Conflict Games with Payoff Uncertainty[1]

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[1]This paper has benefitted from insightful comments by Stephen Morris.

Abstract

Stag hunt and chicken games are canonical representations of two kinds of strategic interactions. In stag hunt, aggression feeds on itself, and mutual fear escalates into conflict. Chicken is a model of preemption and deterrence. With complete information, these games have multiple Nash equilibria. We find sufficient conditions under which payoff uncertainty generates a *unique* equilibrium. These conditions encompass information structures ranging from independent types (as in our previous work) to highly correlated types (as in global games). We use simple, standard arguments from the industrial organization literature to prove uniquness. *Keywords*: conflict, global games, strategic complements, strategic substitutes.

1 Introduction

Simple two-by-two games are frequently used to represent strategic interactions in political science and international relations (see, for example, Jervis [11]). In this literature, the prisoner's dilemma plays a prominent role. But in many instances, stag hunt and chicken games seem more useful metaphors. Stag hunt captures Hobbes's "state of nature", where conflict is caused by lack of trust. Chicken is a model of preemption and deterrence. The prominence of prisoner's dilemma games in the literature may be due to analytical convenience: the prisoner's dilemma has a unique Nash equilibrium, while stag hunt and chicken have multiple equilibria.

The prisoner's dilemma can be thought of as a degenerate stag hunt or chicken game, where extreme levels of hostility have made "war" a dominant strategy. And even if this scenario is not very likely, a player who is himself not intrinsically hostile may be unable to completely rule out the possibility that the opponent is extremely hostile, or that the opponent thinks he is very hostile... As is well known, this type of reasoning may produce "spirals" of fear and aggression. The most useful way to think about these spirals is to formally introduce payoff uncertainty. This not only makes the model more realistic, but it may also generate a unique equilibrium, whether the underlying game is stag hunt or chicken.

Consider a two-player game, where each player must choose either *hawk* (H) or *dove* (D). The hawkish action H might represent an act of war, accumulation of weapons, or some other aggressive action. In the payoff matrix, the row represents the choice of player i, and the column the choice of player j. Only player i's payoff is indicated.

$$\begin{array}{ccc} & H & D \\ H & h_i - c & h_i \\ D & -d & 0 \end{array} \tag{1}$$

The payoff from the peaceful outcome (D, D) is, without loss of generality, normalized to zero. The parameter h_i is incorporates player i's costs and benefits from choosing H. If the opponent chooses H then player i suffers a cost. This cost is c if player i also chooses H, and d if player i chooses D. We assume, for convenience, that the two parameters c and d are the same for both players. However, in general, we will have $h_1 \neq h_2$. We refer to h_i as player i's *hostility parameter* or *type*.

Suppose, for the moment, that there is no payoff uncertainty: the players know everything about the game, including each others types. If $h_i > \max\{c-d, 0\}$ for each $i \in \{1,2\}$ then the game is a *prisoner's dilemma* with a unique Nash equilibrium: (H, H). If $c - d < h_i < 0$ for each $i \in \{1,2\}$ then the game is a *stag-hunt* with two Nash equilibria: (H, H) and (D, D). If $0 < h_i < c - d$ for each $i \in \{1,2\}$ then it is a game of *chicken* with two Nash equilibria: (H, D) and (D, H). In games with multiple Nash equilibria, the criterion of *risk-dominance* is sometimes used to select a unique outcome. In the stag-hunt game, the risk-dominant Nash equilibrium is (H, H) if $h_1 + h_2 > c - d$ and (D, D) if $h_1 + h_2 < c - d$. That is, if the combined hostility levels are not too big, a conflict can be avoided. In chicken, the risk-dominant Nash equilibrium is (H, D) if $h_1 > h_2$ and (D, H) if $h_1 < h_2$. That is, the most hostile player is aggressive, the other backs down.

