected discounted value in the next period, $\frac{\delta}{1+r}\mathbf{q}_C(k, l, s)$. The current payoff to the long-term contract is identical to the short-term contract, since firms cannot default selectively; the only difference is the multiplicative constant $\kappa = \delta c + (1-\delta)$, which captures the fact that long-term contracts are either offering a coupon payment (conditional on not maturing), or paying off their face value of 1 (conditional on maturing).

The expected discounted value of the long-term contract, $\frac{\delta}{1+r}\mathbf{q}_C(k, l, s)$, is the product of the discount rate, the probability of the contract not maturing in the current period, and the expected price in the next period, $\mathbf{q}_C(k, l, s)$. This expected price is given by equation (11). In this expression, the price function $\hat{\mathbf{q}}_L(e, l_m)$ is the price of long-term bond evaluated at the firm's optimal policies, namely:

$$\hat{\mathbf{q}}_L(e, l_m) \equiv \mathbf{q}_L(\ e\ ,\ l_m\ ,\ \mathbf{l}(e, l_m)\ ,\ \mathbf{s}(l, e_m)\ ), \quad (13)$$

and the function $\tilde{\mathbf{e}}$ describes the reinvestment policy of the firm conditional on a particular realization of productivity:

$$\tilde{\mathbf{e}}(\phi, k, s, l) \;\; = \;\; \mathbf{e}' \left( \phi k^{\zeta} + (1-\delta_k)k - (\kappa l + s) \; , \; l \right) \tag{14}$$

The key aspect of equation (11) is that the continuation value of the bond depends on the firm's future policies; in particular, its dividend issuance policy in the current period, and its borrowing and investment policies in the following period. Lack of commitment affects bond prices precisely by reducing the continuation value $\mathbf{q}_C$.

Finally, equation (12) states that the resulting equilibrium prices must be consistent with the investment policy of the firm. This condition is absent from models where investment or output of the borrower is determined independently of the amount of debt raised, as is the case in the bulk of the literature on debt maturity in international finance. I discuss the importance of this difference below.

### 2.3.3 Recursive competitive equilibrium

The definition of an equilibrium in this economy is standard.

**Definition 1** (Recursive competitive equilibrium)**.** *A recursive competitive equilibrium of this economy is a set of value functions* $\{\mathbf{V}, \mathbf{V}^r, \mathbf{V}^c\}$*, policy functions* $\{\mathbf{e}', \mathbf{l}, \mathbf{s}\}$ *and price functions* $\{\mathbf{q}_S, \mathbf{q}_L\}$*, such that:*

- *given* $\{\mathbf{q}_S, \mathbf{q}_L\}$*,* $\{\mathbf{V}, \mathbf{V}^r, \mathbf{V}^c\}$ *and* $\{\mathbf{e}', \mathbf{l}, \mathbf{s}\}$ *are a solution to the firm's problem, i.e. satisfy (1)-(7);*

- *given* $\{\mathbf{V}, \mathbf{V}^r, \mathbf{V}^c\}$ *and* $\{\mathbf{e}', \mathbf{l}, \mathbf{s}\}$*, the price functions* $\{\mathbf{q}_S, \mathbf{q}_L\}$ *are such that the net present value of lending profits is zero, that is, equations (9)-(14) hold for any* $(e, l_m, l, s) \in \mathbb{R}^4_+$*.*

Note two important elements in this definition. First, the zero profit condition must hold for all $(e, l_m, l, s) \in \mathbb{R}^4_+$ — even those that are off the equilibrium path. This effectively makes certain off-equilibrium debt structures too expensive to adopt for the firms; in this sense, this requirement is akin to an endogenous borrowing limit. Second, this definition of equilibrium bypasses the question of entry and exit dynamics. This is because, given the assumption that exit is permanent, the NPV of exit for a firm is independent of the future probability of re-entry, or the value of new entrants. Therefore, entry and exit dynamics do not affect optimal investment and borrowing choices.

### 2.4 Equilibrium default

Before going on to discuss the key aspects of investment and maturity choices in this environment, it is useful to note that the default decision of firms can be simplified to a "threshold strategy". Recall that continuation decision can be state in terms of a continuation set,

$$\mathcal{C}(k,l,s;\mathbf{V}^c) = \left\{ \quad \phi \in \Phi \quad s.t. \quad \begin{array}{l} \phi k^{\zeta} + (1-\delta_k)k - (\kappa l + s) > 0 \\ \mathbf{V}^c\left(\phi k^{\zeta} + (1-\delta_k)k - (\kappa l + s), l\right) > V_d \end{array} \right\}, \tag{15}$$

such that a firm with capital in place $k$ and liabilities $(l,s)$ continues, if and only if, $\phi \in \mathcal{C}(k,l,s;\mathbf{V}^c)$. The following lemma establishes a simpler formulation of this continuation set.

**Lemma 1** (The default boundary)**.** *If $V_d \leq 0$, the continuation set is equal to:*

$$\mathcal{C}(k,l,s;\mathbf{V}^c) = \left\{ \quad \phi \in \Phi \quad s.t. \quad \phi > \boldsymbol{\phi}^d(k, \kappa l + s) \quad \right\}, \tag{16}$$

*where the default boundary $\boldsymbol{\phi}^d$ is given by:*

$$\boldsymbol{\phi}^d(k, \kappa l + s) = \begin{cases} +\infty & \text{if} \quad k = 0 \\ k^{-\zeta}\left(\kappa l + s - (1-\delta_k)k\right) & \text{if} \quad k > 0 \end{cases} \tag{17}$$

The proof of this result (and a more general one, relating to the case of $V_d \geq 0$) is reported in appendix. Two things here are worth noting. First, in general (that is, for any value of $V_d$), the continuation decision can be formulated in terms of a boundary for productivity, below which the firm defaults and above which it continues. The decision to continue is monotonic in productivity because the continuation value, $\mathbf{V}^c$, is monotonic in cash $n$, as established in appendix. Second, one difficulty in formulation 15 is that the continuation set (and the default boundary) can depend on the value function of the firm, $\mathbf{V}^c$.[9] The lemma establishes that when the outside option of the firm $V_d$ is restricted to be (weakly) negative, the continuation decision becomes independent of $\mathbf{V}^c$. This is because in this case, even at the worse possible realization of productivity, $\phi = 0$, the firm still prefers continuation to default. Note that, in this case, $\boldsymbol{\phi}^d$ can be thought of as a "natural default boundary": the firm defaults if and only if its current income falls below its current liabilities.

Given this result, the following assumption will allow for a more compact formulation of the

[9]This is also formally established in the appendix.

individual firm problem and the financial market equilibrium conditions.

**Assumption 1** (Outside option of the firm)**.** *The net present value of defaulting for the firm is* $V_d = 0$*.*

When $V_d = 0$, using the results of lemma (1), the individual firm problem can be rewritten as:

$$\mathbf{V}(e, l_m) = \max_{l,s} \int_{\phi > \boldsymbol{\phi}^d(k,\kappa l+s)} \mathbf{V}^c((\phi - \boldsymbol{\phi}^d(k, \kappa l + s))k^{\zeta}, l)dF(\phi) \tag{18}$$

$$s.t. \quad k = e + q_L(l - \delta l_m) + q_S s \tag{19}$$

$$\mathbf{V}^c(n, l) = n + \max_{0 \le e' \le n} \beta V(e', n) - e' \tag{20}$$

$$l \ge 0 \quad , \quad s \ge 0 \quad , \quad k \ge 0 \tag{21}$$

$$q_S = \mathbf{q}_S(e, l_m, l, s) \quad , \quad q_L = \mathbf{q}_L(e, l_m, l, s) \tag{22}$$

where the function $\boldsymbol{\phi}_d$ is given by (17). The zero-profit conditions of short-term lenders can be rewritten in terms of the default threshold $\boldsymbol{\phi}^d$ as:

$$\mathbf{q}_S(e, l_m, l, s) = \frac{1}{1+r}\left(1 - F\left(\boldsymbol{\phi}^d(\phi, k, \kappa l + s)\right)\right) + \frac{1}{1+r}\int_{\phi \ge \boldsymbol{\phi}^d(k,\kappa l+s)} \frac{(1-\chi)\boldsymbol{\pi}(\phi, k)}{s + \kappa l} dF(\phi), \tag{23}$$

while the continuation value of bonds $\mathbf{q}_C$ can be rewritten as:

$$\mathbf{q}_C(k, l, s) = \int_{\phi > \boldsymbol{\phi}^d(k,\kappa l+s)} \hat{\mathbf{q}}_L\left(\mathbf{e}'\left((\phi - \boldsymbol{\phi}^d(k, \kappa l + s))k^{\zeta}, l\right) \ , \ l\right) dF(\phi) \tag{24}$$

Before moving on to the predictions of the model for optimal debt maturity, I turn to the question of the multiplicity of financial market equilibria in this model.

### 2.5 Equilibrium multiplicity and equilibrium selection

Equilibrium multiplicity can arise in this framework because, given a set of particular firm policy and value functions, the zero-profit conditions of the financial intermediaries may have multiple solutions.[10] The key intuition behind this result is that, all other things equal, an increase in

[10] Given a particular price function for long-term debt, it is straightforward to establish that policy and value functions of individual firms are unique. However, the multiplicity of solution to the risk-neutral pricing condition

investment leads to a lower default boundary and a higher a price of debt; while conversely, a higher price of debt allows the firm to investment more. As a result, for the same level of debt issuance, there can be equilibria with low debt prices and low investment as well as equilibria with high debt prices and high investment. To put it otherwise, the complementarity between investment and debt prices induced by the budget constraint of the firm is what creates scope for multiplicity.

The central point of the following discussion is that while this multiplicity arises even when the firm can only issue short-term debt, the ability to issue long-term debt aggravates the problem by making equilibrium selection criteria more difficult to construct. To clarify this, I start by discussing mutliplicity in the case when the firm issues only short-term debt.[11]

### 2.5.1 Multiplicity when firms only issue short-term debt

**Why does multiplicity arise?** When the firm only issues short-term debt, the zero-profit condition of intermediaries can be rewritten as:

$$k - e = \frac{1}{1+r}\left(1 - F\left(\phi^d(k,s)\right)\right)s. \tag{25}$$

The left hand side of this equation represents the external financing of a firm with equity $e$, issuing $s$ short-term debt contracts. The right-hand side represents the value of these contracts, which depends on the probability of the default of the firm. Thus, given a level of debt issuance $s$, solving for the price function $\mathbf{q}_S(e,s)$ is equivalent to solving for the total investment level of the firm, $k$, consistent with the financial intermediaries breaking even. Equilibrium multiplicity can arise when there are different investment levels $k$ that could satisfy this condition.

Figure (2) illustrates such a situation. In this case, there are three levels of investment that are consistent with lenders breaking even. Recall that all three correspond to the same number of short-term debt contracts issued, $s$. In the first equilibrium, to the left of the graph, debt prices are low, and therefore investment is low. Because investment is low, the default boundary $\phi^d(k,s)$ is high, or equivalently, the current default probability (CDP) is high. This in turns justifies the low debt prices. The second equilibrium features an intermediate level of investment, while the third

means that there may be several recursive competitive equilibria of the form described in section 2.3.3 are not unique; in particular, different equilibria can be constructed by selecting different solutions to the risk-neutral pricing conditions.

[11] Throughout the discussion of equilibrium multiplicity, I assume full liquidation losses, $\chi = 1$; this is not central to the mechanisms described here.

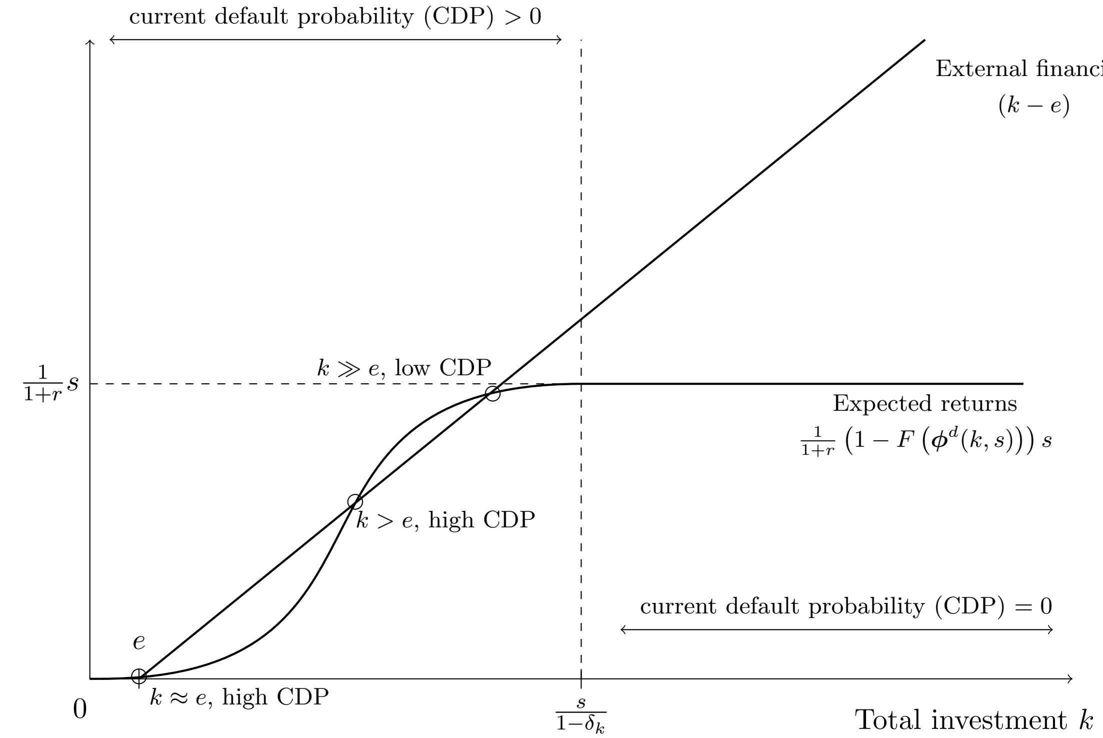


Figure 2: Financial market equilibrium in the short-term debt model.

(to the right of the graph) features high investment, high debt prices, a low default boundary and a low default probability. (Note that default probability can be 0; this occurs whenever the two lines intersect above $\frac{s}{1-\delta_k}$, which is the level of capital that guarantees that debt will be repaid for any productivity realization). Thus, the endogeneity of investment and output to debt prices is at the heart of equilibrium multiplicity in this model.

**Equilibrium selection** In the situation illustrated in figure 2, the three equilibria can be ranked by the amount of capital they provide to the firm. This suggests the following strategy for constructing a particular equilibrium. First, construct the maximum-capital function:

$$\mathbf{k}(e,s) = \arg\max\left\{ \quad k \geq 0 \quad \text{s.t.} \quad k - e = \frac{1}{1+r}\left(1 - F\left(\boldsymbol{\phi}^d(k,s)\right)\right)s \quad \right\}, \tag{26}$$

Then, it is clear that the price schedule defined by:

$$\mathbf{q}_S(e,s) \equiv \frac{1}{1+r}\left(1 - F\left(\boldsymbol{\phi}^d(\mathbf{k}(e,s),s)\right)\right) \tag{27}$$

satisfies the zero-profit condition of intermediaries. Moreover, among all equilibrium price functions, this price functions allows the firm to invest the most, for any capital structure $(e, s)$. As such, from the firm's standpoint, it dominates any other equilibrium price function. Alternatively, if financial intermediaries competed in a Cournot sense, and if there were no entering costs, $\mathbf{q}_S(e, s)$ would be the equilibrium price of debt.[12]

Crucially, note that this Pareto criterion always pins down a unique recursive competitive equilibrium. Indeed, define a price schedule by (26)-(27), and construct the solution to the firm problem associated with this schedule. This defines a recursive competitive equilibrium. Moreover, to the extent that the solution to the firm problem is unique, there cannot be two distinct recursive competitive equilibria with the price schedule (26)-(27).

### 2.5.2 Multiplicity when firms also issue long-term debt

**How does multiplicity change when firms also issue long-term debt?** When firms issue long-term debt, the zero-profit condition of financial intermediaries implies that investment $k$ must satisfy:

$$k - e = \frac{1}{1+r}\left(1 - F\left(\phi^d(k, \kappa l + s)\right)\right)(s + \kappa(l - \delta l_m)) + \frac{\delta}{1+r}\mathbf{q}_C(k, l, s)(l - \delta l_m). \tag{28}$$

In this identity, the left hand-side again represents external financing for a firm with equity $e$, and the right-hand side represents the expected return from lending. Again, one can think of the financial market equilibrium as pinning down the level of investment $k$ that is consistent with financiers breaking even, for a firm with capital structure $(e, l_m, l, s)$.

Two differences with the short-term debt case are worth noting. First, the zero-profit condition implicitly depends on the policy functions of the firm, via the continuation value of long-term bonds $\mathbf{q}_C$. It did not in the short-term debt case, so that financial markets equilibrium and investment choices could be separated. Second, the right-hand side of this equation can be negative. This can occur when $l < \delta l_m$, that is, when the firm chooses to repurchase outstanding debt. In this case, net external financing is negative $(k < e)$; the firm uses its resources in order to extinguish outstanding claims, rather than invest.

[12]The appendix establishes this formally.

