Online Appendix for Bray (2021)
Rust's Algorithm Under a General Sampling Density

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Introduction

My argument seamlessly extends to this case in which the Monte Carlo sample is drawn from a general density function. Regardless of the sampling distribution, Rust's algorithm breaks the curse of dimensionality only if the number of state variables that are meaningfully history dependent—i.e., contingent of the prior period's state s or action a—is $O(\log(d))$. However, the nature of this near memorylessness varies with the sampling distribution. Under the uniform sampling distribution, almost all state variables, t_i, have a distribution that's arbitrarily close to a uniform distribution from almost all states $s \in [0,1]^d$. But under sampling density μ_d, almost all state variables, t_i, have a distribution that's arbitrarily close to the conditional marginal sampling distribution, $\mu_d^i(t_i|t_{<i})$, from almost all states $s \in [0,1]^d$.

Analysis

I will now derive the analog of each of Bray's (2021) results under general sampling density μ_d. Suppose the elements of $m_d = \{m_d^i\}_{i=1}^{b_d}$ are drawn from a distribution with general density function μ_d, which has full support over $[0,1]^d$.

In this case, Rust's random Bellman operator would be

$$(\hat{\Gamma}_d^{b\mu} V)(s) \equiv \max_{a \in \mathbb{a}} u_{da}(s) + \beta \frac{\sum_{i=1}^{b_d} V(m_d^i) p_{da}^{\mu}(m_d^i|s)}{\sum_{i=1}^{b_d} p_{da}^{\mu}(m_d^i|s)},$$
$$\text{where} \quad p_{da}^{\mu}(t|s) \equiv p_{da}(t|s)/\mu_d(t).$$

We will call this operator's fixed point, $\hat{V}_d^{b\mu}$, the *Rust value function under sampling density* μ. Note that we can also express the standard Bellman operator in terms of p_{da}^{μ} and μ_d:

$$(\Gamma_d V)(s) = \max_{a \in \mathbb{a}} u_{da}(s) + \beta \int_{t \in [0,1]^d} V(t) p_{da}^{\mu}(t|s) \mu_d(dt),$$

$$\text{where} \quad \mu_d(dt) \equiv \mu_d(t) \lambda_d(dt)$$

We will now re-express our definitions, assumptions, and propositions in terms of μ_d.

Definition 7. *A sequence of dynamic programs is* strongly Rust solvable under sampling density μ *if for all* $\epsilon > 0$ *there exists* $b \in \mathbb{b}$ *such that* $\sup_{d \in \mathbb{N}} \mathrm{E}(\|\hat{V}_d^{b\mu} - V_d\|) < \epsilon$*.*

Definition 8. *A sequence of dynamic programs is* weakly Rust solvable under sampling density μ *if for all* $\epsilon > 0$ *there exists* $b \in \mathbb{b}$ *such that* $\sup_{d \in \mathbb{N}} \mathrm{E}(\|\hat{V}_d^{b\mu} - V_d\|_1) < \epsilon$*.*

Definition 9. *The* i*th state variable is* ϵ-dependent under sampling density μ *if* $\|V_d^{\mu,-i} - V_d\|_1 < \epsilon$*, where* $V_d^{\mu,-i}$ *is the value function under density function* $p_{da}^{\mu,-i}(t|s) \equiv p_{da}(t|s)\mu_d^i(t_i|t_{<i})/p_d^i(t_i|t_{<i})$*.*

Definition 10. *A sequence of dynamic programs is* nearly memoryless under sampling density μ *if for all* $\epsilon > 0$*, the number of state variables that are not* ϵ*-dependent under sampling density* μ *is* $O(\log(d))$ *as* $d \to \infty$*.*

Definition 11. *A sequence of dynamic programs whose Rust value functions under sampling density* μ *are* $\{\hat{\underline{V}}_d^{b\mu}\}_{d \in \mathbb{N}}$ Rust ϵ-approximates under sampling density μ *a sequence of dynamic programs with value functions* $\{V_d\}_{d \in \mathbb{N}}$ *if there exists* $b \in \mathbb{b}$ *for which* $\sup_{d \in \mathbb{N}} \mathrm{E}(\|\hat{\underline{V}}_d^{b\mu} - V_d\|_1) < \epsilon$*.*

Assumption 7. *The scaled transition density function is Lipschitz continuous in its second argument: For each* $d \in \mathbb{N}$ *and* $t \in [0,1]^d$*, there exists Lipschitz constant* $\ell_d^{\mu}(t) \in \mathbb{R}_+$ *such that* $\max_{a \in \mathbb{a}} \sup_{r \in [0,1]^d} \sup_{s \in [0,1]^d \setminus r} \frac{|p_{da}^{\mu}(t|s) - p_{da}^{\mu}(t|r)|}{\ell_d^{\mu}(t)\|s-r\|_2} \leq 1$*.*

Assumption 8. *The square integral of the scaled transition density Lipschitz function with respect to measure* μ_d *is bounded by a polynomial function of* d*: There exists* $L^{\mu} \in \mathbb{b}$ *such that* $\sup_{d \in \mathbb{N}} \int_{t \in [0,1]^d} \ell_d^{\mu}(t)^2 \mu_d(dt)/L_d^{\mu} \leq 1$*.*

Assumption 9. *At the origin, the scaled transition density function is bounded by a polynomial function of* d*: there exists* $M^{\mu} \in \mathbb{b}$ *such that* $\max_{a \in \mathbb{a}} \sup_{d \in \mathbb{N}} \sup_{t \in [0,1]^d} p_{da}^{\mu}(t|0)/M_d^{\mu} \leq 1$*.*

Assumption 10. *The scaled transition density function is bounded by a polynomial function of* d*: there exists* $M^{\mu} \in \mathbb{b}$ *such that* $\max_{a \in \mathbb{a}} \sup_{d \in \mathbb{N}} \sup_{s,t \in [0,1]^d} p_{da}^{\mu}(t|s)/M_d^{\mu} \leq 1$*.*

Proposition 1. *Any sequence of dynamic programs that satisfies Assumptions 1, 2, 7, 8, and 9 is strongly Rust solvable under sampling density μ.*

Proposition 2. *Any sequence of dynamic programs that satisfies Assumptions 1, 2, and 10 is weakly Rust solvable under sampling density μ.*

Proposition 3. *Every sequence of dynamic programs that satisfies Assumptions 1, 2, 7, 8, and 9 can be ϵ-approximated by a sequence that satisfies Assumptions 1, 2, and 10, for all $\epsilon > 0$.*

Proposition 4. *Any sequence of dynamic programs that satisfies Assumptions 1, 2, and 10 is nearly memoryless under sampling density μ.*

Proposition 5. *Any sequence of dynamic programs that (i) is weakly Rust solvable under sampling density μ and (ii) satisfies Assumptions 1 and 2 can be Rust ϵ-approximated under sampling density μ by a sequence of dynamic programs that is nearly memoryless under sampling density μ, for all $\epsilon > 0$.*

Proofs

Proposition 1. *Any sequence of dynamic programs that satisfies Assumptions 1, 2, 7, 8, and 9 is strongly Rust solvable under sampling density μ.*