Carlsson and van Damme [7] showed that if each player is uncertain about the opponent's type, but types are very highly correlated, then the players coordinate on the risk-dominant outcome. This insight triggered a large literature on *global games*. In reality, it is surely impossible to be certain about the opponent's payoff function. However, if the types are highly correlated, as in the global games literature, then each player is almost perfectly informed about the opponent's type. This is not a plausible assumption for many applications. In reality, payoff uncertainty can be large. Therefore, unlike the global games literature, we do not focus only on "small" perturbations of the payoff matrix (1). We will derive conditions under which a unique equilibrium exists even when there is a significant amount of payoff uncertainty.

Realistically, any parameter in the payoff matrix (1) could be uncertain. For convenience, we will assume the uncertainty only relates to the hostility parameters, while c and d are common knowledge. We consider two possibilities. If $c < d$, then actions are strategic complements, as in a stag-hunt game: each player is more inclined to choose H, the more likely it is that that his opponent will choose H. If $c > d$, then actions are strategic substitutes, as in a chicken game: each player is more inclined to choose H, the more likely it is that the opponent will choose D. Regardless of whether $c < d$ or $c > d$, players with very big hostility parameters are "dominant strategy hawks" who behave as in a prisoner's dilemma, with "war" (H) as a dominant strategy. Conversely, players with very small hostility parameters are "dominant strategy doves" for whom "peace" (D) is dominant. The remaining "moderate" types, who do not have any dominant strategy, must form beliefs about the opponent's action before deciding what to do. With strategic complements, these moderates are coordination types who want to match the action of their opponent. With strategic substitutes, moderates want to choose the opposite of their opponent.

In Industrial Organization, the contrast between strategic complements and strategic substitutes organizes the way we think about strategic interaction (e.g., Bertrand versus Cournot competition). But these concepts serve equally well for the study of international relations and conflict. Two kinds of conflicts are often distinguished in the literature: those where aggression feeds on itself in a cycle of fear, as in stag hunt; and those where toughness forces the opponent to back down, as in chicken.[1] In stag hunt, actions are strategic complements: the incentive to choose H is increasing in the probability that the opponent chooses H. This can trigger an escalating spiral of aggression as in the classic work of Schelling [18] and Jervis [11]. In a chicken game, actions are strategic substitutes: the incentive to choose H is decreasing in the probability that the opponent chooses H. This captures a scenario where players will back down in the face of aggression. For example, suppose the hawkish action represents sending soldiers to a disputed territory. If only one country sends soldiers, then it will control the territory at little cost. But it both countries send their soldiers, a war could easily break out. If the value of the territory is not large enough to justify the risk of war, it is a game of chicken.

By using the standard approach from the Industrial Organization literature, we obtain rather general conditions for a unique equilibrium to exist. The conditions are stated in terms of properties of the distribution of types. But they are equivalent to the well-known condition that the slopes of (appropriately defined) reaction functions should be less than one in absolute value. The conditions are satisfied if types are independently drawn but diffuse - "large idiosyncratic uncertainty". For example, even if the *ordinal* payoffs are those of a stag hunt game with probability close to one, but there is a large amount of uncertainty about idiosyncratic cardinal payoffs, then there is a unique equilibrium. However, independence is not always a reasonable assumption. For example, in a conflict over a common resource, types are determined not only by idiosyncratic preferences, but also by the value

[1] "World Wars I and II are often cast as two quite different models of war.. World War I was an unwanted spiral of hostility... World War II was not an unwanted spiral of hostility-it was a failure to deter Hitler's planned aggression." Joseph Nye (p. 111, [17]). *Understanding International Conflict (6th Edition).*

of the contested resource. In this case, a player's type contains information about his opponent's type, so types are affiliated. Our conditions cover this case as well.