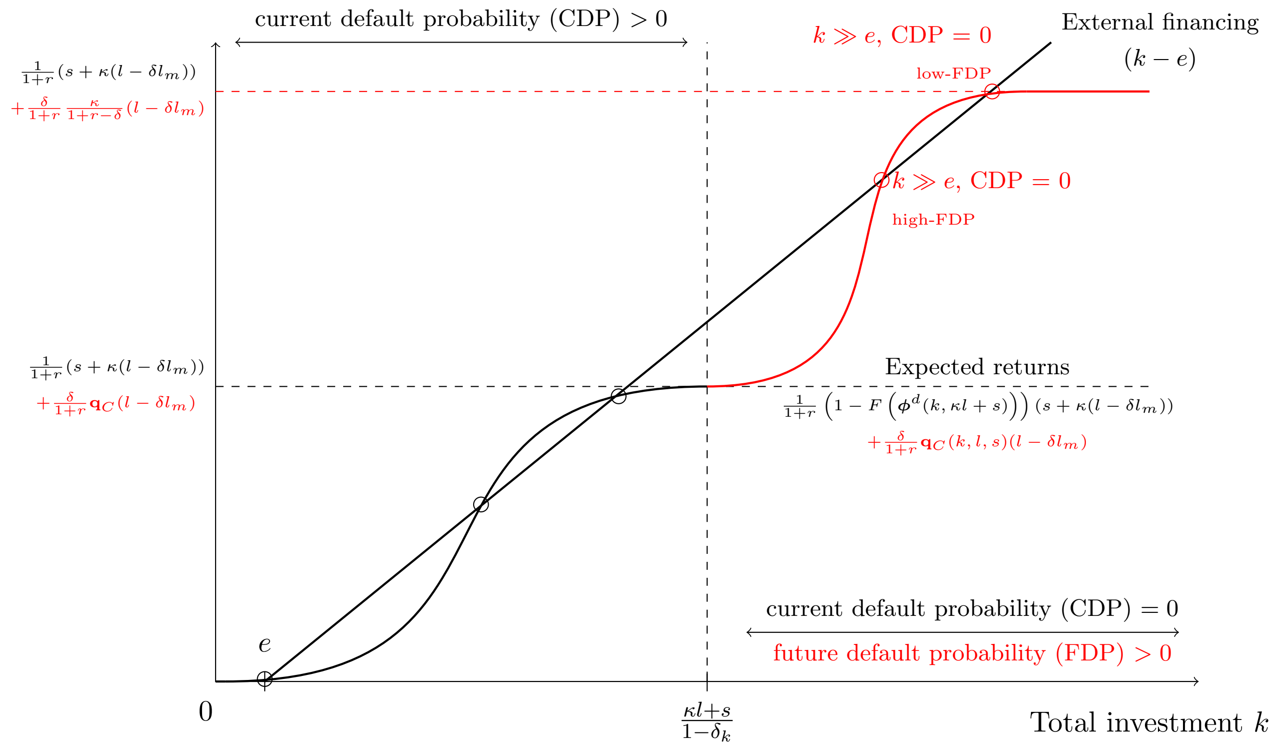


Figure 3: Financial market equilibrium in the model with short- and long-term debt, in the case of net positive long-term debt issuance $l > \delta l_m$.

**Multiplicity when firms are net debt issuers** Figure (3) considers a case in which the firm is a net issuer of long-term debt, that is, when $l > \delta l_m$. The black portions of each line capture expected lending returns in the current period, while the red portions of the lines capture future (more than one-period ahead) lending returns. In the particular example of this graph, there are five different levels of total investment, $k$, that are consistent with zero profits for financial intermediaries. Three of them are similar to the short-term debt case, to the right of the line $k = \frac{\kappa l+s}{1-\delta_k}$. In these equilibria, the firm has a strictly positive current default probability (CDP). However, there are now two additional equilibria, to the right of the line $k = \frac{\kappa l+s}{1-\delta_k}$. In these equilibria, the firm does not default this period (that is, its CDP is zero). However, it may default at some point in the future. As total investment today increases, the net worth of the firm this period increases. As a result, the firm will be able to invest more, and is less likely to default, in future periods. This improves the continuation value of the bonds, $\mathbf{q}_C$. Thus, gross lending returns keep rising even after total investment is large enough for the firm not to be at risk of default in the current period.

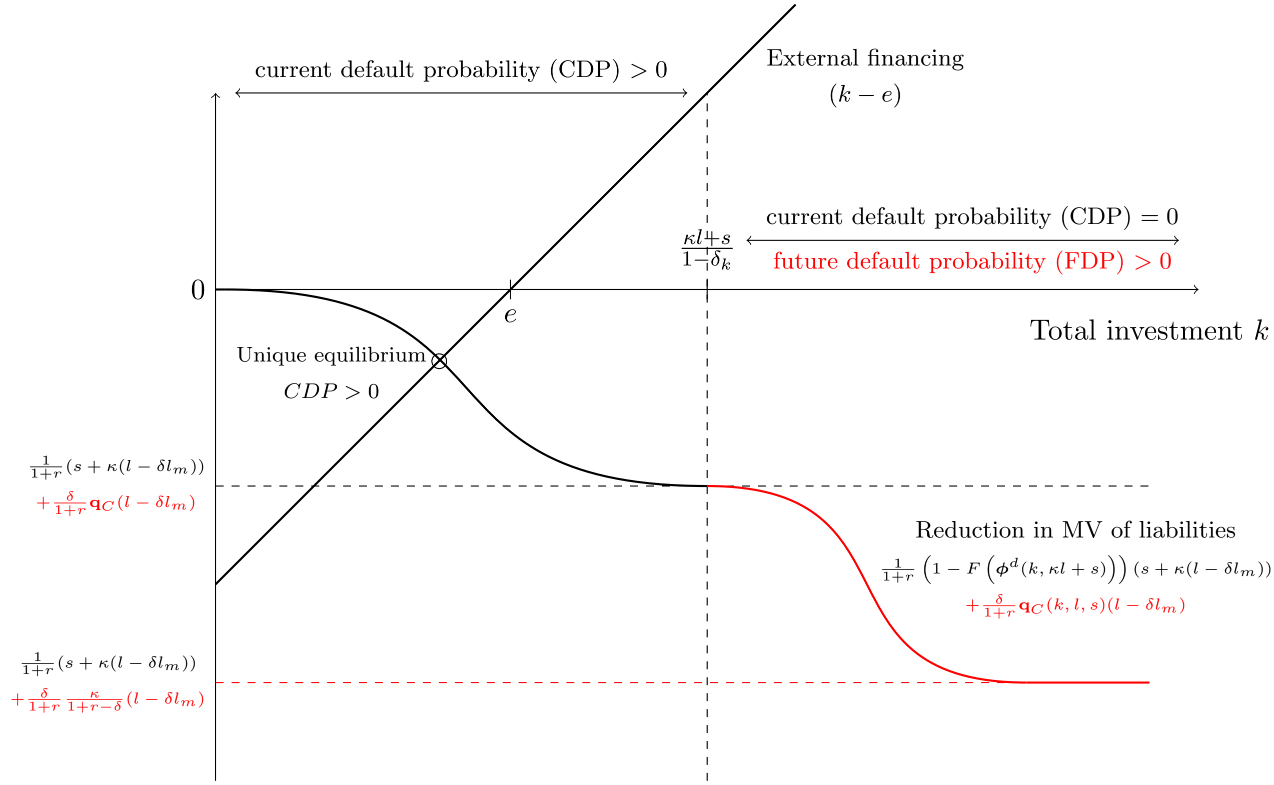


Figure 4: Financial market equilibrium in the model with short- and long-term debt model, in the case of net repurchases of long-term debt $l < \delta l_m$.

The insight that figure (3) illsutrates is that complementarity between *future* continuation probability (and thus the continuation value of long-term debt) and current investment $k$ can lead to *additional* equilibria, relative to the short-term debt case. The two equilibria on the red portion of the graph illustrate this. The lower-k one corresponds to a firm which is likely to default in the future, because current investment is sufficient to guarantee continuation today, but not to allow for substantial accumulation of retained earnings. On the contrary, in the high-k equilibrium, the firm is unlikely to default today or at any point in the future. As a result, it can issue debt at prices close to the risk-free value of the bond, $\mathbf{q}_L^{RF} = \frac{\kappa}{1+r-\delta}$.

**Multiplicity when firms are net debt repurchasers** Figure 4 illustrates the case of debt repurchases. In this case, external financing is negative ($k < e$), and the firm is using up internal cash in order to repurchase outstanding debt. In this case, the right hand side of the zero profit condition (28) is negative; its absolute value is best understood as the net reduction in the market value of outstanding liabilities. Total capital $k$ must now adjust in such a way that the fall in the market value of the firm's liabilities equals net payments to creditors.

In this case, as illustrated by the graph, the financial market equilibrium is always unique. This is because the complementarity between investment and the market value of liabilities is not operating in the case of repurchases. Indeed, as $k$ increases, the firm devotes fewer resources to repurchases, and more to investment. Therefore, the firm is *less* likely to default after the repurchase. This makes remaining the value of remaining liabilities worth *more*. By no-arbitrage, the value of repurchased liabilities also increases. This "substituability" between repurchases and the value of outstanding debt guarantees the unicity of prices in the case of debt repurchases.

**Equilibrium selection** In the example of figure 3, the levels of investment $k$ consistent with the zero-profit condition of the lenders can also be ranked. This suggests following a similar approach to the short-term debt model. Namely, first define a maximum-capital function:

$$\begin{aligned}\mathbf{k}(e,l_m,l,s;\mathbf{q}_C) \quad = \arg\max \Bigg\{ \quad k \geq 0 \quad \text{s.t.} \quad k - e = \; & \frac{1}{1+r}\left(1 - F\left(\phi^d(k,\kappa l + s)\right)\right)(s + \kappa(l - \delta l_m)) \\ + \; & \frac{\delta}{1+r}\mathbf{q}_C(k,l,s)(l - \delta l_m) \quad \Bigg\}\end{aligned} \tag{29}$$

and then, construct price schedules consistent which guarantee that a firm with capital structure $(e, l, l_m, s)$ will have capital $\mathbf{k}(e, l_m, l, s; \mathbf{q}_C)$.

The appendix formally establishes that, given $\mathbf{q}_C$, one can always derive price functions consistent with the maximum-capital function. Thus, *if the continuation value of bonds were exogenously fixed*, then a unique, Pareto-dominating equilibrium could in principle be selected. However, the continuation value of bonds is not exogenously fixed; that is, $\mathbf{q}_C$ depends on the firm's policies, which in turn depend on the equilibrium price functions. Thus, while an equilibrium can, in principle, be constructed using this approach, there is no guarantee that it will globally unique, in constrast to the short-term debt case.

In what follows, when referring to "the equilibrium", I will mean a recursive competitive equilibrium constructed using the maximum-capital function, that is, such that, for each capital structure $(e, l_m, l, s)$, the level of investment consistent with equilibrium in financial markets is the largest possible *given the policy functions of firms*. Appendix C describes the algorithm used to construct the max-capital function, the price function $\mathbf{q}_L(e, l_m, l, s)$, and the associated recursive competitive equilibrium.[13]

# 3 A quantitative evaluation

As a first step towards understanding the predictions of the model, this section reports moments of key financial and real variables in a calibrated version of the model. The key message is that, quantitatively, the optimal maturity of debt tends to be very short, so that a model with endogenous maturity tends to inherit most of the properties of short-term debt models.

## 3.1 Baseline calibration

| Parameter | | Value |
|---|---|---|
| $r$ | Risk-free rate | 0.020 |
| $\beta^{-1} - 1$ | Firm discount factor | 0.030 |
| $\zeta$ | Degree of returns to scale | 0.845 |
| $\delta_k$ | Rate of depreciation | 0.066 |
| $\sigma(\phi)$ | Std. dev. of productivity | 0.126 |
| $\mathbb{E}(\phi)$ | Average productivity | 0.146 |
| $\chi$ | Liquidation losses | 1 |
| $\delta$ | Proxy for debt maturity | 0.93 |
| $c$ | Coupon payment on LT debt | 0 |

Table 1: Structural parameters in the baseline calibration of the ST, EM and LT models.

In order to isolate the role of the endogenous choice of debt maturity on the financial and real choices of the firm, I compare three versions of the model: a model with only long-term debt contracts (LT for short); a model with only short-term debt contracts (ST for short); and the

[13] There are two main difficulties in constructing a numerical solution to the model. The first one is the construction of the menu of prices $\mathbf{q}_S(e, l_m, l, s)$ and $\mathbf{q}_L(e, l_m, l, s)$ that satisfy the equilibrium selection criterium described in section 2.5.2. Appendix C.1 describes a strategy to do this. Because the continuation value of bonds is monotonic, equilibrium multiplicity can only occur in the case of net debt issuances, as discussed above. In this case, the maximum-capital price of long-term bonds can always be located by using the fact that it is the largest fixed point of a contraction mapping. Using this insight, computation of prices is typically fast and precise.

The second difficulty is to construct the relevant state-space on which the firm problem is defined. In particular, because the degree of concavity of the value function $\mathbf{V}(e, l_m)$ varies substantially with $l$, it is important to choose the range of equity levels on which to solve the model for each value of $l_m$ carefully. In order to adress this issue, the algorithm relies on the fact that the net value of equity at the dividend stage, $\beta\mathbf{V}(e, l) - e$, has a global maximum for any value of $l$, denoted by $\bar{\mathbf{e}}(l)$. Because firms never reinvest more than $\bar{\mathbf{e}}(l)$, once $\bar{\mathbf{e}}(l)$ has been identified, it is sufficient to compute the solution for pairs $(e, l)$ such that $e \in [0, \bar{\mathbf{e}}(l)]$. This idea is implemented by solving a finite-horizon economy by backward induction, and adjusting the state-space at each step of the induction. The solution to the original problem obtains as the horizon becomes large. Details are reported in appendix C.2.

endogenous maturity model described in section 2 (EM for short). The LT and the ST models are straightforward restrictions of the EM model: the ST model corresponds to the restriction $l = l_m = 0$, while the LT model corresponds to the restriction $s = 0$. Since none of these restrictions depend on structural parameters, one can compare the three models under a common calibration.

Two model parameters control the duration and cash flow associated with long-term bonds: $\delta$, a proxy for debt maturity, and $\kappa = (1-\delta)c + \delta$, the expected cash flow from a long-term bond with maturity $\delta$ and coupon $c$. These parameters are fixed, respectively, to $\delta = 0.93$ and $\kappa = \delta$. The value of $\delta$ corresponds to a maturity of long-term debt contracts of 15 years, an upper bound on the average maturity of long-term debt contracts documented in the empirical literature.[14] The choice of $\kappa = \delta$ corresponds to the issuance of zero-coupon bonds. Low coupon payments reduce short-run liquidation risk by lowering the immediate repayment to creditors, which should contribute to making long-term debt issuance more attractive for firms. The implications of the model when coupon payments are larger or the maturity of long-term contracts are qualitatively similar.

Given values for $\delta$ and $\kappa$, I calibrate the remaining structural parameters of the model as follows. First, given that the model is calibrated at the annual frequency, the risk-free rate and the discount rate of entrepreneurs are set to $r = 0.02$ and $\beta^{-1} - 1 = 0.030$, respectively. I assume that there is no recovery in default, that is, $\chi = 1$. This assumption can also be relaxed, but it is not central to the results; it is maintained here because of it will prove useful in clarifying the analytical discussion of section 4.

The distribution of productivity shocks is assumed to be Weibull, with a mean normalized so that the unconstrained optimal investment level in a static version of the model is $k^* = \left(\frac{\zeta \mathbb{E}(\phi)}{r+\delta_k}\right)^{\frac{1}{1-\zeta}} = 100$. This leaves three structural parameters to be calibrated: the standard deviation of productivity shocks, $\sigma(\phi)$, the degree of returns to scale, $\zeta$, and the rate of depreciation, $\delta_k$. These parameters are jointly calibrated so that the first and second moments of the investment rate, $\frac{k' - (1-\delta_k)k}{k}$, and the ratio of operating income to book assets, $\frac{\phi k^\zeta}{k}$, in the LT model, match their empirical counterparts in a sample of Compustat firms between 1976 and 2014.[15] Model moments are computed by solving for optimal policies and simulating 100 panels of 5000 firms for 30 periods

[14] Stohs and Mauer (1996) document an average debt maturity of 3.38 years in their sample, which includes short-term debt contracts. Guedes and Opler (1996) report an average time to maturity of 12.2 years at time of issuance in their sample. In more recent work, Harford et al. (2014) find an average maturity of 12.0 years for bond issuances, while the main average maturity measure of Choi et al. (2014) is 9.01 years.

