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Fairness Considerations in Network Flow Problems

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Abstract

In most of the physical networks, such as power, water and transportation systems, there is a system-wide objective function, typically social welfare, and an underlying physics constraint governing the flow in the networks. The standard economics and optimization theories suggest that at optimal operating point, the price in the system should correspond to the optimal dual variables associated with those physical constraint. While this set of prices can achieve the best social welfare, they may feature significant differences even for neighboring agents in the system. This work addresses fairness considerations in network flow problems, where we not only care about the standard social welfare maximization , but also distribution of prices. We first interpret the network flow problem as an economic market problem. We then show that by tuning a design parameter, we can achieve a spectrum of price-fairness, where the gap between prices satisfy certain design objective. We derive the required physical means to implement the fairness adjustment and show that the adjusted optimal solution depends on the original network topology.

1. Introduction

Network flow problem naturally emerges from many important applications, such as communication networks, transportation networks and energy supply networks [7], [11], [4], [6], [1]. This problem of optimally routing and distribution the flow to minimize the transportation cost has been well studied. In this work, we look at the economical interpretation of this problem, where prices associated with the flows arise from the optimal solution as a dual variable associated with the flow constraints. In a competitive market implementation of the network flow problem, the prices at each node correspond to how much each agent should pay/receive for one unit of flow. Due to the cost structure of the arcs, at the optimal solution, a very wide range of optimal prices may arise, resulting in drastically differences from suppliers and consumers as well as among the suppliers and consumers. The important aspect of fairness in the system is often neglected in the existing studies. We first present some background on the studies of fairness and then highlight some of our contributions in this work.

1.1. Literature Review

The question of *fair* allocation of resources has been extensively studied through the years in many areas, notably in social sciences, welfare economics, and engineering. A variety of ways to measure what is fair have been proposed, and no single principle has come out to be universally accepted. However, there are general theories of justice and equity on which most fairness schemes have been based. In this section, we briefly review the most important theories and define special notions of fairness called proportional and max-min fairness, the two criteria that emerge from these foundational theories, which are also widely used in practice. For more details, see [10] and [5].

Among the most prominent theories of justice is Aristotle's equity principle, according to which resources should be allocated in proportion to some preexisting claims, or rights to the resources that each player has. Another theory, widely considered in economics in the 19th century, is classical utilitarianism, which dictates an allocation of resources that maximizes the sum of utilities. A third approach is due to [9], where the key idea is to give priority to the players that are the least well off, so as to guarantee the highest minimum utility level that every player derives. Finally, Nash introduced the Nash standard of comparison, which is the percentage change in a player's utility when he receives a small additional amount of the resources. A transfer of resources between two players is then justified if the gainer's

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utility increases by a larger percentage than the loser's utility decreases. In addition to using theories of justice, there has been literature studying the axiomatic foundations of the concept of fairness. In particular, this set of literature specifies properties that a fairness scheme should ideally satisfy. The main work in this area is within the literature of fair bargains in economics (see [14] and references therein).

In this work, we emphasize the fairness in terms of unit price equality in the framework of network flow problem with cost minimization objective. This work is closely related to [8] and [13]. In [8], the authors proposed axiomatic approach to qualify various fairness metrics in the network resource allocation problem, where the fairness is measured in terms of each user's utility, whereas in our work, we focus on a per-unit cost fairness, which we believe is also a natural consideration. The work in [13] also aims at controlling price differences in a system, but the goal there is to control inter-temporal fluctuations.

1.2. Contributions

Our main contributions are twofold in this paper. We first show that network flow problems can be interpreted as an economic market interaction, with producers and consumers interacting. Based on this observation, we then derive the market clearing prices, and introduce a systematic framework to address the general issue of price fairness by giving the designer an option to balance welfare and fairness. In particular, we refer to fairness in the sense that the prices across from consumers and suppliers should not feature big differences. In this sense, no one is being treated drastically unequally from the counterparts, and the suppliers are not making a huge profit over the consumers. Motivated by the max-min fairness, we have also done a case study where the max-min price difference is considered by the planner and given closed form solution to a physical implementation.