Proof. For $d \in \mathbb{N}$, $V \in \mathbb{V}_d$, $a \in \mathbb{a}$, and $b \in \mathbb{b}$ define

$$\begin{aligned}
(\hat{\Gamma}_{da}^{b\mu} V)(s) \equiv & u_{da}(s) + \beta \frac{\sum_{i=1}^{b_d} V(m_d^i) p_{da}^{\mu}(m_d^i|s)}{\sum_{i=1}^{b_d} p_{da}^{\mu}(m_d^i|s)}, \\
(\tilde{\Gamma}_{da}^{b\mu} V)(s) \equiv & u_{da}(s) + \beta \sum_{i=1}^{b_d} V(m_d^i) p_{da}^{\mu}(m_d^i|s)/b_d, \\
\text{and} \quad Z_{da}^{b\mu}(s) \equiv & \beta \sum_{i=1}^{b_d} g_d^i V(m_d^i) p_{da}^{\mu}(m_d^i|s)/\sqrt{b_d},
\end{aligned}$$

where $\{g_d^i\}_{i=1}^{b_d}$ is a set of independent standard normal random variables. Since $\mathrm{E}((\tilde{\Gamma}_{da}^{b\mu} V)(s)) = (\Gamma_{da} V)(s)$, where Γ_{da} is defined in the proof of Proposition 1, Pollard's (1989) seventh equation implies that

$$\mathrm{E}\left(\|\tilde{\Gamma}_{da}^{b\mu} V - \Gamma_{da} V\|^2\right) \leq \frac{2\pi}{\sqrt{b_d}} \mathrm{E}\left(\sup_{s \in [0,1]^d} Z_{da}^{b\mu}(s)^2\right). \tag{1}$$

We will now bound the expectation on the right. First, note that

$$\begin{aligned}\mathrm{E}(|Z_{da}^{b\mu}(t) - Z_{da}^{b\mu}(s)|^2 : m_d) =& \beta^2 \sum_{i=1}^{b_d} |V(m_d^i)(p_{da}^{\mu}(m_d^i|t) - p_{da}^{\mu}(m_d^i|s))|^2 / b_d \\ \leq& \left(\frac{\beta K_d \|t-s\|_2}{1-\beta} \right)^2 \sum_{i=1}^{b_d} \ell_d^{\mu}(m_d^i)^2 / b_d. \end{aligned} \tag{2}$$

This expression implies that

$$\begin{aligned}\delta_d^m \equiv& \frac{\beta K_d}{1-\beta} \sqrt{\sum_{i=1}^{b_d} d\ell_d^{\mu}(m_d^i)^2 / b_d} \\ \geq& \sup_{s,t\in[0,1]^d} \sqrt{\mathrm{E}(|Z_{da}^{b\mu}(t) - Z_{da}^{b\mu}(s)|^2 : m_d)}. \end{aligned} \tag{3}$$

Now for a given $x > 0$, divide $[0,1]^d$ into $f(x) \equiv \lceil \delta_d^m/(2x) \rceil^d$ equally sized cubes, and define $\{c_x^i\}_{i=1}^{f(x)}$ as the center points of these cubes. By design, these center points satisfy

$$\sup_{s\in[0,1]^d} \min_{i\in\{1,\cdots,f(x)\}} \|s - c_x^i\|_2 \leq \frac{x(1-\beta)}{\beta K_d \sqrt{\sum_{i=1}^{b_d} \ell_d^{\mu}(m_d^i)^2 / b_d}},$$

which with (2) implies that

$$\sup_{s\in[0,1]^d} \min_{i\in\{1,\cdots,f(x)\}} \mathrm{E}(|Z_{da}^{b\mu}(s) - Z_{da}^{b\mu}(c_x^i)|^2 : m_d) \leq x^2. \tag{4}$$

Now combining (3) and (4) with Pollard's (1989) eighth equation yields

$$\begin{aligned}\sqrt{\mathrm{E}\left(\sup_{s\in[0,1]^d} Z_{da}^{b\mu}(s)^2 : m_d \right)} - \sqrt{\mathrm{E}(Z_{da}^{b\mu}(0)^2 : m_d)} & \\ \leq& C \int_0^{\delta_d^m} \sqrt{\log(f(x))} dx \\ =& C\delta_d^m \sqrt{d} \int_0^1 \sqrt{\log(1/u)} du \\ =& C\delta_d^m \sqrt{d\pi}/2, \end{aligned}$$

where $C \geq 1$ is a universal constant that's independent of all model parameters. And since $x - y \leq z$

implies $x^2 \leq 2y^2 + 2z^2$ for all $x, y, z \in \mathbb{R}$, this implies that

$$
\begin{aligned}
\mathrm{E}\Big(\sup_{s\in[0,1]^d} Z_{da}^{b\mu}(s)^2\Big) =& \,\mathrm{E}\Big(\mathrm{E}\Big(\sup_{s\in[0,1]^d} Z_{da}^{b\mu}(s)^2 \,:\, m_d\Big)\Big) \\
\leq& \,\mathrm{E}\Big(2\,\mathrm{E}(Z_{da}^{b\mu}(0)^2 \,:\, m_d) + (C\delta_d^m)^2 d\pi\Big) \\
\leq& \,2\Big(\frac{\beta K_d M_d^\mu}{1-\beta}\Big)^2 + \pi\Big(\frac{d\beta C K_d}{1-\beta}\Big)^2 \mathrm{E}(\ell_d^\mu(m_d^i)^2) \\
\leq& \Big(\frac{\beta K_d}{1-\beta}\Big)^2 (2(M_d^\mu)^2 + \pi d^2 C^2 L_d^\mu).
\end{aligned}
$$

Now combining this with (1) and applying Jensen's inequality yields for all $V \in \mathbb{V}_d$

$$
\begin{aligned}
\mathrm{E}(\|\tilde{\Gamma}_{da}^{b\mu}V - \Gamma_{da}V\|) \leq& \sqrt{\mathrm{E}\left(\|\tilde{\Gamma}_{da}^{b\mu}V - \Gamma_{da}V\|^2\right)} \\
\leq& \sqrt{\frac{2\pi}{\sqrt{b_d}}\,\mathrm{E}\Big(\sup_{s\in[0,1]^d} Z_{da}^{b\mu}(s)^2\Big)} \\
\leq& \frac{2\pi\beta K_d}{b_d^{1/4}(1-\beta)}\sqrt{(M_d^\mu)^2 + d^2C^2L_d^\mu}.
\end{aligned} \tag{5}
$$

Next, define constant function $\iota_d \in \mathbb{V}_d$, where $\iota_d(t) = K_d/(1-\beta)$ for all $t \in [0,1]^d$. With this, (5) yields

$$
\begin{aligned}
\mathrm{E}(\|\hat{\Gamma}_{da}^{b\mu}V_d - \tilde{\Gamma}_{da}^{b\mu}V_d\|) =& \,\mathrm{E}\left(\sup_{s\in[0,1]^d}\left|\Big(1 - \sum_{i=1}^{b_d} p_{da}^\mu(m_d^i|s)/b_d\Big)\beta\frac{\sum_{i=1}^{b_d} V(m_d^i)p_{da}^\mu(m_d^i|s)}{\sum_{i=1}^{b_d} p_{da}^\mu(m_d^i|s)}\right|\right) \\
\leq& \frac{\beta K_d}{1-\beta}\,\mathrm{E}\left(\sup_{s\in[0,1]^d}\left|1 - \sum_{i=1}^{b_d} p_{da}^\mu(m_d^i|s)/b_d\right|\right) \\
=& \,\mathrm{E}\left(\|\tilde{\Gamma}_{da}^{b\mu}\iota_d - \Gamma_{da}\iota_d\|\right) \\
\leq& \frac{2\pi\beta K_d}{b_d^{1/4}(1-\beta)}\sqrt{(M_d^\mu)^2 + d^2C^2L_d^\mu}.
\end{aligned}
$$