[15] The data sources and sample construction are reported in appendix A.

| | | | Data | LT model | ST model | EM model |
|---|---|---|---|---|---|---|
| Investment | $\frac{i}{k}$ | $\mathbb{E}(.)$ | 0.07 | 0.07 | 0.11 | 0.10 |
| | | $\sigma(.)$ | 0.07 | 0.07 | 0.33 | 0.29 |
| Profitability | $\frac{\phi k^{\zeta}}{k}$ | $\mathbb{E}(.)$ | 0.08 | 0.11 | 0.10 | 0.10 |
| | | $\sigma(.)$ | 0.18 | 0.09 | 0.09 | 0.09 |
| Leverage | $\frac{q_L l + q_S s}{k}$ | $\mathbb{E}(.)$ | 0.22 | 0.22 | 0.77 | 0.79 |
| | | $\sigma(.)$ | 0.19 | 0.06 | 0.02 | 0.04 |
| Tobin's Q | $\frac{\mathbf{V} + q_L l + q_S s}{k}$ | $\mathbb{E}(.)$ | 1.36 | 1.55 | 1.46 | 1.37 |
| | | $\sigma(.)$ | 1.40 | 0.07 | 0.49 | 0.23 |
| ST spread | $q_S^{-1} - (1+r)$ | $\mathbb{E}(.)$ | 29bps | | <1bps | <1bps |
| LT spread | $q_L^{-1} - (1+\frac{r}{1-\delta})$ | $\mathbb{E}(.)$ | 265bps | 146bps | | 91bps |
| Default rate | $\mathbf{1}_{\{\phi \in \mathcal{D}\}}$ | $\mathbb{E}(.)$ | 1.52% | 0.54% | <0.05% | <0.05% |

Table 2: Real and financial moments in the baseline calibration.

each, and discarding the first 15 periods.[16]

The rationale for using the LT model to calibrate structural parameters is that, as has been discussed, among others, by Gomes and Schmid (2010), the ST model has difficulty reconciling leverage ratios with observed credit spreads and default rates. That is, in the ST debt model, firms tend to have excessively high leverage ratios, excessively low credit spreads, and excessively low default rates. This will be apparent in the results reported below as well. However, the main point of the dicussion which follows is that optimal debt maturity choices in the EM model tend to induce borrowing and investment behaviors that are very close to those of the ST model. This conclusion would not be changed if one had used the ST model as anchor in the calibration of structural parameters. The resulting structural parameters are reported in table 1.

## 3.2 Real and financial moments

Table 2 reports key real and financial moments in the 3 versions of the model, as well as their empirical counterparts.[17] Both the ST and EM models tend to predict both an excessively volatile investment rate, and excessively high leverage relative to the data, while the LT model does not. Investment is three to four times more volatile in the model than in the data, while leverage is four times larger. As mentioned before, both implications of the ST model have been documented

[16]The simulation is done without replacement; that is, exiting firms from each cohort are not replaced. However, replacing exiting firms by new firms with a common initial size does not affect the qualitative predictions of the model regarding optimal maturity. The results are also not sensitive to the size of the simulation.

[17]Appendix A reports variable definitions in the data.

| | | | Data | EM model |
|---|---|---|---|---|
| Share of short-term debt | $\frac{q_S s}{q_S s + q_L l}$ | $\mathbb{E}(.)$ | 0.30 | 0.85 |
| | | $\sigma(.)$ | 0.32 | 0.06 |
| | | $\% > 0.5$ | 0.19 | 1.00 |
| | | $\% > 0.9$ | 0.09 | 0.34 |
| | | corr( . , $\frac{i}{k}$ ) | -0.10 | -0.20 |
| | | corr( . , lev ) | -0.27 | -0.61 |
| | | corr( . , prof ) | -0.23 | -0.02 |
| | | corr( . , $k$ ) | -0.04 | 0.58 |
| Average maturity of debt | $\frac{\frac{1}{1-\delta} q_L l + q_S s}{q_S s + q_L l}$ | $\mathbb{E}(.)$ | 6.09 | 3.08 |
| | | $\sigma(.)$ | 4.06 | 0.85 |
| | | $\% > 3$ years | 0.74 | 0.43 |
| | | $\% > 5$ years | 0.60 | 0.00 |
| | | corr( . ,$\frac{i}{k}$) | 0.08 | 0.20 |
| | | corr( . , lev ) | 0.26 | 0.61 |
| | | corr( . , prof ) | 0.20 | 0.02 |
| | | corr( . , $k$ ) | 0.06 | -0.58 |

Table 3: Optimal maturity structure in the baseline calibration. For the share of short-term debt, the line "$\% > 0.5$" reports the fraction of firms with a share of short-term debt of at least 50%, and the the line "$\% > 0.9$" is defined similarly. Definitions for the average maturity of debt are similar.

elsewhere; these results show that they also hold for the EM model. In particular, investment is substantially less volatile in the LT than in the ST model because capital accumulation is slower with long-term debt contracts, so that net worth movements of similar magnitudes result in smaller increases in capital. As will be discussed in section 4, this can be thought of as a form of of debt overhang.

These sharp differences between the EM and ST and LT models are also reflected in credit spreads. In particular, ST spreads are virtually zero in both models, whereas LT spreads are much larger. This is a first indication of the fact that "agency costs" of long-term debt issuance, i.e. the discount relative to short-term debt due to an inability to commit to a future path of dividend and investment, is substantial in this model. Consistent with the difference in spreads, rates of default in the EM and ST models are much smaller than in LT model.

### 3.3 Optimal maturity structure

The similarity of financial and real moments between the EM and ST models suggest that the optimal maturity structure of firms in the EM model should be close to short-term. The results reported in table 3 confirm this intuition. The table reports the moments of two measures of debt maturity: the share of short-term debt, and the average maturity of debt. The moments and quantiles of these measures indicate that firms in the EM model use mostly short-term debt. In particular, the average share of short-term debt in the model is 85%, and the average maturity of debt oustanding is 3.08 years (recall that the maturity of the long-term debt contract is calibrated to be 15 years). Virtually all firms have at least 50% of debt outstanding in short-term form, and virtually no firms have an average maturity of debt outstanding over 5 years.

The contrast with the data moments also reported in table 3 is striking. In the sample of firms used for this quantitative exercise, the fraction of short-term debt is, on average, 30%; only about a fifth of all firms have a share of short-term debt above 50%. Likewise, the average maturity of debt outstanding is 6.09 years, and 60% of firms have a maturity of debt outstanding of 5 years or less. The model thus falls far short of being able to account for the typical mix of short and long-term debt used by firms in the data.

While underpredicts the typical maturity of outstanding debt, it does match some of the qualitative features of the relationship between debt maturity, investment, leverage and profitability. In particular, both in the model and in the data, the share of short-term debt is negatively related to the rate of investment and leverage. This is also true for profitability, albeit weakly so in the model. However, as will become clear in the following section, this negative correlation is driven by smaller firms. They are the only net issuers of long-term debt in the model, and they tend to be the firms with the highest leverage and investment rates. This is manifest in the fact that there is a strong positive correlation, in the model, between size (proxied by $k$, total assets) and the share of short-term debt. By contrast, the correlation between size (measured by book assets) and the share of short-term debt in the data is weakly negative.

The upshot of the quantitative exercise of this section is that when debt maturity is endogenous, the optimal maturity structure is close to being purely short-term. As a result, the quantitative predictions of the EM model are close to those of the ST model, with the same shortcomings as the ST model in terms of rationalizing the leverage and spreads observed in the data. The next section explores the mechanisms behind this result.

# 4 Understanding the optimality of short-term debt

This section explores the result that the EM model features mostly short-term in more depth, and attempts to establish which feature of the environment of section 2 are central to this result. I start by a theoretical analysis in section 4.1. While the analytical results provided here do not fully characterize the optimal maturity structure, they help build intuition for the results provided by the numerical calibration. In particular, they help understand the main features of the policy functions of firms, which is the focus of the discussion in section 4.2.[18]

## 4.1 When can short-term financing be optimal, and why?

I now turn to the conditions under which short-term debt financing may be optimal. I start with the case of a firm with no legacy debt ($l_m = 0$). In this case, in certain regions of the state-space of the firm, one can pin down optimal maturity structures as those that maximize the *future* value of long-term debt. Provided that the future value of long-term debt is decreasing with the stock of debt oustanding, the optimal maturity structure is to issue only short-term debt. I then discuss the case of firms with legacy debt ($l_m > 0$). I start by introducing some notation, and stating an assumption that will be useful in the discussion to follow.

### 4.1.1 Preliminaries

**Definition 2** (Net equity value at the dividend stage)**.** *The net value of equity at the dividend stage is defined as:*

$$\mathbf{h}(n, l) = \max_{0 \leq e' \leq n} \beta \mathbf{V}(e', l) - e'.$$

Throughout the discussion, the following property of the value function $\mathbf{V}$ will be assumed to hold. This property requires the value function to be sufficiently concave, so that the the marginal value of reinvesting capital in the firm eventually drops below the marginal value of dividends.

**Assumption 2** (Maximum reinvestment scale)**.** *The value function* $\mathbf{V}$ *is such that, for any* $l$*, the function* $e \to \beta V(e, l) - e$ *has a unique global maximum, denoted by* $\bar{\mathbf{e}}(l)$*.*

With this property, the function $\mathbf{h}(e, l)$ satisfies $\mathbf{h}(e, l) \leq \mathbf{h}(\bar{\mathbf{e}}(l), l)$, with equality for any $e \geq \bar{\mathbf{e}}(l)$. The threshold $\bar{\mathbf{e}}(l)$ plays an important role in what follows: indeed, if a firm reaches the

[18] Throughout the section, I assume full output losses in default, that is, $\chi = 1$. The key qualitative predictions of the model are independent of this assumption; the assumption is instead meant to help clarify the discussion.

dividend issuance stage with debt $l$ but more cash than $\bar{\mathbf{e}}(l)$, it chooses to reinvest exactly $\bar{\mathbf{e}}(l)$, and consume the remainder as dividend.

### 4.1.2 The value of debt and equity

The following lemma provides a decomposition of the value of a firm's debt and equity, between the expected current-period cash flows, and future (or continuation) values. I refer to a firm with internal finance $e$ and initial (or legacy) debt $l_m$, that chooses to issue $s$ short-term debt contracts and $l$ as "a firm with capital structure $(e, l_m, l, s)$." Note that $(l, s)$ may not be the optimal policies, given $(e, l_m)$; in fact, the discussion that follows uses this decomposition in order to study the optimal debt issuance policies.

**Lemma 3.** *Let the functions $\mathbf{W}_e$, $\mathbf{W}_d$, $\mathbf{W}_{cf}$, $\mathbf{W}_e^c$, $\mathbf{W}_d^c$ be defined by:*

$$\mathbf{W}_e(l, s; k, \phi^d) = \int_{\phi > \phi^d} \mathbf{V}^c((\phi - \phi^d)k^\zeta, l)dF(\phi) \tag{30}$$

$$\mathbf{W}_d(l_m, l, s; k, \phi^d) = (\kappa\,(1 - F\,(\phi_d)) + \delta\mathbf{q}_C(k, l, s))\,\delta l_m \tag{31}$$

$$\begin{aligned}\mathbf{W}_{cf}(e; k, \phi_d) &= \mathbb{E}(\phi)k^\zeta - (r + \delta_k)k + (1 + r)e \\ &\quad - F(\phi^d)\left(\mathbb{E}(\phi|\phi \le \phi_d)k^\zeta + (1 - \delta_k)k\right)\end{aligned} \tag{32}$$

$$\mathbf{W}_e^c(l, s; k, \phi_d) = \int_{\phi > \phi^d} \mathbf{h}((\phi - \boldsymbol{\phi}^d(k, \kappa l + s))k^\zeta, l)dF(\phi) \tag{33}$$

$$\mathbf{W}_d^c(l, s; k) = \delta\mathbf{q}_C(k, l, s) \tag{34}$$

*Then, when $k = e + \mathbf{q}_S(e, l_m, l, s)s + \mathbf{q}_L(e, l_m, l, s)(l - \delta l_m)$ and $\phi_d = \boldsymbol{\phi}^d(s + \kappa l, k)$, that is, at a financial market equilibrium, the total value of equity $\mathbf{W}^e$ and legacy debt $\mathbf{W}^d$ is related to expected current income $\mathbf{W}_{cf}$ and the continuation values of equity $\mathbf{W}_e^c$ and debt $\mathbf{W}_d^c$ by:*

$$\mathbf{W}_e(l, s; k, \phi^d) + \mathbf{W}_d(l_m, l, s; k, \phi^d) = \mathbf{W}_{cf}(e; k, \phi_d) + \mathbf{W}_e^c(l, s; k, \phi_d) + \mathbf{W}_d^c(l, s; k).$$

This lemma decomposes of the value of the debt and equity of a firm with a particular capital structure $(e, l_m, l, s)$. This decomposition effectively nets out all transfers between debt and equity holders in the current period, using the zero-profit conditions of lenders. Additionally, it assumes that the firm behaves optimally from the next period onward; this is captured by the use of the function $\mathbf{h} = \beta\mathbf{V} - e$. The first term, $\mathbf{W}_e$, is the value of equity, that is, the objective of the firm in its maximization problem (19). The second term, $\mathbf{W}_d$, is the market value of legacy debt, $l_m$. The third term,

$$\mathbf{W}_{cf}(e;k,\phi_d) = \underbrace{\mathbb{E}(\phi)k^{\zeta} - (r+\delta_k)k + (1+r)e}_{\text{expected income this period}} - \underbrace{F(\phi^d)\left(\mathbb{E}(\phi|\phi \le \phi_d)k^{\zeta} + (1-\delta_k)k\right)}_{\text{expected deadweight losses from bankruptcy}},$$

captures the total expected income generated by the assets of the firm this period, net of bankruptcy costs. Note that, when $\phi^d = 0$, expected income is maximized if and only if $k$ is equal to the optimal static level of investment, $k^*$:

$$k^* = \left(\frac{\zeta\mathbb{E}(\phi)}{r+\delta_k}\right)^{\frac{1}{1-\zeta}}.$$

Finally, $\mathbf{W}_e^c(l,s;k,\phi_d)$ is the expected value of equity net of reinvested earnings, and $\mathbf{W}_d^c(l,s;k)$ is expected value of long-term debt in the following period.

#### 4.1.3 Implementing the optimal static level of investment

The starting point for the analysis of the optimal maturity structure of debt is that, when a firm is either large enough, or has low legacy debt, it may be able to invest at the optimal static level $k^*$ *today*. In this case, long-term debt issuance will be chosen solely to maximize the continuation value of debt and equity, while short-term financing will be arranged so that $k = k^*$. The following proposition describes these situations more formally.

**Proposition 1** (Capital structures implementing $k = k^*$)**.** *Let $(e, l_m, l)$ be given. Assume that:*

$$e \in [\mathbf{e}_{min}(l_m,l), \mathbf{e}_{max}(l_m,l)], \text{ where:}$$

$$\mathbf{e}_{min}(l_m,l) \equiv \frac{r+\delta_k}{1-\delta_k}k^* + \frac{1}{1+r}\bar{\mathbf{e}}(l) + \frac{1}{1+r}\left(\kappa + \delta\hat{\mathbf{q}}_L\left(\bar{\mathbf{e}}(l),l\right)\right)\delta l_m - \frac{\delta}{1+r}\hat{\mathbf{q}}_L\left(\bar{\mathbf{e}}(l),l\right)l, \quad (35)$$

$$\mathbf{e}_{max}(l_m,l) \equiv k^* - \frac{1}{1+r}\left(\kappa + \delta\hat{\mathbf{q}}_L\left(\bar{\mathbf{e}}(l),l\right)\right)(l - \delta l_m). \quad (36)$$

*Then, the following borrowing and investment policies are feasible:*

$$\mathbf{l}^*(e,l_m,l) = l \quad (37)$$

$$\mathbf{s}^*(e,l_m,l) = (1+r)(\mathbf{e}_{max}(l_m,l) - e) \quad (38)$$

$$\mathbf{k}^*(e,l_m,l) = k^* \quad (39)$$

*Under this capital structure, the probability of current default of the firm is* $0$*, and moreover, for any realization of productivity* $\phi \geq 0$*, the firm reinvests the largest possible amount,* $\bar{\mathbf{e}}(l)$*:*

$$\mathbf{e}'\Big( \ \phi(k^*)^\zeta + (1-\delta_k)k^* - (\kappa l + \mathbf{s}^*(e, l_m, l)) \ , \ l \ \Big) = \bar{\mathbf{e}}(l).$$

This proposition establishes that, given a level of legacy debt $l_m$ and desired long-term debt $l$, there is a range of equity levels $e$ such that the firm can implement the investment policy $k^*$, while at the same time having no default risk, and keeping enough cash to reinvest as much as possible in the following period. This policy is feasible whenever $e \geq \mathbf{e}_{min}(l_m, l)$, or:

$$e + \frac{\delta}{1+r}\hat{\mathbf{q}}_L\left(\bar{\mathbf{e}}(l), l\right) l - \frac{1}{1+r}\left(\kappa + \delta\hat{\mathbf{q}}_L\left(\bar{\mathbf{e}}(l), l\right)\right)\delta l_m \geq \frac{r+\delta_k}{1-\delta_k}k^* + \frac{1}{1+r}\bar{\mathbf{e}}(l).$$

Intuitively, the left-hand side of this equation represents the resources available to the firm after long-term debt issuance (in net terms), assuming that the firm is indeed risk-free this period and always reinvests $e' = \bar{\mathbf{e}}(l)$. The right-hand side captures the uses of these funds: investing at scale $k = k^*$, and saving $e' = \bar{e}(l)$ for the following period.