In a stag-hunt game, the fear of dominant strategy hawks triggers "almost dominant strategy hawks" to play H, which triggers "almost-almost dominant strategy hawks" to play H etc. This process is akin to Schelling's [18] "reciprocal fear of surprise attack" and Jervis's [11] "spiral model". A similar but more benign logic causes "almost dominant strategy doves" to play D, followed by "almost-almost dominant strategy doves" etc. In Section 3, we derive a sufficient condition for these spirals to produce a unique equilibrium. In particular, this is the case when types are independent but diffuse, or are determined by a common shock and small noise as in the global games literature. Adding affiliation has an ambiguous effect on these spirals. On the one hand, affiliation makes "almost dominant strategy hawks" think it is quite likely that the opponent is a dominant strategy hawk, which intensifies their fear, and so on. In this sense, affiliation promotes uniqueness when actions are strategic complements. However, if types are strongly affiliated, then there is not much (interim) payoff uncertainty. In this case, multiple equilibria cannot be ruled out.

A different kind of spiral leads to a unique equilibrium in games with strategic substitutes. This spiral is created by types who back down in the face of aggression triggering more aggression which causes more types to back down, etc. In Section 4, we derive a sufficient condition for uniqueness in such games. With strategic complements, the fear of dominant strategy hawks causes "almost dominant strategy doves" to play D. This in turn triggers "almost dominant strategy hawks" to play H etc. Again, with independently drawn diffuse types there is a unique equilibrium. Affiliation prevents this type of spiral from gaining ground, because "almost dominant strategy doves" will think it is quite unlikely that the opponent is a dominant strategy hawk, which mitigates their fear, and so on. Therefore, affiliation always works against uniqueness when actions are strategic substitutes.

We have previously studied the logic of mutual fear and escalation under the assumption that types are independent (Baliga and Sjöström [5], [6], Baliga, Lucca and Sjöström [4]). In complementary work, Chassang and Padro-i-Miguel [8], [9] use the theory of global games and the concept of risk-dominance to formalize the logic of mutual fear. In this article, we consider a more general model of affiliated types, where both independent types and highly correlated types are special cases. Morris and Shin ([15]) study games with strategic substitutes and a broader class of information structures than global games. They argue that uniqueness results from either large idiosyncratic uncertainty (as in Baliga and Sjöström [5], [6], Baliga, Lucca and Sjöström [4]) or from highly correlated types (as in global games). We offer a similar conclusion for the case of strategic complements. As we exploit connections with the Industrial Organization literature, our arguments are different, more familiar and much simpler. Unlike Morris and Shim [15], we also study the case of strategic substitutes. Via the leading example in Carlsson and van Damme [7], we show that our theory of payoff uncertainty and uniqueness differs from global games.

2 The Model

As discussed above, the payoff matrix is

	H	D
H	$h_i - c$	h_i
D	$-d$	0

The two parameters c and d are fixed and the same for both players. However, player i's true hostility parameter (or "type") h_i is his private information.

The hostility parameter h_i has a fixed publicly observed component k_i as well as a random privately observed component η_i. Thus, player i's type is

$$h_i = k_i + \eta_i.$$

The game of incomplete information is played as follows. First η_1 and η_2 are drawn from a symmetric joint distribution with support $[\underline{\eta}, \bar{\eta}] \times [\underline{\eta}, \bar{\eta}]$. Then player 1 is informed about η_1, but not about η_2. Similarly, player 2 is informed about η_2 but not about η_1. Finally, each player makes his choice simultaneously (H or D).

When the players make their choices, everything except η_1 and η_2 is commonly known. In particular, there is no uncertainty about the fixed parameters k_1 and k_2. The introduction of k_1 and k_2 is a convenient way to allow for *ex ante* asymmetries in the distribution of hostilities. If $k_1 = k_2$ then the two players are ex ante symmetric (and it would be without loss of generality to assume $k_1 = k_2 = 0$). But if $k_1 \neq k_2$ then there is a publicly known ex ante asymmetry.

