

# Cost Allocation for a Tree Network with Heterogeneous Customers <sup>\*†</sup>

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## Abstract

We analyze a cost allocation problem which could naturally arise from a situation wherein a tree network  $T = (N \cup \{0\}, E)$ , serving heterogeneous customers, has to be constructed. The customers, located at  $N$ , require some service from a central supplier, located at vertex 0. The customers have heterogeneous preferences for the level or quality of service received from the central supplier.

We formulate the above cost allocation problem as a cooperative game, referred to as an *extended tree* game. The extended tree game is a proper extension of Megiddo's tree game (Math of O.R., 1978), wherein all the customers have identical preferences regarding the level of service received. We prove that an extended tree game is convex, and we show that its Shapley value can be computed in  $\mathcal{O}(p|N|)$  time, where  $p$  is the number of distinct preference levels. We further provide a complete facial description of the core polytope of an extended tree game, and demonstrate that even when there are only two classes of customers, the number of non redundant core constraints could be exponential in  $|N|$ . Nevertheless, we are able to construct an  $\mathcal{O}(p|N|)$  algorithm to check core membership of an arbitrary cost allocation, which can be used to construct an  $\mathcal{O}(p|N|^3)$  combinatorial algorithm to compute the nucleolus of an extended tree game. Finally, we show that the complements of the facet defining coalitions for the core are all connected in an auxiliary tree graph with node set  $N$ .

**Key Words:** cooperative game, convex game, Shapley value, nucleolus, polytope, facets, strongly polynomial algorithm.

## 1 Introduction

We analyze in this paper a cost allocation problem that could naturally arise from a situation wherein a tree network  $T = (N \cup \{0\}, E)$ , serving heterogeneous customers, has to be constructed. The customers, located at the set of nodes  $N$ , are in need of a "service" that can be provided by a central supplier, located at node 0, the root of the tree. For example, one such situation could arise when the customers would like to receive, non simultaneously, some commodity flow which originates at the central supplier. In this case the arcs of the tree can be thought of as "pipelines", and since the flow requirements by the heterogeneous customers could be different, the arcs may have different capacities. The cost allocation problem in this case is concerned with the allocation of the cost of constructing the cheapest network that would satisfy the customers' demand.

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A second example could arise when we view the tree  $T$  as a communication network. Indeed, reliability considerations in communication networks often impose different reliability requirements on different links in the network. Therefore, the heterogeneous users, represented by the set of nodes  $N$ , may impose different reliability requirements on the arcs in the unique path leading from them to the root of the tree  $T$ , which can be thought of as a central communication centre. The cost allocation problem in this case is concerned with the allocation of the cost of construction of the communication network among the customers.

It appears that cooperative game theory can be naturally used to analyze cost allocation problems of the form described above. In such an approach, a cost allocation problem is formulated as a cooperative game in characteristic function form and various solution concepts in cooperative game theory, such as the core, the Shapley value and the nucleolus are examined as possible cost allocation schemes. Indeed, cooperative game theory was used to analyze, for example, cost allocation problems arising in linear programming (Owen (1975), Granot (1986)), network flows (Kalai and Zemel (1982a, 1982b), Granot and Granot (1992), Reijnierse et al. (1996)), cost allocation in communication networks (Bird (1976), Granot and Huberman (1981)), cost allocations in trees (Megiddo (1978), Granot et al. (1996)), cost allocation in airport runways (Littlechild and Owen (1973), Littlechild (1974)), cost allocation in the traveling salesman problem (Tamir (1989), Potters et al. (1992)), cost allocation in routing (Derks and Kuipers (1997)) and cost allocation in the Chinese postman problem (Hamers et al. (1999) and Granot et al. (1999)).

In this paper we formulate the above cost allocation problem as a (cost) cooperative game in characteristic function form, referred to as an extended tree (ET) game. The extended tree game generalizes Megiddo's tree game (Megiddo (1978)), in the sense that in Megiddo's game all users have either identical reliability requirements or identical flow requirements. We prove in Section 4 that an ET game is convex, and thus its core is not empty. We also show in this section that the Shapley value of an ET game can be calculated in  $\mathcal{O}(p|N|)$  time, where  $p$  is the number of distinct requirement levels. In Section 5 we provide a complete characterization of the facial structure of the core of an ET game. Notwithstanding the fact that an ET game may have an exponential number of non redundant core constraints, even when there are only two classes of customers, we are able to construct in Section 6 an  $\mathcal{O}(p|N|)$  time algorithm for computing the minimal excess with respect to an arbitrary cost allocation vector. Hence, it is possible to check core membership in such a game in linear time. Furthermore, the nucleolus of an ET game can be computed in  $\mathcal{O}(p|N|^3)$  time. An algorithm with this complexity bound is presented in a working paper by Granot et al. (2000). In Section 7, we present the construction of an auxiliary tree graph with node set  $N$ , which has the property that the complements of the facet defining coalitions for the core are all connected in the tree graph. As a direct consequence of this property, it can be shown that the nucleolus of an ET game is the unique solution of  $|N| - 1$  nonlinear equations involving excesses.

## 2 Preliminary and notation.

Let  $N = \{1, \dots, n\}$  designate a finite set of users (players, owners, etc.) and let  $c : 2^N \rightarrow \mathbb{R}$  be such that  $c(\emptyset) = 0$ . The pair  $(N; c)$  is called a *cost cooperative game in characteristic function form*, or simply a *game*. The function  $c$  is the *characteristic function* of the game. The game  $(N; c)$  is said to be *monotone* if

$$c(T) \leq c(S)$$

whenever  $T \subseteq S \subseteq N$ . It is said to be *convex* if

$$c(S) + c(T) \geq c(S \cup T) + c(S \cap T)$$

for all  $S, T \subseteq N$ . Equivalently  $(N; c)$  is convex if for all  $i \in N$  and all  $T \subseteq S \subseteq N \setminus \{i\}$ ,

$$c(S \cup \{i\}) - c(S) \leq c(T \cup \{i\}) - c(T).$$

For a vector  $y \in \mathbb{R}^N$  and  $S \subseteq N$  let  $y(S) \equiv \sum_{j \in S} y_j$ .

A vector  $x \in \mathbb{R}^N$  satisfying  $x(N) = c(N)$  will be referred to as a *cost allocation* or *pre-imputation* in  $(N; c)$ , and a cost allocation satisfying  $x_i \leq c(\{i\})$  for  $i = 1, \dots, n$  will be referred to as an *imputation* in the game  $(N; c)$ . The set of cost allocations and the set of imputations are denoted by  $I^*(N; c)$  and  $I(N; c)$ , respectively. For  $x \in I^*(N; c)$ , let  $e(S, x) \equiv c(S) - x(S)$  be the *excess* of coalition  $S$  with respect to  $x$  (in the game  $(N; c)$ ). The *core*,  $\text{Core}(N; c)$ , of the game  $(N; c)$  is given by

$$\text{Core}(N; c) = \{x \in I^*(N; c) \mid e(S, x) \geq 0 \text{ for all } S \subset N\}.$$

Thus the core of a game, if not empty, consists of all cost allocation vectors  $x$  which provide no incentive for any coalition to secede from the rest of the players and act alone.

For  $y \in I^*(N; c)$  and  $i, j \in N$ , the *surplus* of  $i$  with respect to  $j$  is defined as

$$s_{ij}(y) \equiv \min\{c(S) - y(S) \mid i \in S \text{ and } j \notin S\}.$$

The *prekernel*,  $K^*(N; c)$ , introduced by Davis and Maschler (1965), is defined as the set of cost allocations for which all surpluses are "balanced", i.e.

$$K^*(N; c) = \{x \in I^*(N; c) \mid s_{ij}(x) = s_{ji}(x) \text{ for all } i, j \in N\}.$$

For convex games, the *kernel* and prekernel coincide.