Combining this with (5) yields

$$\begin{aligned}
\mathrm{E}\left(\|\hat{\Gamma}_d^{b\mu}V_d - V_d\|\right) =& \mathrm{E}\left(\|\hat{\Gamma}_d^{b\mu}V_d - \Gamma_d V_d\|\right) \\
\leq& \sum_{a\in\mathbb{a}} \mathrm{E}\left(\|\hat{\Gamma}_{da}^{b\mu}V_d - \Gamma_{da}V_d\|\right) \\
\leq& \sum_{a\in\mathbb{a}} \mathrm{E}\left(\|\hat{\Gamma}_{da}^{b\mu}V_d - \tilde{\Gamma}_{da}^{b\mu}V_d\|\right) + \mathrm{E}\left(\|\tilde{\Gamma}_{da}^{b\mu}V_d - \Gamma_{da}V_d\|\right) \\
\leq& \frac{4|\mathbb{a}|\pi\beta K_d}{b_d^{1/4}(1-\beta)}\sqrt{(M_d^\mu)^2 + d^2C^2L_d^\mu}.
\end{aligned}$$

And with this, Rust's (1997) Lemma 2.2 establishes that

$$\begin{aligned}
\mathrm{E}\left(\|\hat{V}_d^{b\mu} - V_d\|\right) \leq& \mathrm{E}\left(\|\hat{\Gamma}_d^{b\mu}V_d - V_d\|\right)/(1-\beta) \\
\leq& \frac{4|\mathbb{a}|\pi\beta K_d}{b_d^{1/4}(1-\beta)^2}\sqrt{(M_d^\mu)^2 + d^2C^2L_d^\mu},
\end{aligned}$$

which is smaller than ϵ when b_d is larger than $\left(\frac{4|\mathbb{a}|\pi\beta K_d}{\epsilon(1-\beta)^2}\sqrt{(M_d^\mu)^2 + d^2C^2L_d^\mu}\right)^4$. □

Proposition 2. *Any sequence of dynamic programs that satisfies Assumptions 1, 2, and 10 is weakly Rust solvable under sampling density μ.*

Proof. Assumptions 2 and 10 ensure that $|V(t)p_{da}^\mu(t|s)| \leq K_d M_d^\mu/(1-\beta)$ for all $V \in \mathbb{V}$. Hence, for $V \in \mathbb{V}$, Popoviciu's inequality implies that $\mathrm{Var}(V(m_d^i)p_{da}^\mu(m_d^i|s)) \leq (K_d M_d^\mu)^2/(1-\beta)^2$, and thus that

$$\mathrm{Var}\left(\sum_{i=1}^{b_d} V(m_d^i)p_{da}^\mu(m_d^i|s)/b_d\right) \leq \frac{(K_d M_d^\mu)^2}{b_d(1-\beta)^2}.$$

Accordingly, Chebyshev's inequality establishes, for a given $\delta > 0$ and $V \in \mathbb{V}$, that

$$\begin{aligned}
\Pr(y_d^b(s) = 1) \leq& \frac{(K_d M_d^\mu)^2}{\delta^2 b_d(1-\beta)^2}, \\
\text{where} \quad y_d^b(s) \equiv& \mathbb{1}\left\{\left|\sum_{i=1}^{b_d} V(m_d^i)p_{da}^\mu(m_d^i|s)/b_d - \int_{t\in[0,1]^d} V_d(t)p_{da}(t|s)\lambda_d(dt)\right| > \delta\right\}.
\end{aligned}$$

And this implies for $V \in \mathbb{V}$ that

$$\begin{aligned}
&\mathrm{E}(|(\tilde{\Gamma}^{b\mu}_{da}V)(s) - (\Gamma_{da}V)(s)|) \\
&= \beta\, \mathrm{E}\left(\left|\sum_{i=1}^{b_d} V(m_d^i)p^{\mu}_{da}(m_d^i|s)/b_d - \int_{t\in[0,1]^d} V_d(t)p_{da}(t|s)\lambda_d(dt)\right|\right) \\
&\leq \Pr(y_d^b(s)=0)\, \mathrm{E}\left(\left|\sum_{i=1}^{b_d} V(m_d^i)p^{\mu}_{da}(m_d^i|s)/b_d - \int_{t\in[0,1]^d} V_d(t)p_{da}(t|s)\lambda_d(dt)\right| : y_d^b(s)=0\right) \\
&\qquad + \Pr(y_d^b(s)=1)\, \mathrm{E}\left(\left|\sum_{i=1}^{b_d} V(m_d^i)p^{\mu}_{da}(m_d^i|s)/b_d - \int_{t\in[0,1]^d} V_d(t)p_{da}(t|s)\lambda_d(dt)\right| : y_d^b(s)=1\right) \\
&\leq 1\cdot\delta + \frac{(K_dM_d^{\mu})^2}{\delta^2 b_d(1-\beta)^2}\cdot 2K_dM_d^{\mu}/(1-\beta) \\
&= \delta + \frac{2(K_dM_d^{\mu})^3}{\delta^2 b_d(1-\beta)^3},
\end{aligned}$$

where $\tilde{\Gamma}^{b\mu}_{da}$ is defined in the proof of Proposition 1 and Γ_{da} is defined in the proof of Proposition 1. Hence, for $V \in \mathbb{V}$ we have

$$\mathrm{E}(\|\tilde{\Gamma}^{b\mu}_{da}V - \Gamma_{da}V\|_1) \leq \delta + \frac{2(K_dM_d^{\mu})^3}{\delta^2 b_d(1-\beta)^3}. \tag{6}$$

And this, in turn, implies that

$$\begin{aligned}
\mathrm{E}(\|\hat{\Gamma}^{b\mu}_{da}V_d - \tilde{\Gamma}^{b\mu}_{da}V_d\|_1) &= \mathrm{E}\left(\int_{s\in[0,1]^d}\left|\left(1-\sum_{i=1}^{b_d} p^{\mu}_{da}(m_d^i|s)/b_d\right)\beta\frac{\sum_{i=1}^{b_d} V_d(m_d^i)p^{\mu}_{da}(m_d^i|s)}{\sum_{i=1}^{b_d} p^{\mu}_{da}(m_d^i|s)}\right|\lambda_d(ds)\right) \\
&\leq \frac{\beta K_d}{1-\beta}\,\mathrm{E}\left(\int_{s\in[0,1]^d}\left|1-\sum_{i=1}^{b_d} p^{\mu}_{da}(m_d^i|s)/b_d\right|\lambda_d(ds)\right) \\
&= \mathrm{E}\left(\|\tilde{\Gamma}^{b\mu}_{da}\iota_d - \Gamma_{da}\iota_d\|_1\right) \\
&\leq \delta + \frac{2(K_dM_d^{\mu})^3}{\delta^2 b_d(1-\beta)^3}
\end{aligned} \tag{7}$$

where $\hat{\Gamma}_{da}^{b\mu}$ and ι_d are defined in the proof of Proposition 1. Now combining (6) and (7) yields

$$\begin{aligned}
\mathrm{E}\left(\|\hat{\Gamma}_{d}^{b\mu}V_d - V_d\|_1\right) =& \mathrm{E}\left(\|\hat{\Gamma}_{d}^{b\mu}V_d - \Gamma_d V_d\|_1\right) \\
\leq& \sum_{a\in\mathrm{a}} \mathrm{E}\left(\|\hat{\Gamma}_{da}^{b\mu}V_d - \Gamma_{da} V_d\|_1\right) \\
\leq& \sum_{a\in\mathrm{a}} \mathrm{E}\left(\|\hat{\Gamma}_{da}^{b\mu}V_d - \tilde{\Gamma}_{da}^{b\mu} V_d\|_1\right) + \mathrm{E}\left(\|\tilde{\Gamma}_{da}^{b\mu}V_d - \Gamma_{da} V_d\|_1\right) \\
\leq& 2|\mathrm{a}|\delta + \frac{4|\mathrm{a}|(K_d M_d^{\mu})^3}{\delta^2 b_d (1-\beta)^3}.
\end{aligned}$$