If η_1 and η_2 are correlated, then player i's knowledge of η_i can be used to update his beliefs about η_j. Formally, the cumulative distribution of η_j conditional on $\eta_i = y$ (where $i \neq j$) is denoted $F(\cdot|y)$. We assume $F(x|y)$ is continuously differentiable, with partial derivatives $F_1(x|y) \equiv \frac{\partial F(x|y)}{\partial x}$ and $F_2(x|y) \equiv \frac{\partial F(x|y)}{\partial y}$. Notice that $F_1(\cdot|y)$ is the density of η_j conditional on $\eta_i = y$. Since player j's hostility is $h_j = k_j + \eta_j$, uncertainty about η_j directly translates into uncertainty about player j's type. The types h_1 and h_2 are correlated if and only if η_1 and η_2 are correlated. If player i's type is $h_i = y$, then player i assigns probability $F(x - k_j|y - k_i)$ to the event that $h_j \leq x$. Indeed, $h_i = y$ if and only if $\eta_i = y - k_i$, and $h_j \leq x$ if and only if $\eta_j \leq x - k_j$. The least (resp. most) hostile type of player i has hostility parameter $\underline{h}_i = k_i + \underline{\eta}$ (resp. $\bar{h}_i = k_i + \bar{\eta}$). Notice that, for any y,

$$F(\underline{h}_j - k_j|y - k_i) = F(\underline{\eta}|y - k_i) = 0$$

and

$$F(\bar{h}_j - k_j|y - k_i) = F(\bar{\eta}|y - k_i) = 1$$

since it is impossible for player j to have hostility parameter below $\underline{h}_j$ or above $\bar{h}_j$.

We make the following assumption:

Assumption 1 (i) $F_1(x|y) > 0$ for all $x, y \in (\underline{\eta}, \bar{\eta})$ and (ii) $F_2(x|y) \leq 0$ for all $x, y \in (\underline{\eta}, \bar{\eta})$.

Part (i) says there is positive density everywhere. Part (ii) says that $F(x|y)$ is not increasing in y. Therefore, as a player becomes more hostile, he becomes no less pessimistic about his opponent's hostility. Part (ii) holds if η_1 and η_2 are affiliated (Milgrom, 2004, Theorem 5.4.3). Affiliation is a natural assumption if the conflict is over some resource such as oil. Of course, Part (ii) also holds if η_1 and η_2 are independent. Independence is a natural assumption if the uncertainty is over the innate attitude towards conflict. For future reference, we note that Assumption 1 implies that if $y > x$ then $F(y|x) - F(x|y) \geq F(y|y) - F(x|y) > 0$.

We classify types into four categories.

Definition 1 *Player i is a* dominant strategy hawk *if $h_i - c \geq -d$ and $h_i \geq 0$ with at least one strict inequality. Player i is a* dominant strategy dove *if $h_i - c \leq -d$ and $h_i \leq 0$ with at least one strict inequality. Player i is a* coordination type *if $c - d \leq h_i \leq 0$. Player i is an* opportunistic type *if $0 \leq h_i \leq c - d$,.*

Notice that coordination types exist only in games with strategic complements. For them, H is a best response to H and D a best response to D. Opportunistic types exist only in games with strategic substitutes. For them, D is a best response to H and H a best response to D

2.1 Bayesian Nash Equilibrium

Suppose player i is of type h_i, and thinks player j will choose D with probability $\delta_j(h_i)$. (This probability can depend on h_i if types are not independent). Type h_i's expected payoff from H is $h_i - (1-\delta_j(h_i))c$, while his expected payoff from D is $-(1-\delta_j(h_i))d$. Thus, if type h_i chooses H instead of D, his *net* gain is

$$h_i + (d-c)(1-\delta_j(h_i)) \tag{2}$$

A *strategy* for player i is a function $\sigma_i : [\underline{h}_i, \bar{h}_i] \to \{H, D\}$ which specifies an action $\sigma_i(h_i) \in \{H, D\}$ for each type $h_i \in [\underline{h}_i, \bar{h}_i]$. In Bayesian Nash equilibrium (BNE), all types maximize their expected payoff. Therefore, $\sigma_i(h_i) = H$ if the expression in (2) is positive, and $\sigma_i(h_i) = D$ if it is negative. (If expression (2) is zero then type h_i is indifferent and can choose either H or D.) We say that player i uses a *cutoff strategy* if there is a *cutoff point* $x \in [\underline{h}_i, \bar{h}_i]$ such that $\sigma_i(h_i) = H$ for all $h_i > x$ and $\sigma_i(h_i) = D$ for all $h_i < x$. A *cutoff equilibrium* is a BNE in cutoff strategies. Cutoff equilibria seem very natural. They capture the intuition that when a player becomes more hostile he becomes more likely to show aggression.