Additionally, the firm's inside equity must satisfy $e \leq \mathbf{e}_{max}(l_m, l)$. This condition guarantees that the level of short-term debt issuance that implements $k = k^*$ and $e' = \bar{e}(l)$ is positive. Otherwise, the constraint that the firm should be a net short-term borrower ($s \geq 0$) could bind.

Note that under the capital structures implementing $k = k^*$, the ratio of the market value of long-term debt to the total market value of debt is given by:

$$\begin{aligned}
\mathbf{m}(e, l_m, l) &\equiv \frac{q_L l}{q_L l + q_S s} \\
&= 1 - \frac{\bar{\mathbf{e}}(l_m, l) - e}{k^* + \frac{1}{1+r}\left(\kappa + \delta\hat{\mathbf{q}}_L\left(\bar{\mathbf{e}}(l), l\right)\right)\delta l_m - e} \\
&\in \left[1 - \frac{\bar{\mathbf{e}}(l_m, l) - \underline{\mathbf{e}}(l_m, l)}{k^* + \frac{1}{1+r}\left(\kappa + \delta\hat{\mathbf{q}}_L\left(\bar{\mathbf{e}}(l), l\right)\right)\delta l_m - \underline{\mathbf{e}}(l_m, l)}, 1\right]
\end{aligned}$$

For a given level of long-term debt issuance, under a policy that implement $k = k^*$, it is clear that average debt maturity should decline as inside equity increases; indeed, as $e$ rises, less total external financing is required; if the optimal amount of long-term financing is fixed (which is the case when $k = k^*$ and $l = l^*$), the amount of short-term financing should therefore fall.

#### 4.1.4 Firms with no legacy debt ($l_m = 0$)

We next turn to a case in which the optimal maturity structure can easibly be pinned down: when the firm has no legacy debt ($l_m = 0$).

**Corollary 4** (Maturity shortening when $l_m = 0$)**.** *Define:*

$$l^* \quad = \quad \arg\max_{l \geq 0} \quad \mathbf{h}(\bar{\mathbf{e}}(l), l) + \delta \hat{\mathbf{q}}_L\left(\bar{\mathbf{e}}(l), l\right) l,$$

*If a firm has no legacy debt ($l_m = 0$), and if $e \in [\mathbf{e}_{min}(l^*, 0), \mathbf{e}_{max}(l^*, 0)]$, then the firm chooses $l = l^*$ and the policy described in proposition 1.*

*Moreover, if the continuation value of bonds, $\hat{\mathbf{q}}_L\left(\bar{\mathbf{e}}(l), l\right) l$, is decreasing with $l$, then $l^* = 0$, and the optimal maturity structure is to issue only short-term debt.*

It is useful to separate the results of this corollary in two parts. First, when the firm has no legacy debt, then for any level of long-term debt issuance $l$, the policy described in proposition 1 is better than any other policies, and implementable provided that the firm has sufficient inside equity. To see why this is the case, recall the decomposition of lemma (1):

$$\mathbf{W}_e(l, s; k, \phi^d) + \mathbf{W}_d(0, l, s; k, \phi^d) = \mathbf{W}_{cf}(e; k, \phi_d) + \mathbf{W}^c_e(l, s; k, \phi_d) + \mathbf{W}^c_d(l, s; k). \tag{40}$$

When the firm has no legacy debt, regardless of the policy implemented:

$$\mathbf{W}_d(0, l, s; k, \phi^d) = 0.$$

Therefore, the firm chooses its capital structure in order to maximize the right-hand side of (40). The expected value of current income $\mathbf{W}_{cf}$, reaches its global maximum when $k = k^*$ and $\phi_d = 0$. Moreover, the future value of debt is increasing in the amount of reinvested income, $e'$, so that setting $e' = \bar{\mathbf{e}}(l)$ maximizes the total value of future claims on the firm.[19] Thus, for any level of long-term debt $l$, a policy that allows the firm to set $k = k^*$ and $e' = \bar{\mathbf{e}}(l)$ while having no risk of default today, $\phi_d = 0$, will be optimal. As established by lemma (1), the policy is feasible when the firm has sufficient equity $e$.

[19] This is formally established in the appendix.

Second, among all possible levels of long-term debt issuance $l$, the corollary states that the firm chooses the one that maximizes the value of future claims on the firm, the long-term debt level denoted by $l^*$. To put it differently, the current decisions of the firm are dissociated from its future decisions; long-term debt issuance is irrelevant to current profits, and is set solely so as to maximize the future value of claims. Note that total issuance of long-term debt is independent from the value of equity, $e$. Therefore, the resulting maturity structure of debt is given by:

$$\begin{aligned}\mathbf{m}(e, 0, l^*) &= 1 - \frac{\bar{\mathbf{e}}(0, l^*) - e}{k^* - e} \\ &\in \left[1 - \frac{\bar{\mathbf{e}}(0, l^*) - \underline{\mathbf{e}}(0, l)}{k^* - \underline{\mathbf{e}}(0, l)}, 1\right]\end{aligned}$$

What determines the value of $l^*$, the optimal level of long-term debt? In general, $l^*$ should maximize the sum of the continuation value of equity, $\mathbf{h}(\bar{\mathbf{e}}(l), l)$, and the continuation value of debt, $\delta \hat{\mathbf{q}}_L (\bar{\mathbf{e}}(l), l)\, l$. However, since the value of the firm, $\mathbf{V}$, is decreasing with $l$, by the envelope theorem, the continuation value of equity is also decreasing with $l$. Therefore, issuance of long-term debt can *only* be positive if continuation value of debt $\hat{\mathbf{q}}_L (\bar{\mathbf{e}}(l), l)\, l$ is increasing with $l$.

Whether the continuation value of debt increases with $l$ cannot be established analytically. However, this term captures the importance of the commitment problem in determining the optimal maturity structure of debt. If the commitment problem is mild, then declines in the long-term price of debt $\hat{\mathbf{q}}_L (\bar{\mathbf{e}}(l), l)$ may be moderate as $l$ increases, and the total continuation value may be increasing with l. However, when they are severe, the long-term price of debt may very sensitive to current long-term debt issuance, and the total continuation value of long-term debt $\hat{\mathbf{q}}_L (\bar{\mathbf{e}}(l), l)\, l$ may decline as the firm issues increasing amounts of long-term debt. In that case, long-term debt issuance is never optimal, and the firm finances itself using only short-term debt.

#### 4.1.5 Firms with legacy debt ($l_m > 0$)

When the firm has legacy debt ($l_m > 0$), no general result can be established regarding the optimal maturity structure of debt. In particular, it may not be the case that the policies described by proposition (1) are optimal, even if they are feasible, that is, even when $e \in [\mathbf{e}_{min}(l_m, l), \mathbf{e}_{max}(l_m, l)]$. To understand when it may be feasible but not optimal to implement $k = k^*$ and $e' = \bar{\mathbf{e}}(l)$, note that value of this policy, relative to another policy that implements some other value of capital $k$

is given by:

$$\begin{aligned}\Delta\mathbf{W}(e,l_m,l;k) &= \tilde{\Pi}k^*g\left(\frac{k}{k^*}\right)+\left(\mathbf{h}(\bar{\mathbf{e}}(l),l)-\int_{\phi>\boldsymbol{\phi}^d(k,\kappa l+s)}\mathbf{h}((\phi-\boldsymbol{\phi}^d(k,\kappa l+s))k^\zeta,l)dF(\phi)\right)\\ &+ F\left(\phi^d(k,\kappa l+s)\right)\left(\mathbb{E}\left[\phi|\phi\leq\phi^d(k,\kappa l+s)\right]k^\zeta+(1-\delta_k)k-\kappa\delta l_m\right)\\ &+ \delta\left(\hat{\mathbf{q}}_L\left(\bar{\mathbf{e}}(l),l\right)-\mathbf{q}_C(k,l,s)\right)(l-\delta l_m).\end{aligned}$$

In this difference, the first two terms capture the gains from investing at $k=k^*$ and $e'=\bar{\mathbf{e}}(l)$. In particular, the function $g$ is positive, convex, and minimized at $\frac{k}{k^*}=1$; it captures the gains in expected current income resulting from setting $k=k^*$. The second term is the gap between the maximized continuation value of equity, and the continuation value of equity under the alternative policy. Both terms are always positive.

However, the terms in the second and third line may not be positive. This can occur when legacy debt $l_m$ is sufficiently large. The first term,

$$F\left(\boldsymbol{\phi}^d(k,\kappa l+s)\right)\left(\mathbb{E}\left[\phi|\phi\leq\phi^d(k,\kappa l+s)\right]k^\zeta+(1-\delta_k)k-\kappa\delta l_m\right),$$

captures liquidation losses, net of current liabilities associated with legacy debt. In effect, a positive default probability $F\left(\boldsymbol{\phi}^d(k,\kappa l+s)\right)$ may be beneficial, because it effectively reduces the value of legacy debt $l_m$. Additionally, the term:

$$\left(\hat{\mathbf{q}}_L\left(\bar{\mathbf{e}}(l),l\right)-\mathbf{q}_C(k,l,s)\right)(l-\delta l_m)$$

is negative, if and only if $l<\delta l_m$, that is, if the firm chooses a policy that involves debt repurchases. In other words, it may sometimes be preferable for the firm to reduce its debt burden. The firm can do so by investing below the optimal size $k<k^*$, and instead use its equity $e$ towards debt repurchases. This may imply a moderate default probability, which in turn further lowers the market value of legacy debt. Clearly, the optimality of such a policy would require legacy debt to be large; indeed, the current gains from a lower debt burden would have to exceed the losses associated with inefficient current investment and lower continuation value of equity (the two terms in the first line).

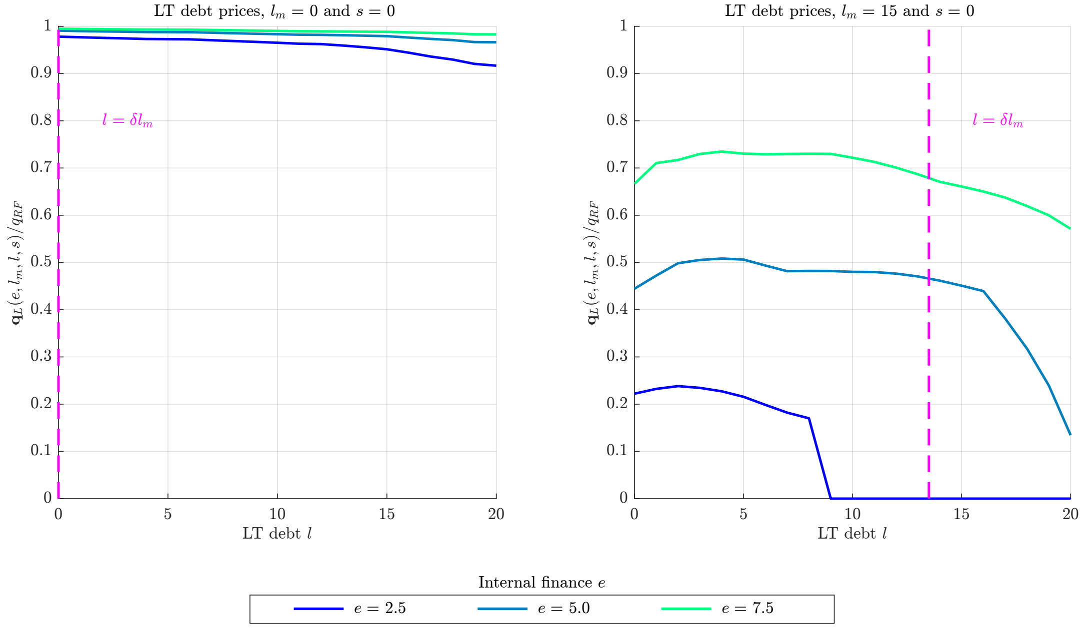


Figure 5: Price schedules $\hat{\mathbf{q}}_{\mathbf{L}}(e, l_m, l, s)$ as a function of desired long-term debt $l$, for different levels of initial outstanding long-term debt $l_m$ and equity $e$, when $s = 0$.

This analysis has emphasized the fact that, even conditional on selecting the high-price equilibria, long-term debt issuance may not be optimal in the model. The numerical analysis of the rest of this section explores the extent to which the properties of the price function that lead short-term financing to dominate are indeed met.

## 4.2 Equilibrium policies in a calibrated version of the model

This section connects the equilibrium policies of firms in a numerical solution to the properties established in the previous analytical discussion. In particular, it shows that, in the numerical solution, firms tend to shorten debt maturity as they accumulate internal resources, even when they have legacy debt.

### 4.2.1 Price schedules for long-term debt

It is useful to first discuss the price schedules implied by the zero-profit conditions of lenders, since the shape of these price schedules constrain firms' borrowing and investment decisions. Examples of these schedules are reported in figure 5. In these graphs, firms do not issue short-term debt ($s = 0$). Each panel compares the long-term bond price schedules of three firms with the same initial level of debt $l_m$ but different internal finance $e$. A dashed line indicates the long-term debt level $l = \delta l_m$. To the left of this line, prices correspond to debt repurchases, and to the right, they correspond to debt issuances.

On the left panel, firms are assumed to have no legacy debt ($l_m = 0$). In this case, firms with sufficient internal finance levels can issue long-term debt at prices that are close to risk-free, that is, $\mathbf{q}_L(e, l_m, l, s) \approx \frac{\kappa}{1+r-\delta}$. (The price functions reported are normalized by this value). At lower levels of internal finance, however, firms face a higher likelihood of current or future default, and accordingly, the price of long-term debt issuances fall. The right panel, by contrast, reports price schedules when firms have a higher level of legacy debt. In this case, even for firms with large levels of internal financing, long-term debt issuance only occurs at a deep discount relative to the risk-free price. For firms with limited internal ressources, the option of issuing long-term debt is essentially unavailable; the price of any long-term debt issuance in excess of current debt levels ($l > \delta l_m$) is zero. This fact will help explain the optimal debt issuance and investment policies which these firms choose in equilibrium.

### 4.2.2 Optimal maturity structure

Figure 6 contrasts the capital structures chosen by two different sets of firms: those with no initial debt ($l_m = 0$, the yellow lines) and those with a large initial stock of debt ($l_m = 100$, the red lines).

**Low legacy debt** In this calibration, firms with no initial debt can access both short- and long-term debt markets on good terms; as a result, they rely highly on external finance, and are very levered. Nevertheless, their probability of default is very low (bottom right panel). For these firms, prices of debt available correspond to the "good equilibria" (low current-default-probability and high-capital) described in figure 3.

The optimal maturity structure of these firms is reported in the bottom two panels. Firms with low internal finance $e$ are unable to raise sufficient external financing via short-term borrowing only in order to reach their desired size. In terms of the discussion of section 4.1, these firms' internal resources are such that:

$$\forall l \geq 0, \quad e \leq \mathbf{e}_{min}(0, l).$$

For these firms, long-term debt issuance provides a way to tap into additional debt capacity and move closer to their desired side — albeit an expensive one. As their internal ressources grow, these

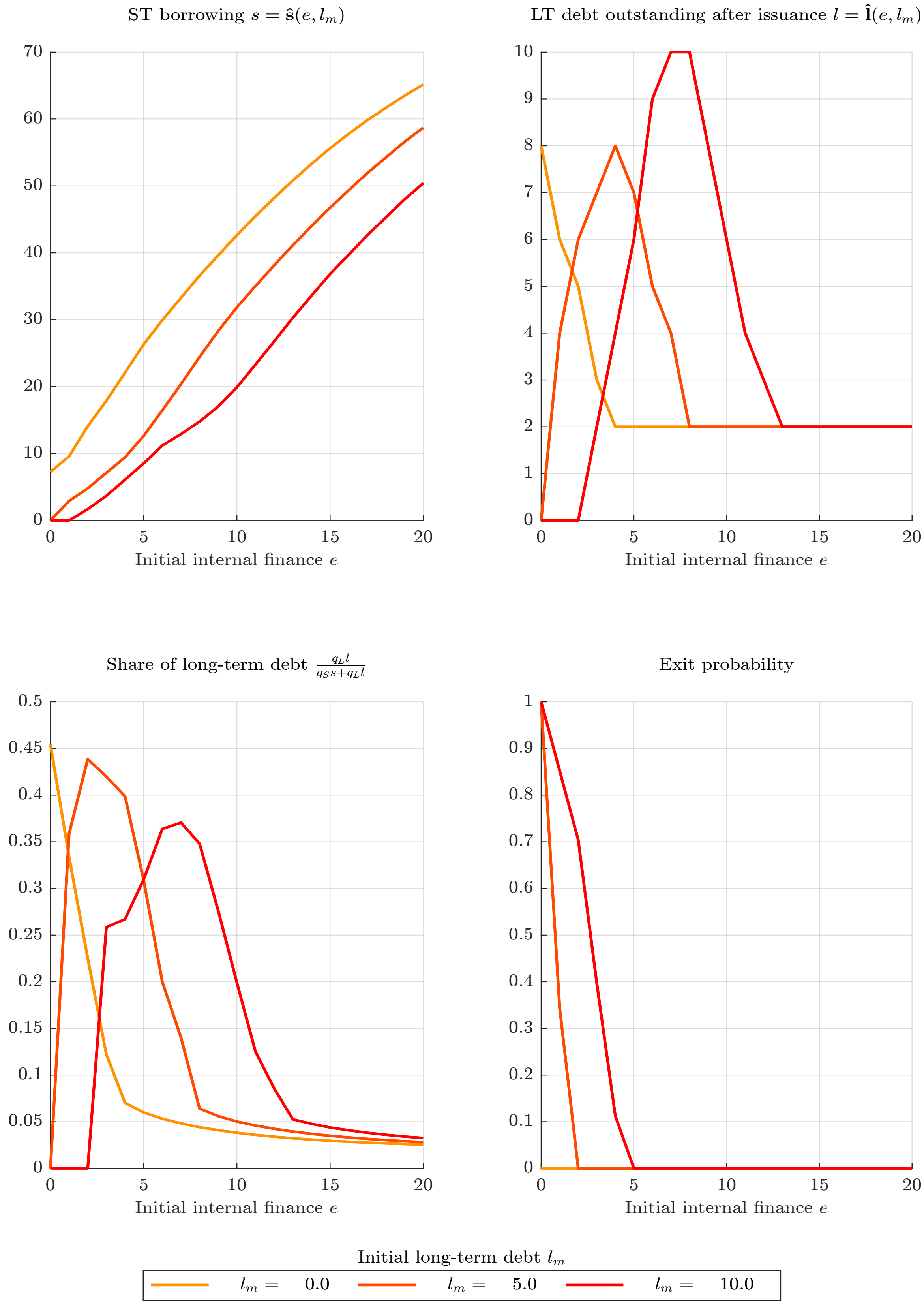


Figure 6: Issuance of short- and long-term debt as a function of initial internal finance $e$, for different levels of initial outstanding long-term debt $l_m$.

firms shift progressively from long-term to short-term borrowing. When $e$ is sufficiently large, these firms issue very little long-term debt. This is consistent with the intuition of corollary 4. Once firms accumulate sufficient amounts of internal finance to implement the maximum-capital/maximum-reinvestment policy, long-term debt issuance is chosen purely to maximize the future value of debt. Since that value, $\hat{\mathbf{q}}_L(\bar{\mathbf{n}}(l), l)l$, is declining in the amount of long-term debt issued, $l$, these firms prefer a shorter-term debt structure. Overall, average maturity of debt outstanding falls with internal financing as these firms grow (bottom left panel).