For a cost allocation  $y$ , let  $\theta(y)$  be the vector in  $\mathbb{R}^{2^n - 2}$  whose entries are the excesses  $e(S, y)$  ( $\emptyset \neq S \neq N$ ), arranged in a nondecreasing order. For  $x, y \in \mathbb{R}^t$  we will say that  $x$  is lexicographically smaller than  $y$ , denoted  $x \prec y$ , if there exists an index  $k$  ( $1 \leq k \leq t$ ), such that  $x_i = y_i$  for  $i < k$ , and  $x_k < y_k$ . If  $x \prec y$  or  $x = y$  we denote it by  $x \preceq y$ . The *prenucleolus*,  $\text{Nu}^*(N; c)$ , of the game  $(N; c)$  is given by

$$\text{Nu}^*(N; c) = \{x \in I^*(N; c) \mid \theta(y) \preceq \theta(x) \text{ for all } y \in I^*(N; c)\}.$$

The prenucleolus was introduced by Schmeidler (1969), who has demonstrated that it is a singleton set contained in the prekernel. The *nucleolus*,  $\text{Nu}(N; c)$ , of the game  $(N; c)$  is given by

$$\text{Nu}(N; c) = \{x \in I(N; c) \mid \theta(y) \preceq \theta(x) \text{ for all } y \in I(N; c)\}.$$

The nucleolus is a singleton set contained in the kernel, provided that the imputation set is non empty. For games with a non empty core, the nucleolus coincides with the prenucleolus, and is contained in the core.

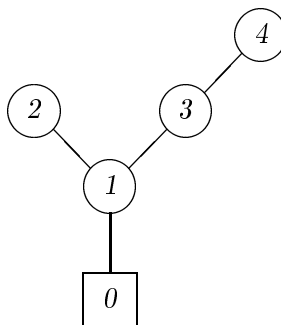
### 3 The extended tree game

Let  $T = (N \cup \{0\}, E)$  be a tree graph, where  $N = \{1, \dots, n\}$  represents the set of players and 0 is the root of the tree. For each node  $i \in N$ , denote by  $e_i$  the unique arc emanating from  $i$  and on the unique path from  $i$  to 0 in  $T$ . Each player (node) can be any one of  $p$  types (or levels),  $p \leq n$ . The type of player  $i$  is denoted by  $\gamma_i$ . Each player requires a connection to the root and the quality or capacity of the connection depends on the type of the player. For a player  $j$  with  $\gamma_j = \ell$ , it is required that each arc on the path from  $j$  to 0 be of type (quality level or capacity level)  $\ell$  or higher. The cost of using an arc  $e_i$  at level  $\ell$  is given by  $a_i^\ell$ . We assume that arc costs are monotone in types. Thus, for each  $i \in N$ ,  $0 \leq a_i^1 \leq a_i^2 \leq \dots \leq a_i^p$ . Further, we define  $a_i^0 := 0$  for all  $i \in N$ . This convention turns out to be convenient in some of the formulae. It also implies that, when we speak of type 0 players, we invariably mean players who do not need a connection to the root (or whose connection is at cost 0).

Given a node  $i \in N$ , let  $T_i$  denote the set of nodes in the subtree of  $T$  rooted at  $i$ . For any subset  $A \subseteq N$ , we let  $\gamma(A) := \max\{\gamma_i \mid i \in A\}$ , where the maximum over the empty set is defined as zero. Then, the quality or capacity of arc  $e_i$  in the optimal tree for a coalition  $S \subseteq N$  must be  $\gamma(T_i \cap S)$  in order to satisfy the requirements of all customers in  $S$  that use arc  $i$ . Note that this is also true when  $\gamma(T_i \cap S) = 0$ , since, by convention,  $a_i^0 = 0$  for all  $i \in N$ . Hence,  $c(S) = \sum_{i \in N} a_i^{\gamma(T_i \cap S)}$  for any  $S \subseteq N$ .

We refer to the pair  $(N; c)$ , where  $c$  is as defined above, as the *extended tree (ET) game*. For  $p = 1$ , i.e., when there is only one type of player, an ET game coincides with Megiddo's tree game (Megiddo (1978)). If  $p = 1$  and  $T$  is a chain, an ET game coincides with the airport game (Littlechild and Owen (1973), Littlechild (1974)).

**Example 3.1:** Consider the tree  $T$  shown in Figure 3.1 below.



**Figure 3.1:** The Tree  $T$

Suppose that player 3 is of type 1, players 1 and 4 are of type 2, and player 2 is of type 3. Then,

$$\begin{array}{ll}
 c(\{1\}) = a_1^2 & c(\{2\}) = a_1^3 + a_2^3 \\
 c(\{3\}) = a_1^1 + a_3^1 & c(\{4\}) = a_1^2 + a_3^2 + a_4^2 \\
 c(\{1, 3\}) = a_1^2 + a_3^1 & c(\{1, 4\}) = a_1^2 + a_3^2 + a_4^2 \\
 c(\{2, 3\}) = a_1^3 + a_2^3 + a_3^1 & c(\{2, 4\}) = a_1^3 + a_2^3 + a_3^2 + a_4^2 \\
 c(\{3, 4\}) = a_1^2 + a_3^2 + a_4^2 & c(\{1, 2, 3\}) = a_1^3 + a_2^3 + a_3^1 \\
 c(\{1, 2, 4\}) = a_1^3 + a_2^3 + a_3^2 + a_4^2 & c(\{1, 3, 4\}) = a_1^2 + a_3^2 + a_4^2 \\
 c(\{2, 3, 4\}) = a_1^3 + a_2^3 + a_3^2 + a_4^2 & c(\{1, 2, 3, 4\}) = a_1^3 + a_2^3 + a_3^2 + a_4^2.
 \end{array}$$

◁

## 4 Basic properties of extended tree games

Some of the basic results for ET games are easily derived by observing that any ET game can be written as a sum of ET games in which there are only players of type 0 and type 1. For an ET game  $(N; c)$ , let  $N^\gamma \subseteq N$  denote the players of type  $\gamma$  ( $\gamma = 0, 1, \dots, p$ ). For  $\gamma = 1, \dots, p$ , define the ET game  $(N; c^\gamma)$  by designating  $N^0 \cup N^1 \cup \dots \cup N^{\gamma-1}$  as 0-players, and  $N^\gamma \cup \dots \cup N^p$  as 1-players, and let the cost of arc  $e_i$  for players of type 1 in this game be defined as  $a_i^\gamma - a_i^{\gamma-1}$ . Now it is straightforward to verify that indeed  $c = \sum_{\gamma=1}^p c^\gamma$ .

Now, let  $(N; c)$  be an ET game for which there are only players of type 0 (in the set  $N^0$ ) and players of type 1 (in the set  $N^1$ ). The *restriction* of  $(N; c)$  to the player set  $N^1$  is defined as the game  $(N^1; \hat{c})$ , where  $\hat{c}(S) = c(S)$  for all  $S \subseteq N^1$ . One can easily see that the restricted game  $(N^1; \hat{c})$  is actually a *standard tree game* (see Granot et al. (1996)), a variant of Megiddo's tree game in which some of the nodes of the network need not be occupied by a player. Since standard tree games are monotone and convex, it follows that an ET game with only two types of players is also monotone and convex. And since the sum of monotone and convex games is again monotone and convex, it follows that every ET game is monotone and convex.