If player j uses a cutoff strategy with cutoff point x, then $\delta_j(y) = F(x - k_j | y - k_i)$, so player i's net gain from choosing H instead of D when his type is $h_i = y$ is

$$\Psi^i(x, y) \equiv y + (d-c)\left(1 - F(x - k_j | y - k_i)\right). \tag{3}$$

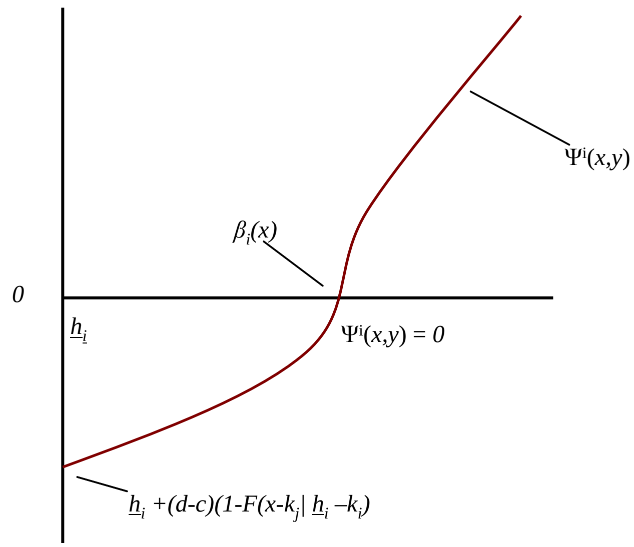
For a cutoff strategy to be a best response, player i should be more inclined to choose H the more hostile he is. That is, $\Psi^i(x, y)$ should be increasing in y :

$$\Psi^i_2(x, y) = 1 - (d-c)F_2(x - k_j | y - k_i) > 0 \tag{4}$$

Figure 1 illustrates this property.

In view of Assumption 1, (4) holds if $d > c$. It also holds if $d < c$ and the two types are not very highly correlated. However, if c is much bigger

Figure 1: $\Psi^i(x,y)$



than d and the two types are highly correlated, then (4) may be violated. The intuition is that, if types are highly correlated and the players use cutoff strategies, then a very hostile type thinks it is very likely that the opponent chooses H. If in addition c is much bigger than d, then the (H, H) outcome is very costly. In this situation, the very hostile type may be inclined to choose D rather than H..

If condition (4) holds then player i's best response to player j's cutoff x is to use a cutoff point denoted $\beta_i(x)$. The best-response function $\beta_i(x)$ is defined as follows. (i) If $\Psi^i(x, \underline{h}_i) \geq 0$ then $\beta_i(x) = \underline{h}_i$ (so player i plays H with probability one). (ii) If $\Psi_i(x, \bar{h}_i) \leq 0$ then $\beta_i(x) = \bar{h}_i$ (so player i plays D with probability one). (iii) Otherwise, $\beta_i(x) \in \left(\underline{h}_i, \bar{h}_i\right)$ is the unique solution to the equation $\Psi^i(x, \beta_i(x)) = 0$ (all types above $\beta_i(x)$ play H, and all types below $\beta_i(x)$ play D). As long as $\Psi^i(x, y)$ is increasing in y, $\beta_i(x)$ is a well-defined continuous function (by the implicit function theorem), and the slope of β_i is obtained by totally differentiating $\Psi^i(x, \beta_i(x)) = 0$. Thus,

$$\beta_i'(x) = -\frac{\Psi_1^i(x, \beta_i(x))}{\Psi_2^i(x, \beta_i(x))} = -\frac{(c-d)F_1(x - k_j|\beta_i(x) - k_i)}{1 - (d-c)F_2(x - k_j|\beta_i(x) - k_i)}. \tag{5}$$

A cutoff equilibrium is an intersection of the two best response curves. That is, the two cutoffs x_1 and x_2 must satisfy $x_1 = \beta_1(x_2)$ and $x_2 = \beta_2(x_1)$. Notice that $\beta_i'(x) > 0$ if $d > c$ (strategic complements) and $\beta_i'(x) < 0$ if $d < c$ (strategic substitutes).