The rest of the paper is organized as follows, Section 2 contains the description of the original problem formulation. In Section 3 we derive a simpler equivalent representations of the problem formulation and show that it has an market interaction interpretation, based on which we derive the fairness adjusted formulation and analyze it in Section 4. Section 4 also contains our case study of max-min price fairness. Section 5 contains our concluding remarks.

Basic Notation:

A vector is viewed as a column vector. For a matrix A, we write $[A]_i$ to denote the i^{th} column of matrix A, and $[A]^j$ to denote the j^{th} row of matrix A. For a vector x, x_i denotes the i^{th} component of the vector. We denote by $I(n)$ the identity matrix of dimension n by n. We use x' and A' to denote the transpose of a vector x and a matrix A respectively. We use standard Euclidean norm (i.e., 2-norm) unless otherwise noted, i.e., for a vector x in $\mathbb{R}^n$, $||x|| = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$. The notation $x \geq 0$ is used to denote element-wise nonnegativity.

2. Model of Network Flow Problem

We consider an incapacitated linear network flow problem over an underlying *directed graph* denote by $\tilde{G} = (\tilde{N}, \tilde{A})$. Set $\tilde{N}$ is the set of *nodes*, $\{1, 2 \ldots, \tilde{n}\}$ and $\tilde{A}$ is the set of *directed arcs*. We use the notation of an ordered pair (i, j) to denote the arc going from node i to node j. Each arc is associated with a positive cost of $\tilde{c}_{ij}$, which represents the cost of transporting one unit of flow over the arc. Each node i is associated with a scalar $\tilde{b}_i$ indicating the external flow coming into node i, where a negative value of b_i is used when there is outgoing flow from node i. The set of nodes with $b_i > 0$ are called *sources*, denoted by S_+, while the set with $b_i < 0$ are called *sinks*, denoted by S_-. The global objective here is to route the incoming flow at the sources to the set of sinks with minimal cost. We can write the problem in the following mathematical form, where we introduce the decision variable $\tilde{x} = [\tilde{x}_{ij}]_{ij}$ to denote the flow across arc (i, j),

$$\min_{\tilde{x} \geq 0} \quad \sum_{(i,j) \in \tilde{A}} \tilde{c}_{ij} \tilde{x}_{ij} \tag{1}$$
$$\text{s.t.} \quad \tilde{b}_i + \sum_{k,(k,i) \in \tilde{A}} \tilde{x}_{ki} - \sum_{j,(i,j) \in \tilde{A}} \tilde{x}_{ij} = 0, \quad \text{for all } i \in \tilde{N}.$$

We adopt the following two standard assumptions:

Assumption 1. *Total incoming and outgoing external flows sum to zero, i.e.,*

$$\sum_{i=1}^{\tilde{n}} \tilde{b}_i = 0.$$

Assumption 2. *The graph is connected.*

The first assumption is necessary for the feasibility of the system. The second assumption is natural also, since otherwise the problem can be broken down to multiple smaller problems to solve.

3. Equivalent Transformations

In this section, we introduce some equivalent transformations of the original problem. The first one in Section 3.1, reduces the complicated network $\tilde{G}$ to a much simpler form, where only sources and sinks are present. This simplification will enable us to focus our development of fairness on a simpler form of the problem. Section 3.2 contains the economic interpretation of the network flow problem, which will be the basis for the fairness considerations in the next section.