Thus, from Shapley (1971) it follows that the core of an ET game is not empty, and the Shapley value is contained in the core. From Maschler et al. (1972) it follows that the bargaining set (see Davis and Maschler (1967)) coincides with the core, and the kernel, prekernel (see Davis and Maschler (1965)), and prenucleolus all coincide with the nucleolus (Maschler et al. (1972)). Note that these results are critical to the fact that an ET game is defined on a tree graph. For example, one can easily construct examples which demonstrate that the core of an ET game could be empty if it were defined on a general graph.

The Shapley value of a standard tree game can be computed in linear time, and since the Shapley value is an additive solution, it follows that the Shapley value of an ET game can be computed in  $\mathcal{O}(p|N|)$  time.

Finally, we note that for the purpose of the analysis carried out in this paper, it can be assumed that there is exactly one node adjacent to the root node 0 in  $T$ . Indeed, assume that there are at least two nodes adjacent to the root node 0 in  $T$ . Let  $i_1$  be one of these adjacent nodes, and let  $N_1$  be the set of nodes in the subtree of  $T$  rooted at  $i_1$ . Define  $N_2 = N \setminus N_1$ , and note that  $(N_1; c_1)$  and  $(N_2; c_2)$  also define extended tree games, where  $c_1$  and  $c_2$  are the restrictions of the cost function to the respective subtrees. Now, for each set  $S \subseteq N$ ,  $c(S) = c_1(S \cap N_1) + c_2(S \cap N_2)$ . Thus,  $(N; c)$  is decomposable with an efficient coalition structure  $\{N_1, N_2\}$ . Each of the extended tree games  $(N_i; c_i)$ ,  $i = 1, 2$ , is convex, and it is easy to show that the core (and nucleolus and Shapley value) of the game  $(N; c)$  is a Cartesian product of the cores (nucleoli, Shapley values) of the subgames  $(N_1; c_1)$  and  $(N_2; c_2)$ . Thus, without loss of generality, we will assume that the tree  $T$  has just one arc incident to the root node.

## 5 The Core of an Extended Tree Game

As discussed earlier, the convexity of an ET game implies that its core is nonempty. In this section we provide a minimal inequality description of the core of an ET game, which will be used in subsequent sections to check core membership in ET games in linear time and to demonstrate that a strongly polynomial combinatorial  $\mathcal{O}(p|N|^3)$  algorithm exists for computing the nucleolus of this class of games.

Recall that the core polytope  $C(N; c)$  is defined as follows:

$$\text{Core}(N; c) = \{x \in \Gamma^*(N; c) \mid e(S, x) \geq 0 \text{ for all } S \subset N\}. \quad (5.1)$$

A minimal inequality description of  $C(N; c)$  is given by the facet defining inequalities (see Nemhauser and Wolsey (1988)).

Given a set  $S$ ,  $S \subseteq N$ , define its completion:

$$S_c = \{i \in N \mid \gamma_i \leq \gamma(T_i \cap S)\},$$

where  $\gamma_j$ , as defined earlier, is the type or level of player  $j$ .

We will say that  $S$ ,  $S \subset N$ , is redundant if the core constraint  $x(S) \leq c(S)$  is redundant in the description of  $C(N; c)$ .

**Lemma 5.1** *If  $S \neq S_c$  for a set  $S \subseteq N$  and  $S \neq N \setminus \{i\}$ ,  $i \in N$ , then  $S$  is redundant.*

*Proof:* Note that  $c(S_c) = c(S)$  by definition of  $S_c$ . For each player  $i \in S_c \setminus S$ , there exists  $j \in T_i \cap S$ , such that  $\gamma_j \geq \gamma_i$ . Thus  $c(N \setminus \{i\}) = c(N)$  for all  $i \in S_c \setminus S$ . The inequality  $x(N \setminus \{i\}) \leq c(N \setminus \{i\})$  along with  $x(N) = c(N)$  imply  $x_i \geq 0 \forall i \in S_c \setminus S$ . The inequality  $x(S) \leq c(S)$  can thus be written as a sum of  $x(S_c) \leq c(S_c)$  and  $-x_i \leq 0$  for  $i \in S_c \setminus S$ .  $\triangleleft$

In the following we will say that  $j$  is a *son* of  $k$  if  $k$  is the node that is visited directly after  $j$  on the path from  $j$  to the root 0 in the network  $T$ . We will say that  $j$  is a *descendant* of  $k$  if  $k$  is visited somewhere on the path from  $j$  to the root.

**Lemma 5.2** *Assume  $S \subseteq N$  is non redundant and let  $\gamma := \gamma(T_k \cap S)$ . Then:*

*i) If  $S$  is a proper subset of  $N \setminus \{k\}$  then  $\gamma < \gamma_k$ .*

*ii) If  $\gamma(T_j) \leq \gamma$  for some son  $j$  of  $k$  and  $S \cup T_j$  is a proper subset of  $N$ , then  $T_j \subseteq S$ .*

*Proof:* i): If  $\gamma \geq \gamma_k$ ,  $S$  is not a convex set. By assumption,  $S$  is a proper subset of  $N \setminus \{k\}$ . Thus, by Lemma 5.1,  $S$  is redundant, which contradicts our assumption that  $S$  is non redundant.

ii): Suppose ii) does not hold. Then, there exists a son  $j$  of  $k$  such that  $\gamma(T_j) \leq \gamma$  and  $T_j \setminus S \neq \emptyset$ . Let  $A = T_j \setminus S$ . We will show  $c(N) + c(S) = c(N \setminus A) + c(A \cup S)$ , which implies that  $S$  is redundant. It will be sufficient to show that

$$\gamma(T_i) + \gamma(T_i \cap S) = \gamma(T_i \setminus A) + \gamma((T_i \cap A) \cup (T_i \cap S)) \quad (5.2)$$

for all  $i \in N$ .

First consider the case where  $i \in T_j$ . Then  $T_i \subseteq T_j$ , and it follows that  $T_i = (T_i \cap A) \cup (T_i \cap S)$  and  $T_i \cap S = T_i \setminus A$ . Thus, (5.2) holds in a trivial way, since the coalitions on the left and right hand sides of (5.2) are identical.

Now consider the case that  $i \in T_\ell$  with  $\ell$  being a son of  $k$ , but  $\ell \neq j$ . Then  $T_i \cap T_j = \emptyset$ , hence  $T_i \cap A = \emptyset$ , and  $T_i \setminus A = T_i$ . Again, we see that the equality trivially holds, since the coalitions on the left and right hand sides of (5.2) are identical.

Next, assume  $k$  is not a descendant of  $i$ , and neither is  $i$  a descendant of  $k$ . Then  $T_i \cap T_k = \emptyset$ , and since  $A \subseteq T_k$ , we have  $T_i \cap A = \emptyset$  and  $T_i \setminus A = T_i$ , and (5.2) holds in this case as well.

Finally, assume  $k$  is a descendant of  $i$ . Then,

$$\gamma(T_i \cap A) \leq \gamma(A) \leq \gamma(T_j) \leq \gamma = \gamma(T_k \cap S) \leq \gamma(T_i \cap S) \leq \gamma(T_i \setminus A).$$

Thus,

$$\gamma(T_i) = \gamma(A \cup (T_i \setminus A)) = \max(\gamma(A), \gamma(T_i \setminus A)) = \gamma(T_i \setminus A),$$

and

$$\gamma((T_i \cap A) \cup (T_i \cap S)) = \max(\gamma(T_i \cap A), \gamma(T_i \cap S)) = \gamma(T_i \cap S).$$

Thus, (5.2) holds in this case as well, and the proof is complete.  $\triangleleft$

For the rest of this section we will assume, for simplicity of exposition, that

$$0 < a_j^1 < \dots < a_j^p \text{ for } j = 1, \dots, n, \quad (5.3)$$

and that there are no type 0 players. This assumption can be relaxed, at the cost of additional notation and more elaborate analysis. However, since the rest of the results in this section are not required for subsequent sections, we chose to impose this non degeneracy assumption.