Proposition 2 *If $\Psi^i(x, y)$ is increasing in y for each $i \in \{1, 2\}$ and all $x \in [\underline{h}_i, \bar{h}_i]$, then a cutoff equilibrium exists.*

Proof. Since the function $\Psi^2(\beta_1(x), x)$ is continuous in x, one of the following three cases must occur:

(i) $\Psi^2(\beta_1(\underline{h}_2), \underline{h}_2) \geq 0$. In this case the cut-off points $(\beta_1(\underline{h}_2), \underline{h}_2)$ form a BNE.

(ii) $\Psi^2(\beta_1(\bar{h}_2), \bar{h}_2) \leq 0$. In this case the cut-off points $(\beta_1(\bar{h}_2), \bar{h}_2)$ form a BNE.

(iii) there is $x \in [\underline{h}_2, \bar{h}_2]$ such that $\Psi^2(\beta_1(x), x) = 0$. In this case the cut-off points $(\beta_1(x), x)$ form a BNE. ■

3 Strategic Complements

Actions are strategic complements when $d > c$. In this case, (4) holds, so a cutoff equilibrium exists by Proposition 2. We first derive a sufficient condition for this to be the *unique* BNE.

3.1 The uniqueness result

Our main result for the case of strategic complements is the following.

Theorem 3 *Suppose $d > c$ and for all $s, t \in (\underline{\eta}, \bar{\eta})$,*

$$F_1(s|t) + F_2(s|t) < \frac{1}{d - c}. \tag{6}$$

There is a unique BNE. This BNE is a cutoff equilibrium.

In the appendix, we prove that all BNE must be cutoff equilibria. At least one cutoff equilibrium exists by Proposition 2. To complete the proof of Theorem 3, we only need to show that there cannot be more than one cutoff equilibrium.

It suffices to show that the slope of the best response function β_i is less than one in absolute value. As is well known, this guarantees that the two best response curves can intersect only once (for example, Vives [21]). Equation (5) can be manipulated as follows:

$$\begin{aligned}\beta_i'(x) &= \frac{(d-c)F_1(x-k_j|\beta_i(x)-k_i)}{1-(d-c)F_2(x-k_j|\beta_i(x)-k_i)} \\ &= 1-\frac{1-(d-c)\left\{F_1(x-k_j|\beta_i(x)-k_i)+F_2(x-k_j|\beta_i(x)-k_i)\right\}}{\Psi_2^i(x,\beta(x))} < 1.\end{aligned}$$

The inequality is due to (4) and (6). Since we already know that $\beta_i'(x) > 0$ when $d > c$, we conclude that $0 < \beta_i'(x) < 1$. Thus, the cutoff equilibrium is unique.[2]

3.2 Diagrammatic illustration

We will illustrate Theorem 3 diagrammatically. Consider the function $Q : [\underline{\eta}, \bar{\eta}] \to [0,1]$ defined by $Q(\eta) \equiv F(\eta|\eta)$. Notice that $Q(\underline{\eta}) = 0$ and $Q(\bar{\eta}) = 1$. By calculus,

$$\frac{\bar{\eta}-\underline{\eta}}{d-c} = \int_{\underline{\eta}}^{\bar{\eta}} \left(\frac{1}{d-c}\right) d\eta > \int_{\underline{\eta}}^{\bar{\eta}} \left\{F_1(\eta|\eta)+F_2(\eta|\eta\right\} d\eta = \int_{\underline{\eta}}^{\bar{\eta}} Q'(\eta)\, d\eta = Q(\bar{\eta})-Q(\underline{\eta}) = 1$$