**High legacy debt** Firms with high initial debt fall into two broad categories. Firms with intermediate and high levels of internal finance, leverage and default probabilities are similar to those of low-debt firms. As in the case of low-debt firms, long-term debt issuance declines as these firms accumulate internal finance. Short-term debt issuance, on the other hand, first increases, then starts declining once the firm has grown to be large enough for self-financing. These leverage and debt issuance policies are thus analogous to those of low-debt firms at with lower levels of internal finance.

There is, however, another group: firms with a combination of high debt and low internal finance. These firms issue little to short-term debt. Some of these firms in fact use their small pool of internal resources to repurchase outstanding debt and deleverage; I come back to discussing this particular prediction of the model below. Those with higher internal finance instead choose to issue *more* long-term debt. These firms are close to default, and discount substantially the possibility of being operating in future periods and thus the future costs of using long-term debt. Additionally, coupon payments on long-term are small relative to short-term debt. Both factors help make long-term debt issuance attractive. As a result, debt maturity increases with internal financing for this group of firms (bottom left panel).

### 4.2.3 Debt overhang, debt repurchases, and investment

**Investment policy and the debt overhang problem** Figure 7 graphs the optimal investment policy function, $\hat{\mathbf{k}}(e, l_m)$, as a function of internal finance, for different levels of long-term debt. Note, first, that regardless of the initial debt level, once firms have sufficient internal ressources (i.e., once $e$ is large enough), investment is equal to its optimal value in the static model, $\hat{\mathbf{k}}(e, l_m) = k^* = 100$. The optimality of this policy was hinted at in corollary 4, although only for the case

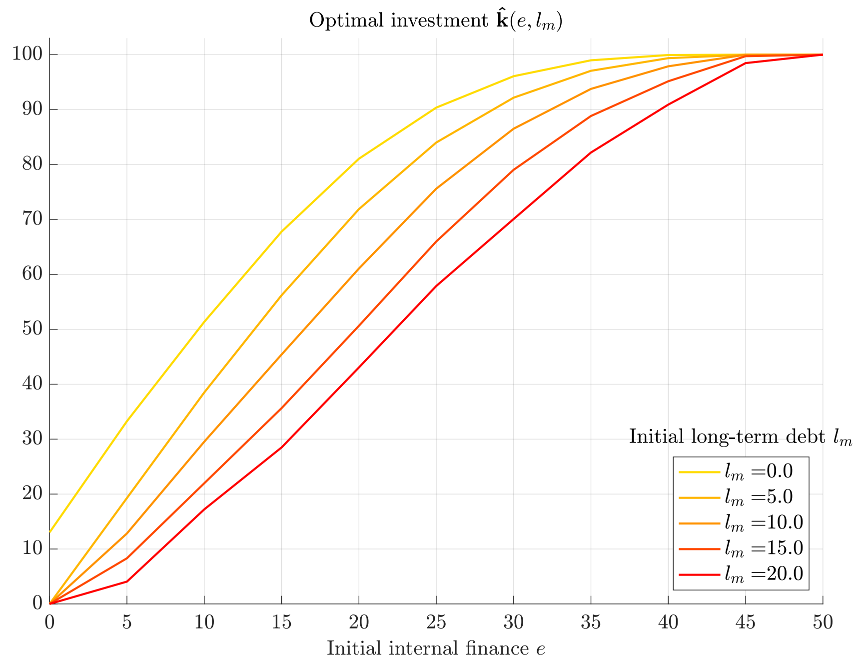


Figure 7: Optimal investment $\hat{\mathbf{k}}(e, l_m)$ as a function of initial internal finance $e$, for different levels of initial outstanding long-term debt $l_m$.

$l_m = 0$. For lower levels of internal finance, however, firms are constrained and operate at a scale below $k^*$.

However, the main message of figure 7 is that the initial indebtdness of the debt has a very strong impact on its investment choices; that is, in this model, the debt overhang problem is fairly severe. Even for modest levels of existing debt, the unconstrained investment level is reached at substantially higher levels of internal financing, relative to the $l_m = 0$ case. To put it differently, for a given level of internal finance $e$, firms with even modest levels of outstanding long-term debt will invest substantially less than firms with no long-term debt oustanding. Note, however, that these firms are still net long-term debt issuers.

This situation changes if the stock of outstanding debt is very large (the case $l_m = 20$ reported in the graph). In this case, as discussed above, firms undertake a different debt and investment policy: they spend their internal ressources repurchasing outstanding debt, rather than investing. (Graphically, this is represented by the fact that the red investment line is close to zero for these high debt levels). As a results, these firms operate with very little fixed assets. Investment only resumes once either sufficient internal finance has been accumulated, or once the firm has sufficiently reduced its outstanding debt stock.

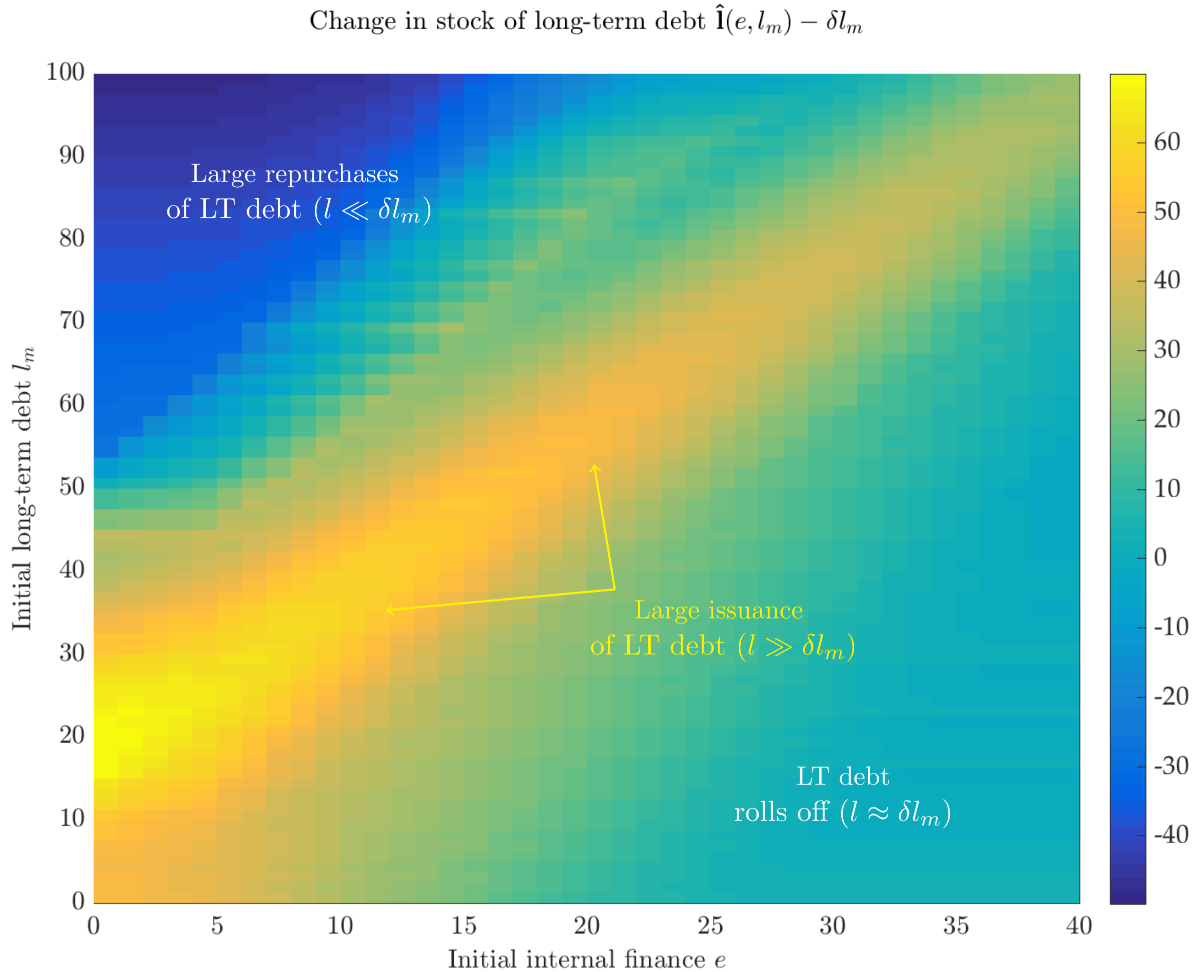


Figure 8: Equilibrium debt policies over firms' state-space $(e, l_m)$. In the blue region, firm repurchase outstanding debt. In the yellow and green regions, firms issue new debt or let existing debt roll off; see text for more details.

**Debt issuances and debt repurchases** How broad is the debt repurchase policy described above, and how does it depend on firms' state variables? Figure (8) answers this question, by reporting the debt issuance and debt repurchase policies of firms. This figure represents the net issuance of long-term debt, $\hat{\mathbf{l}}(e, l_m) - \delta l_m$, as a function of the two state variables of the firm, internal finance $e$ and outstanding debt $b_m$. Darker colors indicate net repurchases, while warmer colors indicate debt issuance. In the top left region, which corresponds to high levels of outstanding debt relative to internal finance ($l_m \gg e$), firms choose to use their cash in order to repurchase debt, instead of investing. In the bottom right region, internal finance is larger relative to long-term debt outstanding, firms tend to either roll-over existing debt ($\hat{\mathbf{l}}(e, l_m) = \delta l_m$), or repurchase small amounts. In that region, firms are typically issuing debt on which they will not default today, and that carries high prices. Finally, in the intermediate yellow region, firms are net long-term debt issuers, albeit at a discount relative to to the bottom-right region.

There are two complementary ways of understanding this graph. Along a particular level of internal finance (that is, along a vertical line), firms tend to undergo large phases of debt deleveraging if their outstanding debt stock is high (the upper parts of the graph). If their debt level is small, they instead tend to slightly reduce the long-term debt stock, i.e., let it roll off progressively. Note that investment is most impaired in the top region: in that case, firms essentially devote all resources to repaying outstanding debt, instead of investing, as discussed previously. In intermediate ranges of current debt, instead, firms extend the maturity of their existing debt; as these firms face high-roll over risk, long-term debt issuance is most attractive for them.

Along a particular level of existing debt (that is, along a horizontal line), different situations are possible. If initial debt is large, firms choose to repurchase long-term debt for low level of internal finance, and only starts issuing long-term debt and investing if internal finance is sufficiently large. Firms with large outstanding debt thus require a sufficiently large "buffer stock" of internal finance in order to undertake investment investment. This results in low investment and low growth, until the "buffer stock" level of internal finance has been reached. On the other hand, firms with initially low long-term debt levels tend to rely on long-term debt only at when they are cash-poor (to the left of the graph); as they accumulate internal finance and grow out of their financial constraint, their long-term debt stock rolls off.[20]

Summarizing, equilibrium policy functions highlight two key features of the model. First, firms tend to converge toward a short maturity structure as they grow. Maturity may however be a non-monotonic function of internal finance when legacy debt is large enough. Second, the debt overhang problem associated with long-term debt issuance is quite severe: firms with large amounts of outstanding debt underinvest, and instead devote some of their resources to debt repurchases. Underinvestment persists untils firms have accumulated a sufficiently large buffer of internal finance.

[20] This latter policy is analogous to the results of Aguiar and Amador (2013), who show that, in the context of a sovereign debt model with shocks to the country's outside value, a policy consisting of financing budgetary shortfalls by short-term debt issuance, and letting initial long-term debt stocks roll off, may be optimal.

# 5 Conclusion

This paper has studied the optimal maturity structure of debt in a neo-classical investment model where debt issuance is frictional. Because firms face roll-over risk, they may want to issue long-term debt; on the other hand, because they are unable to commit to a future path of debt and dividend policies, the issuance of long-term is costly. Despite some simplifying assumptions, this model has many of the key features of standard dynamic investment models. The key departure is to allow firms to issue simultaneously short- and long-term debt.

There are two main findings. First, the optimal maturity structure is often tilted toward short-term debt. In particular, firms with either very low or very high existing leverage tend to either let long-term debt roll off, or repurchase it outright. Long-term debt issuance occurs only for firms with intermediate leverage levels, and tends not to persist over a firm's lifecycle. Second, firms' investment policies are strongly affected by their ability to continuously adjust the maturity structure of debt. In particular, leveraged firms exploit their ability to repurchase outstanding long-term debt in order to shorten debt maturity when the overhang problem is particularly severe.

There are a number of questions left open by this analysis. The short average maturity of debt predicted by the model is a manifestation of the acuteness of the commitment problems associated with long-term debt issuance. Debt covenants, in particular those connected to repurchases or early debt repayments, may help alleviate this problem. Additionally, the model does not explore the relationship between seniority and maturity strucutre. In particular, offering higher seniority to existing bond holders may also help alleviate commitment problems. The framework proposed in this paper hopefully provides a basis from which to study these questions.

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# A Data sources and variable definitions

**Firm-level data** Data moments on investment, profitability, leverage, and Tobin's Q are computed using a merge of the Compustat fundamental annual file with the CRSP dataset. Market value (mv) for a firm-year observationis computed as shares outstanding multiplied by price at closing on the last day of the month of fiscal filing for the firm. Firm-year observations are kept in sample if (1) their 2-digit sic code is not between 60 and 69 (financials) or equal to 49 (utilities); (2) debt in current liabilities (dlc), debt in long-term liabilities (dltt) and market value (mv) are not missing and weakly positive; (3) book assets (at) is not missing and weakly greater than 1m\$; (4) book leverage, the ratio of (dlc+dltt) to at, is between 0 and 1; (5) the variables $\mathrm{dd}_i$, for $i = 2, ..., 5$ (which capture the portion of long-term debt due in 2,...,5 years) are all non-missing and weakly positive; (6) their sum is weakly smaller than 1.01× dltt; (7) operating income before depreciation (oibdp), capital expenditures (capxv) and inventories (invt) are non-missing. The resulting sample years range from 1976 to 2014. There are a total of 129948 firm-year observations and 15998 unique firms (as identified by gvkey) in the final sample.

**Variable definitions in firm-level data** The investment to capital ratio is defined as capxv/at. The leverage ratio is defined as (dlcc+dlt)/at. Profitability is defined as oibdp/at. Tobin's Q is defined as (mv + dltt + dlc - invt)/at. The moments of these variables are all reported for a sample windsorized at the 1st and 99th percentiles. The ratio of short-term to long-term debt is defined as dlc/(dltt+dlc). Finally, there are no direct measure of average time to maturity of outstanding debt, but a proxy can be obtained from the data as: $\frac{1}{\mathrm{dlc}+\mathrm{dltt}}\left(\mathrm{dlc} + \sum_{i=2}^{5} \mathrm{dd}_i \times i + \mathrm{dvlt} \times x\right)$, where dvlt represents long-term debt due in more than five years, and is defined as: $\mathrm{dvlt} = \mathrm{dltt} - \sum_{i=2}^{5} \mathrm{dd}_i$. This definition simply uses the schedule of debt payments due reported in Compustat to obtain a weighted maturity of debt outstanding. In the results of table 3, I use $x = 15$ years, consistent with the baseline calibration of the model. Finally, size ($k$ in table 3) is defined as book assets (at).