3.1. Transportation Reformulation

In this section, we show that the problem in form (1) can be equivalently formulated in a transportation problem format, where all nodes with $b_i = 0$ are removed from the network. We construct a new graph $G = (N, A)$. The nodes in set $N = S_+ \bigcup S_-$ are indexed $\{1, \dots n\}$, which includes all the sources and sinks from the original graph $\tilde{G}$, each associated with an external flow $b_i = \tilde{b}_i$. For each pair of source i and sink j, an arc is present and the cost associated with arc c_{ij} is the minimal cost associated with any path from i to j in the original graph. Formally, we denote the set of paths from i to j by Π_{ij}, where each element is represented by a set of ordered arcs $\{a_{il}, \dots a_{kj}\} \in \Pi_{ij}$ starting from node i to j. Then the cost c_{ij} is defined by

$$c_{ij} = \min_{\{a_{il}, \dots a_{kj}\} \in \Pi_{ij}} \sum_{a \in \{a_{il}, \dots a_{kj}\}} \tilde{c}_a. \tag{2}$$

If the set Π_{ij} is empty in the original graph, then $c_{ij} = \infty$.

The new problem can be represented as follows, where the decision variable $x = [x_{ij}]_{ij}$ for i in S_+ and j in S_- represents the flow from source i to sink j.

$$\begin{aligned} \min_x \quad & \sum_{i \in S_+, j \in S_-} c_{ij} x_{ij} \qquad (3) \\ \text{s.t.} \quad & \sum_{j \in S_-} x_{ij} = b_i, \quad \text{for all } i \in S_+ \\ & -\sum_{i \in S_+} x_{ij} = b_j, \quad \text{for all } j \in S_- \\ & x \geq 0. \end{aligned}$$

The next lemma shows that this transformation is equivalent to the original problem and we will be working with problem (3) for the rest of the paper, due to its simple form.

Lemma 3.1. *Problems (1) and (3) are equivalent, i.e., the optimal objective functions values are equal.*

The proof is based on contradiction and is omitted here due to space constraint.

Remarks: Observe that due to the linearity nature of the problem, we can also deduce that any optimal solution to problem (3) is a tree solution, i.e., at most only $n-1$ arcs will have positive flows. See [3] for details.

3.2. Welfare Equivalence

In this section, inspired by the concept of shadow-price, we use the primal-dual frame work to show that the network flow problem can be equivalently viewed as a social welfare maximization problem with consumer-supplier interactions. This framework will be used as the foundation to develop fairness considerations in the next section.

We first consider the following consumer-supplier interaction, where each source node can be viewed as a supplier and each sink node can be viewed as a consumer. On the supply side, each *supplier (or producer)* i in S_+ receives a price of p_i for one unit of flow transmitted over and has a supply of b_i units needs to be transmitted. The supply side problem can be written as follows, for each $i \in S_+$, the decision variable is y_{ij} for all j, indicating the amount transmitted from supply i to consumer j.

$$\begin{aligned} \max_{y_{ij} \geq 0} \quad & p_i \sum_{j \in S_-} y_{ij} \qquad (4) \\ \text{s.t.} \quad & \sum_{j \in S_-} y_{ij} = b_i, \end{aligned}$$

For *each consumer (or demand)* j in S_-, the decision variables are d_{ij}, which represents the amount of flow consumer j is buying from supplier i. The consumer pays a cost of p_j for consuming each unit of flow. Hence the consumer side problem can be written as

$$\begin{aligned} \max_{d_{ij} \geq 0} \quad & -p_j \sum_{i \in S_+} d_{ij} \qquad (5) \\ \text{s.t.} \quad & -\sum_{i \in S_+} d_{ij} = b_j, \end{aligned}$$

The market reaches equilibrium when both producers and consumers locally solve their problems (4-5) and the market clears, i.e., $y_{ij} = d_{ij}$. We note that due to the linear constraints, the objective value of problems (4-5) are uniquely determined. They are, nevertheless, of interest due to their interpretation associated with the supply and demand interaction as demonstrated below.

We now show that the market interaction is an equivalent way to implement the original network flow problem. We refer to the sum of the objective functions in problems (4) and (5), i.e.,$\sum_{i\in S_+}(p_i\sum_{j\in S_-} y_{ij})-\sum_{j\in S_-}(p_j\sum_{i\in S_+} d_{ij})$, as the *social welfare*.