**Theorem 5.3** *Under the non degeneracy assumption (5.3) we have that  $S \subseteq N$  is a facet if and only if it satisfies the conditions of Lemma 5.2.*

*Proof :* The 'only if'-part of the proof is trivial. To prove the 'if'-part, let  $S \subseteq N$  be a coalition that satisfies the conditions of Lemma 5.2. We intend to show that the inequality  $x(S) \leq c(S)$  is facet defining for the core. To this end, we will identify a core allocation  $\bar{x}$  with  $\bar{x}(S) = c(S)$  and  $\bar{x}(R) < c(R)$  for all  $R \neq \emptyset, S, N$ .

For a permutation  $\pi$  on the player set  $N$ , the *marginal contribution* or *greedy* vector  $m(c, \pi)$  is defined as

$$m_i(c, \pi) = c(P_\pi(i)) - c(P_\pi(i) \setminus \{i\}), (i \in N),$$

with  $P_\pi(i)$  denoting the set of predecessors of player  $i$ , i.e.

$$P_\pi(i) = \{j \in N \mid \pi(j) \leq \pi(i)\}.$$

Say that  $m(c, \pi)$  is  $S$ -greedy if the elements of  $S$  appear first in the permutation  $\pi$ , i.e.  $S = P_\pi(i)$  for some  $i \in N$ . More generally, for disjoint subsets  $A, B, C$ , say that  $m(c, \pi)$  is  $A - B - C$ -greedy if  $A = P_\pi(a)$ ,  $A \cup B = P_\pi(b)$ , and  $A \cup B \cup C = P_\pi(c)$  for  $a, b, c \in N$ .

Define  $\bar{x}$  as the average over all  $|S|!(|N| - |S|)!$   $S$ -greedy vectors. Since an ET game is convex, every greedy vector is an element of the core, hence  $\bar{x}$ , being a convex combination of greedy vectors, is also an element of the core. We also trivially have  $\bar{x}(S) = c(S)$ , and we claim that  $\bar{x}(R) < c(R)$  for all  $R \neq \emptyset, S, N$ .

To this end, we will show that  $x(R) < c(R)$  for at least one  $S$ -greedy vector  $x$ . If  $S \setminus R \neq \emptyset$ , this is true for any  $(S \setminus R) - (S \cap R) - (R \setminus S)$ -greedy vector, which follows immediately from  $x(R) = x(S \cup R) - x(S \setminus R) = x(S \cup R) - c(S \setminus R)$  and the nondegeneracy assumption. So assume that  $R = S \cup A$  with  $A \neq \emptyset$ . Denote  $N \setminus R$  by  $\bar{A}$ . Choose  $k \in N$  such that  $T_k$  contains nodes from  $A$  and  $\bar{A}$ , and such that all sons  $j$  of  $k$  satisfy  $T_j \subseteq S \cup A$  or  $T_j \subseteq S \cup \bar{A}$ . Choose  $a \in T_k \cap A$  and  $\bar{a} \in T_k \cap \bar{A}$  of maximum level. It follows from Lemma 5.2 that  $\gamma_a, \gamma_{\bar{a}} > \gamma(T_k \cap S)$ . Using the assumption of non-degeneracy, it also follows that

$$c(S) + c(S \cup \{a\} \cup \{\bar{a}\}) < c(S \cup \{a\}) + c(S \cup \{\bar{a}\}). \quad (5.4)$$

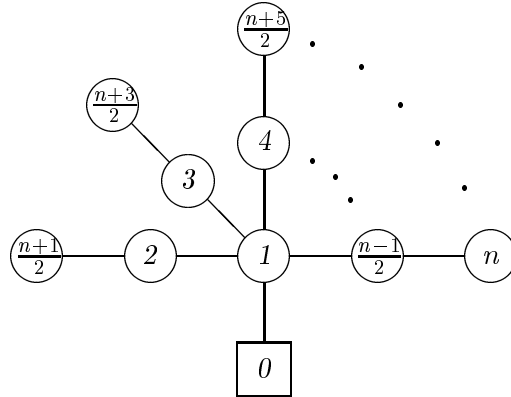
Now, let  $x'$  be any  $S - \{a\} - \{\bar{a}\} - A \setminus \{a\}$ -greedy vector, and let  $x$  be any  $S - \{\bar{a}\} - \{a\} - A \setminus \{a\}$ -greedy vector. Then  $x$  allocates less to  $S \cup A$  than  $x'$  does. Hence,

$$x(S \cup A) < x'(S \cup A) \leq c(S \cup A).$$

◁

Finally, we show below that Theorem 5.3 can be used to establish that an ET game may have an exponential number of non redundant core constraints (i.e. facets of the core polytope) even when there are only two types of players. Indeed, consider the following example.

**Example 5.1:** Consider the tree game induced by the tree depicted in Figure 5.1 below.



**Figure 5.1**

Suppose  $\gamma_j = 1, j = 1, \dots, \frac{n-1}{2}$  and  $\gamma_j = 2, j > \frac{n-1}{2}$ , where  $n$  is odd. Then, each subset  $S, 1 \in S$  and  $S \cap \{\frac{n+1}{2} + 1, \dots, n\} = \emptyset$  is convex and induces  $\frac{n-1}{2}$  connected components  $R_j$  of  $N \setminus S$ , for which  $\gamma(R_j) = 2$ . To show that each such subset  $S$  is a facet, observe that for  $k \notin S$  we have  $0 = \gamma(T_k \cap S) < \gamma_k$  for each such subset  $S$ , thus Condition (i) in Lemma 5.2 is satisfied. Further, for each  $j$  for which  $T_j \setminus S \neq \emptyset$  we have  $2 = \gamma(T_j) > \gamma(T_k \cap S) = 1$ . Thus, Condition (ii) of Lemma 5.2 is also satisfied, and it follows from Theorem 5.3 that  $S$  is indeed a facet. We conclude that there are  $\mathcal{O}(2^{\frac{n}{2}})$  non redundant core constraints for the ET game of this example.  $\triangleleft$

This, somewhat computationally negative result of Example 5.1, should be compared with Megiddo's tree game, which is obtained as a special case of an ET game by having only one type of players and in which the number of non redundant core constraints is linear in the number of nodes in the tree.

## 6 Checking Core Membership

If  $(N; c)$  is an ET game, and  $x$  a cost allocation for this game, then it is natural to ask whether  $x$  is an element of the core or not. As we have seen in the previous section, the number of facets of the core may be exponential in the number of players, even when there are only two types of players. Therefore, simply checking whether  $x(S) \leq c(S)$  for all  $S$  satisfying the conditions of Lemma 5.2 does not yield an efficient (polynomial time) algorithm. On the other hand, we know that polynomial time algorithms for minimizing submodular functions exist (see, e.g., Grötschel et al. (1981) and Iwata et al. (2000)). It follows that a polynomial time algorithm for checking core membership in an ET game exists, since for any cost allocation  $x$ , the excess function  $c(S) - x(S)$  is submodular, and  $x$  is an element of the core if and only if the minimal excess equals 0. In this section we present a procedure that does this task for ET games in  $\mathcal{O}(p|N|)$  time.