**Other data sources** Table 2 also reports moments for the average spread of short-term debt and long-term debt relative to the risk-free rate, and the rate of default. The short-term spread is constructed using commercial paper and T-bill rates. Commercial paper rates are 3-month rates, annualized, and measured at the monthly frequency, obtained from FRED (series CPN3M); this series is only available from 1997 onward. T-bill rates are the annual average rate on the 3-month T-bill, also obtained from FRED. The number reported in table 2 is the average difference between the two series. Long-term spreads are computed using rates on 10-year corporate bonds and rates on 10-year treasuries. An annual time series for the latter is obtained from FRED. For the former, I use the Bank of America-Merrill Lynch 10-year yield on corporate bond series with ratings from AAA to C (the series are indexed by BAML on FRED). For consistency with the short-term debt series, I use only data from 1997 onwards. The series for 10-year yield by rating is weighted by the fraction of firms in Compustat with a credit rating in each category; the credit rating used is the variable splitcrm, obtained from the S&P ratings file on WRDS. Finally, the non-financial corporate default rate is the long-run mean annualized rate of default on non-financial corporate bonds documented by Giesecke et al. (2011).

# B Proofs

## B.1 Proofs ommitted from sections 2 and 4

The following result on the continuation value function $\mathbf{V}^c$ will be used at different stages of the analysis of the model; for clarity, it is summarized here.

**Lemma 5** (Properties of the continuation value $\mathbf{V}^c$)**.** *For every $l \geq 0$, the continuation value function,*

$$\mathbf{V}^c(n,l) = \max_{0 \leq e' \leq n} n - e' + \beta \mathbf{V}(e', l)$$

*is a increasing bijection between $\mathbb{R}_+$ and $[\beta \mathbf{V}(0,l), +\infty[$.*

*Proof.* Fix $l > 0$. Note that $\mathbf{V}^c(n,l)$ can be written as:

$$\mathbf{V}^c(n,l) = n + \max_{0 \leq e' \leq n} \beta \mathbf{V}(e', l) - e',$$

which immediately implies that $\mathbf{V}^c$ is a strictly increasing function of $n$. Moreover, since $\mathbf{V}(e, l_m)$

is a continuous function of $e$, by the theorem of the maximum, $\mathbf{V}^c$ is continuous. Finally,

$$\mathbf{V}^c(n,l) \geq n + \beta \mathbf{V}(0,l),$$

which implies that $\lim_{n\to+\infty} \mathbf{V}^c(n,l) = +\infty$. This establishes the result. $\square$

### B.1.1 The default boundary

Before proving lemma (1), I first establish the following, more general expression of the default boundary.

**Lemma 6** (The general expression of the default boundary)**.** *The continuation set is equal to:*

$$\mathcal{C}(k,l,s;\mathbf{V}^c) = \left\{ \quad \phi \in \Phi \quad s.t. \quad \phi > \boldsymbol{\phi}^d(k, \kappa l + s; \mathbf{V}^c) \quad \right\}, \tag{41}$$

*where the default boundary $\boldsymbol{\phi}^d$ is given by:*

$$\boldsymbol{\phi}^d(k, \kappa l + s; \mathbf{V}^c) = \begin{cases} +\infty & if \quad k = 0 \\ k^{-\zeta}\left(\kappa l + s + \mathbf{n}^d(l;\mathbf{V}^c) - (1-\delta_k)k\right) & if \quad k > 0 \end{cases} \tag{42}$$

*and $\mathbf{n}^d(l;\mathbf{V}^c)$ is given by:*

$$\mathbf{n}^d(l;\mathbf{V}^c) = \begin{cases} 0 & if \quad \mathbf{V}^c(0,l) \geq V_d \\ \textit{unique solution to } \mathbf{V}^c(n,l) = V_d & if \quad \mathbf{V}^c(0,l) < V_d \end{cases} \tag{43}$$

*Proof.* Recall that $\mathcal{C}(k,l,s;\mathbf{V}^c)$ is defined as:

$$\mathcal{C}(k,l,s;\mathbf{V}^c) \equiv \left\{ \quad \phi \in \Phi \quad s.t. \quad \begin{array}{l} \phi k^\zeta + (1-\delta_k)k - (\kappa l + s) > 0 \\ \mathbf{V}^c\left(\phi k^\zeta + (1-\delta_k)k - (\kappa l + s), l\right) > V_d \end{array} \right\}$$

Let :

$$\tilde{\mathcal{C}}(k,l,s;\mathbf{V}^c) \equiv \left\{ \quad \phi \in \Phi \quad s.t. \quad \phi > \boldsymbol{\phi}^d(k, \kappa l + s; \mathbf{V}^c) \quad \right\}, \tag{44}$$

with $\boldsymbol{\phi}^d$ given by (42). First, note that, since $\kappa l + s \geq 0$,

$$\mathcal{C}(0,l,s;\mathbf{V}^c) = \emptyset = \tilde{\mathcal{C}}(0,l,s;\mathbf{V}^c),$$

for any $(l, s) \in \mathbb{R}_+^2$. Next, assume that $k > 0$. Using the results of lemma (5), we can define $\mathbf{n}(l; \mathbf{V}^c)$ as in (43). The second condition in the definition of $\mathcal{C}(k, l, s; \mathbf{V}^c)$ implies:

$$\phi \geq k^{-\zeta} \left( \kappa l + s + \mathbf{n}(l; \mathbf{V}^c) - (1 - \delta_k) k \right),$$

while the first condition implies:

$$\phi \geq k^{-\zeta} \left( \kappa l + s - (1 - \delta_k) k \right).$$

Since $\mathbf{n}(l; \mathbf{V}^c) \geq 0$, the former condition is more restrictive the latter. This establishes that $\mathcal{C}(0, l, s; \mathbf{V}^c) \subseteq \tilde{\mathcal{C}}(0, l, s; \mathbf{V}^c)$, and the reciprocal inclusion is established in a similar way. □

The function $\mathbf{n}(l; \mathbf{V}^c)$ represents the amount of cash in excess of the "natural default boundary", short-term liabilities $\kappa l + s$, which the firm must generate in order to prefer continuation to default. Lemma (1) in the main text is then established by showing that this extra cash is 0 whenever $V_d \leq 0$.

*Proof of lemma (1).* Assume that $V_d \leq 0$. Recall that $\mathbf{V}^c(0, l) = \beta \mathbf{V}(0, l)$. Using the definition of the value function $\mathbf{V}$, equation (4) in the main text, we have that:

$$\mathbf{V}(0, l) \geq \inf_{\phi, k, l, s} \mathbf{V}^r(\phi, k, l, s) = V_d.$$

Thus, $\beta \mathbf{V}(0, l) \geq \beta V_d \geq V_d$, where the last inequality holds because $V_d \leq 0$. Therefore, $\mathbf{V}^c(0, l) \geq V_d$; this establishes the result. □

### B.1.2 Equilibrium multiplicity

This section discusses the construction of price functions consistent with the maximum-capital functions. I start with the case of the short-term debt model.

**Lemma 7** (Equilibrium selection in the short-term debt model)**.** *Let the maximum-capital function be defined by:*

$$\mathbf{k}(e, s) = \max \left\{ \quad k \geq 0 \quad s.t. \quad k - e = \frac{1}{1 + r} \left( 1 - F\left( \boldsymbol{\phi}^d(k, s) \right) \right) s \quad \right\}. \tag{45}$$

*Then, the price function:*

$$\mathbf{q}_S(e,s) \equiv \frac{1}{1+r}\left(1 - F\left(\boldsymbol{\phi}^d\left(\mathbf{k}(e,s),s\right)\right)\right) \tag{46}$$

*satisfies the zero-profit condition of financial intermediaries, and imply that a firm with capital structure* $(e,s)$ *operate with capital* $\mathbf{k}(e,s)$*.*

*Proof.* It is necessary and sufficient to establish that $\mathbf{q}_S$ defined by (46) satisfies the functional equation:

$$\mathbf{q}_S(e,s) = \frac{1}{1+r}\left(1 - F\left(\boldsymbol{\phi}^d\left(e + \mathbf{q}_S(e,s)s, s\right)\right)\right).$$

To see why this is true, note that:

$$\begin{aligned}
\mathbf{q}_S(e,s) &\equiv \frac{1}{1+r}\left(1 - F\left(\boldsymbol{\phi}^d\left(\mathbf{k}(e,s),s\right)\right)\right) \\
&= \frac{1}{1+r}\left(1 - F\left(\boldsymbol{\phi}^d\left(e + \frac{1}{1+r}\left(1 - F\left(\boldsymbol{\phi}^d(\mathbf{k}(e,s),s)\right)\right), s\right)\right)\right) \\
&= \frac{1}{1+r}\left(1 - F\left(\boldsymbol{\phi}^d\left(e + \mathbf{q}_S(e,s)s, s\right)\right)\right).
\end{aligned}$$

The first line uses the definition of $\mathbf{q}_S$, the second line uses the fact that $\mathbf{k}(e,s)$ satisfies the condition in (45), and the last line uses again the definition of $\mathbf{q}_S$. □

The construction of equilibrium price functions from the max-capital function in the case of the model with both short- and long-term debt is very similar.

**Lemma 8** (Equilibrium selection in the general model)**.** *Fix* $\mathbf{q}_C$*, the continuation value function for long-term debt. Let the maximum-capital function be defined by:*

$$\begin{aligned}
\mathbf{k}(e,l_m,l,s;\mathbf{q}_C) \equiv \arg\max \Bigg\{ \quad k \geq 0 \quad s.t. \quad k - e = &\frac{1}{1+r}\left(1 - F\left(\boldsymbol{\phi}^d(k, \kappa l + s)\right)\right)(s + \kappa(l - \delta l_m)) \\
+ &\frac{\delta}{1+r}\mathbf{q}_C(k,l,s)(l - \delta l_m) \quad \Bigg\}
\end{aligned} \tag{47}$$

*Then, the price functions:*

$$\mathbf{q}_S(e,l_m,l,s) \equiv \frac{1}{1+r}\left(1 - F\left(\boldsymbol{\phi}^d\left(\mathbf{k}(e,l_m,l,s;\mathbf{q}_C), \kappa l + s\right)\right)\right) \tag{48}$$

$$\mathbf{q}_L(e,l_m,l,s) \equiv \kappa\mathbf{q}_S(e,l_m,l,s) + \frac{\delta}{1+r}\mathbf{q}_C(\mathbf{k}(e,l_m,l,s;\mathbf{q}_C),l,s) \tag{49}$$

*satisfy the zero-profit condition of financial intermediaries, and imply that a firm with capital structure* $(e, l_m, l, s)$ *operates with capital* $\mathbf{k}(e, l_m, l, s; \mathbf{q}_C)$*.*

*Proof.* Let $\mathbf{q}_S$ and $\mathbf{q}_L$ be given by (48)-(49). We need to show that, for any $(e, l_m, l, s)$, these functions sastify:

$$\begin{aligned} \mathbf{q}_S(e, l_m, l, s) &\equiv \frac{1}{1+r}\left(1 - F\left(\boldsymbol{\phi}^d\left(k, \kappa l + s\right)\right)\right) &(50)\\ \mathbf{q}_L(e, l_m, l, s) &\equiv \kappa \mathbf{q}_S(e, l_m, l, s) + \frac{\delta}{1+r}\mathbf{q}_C(k, l, s) &(51) \end{aligned}$$

where:

$$k = e + \mathbf{q}_S(e, l_m, l, s)s + \mathbf{q}_L(e, l_m, l, s)(l - \delta l_m).$$

Fix $(e, l_m, l, s)$ and define $k \equiv e + \mathbf{q}_S(e, l_m, l, s)s + \mathbf{q}_L(e, l_m, l, s)(l - \delta l_m)$. Then, since $\mathbf{k}(e, l_m, l, s; \mathbf{q}_C)$ satisfies the equality in (47), we have:

$$\begin{aligned} \mathbf{k}(e, l_m, l, s; \mathbf{q}_C) &= e + \mathbf{q}_S(e, l_m, l, s)s + \mathbf{q}_L(e, l_m, l, s)(l - \delta l_m) \\ &= k, \end{aligned}$$

where the first line makes use of the definitions of $\mathbf{q}_S$ and $\mathbf{q}_L$. Thus, $k = \mathbf{k}(e, l_m, l, s; \mathbf{q}_C)$. Replacing $\mathbf{k}(e, l_m, l, s; \mathbf{q}_C)$ by $k$ in (48)-(49), we see that the zero-profit conditions (50)-(51) are satisfied. $\square$

### B.1.3 The optimality of short-term debt

*Proof of lemma 3.* First, note that since $k$ satisfies the zero-profit conditions of lenders, we have that:

$$\begin{aligned} (1+r)k &= (1+r)e + \left(1 - F\left(\boldsymbol{\phi}^d(k, \kappa l + s)\right)\right)(s + \kappa(l - \delta l_m)) + \delta \mathbf{q}_C(k, l, s)(l - \delta l_m) \\ &= (1+r)e + \left(1 - F\left(\boldsymbol{\phi}^d(k, \kappa l + s)\right)\right)\left(\boldsymbol{\phi}^d(k, \kappa l + s)k^\zeta + (1 - \delta_k)k\right) \\ &\quad - \left(\kappa\left(1 - F\left(\boldsymbol{\phi}^d(k, \kappa l + s)\right)\right) + \delta \mathbf{q}_C(k, l, s)\right)\delta l_m \\ &\quad + \delta \mathbf{q}_C(k, l, s)l. \end{aligned}$$

Second, note that $\mathbf{V}^c(n,l) = n + \mathbf{h}(n,l)$ so that:

$$\mathbf{W}_e(l,s;k,\phi^d) = \int_{\phi>\phi^d} (\phi-\phi^d)k^\zeta dF(\phi) + \int_{\phi>\phi^d} \mathbf{h}((\phi-\phi^d)k^\zeta, l)dF(\phi)$$

Therefore, when $k$ satisfies the zero-profit conditions of lenders, and $\phi = \boldsymbol{\phi}^d(k,\kappa l+s)$, we have that:

$$\begin{aligned}\mathbf{W}_e(l,s;k,\phi_d) + (1+r)k &= (1+r)e\left(\int_{\phi>\boldsymbol{\phi}^d(k,\kappa l+s)} \phi dF(\phi)\right)k^\zeta + \left(1-F\left(\boldsymbol{\phi}^d(k,\kappa l+s)\right)\right)(1-\delta_k)k \\ &\quad - \left(\kappa\left(1-F\left(\boldsymbol{\phi}^d(k,\kappa l+s)\right)\right) + \delta\mathbf{q}_C(k,l,s)\right)\delta l_m \\ &\quad + \int_{\phi>\boldsymbol{\phi}^d(k,\kappa l+s)} \mathbf{h}((\phi-\boldsymbol{\phi}^d(k,\kappa l+s))k^\zeta, l)dF(\phi) + \delta\mathbf{q}_C(k,l,s)l\end{aligned}$$

Substituting $\left(\int_{\phi>\boldsymbol{\phi}^d(k,\kappa l+s)} \phi dF(\phi)\right) = \mathbb{E}(\phi) - F(\boldsymbol{\phi}^d(k,\kappa l+s)\mathbb{E}\left[\phi|\phi \leq \phi^d(k,\kappa l+s)\right]$, and rearranging terms, one obtains the expression of the lemma. $\square$

*Proof of proposition 1.* Let $(l,l_m)$ be given, and assume that $e \in [\mathbf{e}_{min}(l,l_m), \mathbf{e}_{max}(l,l_m)]$. Let:

$$\begin{aligned} s &\equiv (1+r)(\mathbf{e}_{max}(l,l_m) - e) \\ &= (1+r)(k^* - e) - (\kappa + \delta\hat{\mathbf{q}}_L(\bar{\mathbf{e}}(l),l))(l-\delta l_m).\end{aligned}$$

Then, we have that:

$$k^* = e + \frac{1}{1+r}(s + \kappa(l-\delta l_m)) + \frac{\delta}{1+r}\hat{\mathbf{q}}_L(\bar{\mathbf{e}}(l),l)(l-\delta l_m) \tag{52}$$

First, note that since $e \leq \mathbf{e}_{max}(l,l_m)$, $s \geq 0$. Second, note that:

$$\begin{aligned}\boldsymbol{\phi}^d(k^*,\kappa l+s) &= (k^*)^{-\zeta}(\kappa l + s - (1-\delta_k)k^*) \\ &= (k^*)^{-\zeta}((r+\delta_k) - (1+r)e + (\kappa + \delta\hat{\mathbf{q}}_L(\bar{\mathbf{e}}(l),l))\delta l_m - \delta\hat{\mathbf{q}}_L(\bar{\mathbf{e}}(l),l)l) \\ &\leq -(k^*)^{-\zeta}\bar{\mathbf{e}}(l)\end{aligned}$$

where in order to move from the second to the third line, we use the fact that$e \geq \mathbf{e}_{min}(l,l_m)$. Therefore,

$$\forall \phi \geq 0, \quad \phi k^\zeta + (1-\delta_k)k - (\kappa l + s) = (\phi - \boldsymbol{\phi}^d(k^*,\kappa l+s))(k^*)^\zeta \geq \bar{\mathbf{e}}(l),$$

so that, using assumption (2), we obtain:

$$\forall \phi \geq 0, \quad \tilde{\mathbf{e}}(\phi, k^*, s, l) = \mathbf{e}'\left(\phi(k^*)^\zeta + (1-\delta_k)k^* - (\kappa l + s), l\right) = \bar{\mathbf{e}}(l).$$