Theorem 3.1. *The optimal primal-dual solution pair to problem 3 and the market equilibrium of problems (under market clearing condition) (4) and (5) are equivalent with* $p_i-p_j=c_{ij}$ *for* $x_{ij}>0$.

Proof. Due to the equality constraints in problems (4) and (5), for each price vector p, the objective function value is fixed. The market equilibrium emerges when we have market clearing, i.e., $d_{ij}=y_{ij}$. We note that the optimal dual variable q associated with equality constraints of problem (3) solves the following problem, $\max_q \quad \min_{x\geq 0}\sum_{i\in S_+,j\in S_-} c_{ij}x_{ij}-\sum_{i\in S_+} q_i(\sum_{j\in S_-} x_{ij}-b_i)-\sum_{j\in S_-} q_j(-\sum_{i\in S_+} x_{ij}-b_j)$.

By rearranging the terms, we have that the optimal primal-dual pair solves $\max_q\sum_{i\in S_+} q_ib_i+\sum_{i\in S_-} q_jb_j+\min_{x\geq 0}\sum_{i\in S_+,j\in S_-} x_{ij}(c_{ij}-q_i+q_j)$. From LP duality, we have that the above expression h implies that $c_{ij}-q_i+q_j=0$ for any $x_{ij}>0$. Hence, if we assign $p_i=q_i$ and $p_j=q_j$ to problems (4) and (5), we have that at market equilibrium, the social welfare is equivalent to the optimal objective function value of problem (3), also they share the same solution. □

From complementary slackness for problem (3), we also have that $c_{ij}\geq p_i-p_j$, for all pairs of i,j.

We note that the equality constraint in the supplier and consumer problems can be applicable in many real time transportation systems, where the supply and demand has to be met strictly. Examples include taxi dispatch system, water, oil and natural gas flow systems.

During the preceding analysis, we interpreted the dual variable associated with the equality constraints in problem (3) as the price and derived the producer price to sell and consumer cost for using based on these dual variables. Based on these equivalent transformations, we have that the network flow problem can be equivalently implemented using market dynamics with appropriate prices for accessing the market.

4. Fairness Adjusted Formulation and Analysis

This section is built upon the previous discussion where prices for accessing the market naturally arises as the dual variable to problem (3). Since these prices are the optimal dual variables associated with an equality constraint, they are unconstrained and a price where one consumer pays can differently from another. Similarly, the producers can face significantly gap in prices and the prices between producer and consumer can also be very different, based on the distribution of c_{ij}. While the optimal solution for problem (3) does solve the social welfare problem optimally (based on problems (4), (5)), it ignores an important issue in the society, fairness. In the long run, this difference in prices may give economic incentives to lower the cost of certain source-sink paths or even relocation of producers and consumers. However, in the short run, this can be perceived as unfairness in the system. Hence, our goal here is to introduce a systematic way of reducing the price spread in the system and derive a physical implementation. Section 4.1 derives the problem formulation when we add a general form of price fairness consideration into the previous problem. Section 4.2 is our case study, where we analyze the case when we penalize the maximum difference in price inequalities.

4.1. Price Fairness Aware Formulation

We will work with the original problem (3) and based on the equivalence established in the previous section, the result here will shed light on how to implement the fairness-aware system in the market environment. We first introduce some short hand notations. We use vectors $x=[x_{ij}]_{ij}$ and $c=[c_{ij}]_{ij}$ to represent all the flows and cost in the system. We also combine the two equality constraint into a compact matrix form of $Bx=b$, with B in $\mathbb{R}^{|N|\times|A|}$ composed of elements $0,1,-1$. Hence, problem (3) can be compactly written as $\quad \min_{x\geq 0,Bx=b} \quad c'x$.