By  $C_k$  we denote the set of sons of a node  $k$  in  $T$ . For any  $S \subseteq N, S \neq \emptyset$  there is a unique node  $r$  such that  $S \subseteq T_r$ , and such that no son  $j \in C_r$  exists for which  $S \subseteq T_j$ . Denote this node by  $r_S$ , and let us call it the *root* of coalition  $S$ . So  $T_{r_S}$  is the smallest subtree that contains  $S$ . For a nonempty  $S \subseteq N$  and for



every  $j \in C_{r_S}$  we define  $S_j = T_j \cap S$ . Clearly, the sets  $S_j$ ,  $j \in C_{r_S}$ , form a partition of  $S \setminus \{r_S\}$ . Finally, by  $\mathcal{B}$  we denote the collection of coalitions satisfying conditions *i*) and *ii*) of Lemma 5.2.

**Lemma 6.1** *Let  $A, B \neq \emptyset$  be coalitions such that  $N \setminus A \in \mathcal{B}$ ,  $N \setminus B \in \mathcal{B}$ ,  $r_A = r_B =: r$  and  $\gamma(T_r \setminus A) = \gamma(T_r \setminus B) =: \gamma$ . Then:*

$$c(N \setminus A) - \sum_{j \in C_r} c(N \setminus A_j) = c(N \setminus B) - \sum_{j \in C_r} c(N \setminus B_j).$$

*Proof :* Suppose  $A$  is of cardinality one. Thus  $A = \{r\}$  and  $\gamma(T_r \setminus \{r\}) = \gamma$ . If  $r \in B$ , then by Lemma 5.2(ii),  $B = \{r\}$ . If  $r \notin B$ , then by Lemma 5.2(ii), there exists only one son  $j$  of  $r$  for which  $T_j \cap B \neq \emptyset$ . But then  $r_B \neq r$ . Thus, if  $A$  is of cardinality 1, then  $A = B = \{r\}$ , and the claim of the lemma is trivially true.

Next assume that  $A$  and  $B$  are of cardinality at least two. To prove the lemma, it is sufficient to show that for all  $i \in N$ ,

$$\gamma(T_i \setminus A) - \sum_{j \in C_r} \gamma(T_i \setminus A_j) = \gamma(T_i \setminus B) - \sum_{j \in C_r} \gamma(T_i \setminus B_j). \quad (6.5)$$

First consider the case that  $i$  is a descendant of  $r$ , i.e.  $i \in T_{j^*}$  for some son  $j^*$  of  $r$ . Then  $T_i \subseteq T_{j^*}$  and  $T_i \cap T_j = \emptyset$  for  $j \in C_r$ ,  $j \neq j^*$ . It follows that *i*)  $T_i \setminus A = T_i \setminus A_{j^*}$ , *ii*)  $T_i \setminus B = T_i \setminus B_{j^*}$ , and *iii*)  $T_i \setminus A_j = T_i \setminus B_j = T_i$  for  $j \in C_r$ ,  $j \neq j^*$ . Equation (6.5) follows directly.

Next assume that neither is  $i$  a descendant of  $r$ , nor is  $r$  a descendant of  $i$ . Then  $T_i \cap T_j = \emptyset$  for all  $j \in C_r$ . It follows that *i*)  $T_i \setminus A = T_i \setminus B = T_i$ , and *ii*)  $T_i \setminus A_j = T_i \setminus B_j = T_i$  for all  $j \in C_r$ , and Equation (6.5) follows directly.

Finally, assume that  $r$  is a descendant of  $i$  (hence  $T_r \subseteq T_i$ ). By assumption we have  $\gamma(T_r \setminus A) = \gamma(T_r \setminus B) = \gamma$ . Hence,

$$\begin{aligned} \gamma(T_i \setminus A) &= \gamma((T_r \setminus A) \cup (T_i \setminus T_r)) \\ &= \max(\gamma(T_r \setminus A), \gamma(T_i \setminus T_r)) \\ &= \max(\gamma, \gamma(T_i \setminus T_r)). \end{aligned}$$

Similarly one shows that  $\gamma(T_i \setminus B) = \max(\gamma, \gamma(T_i \setminus T_r))$ . Thus,  $\gamma(T_i \setminus A) = \gamma(T_i \setminus B)$ .

Now, for a fixed  $j^* \in C_r$  we have

$$\begin{aligned} \gamma(T_r \setminus A_{j^*}) &= \max(\gamma_r, \max\{\gamma(T_j \setminus A_{j^*}) \mid j \in C_r\}) \\ &= \max(\gamma_r, \gamma(T_{j^*} \setminus A_{j^*}), \max\{\gamma(T_j) \mid j \neq j^*, j \in C_r\}). \end{aligned} \quad (6.6)$$

We claim that

$$\gamma(T_{j^*} \setminus A_{j^*}) \leq \max(\gamma_r, \max\{\gamma(T_j) \mid j \neq j^*, j \in C_r\}). \quad (6.7)$$

Indeed, recall that  $T_r$  is the smallest subtree that contains  $A$ . Hence,  $r \in A$  or there are at least two sons  $j$  of  $r$  for which  $T_j \setminus (N \setminus A) \neq \emptyset$ . If  $r \in A$ , then, since it is assumed that  $|A| \geq 2$ , it follows from Lemma 5.2(ii) that  $\gamma < \gamma_r$ , and (6.7) follows since  $\gamma(T_{j^*} \setminus A_{j^*}) \leq \gamma$ . If, on the other hand we have two sons of  $r$  for which  $T_j \setminus (N \setminus A) \neq \emptyset$ , then according to Lemma 5.2(ii),  $\gamma(T_j) > \gamma$  for both sons. Since at least one of these sons is not  $j^*$ , (6.7) follows from  $\max\{\gamma(T_j) \mid j \neq j^*, j \in C_r\} > \gamma \geq \gamma(T_{j^*} \setminus A_{j^*})$ .

Combining (6.6) and (6.7) we obtain that for a fixed son  $j^*$  of  $r$ :

$$\gamma(T_r \setminus A_{j^*}) = \max(\gamma_r, \max\{\gamma(T_j) \mid j \in C_r, j \neq j^*\}) = \gamma(T_r \setminus T_{j^*}). \quad (6.8)$$

Hence, for all  $j \in C_r$  we have

$$\begin{aligned}\gamma(T_i \setminus A_j) &= \gamma((T_i \setminus T_r) \cup (T_r \setminus A_j)) \\ &= \max(\gamma(T_i \setminus T_r), \gamma(T_r \setminus A_j)).\end{aligned}$$

Similarly one shows that

$$\gamma(T_i \setminus B_j) = \max(\gamma(T_i \setminus T_r), \gamma(T_r \setminus B_j)),$$

which implies that  $\gamma(T_i \setminus A_j) = \gamma(T_i \setminus B_j)$  for all  $j \in C_r$ . We conclude that 6.5 holds in this case as well, and the proof of Lemma 6.1 is complete.  $\triangleleft$

By Lemma 6.1, the value  $c(N \setminus S) - \sum_{j \in C_r} c(N \setminus S_j)$  for coalitions  $S \neq \emptyset$  that satisfy  $N \setminus S \in \mathcal{B}$ ,  $r_S = r$  and  $\gamma(T_r \setminus S) = \gamma$  is a constant. Let  $\alpha(r, \gamma)$  denote this constant.

For a fixed  $r$ ,  $\alpha(r, \gamma)$  is a non-decreasing function in  $\gamma$ . The proof of this result is similar to the proof of Lemma 6.1: for two coalitions  $A$  and  $B$  with  $N \setminus A, N \setminus B \in \mathcal{B}$  and with  $r_A = r_B = r$ , one can show that  $c(N \setminus A) - \sum_{j \in C_r} c(N \setminus A_j) \geq c(N \setminus B) - \sum_{j \in C_r} c(N \setminus B_j)$  whenever  $\gamma(T_r \setminus A) \geq \gamma(T_r \setminus B)$ .