We can use this to compute $\mathbf{q}_C(k^*, l, s)$:

$$\begin{aligned}
\mathbf{q}_C(k^*, l, s) &= \int_{\phi > \phi^d(k, \kappa l + s)} \hat{\mathbf{q}}_L\left(\tilde{\mathbf{e}}(\phi, k, s, l), l\right) dF(\phi) \\
&= \int_{\phi > 0} \hat{\mathbf{q}}_L\left(\tilde{\mathbf{e}}(\phi, k, s, l), l\right) dF(\phi) \\
&= \int_{\phi > 0} \hat{\mathbf{q}}_L\left(\bar{\mathbf{e}}(l), l\right) dF(\phi) \\
&= \hat{\mathbf{q}}_L\left(\bar{\mathbf{e}}(l), l\right).
\end{aligned}$$

We can replace this into equation (52) to obtain:

$$k^* = e + \frac{1}{1+r}\left(s + \kappa(l - \delta l_m)\right) + \frac{\delta}{1+r}\mathbf{q}_C(k^*, l, s)(l - \delta l_m).$$

Therefore, for the capital structure $(e, l_m, l, s)$, the zero-profit conditions of lenders are satisfied when $k = k^*$, $\mathbf{q}_S(e, l_m, l, s) = \frac{1}{1+r}$, and $\mathbf{q}_L(e, l_m, l, s) = \frac{1}{1+r}\left(\kappa + \hat{\mathbf{q}}_L\left(\bar{\mathbf{e}}(l), l\right)\right)$. This establishes the feasibility of that capital structure. $\square$

*Proof of corollary 4.* For any $(l_m, l)$, the value of the policy described in proposition 1 is given by:

$$\mathbf{W}_e(l^*(e, l_m, l), s^*(e, l_m, l); k^*, 0) = (1+r)e + \tilde{\Pi}k^* - \kappa\delta l_m + \mathbf{h}(\bar{\mathbf{e}}(l), l) + \delta\hat{\mathbf{q}}_L(\bar{\mathbf{e}}(l), l)(l - \delta l_m),$$

where $\tilde{\Pi} = \left(\frac{1}{\zeta} - 1\right)(r + \delta_k)$. Moreover, given $(e, l_m, l)$, the gap between the value of the firm under this policy, and another policy $s$ with an associated level of capital $k = \mathbf{k}(e, l_m, l, s)$ is given by:

$$\begin{aligned}
\Delta\mathbf{W}_e(e, l_m, l) &= \tilde{\Pi}k^* g\left(\frac{k}{k^*}\right) && (53)\\
&+ F\left(\phi^d(k, \kappa l + s)\right)\left(\mathbb{E}\left[\phi | \phi \leq \phi^d(k, \kappa l + s)\right] k^\zeta + (1-\delta_k)k - \kappa\delta l_m\right) && (54)\\
&+ \left(\mathbf{h}(\bar{\mathbf{e}}(l), l) - \int_{\phi > \phi^d(k, \kappa l + s)} \mathbf{h}((\phi - \phi^d(k, \kappa l + s))k^\zeta, l) dF(\phi)\right) && (55)\\
&+ \delta\left(\hat{\mathbf{q}}_L\left(\bar{\mathbf{e}}(l), l\right) - \mathbf{q}_C(k, l, s)\right)(l - \delta l_m), && (56)
\end{aligned}$$

where $g(x) = \frac{\zeta}{1-\zeta} x \left( \frac{1}{\zeta} x^{\zeta-1} - 1 \right)$ is strictly positive, convex, and minimized at $x = 1$. When $l_m = 0$, for any $l$, this difference boils down to:

$$
\begin{aligned}
\Delta \mathbf{W}_e(e, 0, l) &= \tilde{\Pi} k^* g\left(\frac{k}{k^*}\right) && (57)\\
&+ \;\; F\left(\phi^d(k, \kappa l + s)\right)\left(\mathbb{E}\left[\phi | \phi \leq \phi^d(k, \kappa l + s)\right] k^\zeta + (1 - \delta_k) k\right) && (58)\\
&+ \left(\mathbf{h}(\bar{\mathbf{e}}(l), l) - \int_{\phi > \boldsymbol{\phi}^d(k, \kappa l + s)} \mathbf{h}((\phi - \boldsymbol{\phi}^d(k, \kappa l + s)) k^\zeta, l) dF(\phi)\right) && (59)\\
&+ \delta\left(\hat{\mathbf{q}}_L\left(\bar{\mathbf{e}}(l), l\right) - \mathbf{q}_C(k, l, s)\right) l, && (60)
\end{aligned}
$$

First, by definition, the term (57) is positive. Second, the term (58), corresponding to bankruptcy losses, is also (weakly) positive. Second, note that by assumption (2),

$$
\int_{\phi > \boldsymbol{\phi}^d(k, \kappa l + s)} \mathbf{h}((\phi - \boldsymbol{\phi}^d(k, \kappa l + s)) k^\zeta, l) dF(\phi) \leq \mathbf{h}(\bar{e}(l), l),
$$

so that the term (59) is also positive. Third, we have that:

$$
\begin{aligned}
\delta \mathbf{q}_C(k, l, s) l &= \delta \int_{\phi \geq \boldsymbol{\phi}^d(\kappa l + s, l)} \hat{\mathbf{q}}_L\left(\mathbf{e}'\left((\phi - \boldsymbol{\phi}^d(\kappa l + s, l)) k^\zeta, l\right), l\right) dF(\phi)\\
&= \delta \int_{\phi = \boldsymbol{\phi}^d(\kappa l + s, l)}^{\phi = \boldsymbol{\phi}^d(\kappa l + s, l) + k^{-\zeta} \bar{\mathbf{e}}(l)} \hat{\mathbf{q}}_L\left(\left(\phi - \boldsymbol{\phi}^d(\kappa l + s, l)\right) k^\zeta, l\right) dF(\phi)\\
&\quad + \delta\left(1 - F(\boldsymbol{\phi}^d(\kappa l + s, l) + k^{-\zeta} \bar{\mathbf{e}}(l))\right) \hat{\mathbf{q}}_L\left(\bar{\mathbf{e}}(l), l\right) l\\
&\leq \delta \hat{\mathbf{q}}_L\left(\bar{\mathbf{e}}(l), l\right) l,
\end{aligned}
$$

where the second line uses assumption (2), and the third line uses the fact that $\hat{\mathbf{q}}_L(e, l_m)$ is increasing in $e$. Thus, the term (60) is also strictly positive. Therefore, $\Delta \mathbf{W}(e, 0, l, s) \geq 0$ for any $(l, s)$.

This means that, when $l_m = 0$, for any $(e, l)$ such that $e \in [\underline{\mathbf{e}}(0, l), \bar{\mathbf{e}}(0, l)]$, the policy described in proposition (1) achieves a larger value than all other policies. Recall that this value is given by:

$$
\begin{aligned}
\mathbf{W}^m(e, 0, l) &= (1 + r) e + \tilde{\Pi} k^* + \mathbf{h}(\mathbf{n}(l), l) + \delta \hat{\mathbf{q}_L}(\bar{\mathbf{e}}(l), l) l\\
&\leq (1 + r) e + \tilde{\Pi} k^* + \max_{l \geq 0} \left\{\mathbf{h}(\bar{\mathbf{e}}(l), l) + \delta \hat{\mathbf{q}}_L\left(\bar{\mathbf{e}}(l), l\right) l\right\}\\
&= \mathbf{W}^m(e, 0, l^*),
\end{aligned}
$$

where $l^* \equiv \arg\max_{l\geq 0} \{\mathbf{h}(\bar{\mathbf{e}}(l), l) + \delta \hat{\mathbf{q}}_L (\bar{\mathbf{e}}(l), l)\, l\}$.

Finally, we need to establish that when $\hat{\mathbf{q}}_L(\bar{\mathbf{e}}(l), l)l$ is decreasing in $l$, the optimal debt structure features only long-term debt: $l^* = 0$. Note that, because $\mathbf{V}(n, l)$ is decreasing in $l$ for each $l$, then by the envelope theorem, $\mathbf{h}(\bar{\mathbf{e}}(l), l)$ is decreasing in $l$. Thus, if the continuation value of debt is decreasing in $l$, $\mathbf{h}(\bar{\mathbf{e}}(l), l) + \delta \hat{\mathbf{q}}_L (\bar{\mathbf{e}}(l), l)\, l$ is a decreasing function of $l$, so that $l^* = 0$.

□

# C Numerical appendix

## C.1 Computation of the price function

I first discuss the numerical construction of the capital function:

$$\mathbf{k}(e, l_m, l, s) = \max \left\{ k - e = \frac{1}{1+r} \left(1 - F\left(\boldsymbol{\phi}^d(k, \kappa l + s)\right)\right) (s + \kappa(l - \delta l_m)) + \frac{\delta}{1+r} \mathbf{q}_C(k, l, s)(l - \delta l_m) \right\}, \tag{61}$$

for a given continuation value function $\mathbf{q}_C$. Recall that this function can then be used to construct the price functions $\mathbf{q}_S$ and $\mathbf{q}_L$ consistent with the zero-profit conditions. Define:

$$\begin{aligned}
\mathbf{R}(k) &= \frac{1}{1+r} \left(1 - F\left(\boldsymbol{\phi}^d(k, \kappa l + s)\right)\right) (s + \kappa(l - \delta l_m)) + \frac{\delta}{1+r} \mathbf{q}_C(k, l, s)(l - \delta l_m) \\
\mathbf{L}(k) &= k - e \\
\boldsymbol{\Delta}(k) &= \mathbf{R}(k) - \mathbf{L}(k)
\end{aligned}$$

where the dependence on other variables is omitted for brevity. The solution method distinguishes between two cases: $l > 0$ or $s > 0$, and $l = s = 0$.

### C.1.1 The case $l > 0$ or $s > 0$

When $l > 0$ or $s > 0$, total short-term liabilities, $\kappa l + s > 0$, are strictly positive, so that:

$$\lim_{k \to 0^+} \boldsymbol{\phi}^d(\kappa l + s, k) = +\infty = \boldsymbol{\phi}^d(\kappa l + s, 0).$$

This implies that $\lim_{k \to 0^+} \mathbf{R}(k) = 0 = \mathbf{R}(0)$, so that $\mathbf{R}(k)$ is continuous at $k = 0$. Therefore, $\boldsymbol{\Delta}(k)$ is continuous on $\mathbb{R}_+$. Additionally, note that:

$$\begin{aligned} &\forall k \geq \tfrac{1}{1-\delta_k} \left( \overline{\mathbf{n}}(l) + \kappa l + s \right) \equiv k_\infty, \\ &\qquad \mathbf{R}(k) = \tfrac{1}{1+r} \left( s + \kappa(l - \delta l_m) \right) + \hat{\mathbf{q}}_L(\overline{\mathbf{n}}(l), l)(l - \delta l_m) \equiv \mathbf{R}_\infty < +\infty \end{aligned}$$

so that $\lim\limits_{k \to +\infty} \boldsymbol{\Delta}(k) = -\infty$. Thus, by the intermediate value theorem, the zero-profit condition (ZPC) in (61) has a least one solution.

If $k_\infty \leq \mathbf{R}_\infty + e$, then $k = \mathbf{R}_\infty + e$ satisfies $\boldsymbol{\Delta}(k) = \mathbf{R}(k) - (k - e) = \mathbf{R}_\infty - \mathbf{R}_\infty = 0$. Additionally, $\forall k' \geq \mathbf{R}_\infty + e$, $\boldsymbol{\Delta}(k) = \mathbf{R}(k) - (k-e) = \mathbf{R}_\infty - (k-e) < 0$. Thus, $\mathbf{k}(e, l_m, l, s) = \mathbf{R}_\infty + e$.

If $k_\infty < \mathbf{R}_\infty + e$, then, $\forall k \geq k_\infty$ $\Delta(k) = \mathbf{R}_\infty - (k-e) \leq \mathbf{R}_\infty - (k_\infty - e) < 0$. Since $\boldsymbol{\Delta}(0) = e \geq 0$, by the intermediate value theorem, it must be that:

$$\mathbf{k}(e, l_m, l, s) \in [0, k_\infty[\,.$$

In order to solve numerically for the price function, I then guess that $\mathbf{R}$ has the following property:

**Assumption 3** (Monotonicity of $\mathbf{R}$). *If $\mathbf{R}_\infty \geq 0$, then $\mathbf{R}(k)$ is increasing with $k$. Otherwise, $\mathbf{R}(k)$ is decreasing with $k$.*

In particular, note that in order for $\mathbf{R}_\infty$ to be negative, it must be the case that $l < \delta l_m$, that is, the firm must be a net repurchaser of debt. Under this assumption, two situations are possible:

- **If $\mathbf{R}_\infty < 0$ and $\mathbf{R}$ is decreasing**, then $\boldsymbol{\Delta}$ is also decreasing. The equation $\boldsymbol{\Delta}(k) = 0$ then has a unique solution on $[0, k_\infty[$. In this case, $\mathbf{k}(e, l_m, l, s)$ can be found using the bisection method on the interval $[0, k_\infty]$.

- **If $\mathbf{R}_\infty \geq 0$ and $\mathbf{R}$ is increasing**, then the equation $\boldsymbol{\Delta}(k)$ may have multiple solutions. However, the largest solution to $\boldsymbol{\Delta}(k) = 0$ can be found by constructing a sequence $\{k_{j+1} = e + \mathbf{R}(k_j)\}_{j \geq 0}$, with $k_0 = k_\infty$. Because $\mathbf{R}$ is increasing, this sequence will converge to $\mathbf{k}(e, l_m, l, s)$ as $j \to +\infty$.

Once the equilibrium price function has been obtained, assumption (3) can be verified numerically, by computing the numerical derivative of $\Delta$.

### C.1.2 The case $l = s = 0$

When $\kappa l + s = 0$ (which occurs if and only if $l = s = 0$), $\boldsymbol{\phi}^d$ is not continuous at $k = 0$, since $\boldsymbol{\phi}^d(0,0) = +\infty$, but:

$$\lim_{k \to 0^+} \boldsymbol{\phi}^d(k,0) = \lim_{k \to 0^+} -(1-\delta_k)k^{1-\zeta} = 0.$$

In turn, this implies that:

$$\lim_{k \to 0^+} \mathbf{R}(k) = -\frac{\delta l_m}{1+r}\left(\kappa + \delta \hat{\mathbf{q}}_{\mathbf{L}}(\mathbf{0},\mathbf{0})\right) \equiv \mathbf{R}(0^+).$$

When $e > -\mathbf{R}(0^+)$, there are unique solutions to $\Delta(k) = 0$, and the largest one can be found using the same strategy as in the case $l = s = 0$. On the other hand, when $e \leq -\mathbf{R}(0^+)$, there may not be a solution to $\boldsymbol{\Delta}(k) = 0$. However, in this case, the capital function $\mathbf{k}(e, l_m, l, s)$ has a well-defined right-limit at $s = l = 0$, namely:

$$\lim_{\kappa l + s \to 0^+} \mathbf{k}(e, l_m, l, s) = 0.$$

This suggests continuously extending $\mathbf{k}$ at the boundary $l = s = 0$:

$$\mathbf{k}(e, l_m, l, s) = 0 \quad \text{if} \quad l = s = 0 \text{ and } e \leq -\mathbf{R}(0^+).$$

While this extension of the capital function is continuous, it is not a solution to the ZPC, so that the derivation of price functions from $\mathbf{k}$, as described in lemma 47, is not possible. Instead, the following lemma provides a continuous extension of the price functions at the $l = s = 0$ boundary.

**Lemma 9** (Continuous extension for prices)**.** *Let $\mathbf{q}_C$ and $\hat{\mathbf{q}}_{\mathbf{Y}}$, and $(e, l_m) \in \mathbb{R}^2_+$, $l_m > 0$, be given and such that:*

$$e \leq \frac{\delta l_m}{1+r}\left(\kappa + \delta \hat{\mathbf{q}}_{\mathbf{L}}(\mathbf{0},\mathbf{0})\right).$$

*Define:*

$$\mathbf{q}_S(e, l_m, 0, 0) \equiv \frac{e}{\delta l_m} \frac{1}{\kappa + \delta \mathbf{q}_C(0^+, 0, 0)}, \tag{62}$$

$$\mathbf{q}_L(e, l_m, 0, 0) \equiv \frac{e}{\delta l_m}. \tag{63}$$

*Then,* $k=0$ *satisfies* $k-e=\mathbf{q}_S(e,l_m,0,0)\times 0+\mathbf{q}_L(e,l_m,0,0)\times(-\delta l_m)$*, and moreover:*

$$\lim_{\kappa l+s\to 0^+}\mathbf{q}_S(e,l_m,l,s) \;=\; \mathbf{q}_S(e,l_m,0,0), \tag{64}$$

$$\lim_{\kappa l+s\to 0^+}\mathbf{q}_L(e,l_m,l,s) \;=\; \mathbf{q}_L(e,l_m,0,0). \tag{65}$$

Note that the resulting prices and capital level satisfy "a" zero-profit condition, but not the correct one. Indeed, the short-term debt price is not consistent with the default boundary implied by the fact that $k=0$. In other words, it is the default probability at $k=0$ which is problematic. One can however construct a continuous extension of the default boundary, via:

$$\boldsymbol{\phi}_0^d(e,l_m)\equiv F^{-1}\left(1-(1+r)\mathbf{q}_S(e,l_m,0,0)\right).$$

This default boundary satisfies is the left-hand limit of the default boundary when $\kappa l+s>0$, that is:

$$\boldsymbol{\phi}_0^d(e,l_m)=\lim_{\kappa l+s\to 0^+}\boldsymbol{\phi}^d\left(\mathbf{k}(e,l_m,l,s),\kappa l+s\right).$$

Finally, by continuity in $l_m$ when $e=0$, note that the lemma suggests that a continuous extension of the capital level and debt prices for the "empty" capital structure $e=l_m=l=s=0$ is:

$$\mathbf{k}(0,0,0,0)=0 \quad \text{and} \quad \mathbf{q}_S(0,0,0,0)=\mathbf{q}_L(0,0,0,0)=0,$$

with the default boundary $\boldsymbol{\phi}^d\left(\mathbf{k}(0,0,0,0),0\right)=+\infty$.