And with this, Rust's (1997) Lemma 2.2 establishes that

$$\begin{aligned}
\mathrm{E}\left(\|\hat{V}_{d}^{b\mu} - V_d\|_1\right) \leq& \mathrm{E}\left(\|\hat{\Gamma}_{d}^{b\mu}V_d - V_d\|_1\right)/(1-\beta) \\
\leq& 2|\mathrm{a}|\delta/(1-\beta) + \frac{4|\mathrm{a}|(K_d M_d^{\mu})^3}{\delta^2 b_d (1-\beta)^4},
\end{aligned}$$

which is less than ϵ when $\delta < \frac{\epsilon(1-\beta)}{4|\mathrm{a}|}$ and $b_d > \frac{8|\mathrm{a}|(K_d M_d^{\mu})^3}{\delta^2 \epsilon (1-\beta)^4}$. □

Proposition 3. *Every sequence of dynamic programs that satisfies Assumptions 1, 2, 7, 8, and 9 can be ϵ-approximated by a sequence that satisfies Assumptions 1, 2, and 10, for all $\epsilon > 0$.*

Proof. First, choose $\epsilon > 0$ small enough so that $\int_{t\in[0,1]^d} \ell_d^{\mu}(t)\mu_d(dt) > \delta_d \equiv \frac{\epsilon(1-\beta)^2}{2\mathrm{a}\beta K_d \sqrt{d}}$. Second, set $\gamma_d \in \mathbb{R}$ such that $\int_{t\in[0,1]^d} \max(0, \ell_d^{\mu}(t) - \gamma_d)\mu_d(dt) = \delta_d$. Third, define probability density function

$$\omega_d^{\mu}(t) \equiv \frac{\mathbb{1}\left\{\ell_d^{\mu}(t) \geq \gamma_d\right\}}{\int_{r\in[0,1]^d} \mathbb{1}\left\{\ell_d^{\mu}(r) \geq \gamma_d\right\}\mu_d(dr)}.$$

Fourth, Jensen's inequality yields

$$
\begin{aligned}
\int_{t\in[0,1]^d} \ell_d^\mu(t)^2\mu_d(dt) \geq& \int_{t\in[0,1]^d} \mathbb{1}\big\{\ell_d^\mu(t)\geq\gamma_d\big\}\ell_d^\mu(t)^2\mu_d(dt)\\
=&\int_{t\in[0,1]^d}\omega_d^\mu(t)\ell_d^\mu(t)^2\mu_d(dt)\int_{r\in[0,1]^d}\mathbb{1}\big\{\ell_d^\mu(r)\geq\gamma_d\big\}\mu_d(dr)\\
\geq&\left(\int_{t\in[0,1]^d}\omega_d^\mu(t)\ell_d^\mu(t)\mu_d(dt)\right)^2\int_{r\in[0,1]^d}\mathbb{1}\big\{\ell_d^\mu(r)\geq\gamma_d\big\}\mu_d(dr)\\
=&\left(\gamma_d+\frac{\int_{t\in[0,1]^d}\max(0,\ell_d^\mu(t)-\gamma_d)\mu_d(dt)}{\int_{r\in[0,1]^d}\mathbb{1}\big\{\ell_d^\mu(r)\geq\gamma_d\big\}\mu_d(dr)}\right)^2\int_{r\in[0,1]^d}\mathbb{1}\big\{\ell_d^\mu(r)\geq\gamma_d\big\}\mu_d(dr)\\
=&\left(\gamma_d+\frac{\delta_d}{\int_{r\in[0,1]^d}\mathbb{1}\big\{\ell_d^\mu(r)\geq\gamma_d\big\}\mu_d(dr)}\right)^2\int_{r\in[0,1]^d}\mathbb{1}\big\{\ell_d^\mu(r)\geq\gamma_d\big\}\mu_d(dr)\\
\geq&\min_{y\in\mathbb{R}_+}(\gamma_d+\delta_d/y)^2y\\
=&2\gamma_d\delta_d,
\end{aligned}
$$

which with Assumption 8 implies

$$
\gamma_d\leq L_d^\mu/(2\delta_d).
$$

Fifth, define probability density function

$$
\begin{aligned}
\underline{p}_{da}^\mu(t|s)\equiv&\overline{p}_{da}^\mu(t|s)+1-\int_{r\in[0,1]^d}\overline{p}_{da}^\mu(r|s)\mu_d(dr),\\
\text{where}\quad \overline{p}_{da}^\mu(t|s)\equiv&\max\left(p_{da}^\mu(t|0)+\sqrt{d}\gamma_d,p_{da}^\mu(t|s)\right).
\end{aligned}
$$

This density function satisfies Assumption 10, since

$$
\begin{aligned}
\underline{p}_{da}^\mu(t|s)\leq&\overline{p}_{da}^\mu(t|s)+1\\
\leq&p_{da}^\mu(t|0)+\sqrt{d}\gamma_d+1\\
\leq&M_d^\mu+\sqrt{d}L_d^\mu/(2\delta_d)+1\\
=&M_d^\mu+\frac{\mathrm{a}\beta K_dL_d^\mu d}{\epsilon(1-\beta)^2}+1,
\end{aligned}
$$

which is polynomially bounded. Also, by design we have

$$\begin{aligned}
&\int_{t\in[0,1]^d} |\underline{p}^{\mu}_{da}(t|s) - p^{\mu}_{da}(t|s)|\mu_d(dt) \\
&\quad= \int_{t\in[0,1]^d} \left|\overline{p}^{\mu}_{da}(t|s) + 1 - \int_{r\in[0,1]^d} \overline{p}^{\mu}_{da}(r|s)\mu_d(dr) - p^{\mu}_{da}(t|s)\right|\mu_d(dt) \\
&\quad= \int_{t\in[0,1]^d} \left|\overline{p}^{\mu}_{da}(t|s) + \int_{r\in[0,1]^d} \left(p^{\mu}_{da}(r|s) - \overline{p}^{\mu}_{da}(r|s)\right)\mu_d(dr) - p^{\mu}_{da}(t|s)\right|\mu_d(dt) \\
&\quad\leq \int_{t\in[0,1]^d} |\overline{p}^{\mu}_{da}(t|s) - p^{\mu}_{da}(t|s)|\mu_d(dt) + \int_{r\in[0,1]^d} |p^{\mu}_{da}(r|s) - \overline{p}^{\mu}_{da}(r|s)|\mu_d(dr) \\
&\quad=2 \int_{t\in[0,1]^d} |p^{\mu}_{da}(t|s) - \overline{p}^{\mu}_{da}(t|s)|\mu_d(dt) \\
&\quad\leq 2 \int_{t\in[0,1]^d} \max\left(0, \ell^{\mu}_d(t)\|t\| - \sqrt{d}\gamma_d\right)\mu_d(dt) \\
&\quad\leq 2\sqrt{d} \int_{t\in[0,1]^d} \max\left(0, \ell^{\mu}_d(t) - \gamma_d\right)\mu_d(dt) \\
&\quad=2\sqrt{d}\delta_d. \qquad (8)
\end{aligned}$$