The result of Lemma 6.1 can be used to design a linear time algorithm for checking core membership in an ET game. Define  $\mathcal{E}(k, \gamma)$  as the collection of coalitions  $S$  satisfying Lemma 5.2,  $r_{N \setminus S} = k$  and  $\gamma(T_k \cap S) \leq \gamma$ , and define  $\mathcal{F}(k, \gamma)$  as the union of all collections  $\mathcal{E}(j, \gamma)$  for  $j \in T_k$ . Further, with respect to an arbitrary allocation  $x$ , define

$$e(k, \gamma, x) = \min\{c(S) - x(S) \mid S \in \mathcal{E}(k, \gamma)\},$$

and

$$f(k, \gamma, x) = \min\{c(S) - x(S) \mid S \in \mathcal{F}(k, \gamma)\}.$$

Finally, define

$$S(k, \gamma) = \{j \in C_k \mid \gamma(T_j) > \gamma\},$$

and define the auxiliary variable  $g(k, \gamma, x)$  by

$$g(k, \gamma, x) = \alpha(k, \gamma) + (|C_k| - 1)c(N) + \sum_{j \in S(k, \gamma)} f(j, \gamma, x).$$

We will assume that there are no type 0 players. This restriction is not essential: if there are type 0 players we can eliminate them by incrementing the type of each player by one, as well as the type-index associated with the arc-costs. That is, the cost  $a_i^j$  of arc  $e_i$  incurred by a player of type  $j$  is now denoted by  $a_i^{j+1}$  for all  $i$  and  $j$ . This transformation does not change the ET game, but it simplifies the initialization step of the procedure. Now, the procedure **Minimal Excess** that is presented below provides recursive formulae for computing the numbers  $e(k, \gamma, x)$  and  $f(k, \gamma, x)$ .

## Procedure Minimal Excess

### Initialization Step

**for every leaf  $k$  and every type  $\gamma$  do begin**

$$e(k, \gamma, x) := c(N \setminus \{k\}) - x(N \setminus \{k\});$$

$$f(k, \gamma, x) := e(k, \gamma, x);$$

**end;**

**for every node  $k$  do begin**

$$e(k, 0, x) := c(N \setminus T_k) - x(N \setminus T_k);$$

$$f(k, 0, x) := \min(e(k, 0, x), \min\{f(j, 0, x) \mid j \in C_k\});$$

**end;**

## Recursive Step

**for every non leaf  $k$  and  $\gamma \geq 1$  do begin**  
   **if  $\gamma_k > \gamma$  then**  
      $e(k, \gamma, x) := \min(e(k, \gamma - 1, x), x_k + g(k, \gamma, x));$   
   **if  $\gamma_k \leq \gamma$  and  $|S(k, \gamma)| > 1$  then**  
      $e(k, \gamma, x) := \min(e(k, \gamma - 1, x), g(k, \gamma, x));$   
   **if  $\gamma_k \leq \gamma$  and  $|S(k, \gamma)| \leq 1$  and  $\gamma(T_k \setminus \{k\}) = \gamma$  then**  
      $e(k, \gamma, x) := \min(e(k, \gamma - 1, x), c(N \setminus \{k\}) - x(N \setminus \{k\}));$   
   **if  $\gamma_k \leq \gamma$  and  $|S(k, \gamma)| \leq 1$  and  $\gamma(T_k \setminus \{k\}) \neq \gamma$  then**  
      $e(k, \gamma, x) := e(k, \gamma - 1, x);$   
      $f(k, \gamma, x) := \min(e(k, \gamma, x), \min\{f(j, \gamma, x) \mid j \in C_k\});$   
**end;**

**Theorem 6.2** *Procedure Minimal Excess correctly computes the numbers  $e(k, \gamma, x)$  and  $f(k, \gamma, x)$  for all  $k \in N$  and all  $\gamma \in \{0, 1, \dots, p\}$ .*

*Proof :* The first three lines of the initialization are explained by the fact that, for all leaves  $k$  of the network and all  $\gamma$ , we have  $\mathcal{E}(k, \gamma) = \{N \setminus \{k\}\}$ . The correctness of the next three lines follows from the fact that, for all nodes  $k \in N$ , we have  $\mathcal{E}(k, 0) = \{N \setminus T_k\}$  (since we assume that there are no type 0 players).

Let us now explain the recursive step. The formula for  $f(k, \gamma, x)$  is self-explanatory, so we will explain only the computation of  $e(k, \gamma, x)$ . Assume that  $k$  is an arbitrary non leaf node of the network and that  $\gamma \geq 1$ . Further, assume that  $e(j, \gamma, x)$  and  $f(j, \gamma, x)$  have already been determined for all  $j \in C_k$ , as well as  $e(k, \gamma - 1, x)$  and  $f(k, \gamma - 1, x)$ .

In the following, let  $E(k, \gamma, x)$  denote a coalition in  $\mathcal{E}(k, \gamma)$  such that its excess equals  $e(k, \gamma, x)$ , and let  $F(k, \gamma, x)$  denote a coalition in  $\mathcal{E}(k, \gamma)$  such that its excess equals  $f(k, \gamma, x)$ .

**Case 1:**  $\gamma_k > \gamma$ .

We will show that

$$e(k, \gamma, x) \leq \min(e(k, \gamma - 1, x), x_k + g(k, \gamma, x)). \quad (6.9)$$

To see that  $e(k, \gamma, x) \leq e(k, \gamma - 1, x)$  it is sufficient to note that  $\mathcal{E}(k, \gamma) \supseteq \mathcal{E}(k, \gamma - 1)$ .

To see that  $e(k, \gamma, x) \leq x_k + g(k, \gamma, x)$ , construct an element of  $\mathcal{E}(k, \gamma)$  as follows. For each  $j \in S(k, \gamma)$ , let  $S_j = N \setminus F(j, \gamma, x)$ , and define  $S = \cup_{j \in S(k, \gamma)} S_j \cup \{k\}$ . Now one can easily verify that  $N \setminus S$  satisfies the conditions of Lemma 5.2, that  $r_S = k$  and that  $\gamma(T_k \setminus S) \leq \gamma$ . Hence,  $N \setminus S \in \mathcal{E}(k, \gamma)$ . Denoting  $\tilde{\gamma} = \gamma(T_k \setminus S)$ , it now follows from Lemma 6.1 that

$$c(N \setminus S) - x(N \setminus S) = \alpha(k, \tilde{\gamma}) + x_k + (|C_k| - 1)c(N) + \sum_{j \in S(k, \gamma)} f(j, \gamma, x),$$

hence

$$e(k, \gamma, x) \leq \alpha(k, \tilde{\gamma}) + x_k + (|C_k| - 1)c(N) + \sum_{j \in S(k, \gamma)} f(j, \gamma, x).$$

Since  $\alpha(k, \tilde{\gamma}) \leq \alpha(k, \gamma)$ , (6.9) now follows.

We will next show that equality holds in (6.9). To this end, let  $S = N \setminus E(k, \gamma, x)$ . If  $\gamma(T_k \setminus S) < \gamma$ , then we trivially have

$$e(k, \gamma, x) = e(k, \gamma - 1, x),$$

and equality in (6.9) follows immediately. Thus, consider the case  $\gamma(T_k \setminus S) = \gamma$ . Then it follows from Lemma 6.1 that

$$c(N \setminus S) - x(N \setminus S) = x_k + \alpha(k, \gamma) + (|C_k| - 1)x(N) + \sum_{j \in C_k} (c(N \setminus S_j) - x(N \setminus S_j)),$$

where  $S_j = T_j \cap S$ . It follows from Lemma 5.2(ii) that  $T_j \subseteq N \setminus S$  whenever  $\gamma(T_j) \leq \gamma$ , hence  $S_j \neq \emptyset$  is only possible for  $j \in S(k, \gamma)$ . Therefore, the above equation is reduced to:

$$c(N \setminus S) - x(N \setminus S) = x_k + \alpha(k, \gamma) + (|C_k| - 1)x(N) + \sum_{j \in S(k, \gamma)} (c(N \setminus S_j) - x(N \setminus S_j)).$$

Moreover, for each  $j \in S(k, \gamma)$ , the coalition  $N \setminus S_j$  is a member of  $\mathcal{F}(j, \gamma)$ , hence

$$c(N \setminus S) - x(N \setminus S) \geq x_k + \alpha(k, \gamma) + (|C_k| - 1)x(N) + \sum_{j \in S(k, \gamma)} f(j, \gamma, x).$$

We have proved equality in (6.9) also for this case.