## C.2 Numerical construction of a recursive competitive equilibrium

**General approach** I solve for a recursive competitive equilibrium by computing the limit of a finite-horizon economy. An important feature of the solution method is that the state-space for $(e,l_m)$ is adjusted endogenously at each horizon. Specifically, at each horizon $t$, two boundary functions for inside equity, $\underline{\mathbf{n}}_t(l)$ and $\overline{\mathbf{n}}_t(l)$, are constructed as follows:

$$\begin{aligned}\underline{\mathbf{n}}_t(l) &= \inf\left\{n\geq 0 \text{ s.t. } \beta\mathbf{V}_t(n,l)-n\geq\beta\mathbf{V}_t(0,l)\right\}\\ \overline{\mathbf{n}}_t(l) &= \arg\sup_{n\geq\underline{\mathbf{n}}_t(l)}\left\{\beta\mathbf{V}_t(n,l)-n\right\}\end{aligned}$$

Note that these functions are always well-defined, and satisfy $\underline{\mathbf{n}}_t(l) \leq \overline{\mathbf{n}}(l)$ (although both may be $+\infty$). The second boundary is similar to the one used in the main text to discuss the cases in which short-term borrowing is optimal. These boundaries are useful because they imply that the dividend issuance policy period $t-1$ is given by:

$$\mathbf{e}'_{t-1}(n,l) = \begin{cases} n & \text{if } 0 \leq n < \underline{\mathbf{n}}_t(l) \\ 0 & \text{if } \underline{\mathbf{n}}_t(l) \leq n < \overline{\mathbf{n}}_t(l) \\ \overline{\mathbf{n}}_t(l) & \text{if } \overline{\mathbf{n}}_t(l) \leq n \end{cases}$$

This, in turn, simplifies the expression of the continuation value of debt or equity at time at the dividend issuance stage $t-1$. For example, the continuation value of equity will satisfy:

$$\mathbf{V}_{t-1}(n,l) - n = \begin{cases} \beta \mathbf{V}_t(0,l) & \text{if } 0 \leq n < \underline{\mathbf{n}}_t(l) \\ \beta \mathbf{V}_t(n,l) - n & \text{if } \underline{\mathbf{n}}_t(l) \leq n < \overline{\mathbf{n}}_t(l) \\ \beta \mathbf{V}_t(\overline{\mathbf{n}}_t(l),l) - \overline{\mathbf{n}}_t(l) & \text{if } \overline{\mathbf{n}}_t(l) \leq n \end{cases}$$

The fact that $\mathbf{V}_{t-1}(n,l) - n$ is constant outside the boundaries can be used to approximate the value function, as described next.

**Approximation of integrals** In computing both the value function of the firm and the price of debt contracts, integrals of the form:

$$\mathcal{I}(k, \phi^d, \mathbf{h}) = \int_{\phi \geq \phi^d} \mathbf{h}\left(\left(\phi - \phi^d\right) k^{\zeta}\right) dF(\phi)$$

need to be approximated. Here, as discussed above, the function $\mathbf{h}$ is assumed to satisfy:

$$\mathbf{h}(n) = \begin{cases} \mathbf{h}(0) & \text{if } 0 \leq n < \underline{n} \\ \mathbf{h}(n) & \text{if } \underline{n} \leq n < \overline{n} \\ \mathbf{h}(\overline{n}) & \text{if } \overline{n} \leq n \end{cases}$$

Let $\{n_i\}_{i=1}^{N+1}$ be a grid of $N+1$ points, with $n_1 = 0$, $n_2 = \underline{n}$ and $n_{N+1} = \overline{n}$; assume that $h_i \equiv \mathbf{h}(n_i)$ is known for $1 \leq i \leq N+1$. When $k > 0$ and $\phi^d < +\infty$, the integral is approximated using the trapezoidal method on the disjoint intervals $\left[\phi^d + k^{-\zeta} n_i, \phi^d + k^{-\zeta} n_{i+1}\right]$, $i = 1, ..., N$. This approximation is given by:

$$
\begin{aligned}
\mathcal{I}(k, \phi^d, \mathbf{h}) \quad \approx & \quad \left(F\left(\phi^d + k^{-\zeta} n_2\right) - F\left(\phi^d\right)\right) h_1 \\
+ & \quad \sum_{i=2}^{N} \left( \left[F\left(\phi^d + k^{-\zeta} n_{i+1}\right) - F\left(\phi^d + k^{-\zeta} n_i\right)\right] \left(\alpha_i - \beta_i \phi^d k^{\zeta}\right) \right. \\
& \qquad \left. + \left[\int_{\phi=\phi^d + k^{-\zeta} n_i}^{\phi=\phi^d + k^{-\zeta} n_{i+1}} \phi dF(\phi)\right] \beta_i k^{\zeta} \right) \\
+ & \quad \left(1 - F\left(\phi^d + k^{-\zeta} n_{N+1}\right)\right) h_{N+1},
\end{aligned}
$$

where the coefficients $\{\alpha_i, \beta_i\}_{i=2,\ldots,N}$ are given by:

$$
\begin{aligned}
\beta_i &= \frac{h_{i+1} - h_i}{n_{i+1} - n_i} \\
\alpha_i &= h_i - \beta_i n_i.
\end{aligned}
$$

Note that this approximation is valid for both $\phi^d > 0$ and $\phi^d \leq 0$, with the abuse of notation that $F(x) = 0$ when $x < 0$. In the case $\phi^d \leq 0$, some or all of integrals on the disjoint intervals may be 0. If $k = 0$ or $\phi^d = +\infty$, the integral is equal to $\mathcal{I}(k, \phi^d, \mathbf{h}) = 0$.

**Detailed algorithm** The algorithm proceeds as follows:

1. **Initialization:** Choose a maximum level of long-term debt, $\bar{l}$, and a grid for long-term debt issuance $\{l_i\}_{i=1}^{N_l}$, such that $l_1 = 0$ and $l_{N_l} = \bar{l}$.

2. **Solution to the last-period problem:** Let $\mathbf{V}_T$ be given by:

$$
\begin{aligned}
\mathbf{V}_T(e, l_m) \quad &= \quad \max_{s \geq 0} \quad \int_{\phi > \boldsymbol{\phi}^d(k,s)} (\phi - \boldsymbol{\phi}^d(k, s)) k^{\zeta} dF(\phi) \\
s.t. \quad & \quad k = \mathbf{k}_T(e, l_m, s)
\end{aligned}
$$

$$
\mathbf{k}_T(e, l_m, s) = \max \left\{ \quad k \geq 0 \quad \text{s.t.} \quad k - e = \frac{1}{1+r} \left(1 - F\left(\boldsymbol{\phi}^d(k, s)\right)\right) (s - \kappa \delta l_m) \quad \right\}.
$$

The solution to the last period problem is constructed in three steps:

Step 1: Using the definition of $\mathbf{V}_T$, construct the boundaries $\left\{\underline{n}_{i,T-1}\right\}_{i=1}^{N_l}$ and $\{\overline{n}_{i,T-1}\}_{i=1}^{N_l}$:

$$\begin{aligned}\underline{n}_{i,T-1} &= \inf\left\{n \geq 0 \text{ s.t. } \beta\mathbf{V}_t(n,l) - n \geq \beta\mathbf{V}_T(0,l)\right\} \\ \overline{n}_{i,T-1} &= \arg\sup_{n \geq \underline{n}_{i,T-1}} \{\beta\mathbf{V}_T(n,l) - n\}\end{aligned}$$

Step 2: Construct grids $\{e_{T,i,j}, l_{T,i,j}\}_{i,j\in[1,N_l]\times[1,N+1]}$ for equity and debt given by:

$$e_{T,i,1} = 0, \quad e_{T,i,2} = \underline{n}_{i,T-1}, \quad ..., \quad e_{T,i,N+1} = \overline{n}_{i,T-1}$$

$$l_{T,i,j} = l_i \quad \forall j \in [1, N+1]$$

Step 3: Compute the value of the firm at those points, that is, $V_{T,i,j} = \mathbf{V}_T\left(e_{T,i,j}, l_{T,i,j}\right)$, as given by the expression above. Additionally, compute the value of long-term debt at the firm's optimal policies, which is given by:

$$q_{T,i,j} = \frac{1}{1+r}\left(1 - F\left(\boldsymbol{\phi}^d(\mathbf{k}_T(e_{i,j}, l_{i,j}, \mathbf{s}(e_{i,j}, l_{i,j})))\right)\right)\kappa$$

where $\mathbf{s}$ is the policy function for short-term borrowing that solves the last period problem.

3. **Solution to interim period problems:** Let $t \leq T$ and suppose $\{e_{t,i,j}, l_{t,i,j}, V_{t,i,j}, q_{t,i,j}\}$ have been constructed. Define:

$$\begin{aligned}\mathbf{V}_{t-1}(e, l_m) \quad &= \quad \max_{l,s\geq 0} \int_{\phi>\boldsymbol{\phi}^d(k,s)} \mathbf{V}^c\left((\phi - \boldsymbol{\phi}^d(k, \kappa l + s))k^\zeta, l\right) dF(\phi) \\ s.t. \quad & k = \mathbf{k}_{t-1}(e, l_m, l, s)\end{aligned}$$

where:

$$\begin{aligned}\mathbf{k}_{t-1}(e, l_m, l, s) \quad = \arg\max\Big\{ \quad & k \geq 0 \text{ s.t. } k - e = \frac{1}{1+r}\left(1 - F\left(\boldsymbol{\phi}^d(k, \kappa l + s)\right)\right)(s + \kappa(l - \delta l_m)) \\ & + \frac{\delta}{1+r}\mathbf{q}_{C,t-1}(k, l, s)(l - \delta l_m) \quad \Big\}\end{aligned}$$

Note that, for each $i = 1, ..., N_l$, the objective function of the firm takes the form:

$$\mathcal{W}(k, \phi^d, l_i) \quad = \quad \int_{\phi > \boldsymbol{\phi}^d} (\phi - \boldsymbol{\phi}^d) k^\zeta dF(\phi) + \int_{\phi > \boldsymbol{\phi}^d} \mathbf{h}_t^V \left( (\phi - \boldsymbol{\phi}^d) k^\zeta, l_i \right) dF(\phi)$$

where the function $\mathbf{h}^V$ is given by:

$$\mathbf{h}_t^V(n, l_i) = \begin{cases} \beta \mathbf{V}_t(0, l_i) & \text{if } 0 \leq n < \underline{n}_{i,t} \\ \beta \mathbf{V}_t(n, l_i) - n & \text{if } \underline{n}_{i,t} \leq n < \overline{n}_{i,t} \\ \beta \mathbf{V}_t(\underline{n}_{i,t}, l_i) - \underline{n}_{i,t} & \text{if } \overline{n}_{i,t} \leq n \end{cases}$$

Given that the values of $\mathbf{V}_t(., l_i)$ at $\left\{e_{i,1,t} = 0, e_{i,2,t} = \underline{n}_{i,t}, ..., e_{i,N,t} = \overline{n}_{i,t}\right\}$ are given by $V_{t,i,j}$, the values of $\mathbf{h}_t^V$ at these points is also known. Thus, we can use the method described above to approximate that portion of the integral.

Likewise, note that, for any $i = 1, ..., N_l$, the continuation value of the bond is given by:

$$\mathbf{q}_{C,t-1}(k, l, s_i) = \int_{\phi \geq \phi_d} \mathbf{h}_t^q \left( (\phi - \boldsymbol{\phi}^d) k^\zeta, l_i \right),$$

where the function $\mathbf{h}_t^q$ is given by:

$$\mathbf{h}_t^q(n, l_i) = \begin{cases} \hat{\mathbf{q}}_{L,t}(0, l_i) & \text{if } 0 \leq n < \underline{n}_{i,t} \\ \hat{\mathbf{q}}_{L,t}(n, l_i) & \text{if } \underline{n}_{i,t} \leq n < \overline{n}_{i,t} \\ \hat{\mathbf{q}}_{L,t}(\underline{n}_{i,t}, l_i) & \text{if } \overline{n}_{i,t} \leq n \end{cases}$$

Given that the values of $\hat{\mathbf{q}}_t(., l_i)$ at $\left\{e_{i,1,t} = 0, e_{i,2,t} = \underline{n}_{i,t}, ..., e_{i,N,t} = \overline{n}_{i,t}\right\}$ are given by $q_{t,i,j}$, the values of $\mathbf{h}_t^q$ at these points is also known. Thus, we can again use the method described above to approximate that portion of the integral. Using these approximations, the interim period problem at time $t-1$ is solved in three steps:

Step 1: Given the approximation method for $\mathbf{V}_{t-1}$, construct the boundaries $\left\{\underline{n}_{i,t-2}\right\}_{i=1}^{N_l}$ and $\{\overline{n}_{i,t-2}\}_{i=1}^{N_l}$:

$$\begin{aligned} \underline{n}_{i,t-2} &= \inf \left\{ n \geq 0 \text{ s.t. } \beta \mathbf{V}_{t-1}(n, l) - n \geq \beta \mathbf{V}_{t-1}(0, l) \right\} \\ \overline{n}_{i,t-2} &= \arg \sup_{n \geq \underline{n}_{i,T-1}} \left\{ \beta \mathbf{V}_{t-1}(n, l) - n \right\} \end{aligned}$$

Step 2: Construct grids $\{e_{t-1,i,j}, l_{t-1,i,j}\}_{i,j\in[1,N_l]\times[1,N+1]}$ for equity and debt given by:

$$e_{t-1,i,1} = 0, \quad e_{t-1,i,2} = \underline{n}_{i,t-2}, ..., \quad e_{t-1,i,N+1} = \overline{n}_{i,t-2}$$

$$l_{t-1,i,j} = l_i \quad \forall j \in [1, N+1]$$

Step 3: Compute the value of the firm at those points, that is, $V_{t-1,i,j} = \mathbf{V}_{t-1}\left(e_{t-1,i,j}, l_{t-1,i,j}\right)$, as given by the expression above. Additionally, compute the value of long-term debt at the firm's optimal policies, which is given by:

$$\begin{aligned} q_{t-1,i,j} \quad &= \quad \frac{1}{1+r}\left(1 - F\left(\boldsymbol{\phi}^d\left(k_{t-1,i,j}, \mathbf{s}(e_{t-1,i,j}, l_{t-1,i,j}) + \kappa\mathbf{l}(e_{t-1,i,j}, l_{t-1,i,j})\right)\right)\right)\kappa \\ &+ \quad \frac{\delta}{1+r}\mathbf{q}_C\left(k_{t-1,i,j}, \boldsymbol{\phi}^d\left(k_{t-1,i,j}, \mathbf{s}(e_{t-1,i,j}, l_{t-1,i,j}) + \kappa\mathbf{l}(e_{t-1,i,j}, l_{t-1,i,j})\right), l_{t-1,i,j}\right) \end{aligned}$$

where $k_{t-1,i,j} = \mathbf{k}_{t-1}\left(e_{t-1,i,j}, l_{t-1,i,j}, \mathbf{s}(e_{t-1,i,j}, l_{t-1,i,j}), \mathbf{l}(e_{t-1,i,j}, l_{t-1,i,j})\right)$, and $\mathbf{s}$ and $\mathbf{l}$ are the policy functions in the last-period problem.

Let $\epsilon > 0$ be the stopping criterion. If

$$\max_{i,j} \quad |x_{t-1,i,j} - x_{t,i,j}| \quad \geq \epsilon$$

for $x \in \{e, V, q\}$, repeat the step given the new grids and value functions $\{e_{t,i,j}, l_{t,i,j}, V_{t,i,j}, q_{t,i,j}\}$. Otherwise, stop constructing the interim period problem.

4. **Convergence:** Check whether the resulting policy function for long-term borrowing is such that:

$$\mathbf{l}(e_{t-1,i,j}, l_{t-1,i,j}) < \bar{l} \quad \forall i, j.$$

If not, go back to the first step, extend the grid for $l$, and repeat steps 1-4.