The price vector p now is associated with the constraint $Bx=b$. In order to study fairness in the system, we need to rewrite the price vector p. We focus on the case where an optimal primal-dual solution (x,p) to the original problem (3) is known and study the effect of fairness in price consideration on the problem. Without loss of generality, we order the nodes by decreasing order of optimal dual variables p_i. We let p_n be the price associated with the last node n. We denote by e the vector of all 1 in $\mathbb{R}^{|N|}$ and by E the matrix in $\mathbb{R}^{|N|\times(|N|-1)}$ with

$$E = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ & & \vdots & 1 \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix}, \tag{6}$$

such that $p = ep_n + E\Delta p$.

To introduce a fairness consideration in the system, we follow the same approach as in our previous works [13] and [12]. We introduce an additional term for fairness in addition to the standard welfare consideration and show how this can be implemented.

Theorem 4.1. *The following two problems are equivalent*

$$\max_{p_T,\Delta p} \quad \min_{x\geq 0} c'x - (ep_n + E\Delta p)'(Bx-b) - S\,||\Delta p||_w. \tag{7}$$

$$\begin{aligned} \min_{x\geq 0} \quad & c'x, \\ s.t. \quad & \left|\left|E'(Bx-b)\right|\right|_q \leq S, \qquad e'(Bx-b) = 0, \end{aligned} \tag{8}$$

where the two norms $||\cdot||_w$ and $||\cdot||_q$ are dual operators and S is a nonnegative scalar.

Proof. We first rearrange the terms in (7) as

$$\max_{p_T,\Delta p} \quad \min_{x\geq 0} c'x - p_n e'(Bx-b) - \Delta p' E'(Bx-x) - S\,||\Delta p||_w.$$

By Saddle Point Theorem (Proposition 2.6.4 in [2]) and linearity of the problem, we can exchange the minimization and maximization, to express the problem equivalently as

$$\min_{x\geq 0} \quad \max_{p_T,\Delta p} c'x - p_n e'(Bx-b) - \Delta p' E'(Bx-x) - S\,||\Delta p||_w.$$

For any solution pair with $e'(Bx-b) \neq 0$, there exists p_n such that the value of the previous problem attains infinity and thus cannot be the optimal solution. Therefore we conclude $e'(Bx-b) = 0$.

We next analyze the terms involving Δp: $-\Delta p'E'(Bx-x) - S\,||\Delta p||_w$. From Höler's inequality, we have

$$-\Delta p'E'(Bx-b) - S\,||\Delta p||_w \leq ||\Delta p||_w \left(\left|\left|E'(Bx-b)\right|\right|_q - S\right),$$

i.e., $\max_{\Delta p} -\Delta p'E'(Bx-b) - S\,||\Delta p||_w = ||\Delta p^*||_w\,(||E'(Bx-b)||_q - S)$, where Δp^* is the maximizer. Therefore problem (7) is equivalent to

$$\max_{\Delta p} \quad \min_{x\geq 0} c'x + ||\Delta p||_w \left(\left|\left|E'(Bx-b)\right|\right|_q - S\right).$$

Since Δp is unconstrained, the scalar $||\Delta p||$ can take any nonnegative values, which yields another equivalent formulation of

$$\max_{\alpha\geq 0} \quad \min_{x\geq 0} c'x + \alpha\left(\left|\left|E'(Bx-b)\right|\right|_q - S\right).$$

If $||E'(Bx-b)||_q - S \geq 0$, then the problem attains value of infinity and thus cannot be optimal, and therefore the preceding formulation is equivalent to (8). □

The preceding theorem shows that adding a term to control price spread is equivalent to relax the constraint of $Bx = b$, where the design parameter S is used to control how much emphasis the fairness consideration term obtains compared with the original social welfare. A value of $S = 0$ recovers the original problem, and $S = \infty$ puts entire weight on the fairness and ignores the welfare. The equality constraint can be viewed as strictly satisfying supply and demand, while the relaxed version can be viewed as if there is storage (with nonempty initial level) to compensate for the supply-demand (incoming-outgoing) mismatch at each node. The constraint $e'(Bx-b) = 0$ implies that across the entire system, total incoming and outgoing flows are still balanced.