Now let $\underline{\Gamma}^{\mu}_d$ represent the Bellman operator under density function $\underline{p}^{\mu}_{da}$. With (8), we find that this operator satisfies

$$\begin{aligned}
\|\underline{\Gamma}^{\mu}_d V_d - V_d\| &= \|\underline{\Gamma}^{\mu}_d V_d - \Gamma_d V_d\| \\
&\leq \sum_{a\in\mathrm{a}} \sup_{s\in[0,1]^d} \beta \left| \int_{t\in[0,1]^d} V_d(t)(\underline{p}^{\mu}_{da}(t|s) - p^{\mu}_{da}(t|s))\mu_d(dt)\right| \\
&\leq \frac{\beta K_d}{1-\beta} \sum_{a\in\mathrm{a}} \sup_{s\in[0,1]^d} \int_{t\in[0,1]^d} |\underline{p}^{\mu}_{da}(t|s) - p^{\mu}_{da}(t|s)|\mu_d(dt) \\
&\leq \frac{\beta K_d}{1-\beta} \sum_{a\in\mathrm{a}} \sup_{s\in[0,1]^d} 2\sqrt{d}\delta_d \\
&= \frac{2\mathrm{a}\beta\delta_d K_d\sqrt{d}}{1-\beta} \\
&= \epsilon(1-\beta).
\end{aligned}$$

And with this, Rust's (1997) Lemma 2.2 establishes that $\|\underline{V}^{\mu}_d - V_d\| \leq \epsilon$, where $\underline{V}^{\mu}_d$ is the fixed point of $\underline{\Gamma}^{\mu}_d$. □

Proposition 4. *Any sequence of dynamic programs that satisfies Assumptions 1, 2, and 10 is nearly memoryless under sampling density μ.*

Proof. Pinsker's inequality establishes that $\|\lambda_1 - p^i_{da}(\cdot|s,t_{<i})\|_1 \leq \sqrt{2\kappa(p^i_{da}(\cdot|s,t_{<i}),\lambda_1)}$. And this implies that

$$
\begin{aligned}
&\int_{t_{\geq i}\in[0,1]^{d-i+1}} \left|p^{-i}_{da}(t_{<i},t_{\geq i}|s) - p_{da}(t_{<i},t_{\geq i}|s)\right|\lambda_{d-i+1}(dt_{\geq i}) \\
&\qquad\qquad = \int_{t_i\in[0,1]}\int_{t_{\geq i+1}\in[0,1]^{d-i}} \left|p^{-i}_{da}(t_{<i},t_i,t_{\geq i+1}|s) - p_{da}(t_{<i},t_i t_{\geq i+1}|s)\right|\lambda_{d-i}(dt_{\geq i+1})\lambda_1(dt_i) \\
&\qquad\qquad = \int_{t_i\in[0,1]} \left|1 - p^i_{da}(t_i|s,t_{<i})\right| \int_{t_{\geq i+1}\in[0,1]^{d-i}} p^{-i}_{da}(t_{<i},t_i,t_{\geq i+1}|s)\lambda_{d-i}(dt_{\geq i+1})\lambda_1(dt_i) \\
&\qquad\qquad = \int_{t_i\in[0,1]} \left|1 - p^i_{da}(t_i|s,t_{<i})\right| \frac{\int_{t_{\geq i+1}\in[0,1]^{d-i}} p_{da}(t_{<i},t_i,t_{\geq i+1}|s)\lambda_{d-i}(dt_{\geq i+1})}{p^i_{da}(t_i|s,t_{<i})}\lambda_1(dt_i) \\
&\qquad\qquad = \int_{t_i\in[0,1]} \left|1 - p^i_{da}(t_i|s,t_{<i})\right| \frac{p^{<i+1}_d(t_{<i},t_i)}{p^i_{da}(t_i|s,t_{<i})}\lambda_1(dt_i) \\
&\qquad\qquad = p^{<i}_d(t_{<i}) \int_{t_i\in[0,1]} \left|1 - p^i_{da}(t_i|s,t_{<i})\right|\lambda_1(dt_i) \\
&\qquad\qquad = p^{<i}_d(t_{<i})\|\lambda_1 - p^i_{da}(\cdot|s,t_{<i})\|_1 \\
&\qquad\qquad \leq p^{<i}_d(t_{<i})\sqrt{2\kappa(p^i_{da}(\cdot|s,t_{<i}),\lambda_1)}.
\end{aligned}
$$

And with this, Jensen's inequality yields

$$
\begin{aligned}
\|p^{-i}_{da}(\cdot|s) - p_{da}(\cdot|s)\|_1 &= \int_{t\in[0,1]^d} \left|p^{-i}_{da}(t|s) - p_{da}(t|s)\right|\lambda_d(dt) \\
&= \int_{t_{<i}\in[0,1]^{i-1}}\int_{t_{\geq i}\in[0,1]^{d-i+1}} \left|p^{-i}_{da}(t_{<i},t_{\geq i}|s) - p_{da}(t_{<i},t_{\geq i}|s)\right|\lambda_{d-i+1}(dt_{\geq i})\lambda_{i-1}(dt_{<i}) \\
&\leq \int_{t_{<i}\in[0,1]^{i-1}} p^{<i}_d(t_{<i})\sqrt{2\kappa(p^i_{da}(\cdot|s,t_{<i}),\lambda_1)}\lambda_{i-1}(dt_{<i}) \\
&\leq \sqrt{2\int_{t_{<i}\in[0,1]^{i-1}} p^{<i}_d(t_{<i})\kappa(p^i_{da}(\cdot|s,t_{<i}),\lambda_1)\lambda_{i-1}(dt_{<i})} \\
&= \sqrt{2\,\mathrm{E}\left(\kappa(p^i_{da}(\cdot|s,t_{<i}),\lambda_1)\right)}.
\end{aligned}
$$

Next, Lemma 1 implies that there exists $m,n\in\mathbb{N}$ such that $\sum_{i=1}^d \mathrm{E}\left(\kappa(p^i_{da}(\cdot|s,t_{<i}),\lambda_1)\right) < m + n\log(d)$, for all $d\in\mathbb{N}$. Since $\kappa(p^i_{da}(\cdot|s,t_{<i}),\lambda_1)) \geq 0$, this implies that for a given $\gamma > 0$ the inequality $\sqrt{2\,\mathrm{E}\left(\kappa(p^i_{da}(\cdot|s,t_{<i}),\lambda_1)\right)} \leq \gamma$ holds for at least $d - 2(m+n\log(d))/\gamma^2$ values of i. And with the result above, this implies that $\|p^{-i}_{da}(\cdot|s) - p_{da}(\cdot|s)\|_1 < \gamma$ holds for at least $d-2(m+n\log(d))/\gamma^2$ values of i. Now define $\Omega^i_{da} = \{s\in[0,1]^d : \|p^{-i}_{da}(\cdot|s) - p_{da}(\cdot|s)\|_1 < \gamma\}$ as the set points that satisfy this inequality for a given $i\in\{1,\cdots,d\}$. The Lebesgue measures of these sets satisfy

$$\begin{aligned}\sum_{i=1}^{d}\lambda_d(\Omega_{da}^i) =& \sum_{i=1}^{d}\int_{s\in[0,1]^d}\mathbb{1}\{s\in\Omega_{da}^i\}\lambda_d(ds)\\ =& \int_{s\in[0,1]^d}\sum_{i=1}^{d}\mathbb{1}\{s\in\Omega_{da}^i\}\lambda_d(ds)\\ \geq& \int_{s\in[0,1]^d}\big(d-2(m+n\log(d))/\gamma^2\big)\lambda_d(ds)\\ =& d-2(m+n\log(d))/\gamma^2.\end{aligned}$$