**Case 2:**  $\gamma_k \leq \gamma$  and  $|S(k, \gamma)| \geq 2$ .

Again we have the trivial upper bound  $e(k, \gamma - 1, x)$  for  $e(k, \gamma, x)$ , i.e.  $e(k, \gamma, x) \leq e(k, \gamma - 1, x)$ . Another upper bound is found as follows.

For each  $j \in S(k, \gamma)$ , let  $S_j = N \setminus F(j, \gamma, x)$ , and define  $S = \cup_{j \in S(k, \gamma)} S_j$ . One can easily verify that  $N \setminus S$  satisfies the conditions of Lemma 5.2, that  $r_S = k$  (since at least two of the  $S_j$ 's are non empty), and that  $\gamma(T_k \setminus S) \leq \gamma$ . Hence,  $N \setminus S \in \mathcal{E}(k, \gamma)$ . Then, using essentially the same argument as that used in Case 1, it follows from Lemma 6.1 that:

$$e(k, \gamma, x) \leq g(k, \gamma, x).$$

Combining the upper bounds we obtain

$$e(k, \gamma, x) \leq \min(e(k, \gamma - 1, x), g(k, \gamma, x)). \quad (6.10)$$

With a similar argument as the one used in Case 1, one shows that equality holds in (6.10).

**Case 3:**  $\gamma_k \leq \gamma$  and  $|S(k, \gamma)| \leq 1$  and  $\gamma(T_k \setminus \{k\}) = \gamma$ .

We claim that  $\mathcal{E}(k, \gamma) = \mathcal{E}(k, \gamma - 1) \cup \{N \setminus \{k\}\}$ . Since we trivially have  $\mathcal{E}(k, \gamma - 1) \subseteq \mathcal{E}(k, \gamma)$ , it suffices to show that  $N \setminus \{k\}$  is the only coalition in  $\mathcal{E}(k, \gamma) \setminus \mathcal{E}(k, \gamma - 1)$ . Suppose to the contrary that  $S \neq N \setminus \{k\}$  is a coalition that satisfies the conditions of Lemma 5.2 and for which  $r_{N \setminus S} = k$  and  $\gamma(T_k \cap S) = \gamma$ .

Observe that  $k \in S$ ; otherwise, since  $S \neq N \setminus \{k\}$ ,  $S \cup \{k\}$  is a proper subset of  $N$  and by Lemma 5.2(i)  $\gamma < \gamma_k$ , which contradicts our assumption that  $\gamma_k \leq \gamma$ . Now, since  $k \in S$  and  $r_{N \setminus S} = k$ , there are at least two sons of  $k$ , say  $i$  and  $j$ , such that  $T_i \setminus S \neq \emptyset$  and  $T_j \setminus S \neq \emptyset$ . According to Lemma 5.2(ii), we must have  $\gamma(T_i) > \gamma$  and  $\gamma(T_j) > \gamma$ . This however contradicts the fact that  $|S(k, \gamma)| \leq 1$ .

**Case 4:**  $\gamma_k \leq \gamma$  and  $|S(k, \gamma)| \leq 1$  and  $\gamma(T_k \setminus \{k\}) \neq \gamma$ .

In this case we have  $\mathcal{E}(k, \gamma) = \mathcal{E}(k, \gamma - 1)$ . The proof for Case 3, which shows that  $N \setminus \{k\}$  is the only possible candidate for a coalition in  $\mathcal{E}(k, \gamma) \setminus \mathcal{E}(k, \gamma - 1)$  is also valid in this case. Contrary to Case 3 however, one can verify here that  $N \setminus \{k\}$  is *not* a member of  $\mathcal{E}(k, \gamma) \setminus \mathcal{E}(k, \gamma - 1)$ .  $\triangleleft$

Note that  $f(r_N, p, x)$  is the minimal excess over all non redundant coalitions except  $N$ , where  $p$  denotes the highest possible quality of a node. The exclusion of  $N$  is due to the fact that the root of  $\emptyset$  is not defined, hence it is not in any of the collections  $\mathcal{E}(k, \gamma)$ . Thus, a cost allocation  $x$  is in the core if and only if  $f(r_N, p, x)$  is non negative. Clearly, for a fixed  $x$ ,  $f(r_N, p, x)$  can be computed in  $\mathcal{O}(p|N|)$  time, provided that the constants  $\alpha(k, \gamma)$  and the sets  $S(k, \gamma)$  are known. One can check that these constants and sets can be calculated in a preprocessing procedure in  $\mathcal{O}(p|N|)$  time. Thus, core membership can be verified in  $\mathcal{O}(p|N|)$  time, and if  $p$  is a part of the input, it can be verified in  $\mathcal{O}(|N|^2)$  time, since  $p \leq |N|$ .

## 7 The nucleolus of an ET game

In this section we prove that for an ET game  $(N; c)$  it is possible to construct an auxiliary tree graph  $\Gamma(N; c)$  with node set  $N$ , such that the complement of every coalition satisfying the conditions of Lemma 5.2 is connected in  $\Gamma(N; c)$ . The fact that the complements of the facet defining coalitions are tree-connected is a property that ET games have in common with, for example, MCST games (Granot and Huberman (1984)) and routing games (Derks and Kuipers (1997)). Furthermore, like in the case of MCST games and routing games, the nucleolus of an ET game is completely determined by its facet defining coalitions.

In Kuipers (1994) it was shown that this property can be exploited to compute the prenucleolus in  $\mathcal{O}(n^4|\mathcal{B}|)$  time, where  $n$  is the number of players in the game, and  $\mathcal{B}$  denotes the collection of facet defining coalitions. The algorithm was subsequently improved to run in  $\mathcal{O}(n^3|\mathcal{B}|)$  time by Kuipers et al. (2000). Note that this result does not imply that the (pre)nucleolus of an ET game can be computed in polynomial time, since the number of facet defining coalitions grows exponentially with the number of players. It is possible though to construct an  $\mathcal{O}(n^3p)$  algorithm for computing the nucleolus of an ET game. This was done in a working paper by Granot et al. (2000). The techniques that are used in the working paper are similar to the ones used for the algorithm developed by Kuipers et al. (2000).

As is well known, the nucleolus coincides with the kernel in convex games (Maschler et al. (1972)). Therefore, in convex games, the nucleolus is the unique core point at which every pair of players is balanced in the sense of the kernel (see Maschler et al. (1979)). Thus, it follows that the nucleolus of an ET game is determined by  $\frac{1}{2}n(n-1)$  balanced equations, where  $n$  is the number of players. The property that all complements of facet defining coalitions are tree connected has also consequences for this geometrical characterization of the nucleolus. Potters and Reijnierse (1995) showed that this property ensures that only the balanced equations corresponding to pairs of players who are neighbors in the tree are needed to characterize the prenucleolus. Thus, for an ET game,  $n-1$  balanced equations are already sufficient for the characterization of the nucleolus.