4.2. Minimize Maximum Price Inequality

To study this fairness adjustment and the physical implementation thereof further, for the rest of the paper, we focus on the scenario where w-norm is the L_1 and q-norm is its dual norm, L_∞ norm. This is a natural choice, since the nodes are ordered in the decreasing order of prices and L_1 norm corresponds to

$$||\Delta p||_1 = p_1 - p_2 + p_2 - p_3 + \ldots + p_{n-1} - p_n = p_1 - p_n.$$

A penalty on $||\Delta p||_1$ aims at reduce the difference between the maximum and the minimum prices in the system. The problem considered in this section is given by

$$\begin{aligned} \min_{x\geq 0} \quad & c'x, \\ \text{s.t.} \quad & \left|\left|E'(Bx-b)\right|\right|_\infty \leq S, \qquad e'(Bx-b) = 0, \end{aligned} \tag{9}$$

Recall definition of matrix E in Eq. (6), the i^{th} element of the vector $E'(Bx-b)$ is given by $[E'(Bx-b)]_i = \sum_{t=1}^{i}[Bx-b]_t$, which represent the accumulative incoming-outgoing flow mismatch starting from 1 up to i. This can be physically implemented by introduce a storage of size $2S$ at each node i, with level S without any knowledge of the original welfare maximization problem. We next show that we can derive how to implement this price fairness aware system with much smaller storage requirement, if we know the optimal primal-dual pair (x, p) to the original problem and the original problem has either only one source or one sink. In the following, we will do a case study and give an explicit solution in how to allocate a total size of $2S$ storage access (with S unit for incoming flow and another S unit for outgoing flow) to a problem where only one sink is present.[1]

By Assumption 1, we have $e'b = 0$. Also, due to the structure of matrix B, each column of B has exactly a 1 and a -1, the rest being 0, we have $e'B = 0$. Therefore the equality constraint above is redundant and can be removed. This corresponds to our freedom in choosing p_n, and without loss of generality, we set $p_n = 0$. We have the following lemma to characterize effects of S on the system.

Lemma 4.1. *We form new vectors y, $\tilde{c}$, by ordering the components of original optimal solution of problem (3), x_{ij}, and associated c_{ij}, by their corresponding c_{ij} in decreasing order, i.e., if $y_1 = x_{vu}$, then $\tilde{c}_1 = c_{vu}$, which is the largest cost associated with arcs with positive flow. For problem (9), we let index t be the largest integer satisfying $S \geq \sum_{i=1}^{j} y_i$ (when this condition does not hold for any j, we let $t = 0$). Then we have the optimal solution satisfies either $||\Delta p||_1 = \tilde{c}_{t+1}$, or when t =length of $\tilde{c}$, i.e., $S \geq \sum_{i \in S_+} b_i$, $||\Delta p||_1 = 0$.*

Due to space limit, we only highlight the intuition of the proof here. By complementary slackness and LP duality, we note that the price differences $p_i - p_j$ are associated with arcs with positive flow and the differences are precisely the corresponding c_{ij}. This theorem shows that the required storage size would depend on the original network topology and its associated costs.

5. Conclusions

In this paper, we studied the network flow problem and analyzed the variant where instead of cost minimization (or social welfare maximization), we also considered the issue of fairness in the price inequality. We first presented an economic interpretation of the network flow problem and then introduced a penalty term of price inequality in the system wide objective and derived an equivalent physical implementation. We analyzed the case of minimizing maximum price inequality as a case study. Future work directions include to characterize storage distribution to general setup (other variations of price inequality and network topology) and nonlinear objectives, where the allocation of storage will depend on the underlying network topology.

[1] We use the term "storage" loosely here to refer to a device that is nonempty, and can either hold extra flow or output extra flow.

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