Since $\lambda_d(\Omega_{da}^i)\leq 1$ this implies that

$$\lambda_d(\Omega_{da}^i)\geq 1-\delta \tag{9}$$

holds for at least $d-2(m+n\log(d))/(\delta\gamma^2)$ values of i. Thus, for i that satisfies (10), we have

$$\begin{aligned}\|\Gamma_{da}^{-i}V_d-\Gamma_{da}V_d\|_1 =& \int_{s\in[0,1]^d}\Bigg|u_{da}(s)+\beta\int_{t\in[0,1]^d}V(t)p_{da}^{-i}(t|s)\lambda_d(dt)\\ & \qquad -u_{da}(s)-\beta\int_{t\in[0,1]^d}V(t)p_{da}(t|s)\lambda_d(dt)\Bigg|\lambda_d(ds)\\ =&\beta\int_{s\in[0,1]^d}\left|\int_{t\in[0,1]^d}V_d(t)(p_{da}^{-i}(t|s)-p_{da}(t|s))\lambda_d(dt)\right|\lambda_d(ds)\\ \leq&\beta\|V_d\|\int_{s\in[0,1]^d}\|p_{da}^{-i}(\cdot|s)-p_{da}(\cdot|s)\|_1\lambda_d(ds)\\ \leq&\beta\|V_d\|\Bigg(\int_{s\in\Omega_{da}^i}\|p_{da}^{-i}(\cdot|s)-p_{da}(\cdot|s)\|_1\lambda_d(ds)\\ & \qquad +\int_{s\in[0,1]^d\setminus\Omega_{da}^i}\big(\|p_{da}^{-i}(\cdot|s)\|_1+\|p_{da}(\cdot|s)\|_1\big)\lambda_d(ds)\Bigg)\\ \leq&\beta\|V_d\|\left(\int_{s\in\Omega_{da}^i}\gamma\lambda_d(ds)+\int_{s\in[0,1]^d\setminus\Omega_{da}^i}2\lambda_d(ds)\right)\\ =&\beta\|V_d\|\big(\gamma\lambda_d(\Omega_{da}^i)+2(1-\lambda_d(\Omega_{da}^i))\big)\\ \leq&\beta(K_d/(1-\beta))(\gamma+2\delta),\end{aligned}$$

where Γ_{da} is the action-a Bellman operator defined in the proof of Proposition 1, and Γ_{da}^{-i} is the analogous operator under p_{da}^{-i}. Thus, for i that satisfies (10), we have

$$\begin{aligned}\|\Gamma_d^{-i}V_d - V_d\|_1 =& \|\Gamma_d^{-i}V_d - \Gamma_d V_d\|_1 \\ \leq& \sum_{a\in\mathbb{a}} \|\Gamma_{da}^{-i}V_d - \Gamma_{da}V_d\|_1 \\ \leq& |\mathbb{a}|\beta(K_d/(1-\beta))(\gamma+2\delta),\end{aligned}$$

where Γ_d^{-i} is the analog of Γ_d under p_d^{-i}. And with this, Rust's (1997) Lemma 2.2 implies that $\|V_d^{-i} - V_d\|_1 < |\mathbb{a}|\beta K_d(\gamma+2\delta)/(1-\beta)^2$ holds for i that satisfies (10). Hence, setting $\gamma = \delta = \frac{\epsilon(1-\beta)^2}{4|\mathbb{a}|\beta K_d}$, we find that $\|V_d^{-i} - V_d\|_1 < \epsilon$ holds for at least $d - 2(m+n\log(d))/(\delta\gamma^2) = d - 128\left(\frac{|\mathbb{a}|\beta K_d}{\epsilon(1-\beta)^2}\right)^3(m + n\log(d))$ values of i. □

Lemma 1. *Assumption 10 implies that there exist $m, n \in \mathbb{N}$ such that* $\max_{a\in\mathbb{a}} \sup_{d\in\mathbb{N}} \sup_{s\in[0,1]^d} \frac{\sum_{i=1}^d \mathrm{E}(\kappa(p_d^i(\cdot|t_{<i}), \mu_d^i(\cdot|t_{<i}))}{m+n\log(d)} < 1.$

Proof. The following implies the result:

$$\begin{aligned}\sup_{t\in[0,1]^d} p_{da}^{\mu}(t|s) =& \exp\Big(\sup_{t\in[0,1]^d} \log(p_{da}^{\mu}(t|s))\Big) \\ \geq& \exp\Big(\int_{t\in[0,1]^d} \log(p_{da}^{\mu}(t|s)) p_{da}(t|s)\lambda_d(dt)\Big) \\ =& \exp\Big(\sum_{i=1}^d \int_{t\in[0,1]^d} \log\Big(\frac{p_{da}^i(t_i|s,t_{<i})}{\mu_d^i(t_i|t_{<i})}\Big) p_{da}(t|s)\lambda_d(dt)\Big) \\ =& \exp\Big(\sum_{i=1}^d \mathrm{E}\left(\kappa(p_{da}^i(\cdot|s,t_{<i}), \mu_d^i(\cdot|t_{<i}))\right)\Big).\end{aligned}$$

□

Proposition 4. *Any sequence of dynamic programs that satisfies Assumptions 1, 2, and 10 is nearly memoryless under sampling density μ.*

Proof. Pinsker's inequality establishes that $\|p_{da}^i(\cdot|s,t_{<i}) - \mu_d^i(\cdot|t_{<i})\|_1 \leq \sqrt{2\kappa(p_{da}^i(\cdot|s,t_{<i}), \mu_d^i(\cdot|t_{<i}))}$.

And this implies that

$$\begin{aligned}
&\int_{t_{\geq i}\in[0,1]^{d-i+1}} \left|p_{da}^{\mu,-i}(t_{<i}, t_{\geq i}|s) - p_{da}(t_{<i}, t_{\geq i}|s)\right| \lambda_{d-i+1}(dt_{\geq i}) \\
&\qquad = \int_{t_{\geq i}\in[0,1]^{d-i+1}} \left|\mu_d^i(t_i|t_{<i})/p_{da}^i(t_i|s,t_{<i}) - 1\right| p_{da}(t_{<i}, t_i, t_{\geq i+1}|s)\lambda_{d-i+1}(dt_{\geq i}) \\
&\qquad = \int_{t_i\in[0,1]} \left|\mu_d^i(t_i|t_{<i})/p_{da}^i(t_i|s,t_{<i}) - 1\right| p_d^{<i+1}(t_{<i}, t_i)\lambda_1(dt_i) \\
&\qquad = p_d^{<i}(t_{<i}) \int_{t_i\in[0,1]} \left|\mu_d^i(t_i|t_{<i}) - p_{da}^i(t_i|s,t_{<i})\right| \lambda_1(dt_i) \\
&\qquad = p_d^{<i}(t_{<i}) \|\mu_d^i(\cdot|t_{<i}) - p_{da}^i(\cdot|s,t_{<i})\|_1 \\
&\qquad \leq p_d^{<i}(t_{<i}) \sqrt{2\kappa(p_{da}^i(\cdot|s,t_{<i}), \mu_d^i(\cdot|t_{<i}))}.
\end{aligned}$$