Finally, let us see how the auxiliary tree  $\Gamma(N; c)$  is constructed, and prove that indeed the complements of all facet defining coalitions are connected in  $\Gamma(N; c)$ .

Let  $(N; c)$  be an ET game defined on a tree-network  $T$ . Consider the following procedure to number (i.e. order) the players. In the first round we number the players of type 0. This is done as follows. We start at the root and visit all nodes according to a breadth-first procedure in the tree. If a node of type 0 is visited, we assign it the first available natural number. In the second round we number the nodes of type 1. We return to the root and again we visit all nodes according to the breadth-first procedure. This time we assign the first available natural number whenever a node of type 1 is visited. This process is repeated until all nodes are numbered.

Now, for every  $k \in N$ , define  $m(k)$  as the highest node in  $T_k$ . Let us call node  $k$  an *endpoint* if  $m(k) = k$ , and let us denote the set of endpoints by  $M$ . Consider the following procedure to construct a directed graph with  $M$  as its node set.

### Procedure Auxiliary Tree

**Case 1:**  $k \notin M$ . For every son  $j$  of  $k$  (with respect to the network  $T$ ), let  $\ell_j$  be the highest node in  $T_j$ , i.e.  $\ell_j = m(j)$ . Let  $M(k) = \{\ell_j \mid j \text{ is a son of } k\}$ . If  $|M(k)| \leq 1$ , we define  $A(k) = \emptyset$ . Otherwise, let  $\bar{\ell}$  denote the highest node in  $M(k)$ , and define  $A(k)$  as the set of all arcs  $(\bar{\ell}, \ell_j)$  for  $\ell_j \in M(k)$ ,  $\ell_j \neq \bar{\ell}$ .

**Case 2:**  $k \in M$ . For every son  $j$  of  $k$  (with respect to the network  $T$ ), let  $\ell_j$  be the highest node in  $T_j$ . Let  $M(k) = \{\ell_j \mid j \text{ is a son of } k\}$ , and define  $A(k)$  as the set of all arcs  $(k, \ell_j)$  for all  $\ell_j \in M(k)$ .

Let  $A$  denote the union of all arc sets  $A(k)$ .

**Lemma 7.1** *The graph  $(M, A)$  is a tree directed away from  $n$ , such that the node numbers on any directed path are decreasing.*

*Proof:* Define  $B(k) := \cup_{j \in T_k} A(j)$ . We claim that  $(T_k \cap M, B(k))$  is a directed tree rooted away from  $m(k)$  for all  $k$ . Clearly,  $A(k) = \emptyset$  whenever  $|T_k \cap M| \leq 1$ . Hence,  $B(k) = \emptyset$  whenever  $|T_k \cap M| \leq 1$ , and our claim is trivially true when  $|T_k \cap M| = 1$ .

Now assume that  $(T_k \cap M, B(k))$  is a directed tree rooted away from  $m(k)$  whenever  $1 \leq |T_k \cap M| < \ell$  and suppose that  $|T_k \cap M| = \ell$ . If  $k$  has only one son, say  $j$ , and  $k$  is not an endpoint, then  $T_j \cap M = T_k \cap M$ . In this case  $B(j) = B(k)$  and  $(T_k \cap M, B(k)) = (T_j \cap M, B(j))$ . Hence, we may assume without loss of generality that  $T_j \cap M$  is a proper subset of  $T_k \cap M$  for all sons  $j$  of  $k$ . By the induction assumption, every graph  $(T_j \cap M, B(j))$  is a directed tree rooted away from  $m(j)$ . Hence, the graph  $(T_k \cap M, \cup(B(j): j \text{ is a son of } k))$  is a forest. If  $k \in M$ , the components of this forest are the singleton  $k$  and the trees  $(T_j \cap M, B(j))$  for the sons  $j$  of  $k$ . If  $k \notin M$ , then the components are only the trees  $(T_j \cap M, B(j))$  for the sons  $j$  of  $k$ . In either case, the arc set  $A(k)$  consists of the arcs from the highest node in  $T_k \cap M$ , i.e.  $m(k)$ , to the highest node in each component  $(T_j \cap M, B(j))$ , i.e.  $m(j)$ . Hence,  $(T_k \cap M, B(k))$  is a directed tree rooted away from  $m(k)$ .

Now, suppose that, with respect to the network  $T$ , node 1 is adjacent to the root 0. Then,  $T_1 = N$  and  $B(1) = A$ , the set of all arcs that were generated in the construction process of  $(M, A)$ . We conclude that  $(M, A) = (T_1 \cap M, B(1))$  is a directed tree rooted away from  $m(1) = n$ .  $\triangleleft$

The directed tree that was constructed by procedure **Auxiliary Tree** can be extended to a directed tree with node set  $N$  by adding node  $i$  together with arc  $(m(i), i)$  for every  $i \in N \setminus M$ . We denote this tree by  $\Gamma(N; c)$ .

**Lemma 7.2** *Let  $(N; c)$  be an ET game, and let  $\mathcal{B}$  denote the collection of coalitions satisfying the conditions of Lemma 5.2. Then the complement of every coalition in  $\mathcal{B}$  is connected in  $\Gamma(N; c)$ .*

*Proof:* Let  $S \in \mathcal{B}$ , and let  $a, b \in N \setminus S$ . We claim that  $c \in N \setminus S$  for every  $c$  on the path from  $a$  to  $b$  in  $\Gamma(N; c)$ . Let us refer to the *distance* between  $a$  and  $b$  as the number of arcs in the path from  $a$  to  $b$  in the  $\Gamma(N; c)$ . Our claim is trivially true if the distance is 1. Let us assume that the claim is true whenever the distance is less than, say  $d$ . Now, let  $a$  and  $b$  be such that the distance is exactly  $d$ . Assume without loss of generality that  $a > b$ .

Let  $c$  be the node adjacent to  $b$  on this path. Note that  $c \in M$ , since  $c$  is not a leaf of  $\Gamma(N; c)$ . Recall that the tree  $\Gamma(N; c)$  is directed away from the root  $n$  and node numbers are decreasing along directed paths. Therefore, the path from  $a$  to  $b$  in this tree can be seen as consisting of (maximally) two parts. The first part goes against the direction of the arcs and node numbers are increasing. The second part goes in the direction of the arcs and node numbers are decreasing. It may happen that this second part starts immediately at node  $a$ , in which case the whole path is decreasing. In either case, we see that  $(c, b)$  is an arc of  $\Gamma(N; c)$  (and not  $(b, c)$ ).

Assume first that  $b \notin M$ . Then it is immediate from the construction of  $\Gamma(N; c)$  that  $m(b) = c$ . It follows from Lemma 5.1 that  $b \in S$  if  $m(b) \in S$ , so we conclude that  $c = m(b) \notin S$ .

Now assume that  $b \in M$ . Then let  $k$  denote the node for which  $(c, b) \in A(k)$ , where  $A(k)$  is defined as in procedure  $B$ . Further, let  $p$  denote the son of  $k$  for which  $b$  is the highest node in  $T_p$ . Note that  $T_p \not\subseteq S$ , since  $b \notin S$ . Moreover,  $a \notin T_p$ , as  $b$  is the highest node in  $T_b$ . Hence  $S \cup T_p$  is a proper subset of  $N$ . Then, by Lemma 5.2, we have  $\gamma_b = \gamma(T_p) > \gamma(T_k \cap S)$ . Since  $c > b$  we have  $\gamma_c \geq \gamma_b$ . It follows that  $\gamma_c > \gamma(T_k \cap S)$ , and hence  $c \notin T_k \cap S$ . On the other hand, we have  $c \in T_k$ . We conclude that  $c \notin S$ .  $\triangleleft$

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