And with this, Jensen's inequality yields

$$\begin{aligned}
\|p_{da}^{\mu,-i}(\cdot|s) - p_{da}(\cdot|s)\|_1 &= \int_{t\in[0,1]^d} \left|p_{da}^{\mu,-i}(t|s) - p_{da}(t|s)\right| \lambda_d(dt) \\
&\leq \int_{t_{<i}\in[0,1]^{i-1}} p_d^{<i}(t_{<i}) \sqrt{2\kappa(p_{da}^i(\cdot|s,t_{<i}), \mu_d^i(\cdot|t_{<i}))} \lambda_{i-1}(dt_{<i}) \\
&\leq \sqrt{2 \int_{t_{<i}\in[0,1]^{i-1}} p_d^{<i}(t_{<i}) \kappa(p_{da}^i(\cdot|s,t_{<i}), \mu_d^i(\cdot|t_{<i})) \lambda_{i-1}(dt_{<i})} \\
&= \sqrt{2\,\mathrm{E}\left(\kappa(p_{da}^i(\cdot|s,t_{<i}), \mu_d^i(\cdot|t_{<i}))\right)}.
\end{aligned}$$

Next, Lemma 1 implies that there exists $m, n \in \mathbb{N}$ such that $\sum_{i=1}^d \mathrm{E}\left(\kappa(p_{da}^i(\cdot|s,t_{<i}), \mu_d^i(\cdot|t_{<i}))\right) < m + n\log(d)$, for all $d \in \mathbb{N}$. Since $\kappa(p_{da}^i(\cdot|s,t_{<i}), \mu_d^i(\cdot|t_{<i})) \geq 0$, this implies that for a given $\gamma > 0$ the inequality $\sqrt{2\,\mathrm{E}\left(\kappa(p_{da}^i(\cdot|s,t_{<i}), \mu_d^i(\cdot|t_{<i}))\right)} \leq \gamma$ holds for at least $d - 2(m + n\log(d))/\gamma^2$ values of i. And with the result above, this implies that $\|p_{da}^{\mu,-i}(\cdot|s) - p_{da}(\cdot|s)\|_1 < \gamma$ holds for at least $d - 2(m + n\log(d))/\gamma^2$ values of i. Now define $\Omega_{da}^{\mu,i} = \{s \in [0,1]^d : \|p_{da}^{\mu,-i}(\cdot|s) - p_{da}(\cdot|s)\|_1 < \gamma\}$ as the set points that satisfy this inequality for a given $i \in \{1, \cdots, d\}$. The Lebesgue measures of these sets satisfy

$$
\begin{aligned}
\sum_{i=1}^{d} \lambda_d(\Omega_{da}^{\mu,i}) =& \sum_{i=1}^{d} \int_{s\in[0,1]^d} \mathbb{1}\{s \in \Omega_{da}^{\mu,i}\} \lambda_d(ds) \\
=& \int_{s\in[0,1]^d} \sum_{i=1}^{d} \mathbb{1}\{s \in \Omega_{da}^{\mu,i}\} \lambda_d(ds) \\
\geq& \int_{s\in[0,1]^d} \big(d - 2(m + n\log(d))/\gamma^2\big) \lambda_d(ds) \\
=& d - 2(m + n\log(d))/\gamma^2.
\end{aligned}
$$

Since $\lambda_d(\Omega_{da}^{\mu,i}) \leq 1$ this implies that

$$
\lambda_d(\Omega_{da}^{\mu,i}) \geq 1 - \delta \tag{10}
$$

holds for at least $d - 2(m + n\log(d))/(\delta\gamma^2)$ values of i. Thus, for i that satisfies (10), we have

$$
\begin{aligned}
\|\Gamma_{da}^{\mu,-i} V_d - \Gamma_{da} V_d\|_1 =& \int_{s\in[0,1]^d} \Bigg| u_{da}(s) + \beta \int_{t\in[0,1]^d} V(t) p_{da}^{\mu,-i}(t|s) \lambda_d(dt) \\
& \qquad - u_{da}(s) - \beta \int_{t\in[0,1]^d} V(t) p_{da}(t|s) \lambda_d(dt) \Bigg| \lambda_d(ds) \\
=& \beta \int_{s\in[0,1]^d} \left| \int_{t\in[0,1]^d} V_d(t) (p_{da}^{\mu,-i}(t|s) - p_{da}(t|s)) \lambda_d(dt) \right| \lambda_d(ds) \\
\leq& \beta \|V_d\| \int_{s\in[0,1]^d} \|p_{da}^{\mu,-i}(\cdot|s) - p_{da}(\cdot|s)\|_1 \lambda_d(ds) \\
\leq& \beta \|V_d\| \Bigg(\int_{s\in\Omega_{da}^{\mu,i}} \|p_{da}^{\mu,-i}(\cdot|s) - p_{da}(\cdot|s)\|_1 \lambda_d(ds) \\
& \qquad + \int_{s\in[0,1]^d \setminus \Omega_{da}^{\mu,i}} \big(\|p_{da}^{\mu,-i}(\cdot|s)\|_1 + \|p_{da}(\cdot|s)\|_1\big) \lambda_d(ds) \Bigg) \\
\leq& \beta \|V_d\| \left(\int_{s\in\Omega_{da}^{\mu,i}} \gamma \lambda_d(ds) + \int_{s\in[0,1]^d \setminus \Omega_{da}^{\mu,i}} 2\lambda_d(ds) \right) \\
=& \beta \|V_d\| \big(\gamma \lambda_d(\Omega_{da}^{\mu,i}) + 2(1 - \lambda_d(\Omega_{da}^{\mu,i}))\big) \\
\leq& \beta (K_d/(1-\beta))(\gamma + 2\delta),
\end{aligned}
$$

where Γ_{da} is the action-a Bellman operator defined in the proof of Proposition 1, and $\Gamma_{da}^{\mu,-i}$ is the analogous operator under $p_{da}^{\mu,-i}$. Thus, for i that satisfies (10), we have

$$\begin{aligned}\|\Gamma_d^{\mu,-i}V_d - V_d\|_1 =& \|\Gamma_d^{\mu,-i}V_d - \Gamma_d V_d\|_1 \\ \leq& \sum_{a\in\mathbb{a}} \|\Gamma_{da}^{\mu,-i}V_d - \Gamma_{da}V_d\|_1 \\ \leq& |\mathbb{a}|\beta(K_d/(1-\beta))(\gamma+2\delta),\end{aligned}$$

where $\Gamma_d^{\mu,-i}$ is the analog of Γ_d under $p_d^{\mu,-i}$. And with this, Rust's (1997) Lemma 2.2 implies that $\|V_d^{\mu,-i} - V_d\|_1 < |\mathbb{a}|\beta K_d(\gamma+2\delta)/(1-\beta)^2$ holds for i that satisfies (10). Hence, setting $\gamma = \delta = \frac{\epsilon(1-\beta)^2}{4|\mathbb{a}|\beta K_d}$, we find that $\|V_d^{\mu,-i} - V_d\|_1 < \epsilon$ holds for at least $d - 2(m + n\log(d))/(\delta\gamma^2) = d - 128\left(\frac{|\mathbb{a}|\beta K_d}{\epsilon(1-\beta)^2}\right)^3(m + n\log(d))$ values of i. □

Proposition 5. *Any sequence of dynamic programs that (i) is weakly Rust solvable under sampling density μ and (ii) satisfies Assumptions 1 and 2 can be Rust ϵ-approximated under sampling density μ by a sequence of dynamic programs that is nearly memoryless under sampling density μ, for all $\epsilon > 0$.*

Proof. Define Ξ_d^μ as a general set with measure b_d^{-2} under sampling density μ: $\int_{t\in\Xi_d^\mu}\mu_d(dt) = b_d^{-2}$. And use this to define the following probability transition density function:

$$\begin{aligned}\underline{p}_{da}^{\mu b}(t|s) \equiv& \mathbb{1}\{p_{da}^{\mu}(t|s) \leq b_d^2\}p_{da}^{\mu}(t|s) + b_d^2\mathbb{1}\{t\in\Xi_d^\mu\}\overline{p}_{da}^{\mu}(s), \\ \text{where}\quad \overline{p}_{da}^{\mu}(s) \equiv& \int_{r\in[0,1]^d}\mathbb{1}\{p_{da}^{\mu}(r|s) > b_d^2\}p_{da}^{\mu}(r|s)\mu_d(dr).\end{aligned}$$

This density function funnels all the mass that exceeds to b_d^2 into Ξ_d^μ. Since it never exceeds $2b_d^2$, this density function satisfies Assumption 10. So Proposition 4 establishes that this density function corresponds with a sequence of dynamic programs that's nearly memoryless under sampling density μ (given Assumptions 1 and 2).

Define the following as the set of points for which the new density function equals the old density function:

$$\Omega_d^{\mu b}(s) \equiv \{t\in[0,1]^d : \underline{p}_{da}^{\mu b}(t|s) = p_{da}^{\mu}(t|s)\ \forall\ a\in\mathbb{a}\}.$$

The measure of this set under sampling density μ satisfies

$$\begin{aligned}\int_{t\in\Omega_d^\mu(s)}\mu_d(dt) \geq& 1-\int_{t\in\Xi_d^\mu}\mu_d(dt)-\sum_{a\in\mathbb{a}}\int_{t\in[0,1]^d}\mathbb{1}\{p_{da}^\mu(r|s)>b_d^2\}\big)\mu_d(dt)\\ \geq& 1-b_d^{-2}-b_d^{-2}\sum_{a\in\mathbb{a}}\int_{t\in[0,1]^d}\mathbb{1}\{p_{da}^\mu(r|s)>b_d^2\}p_{da}^\mu(r|s)\mu_d(dt)\\ \geq& 1-b_d^{-2}-b_d^{-2}\sum_{a\in\mathbb{a}}\int_{t\in[0,1]^d}p_{da}^\mu(r|s)\mu_d(dt)\\ \geq& 1-(1+|\mathbb{a}|)/b_d^2.\end{aligned}$$

The probability that this set contains all m_d^i values satisfies

$$\begin{aligned}\Pr\big(\cup_{i=1}^{b_d}m_d^i\subset\Omega_d^b(s)\big)=&\Big(\int_{t\in\Omega_d^\mu(s)}\mu_d(dt)\Big)^{b_d}\\ \geq&(1-(1+|\mathbb{a}|)/b_d^2)^{b_d}\\ =&\exp\big(b_d\log(1-(1+|\mathbb{a}|)/b_d^2)\big)\\ =&\exp\big(b_d(-(1+|\mathbb{a}|)/b_d^2-(1+|\mathbb{a}|)^2/(2b_d^4)-(1+|\mathbb{a}|)^3/(3b_d^6)-\cdots)\big)\\ >&\exp(-(1+|\mathbb{a}|)/b_d)\\ >&1-(1+|\mathbb{a}|)/b_d.\end{aligned}\tag{11}$$

Next, define $\hat{\underline{\Gamma}}_d^{\mu b}$ as Rust's random Bellman operator evaluated under density function $\underline{p}_{da}^{\mu b}$. Since $\hat{V}_d^{\mu b}\in\mathbb{V}_d$, we have

$$|(\hat{\underline{\Gamma}}_d^{\mu b}\hat{V}_d^{\mu b})(s)-(\hat{\Gamma}_d^{\mu b}\hat{V}_d^{\mu b})(s)|\leq\beta K_d/(1-\beta).\tag{12}$$

And if $\cup_{i=1}^{b_d}m_d^i\subset\Omega_d^b(s)$ then

$$|(\hat{\underline{\Gamma}}_d^{\mu b}\hat{V}_d^{\mu b})(s)-(\hat{\Gamma}_d^{\mu b}\hat{V}_d^{\mu b})(s)|=0.\tag{13}$$

Now combining (11)–(13) yields

$$\begin{aligned}\mathrm{E}\left(\|\hat{\underline{\Gamma}}_d^{\mu b}\hat{V}_d^{\mu b} - \hat{V}_d^{\mu b}\|_1\right) =& \mathrm{E}\left(\|\hat{\underline{\Gamma}}_d^{\mu b}\hat{V}_d^{\mu b} - \hat{\Gamma}_d^{\mu b}\hat{V}_d^{\mu b}\|_1\right)\\ =& \int_{s\in[0,1]^d} \mathrm{E}\left(|(\hat{\underline{\Gamma}}_d^{\mu b}\hat{V}_d^{\mu b})(s) - (\hat{\Gamma}_d^{\mu b}\hat{V}_d^{\mu b})(s)|\right)\lambda_d(ds)\\ \leq& \int_{s\in[0,1]^d} \beta K_d/(1-\beta)\big(1 - \Pr\left(\cup_{i=1}^{\mu b_d} m_d^i \subset \Omega_d^{\mu b}(s)\right)\big)\lambda_d(ds)\\ <& \int_{s\in[0,1]^d} \frac{\beta K_d(1+|\mathbb{a}|)}{b_d(1-\beta)}\lambda_d(ds)\\ =& \frac{\beta K_d(1+|\mathbb{a}|)}{b_d(1-\beta)}.\end{aligned}$$

And with this, we can use Rust's (1997) Lemma 2.2 to establishes that

$$\mathrm{E}\left(\|\underline{\hat{V}}_d^{\mu b} - \hat{V}_d^{\mu b}\|_1\right) \leq \mathrm{E}\left(\|\hat{\underline{\Gamma}}_d^{\mu b}\hat{V}_d^{\mu b} - \hat{V}_d^{\mu b}\|_1\right)/(1-\beta) < \frac{\beta K_d(1+|\mathbb{a}|)}{b_d(1-\beta)^2}. \tag{14}$$

Now for a given $\epsilon > 0$ choose $b \in \mathbb{b}$ such that $\frac{\beta K_d(1+|\mathbb{a}|)}{b_d(1-\beta)^2} < \epsilon/2$ and $\mathrm{E}(\|\hat{V}_d^{\mu b} - V_d\|_1) < \epsilon/2$. And with this, (14) yields $\mathrm{E}\left(\|\underline{\hat{V}}_d^{\mu b} - V_d\|_1\right) \leq \mathrm{E}\left(\|\underline{\hat{V}}_d^{\mu b} - \hat{V}_d^{\mu b}\|_1\right) + \mathrm{E}(\|\hat{V}_d^{\mu b} - V_d\|_1) < \epsilon$. □

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