where the inequality is due to (6). Therefore, for each $i \in \{1,2\}$,

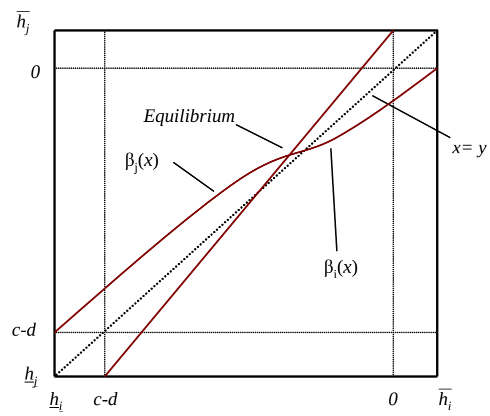
$$\bar{h}_i - \underline{h}_i = \bar{\eta} - \underline{\eta} > d - c.$$

This implies that either $\bar{h}_i > 0$ or $\underline{h}_i < c - d$ (or both). That is, the support includes dominant strategy types.

Suppose $\underline{h}_i < c - d < 0 < \bar{h}_i$ for each $i \in \{1,2\}$ so each player's possible types include both both dominant strategy hawks and dominant strategy doves. Notice that $\Psi^i(\underline{h}_j, y) = y + d - c$, so $\beta_i\left(\underline{h}_j\right) = c - d$. That is, if player j plays H with probability 1 (his cutoff point is $\underline{h}_j$) then player

[2] It is possible to show, generalizing the results in Baliga and Sjostrom, that another sufficient condition for a unique equilibrium in cut off strategies is that ϕ is concave. To prove this, define the modified best response function $\hat{\beta}$ by using the function $\hat{\Psi}(x,y) \equiv y + (d-c)(1-\phi(x))$ instead of the function Ψ. If ϕ is concave, then $\hat{\beta}$ intersects the 45 degree line exactly once. Moreover, β coincides with $\hat{\beta}$ on the 45 degree line, so β intersects the 45 degree line in a unique point.

Figure 2: Equilibrium for Strategic Complements



i's best response is to choose H whenever he is not a dominant strategy dove. Similarly, $\Psi^i(\bar{h}_j, y) = y$, so $\beta_i\left(\bar{h}_j\right) = 0$. That is, if player j plays H with probability 0 (his cutoff point is $\bar{h}_j$) then player i's best response is to choose H whenever he is a dominant strategy hawk. If $\underline{h}_j < x < \bar{h}_j$, then $\beta_i(x) \in \left[\underline{h}_i, \bar{h}_i\right]$ is the unique solution to the equation $\Psi^i(x, \beta_i(x)) = 0.$, and we know that $0 < \beta_i'(x) < 1$.

Notice that, if there are both dominant strategy hawks and dominant strategy doves, then certainly some types choose H, some types choose D. Hence, the unique equilibrium must be interior: each player i chooses a cutoff point $h_i^* \in \left(\underline{h}_i, \bar{h}_i\right)$ which solves

$$h_i^* + (d - c)\left(1 - F(h_j^* - k_j | h_i^* - k_i)\right) = 0. \tag{7}$$

The equilibrium (h_1^*, h_2^*) is illustrated in Figure 2.

Now suppose some player, say player 1, can be a dominant strategy hawk but not a dominant strategy dove: $c - d < \underline{h}_1 < 0 < \bar{h}_1$. As indicated in Figure 3, it is now possible that in equilibrium player 1 chooses H with probability one, but the equilibrium is still unique.

If both players can be dominant strategy hawks but not dominant strategy doves, i.e., $\underline{h}_i \geq c - d$ for $i \in \{1, 2\}$, then there surely exists an equilibrium where each player chooses H, regardless of type. This is illustrated in Figure 4. By Theorem 3, there can be no other equilibrium in this case. Thus, if each player can be a dominant strategy hawk, but not a dominant strategy dove, then peaceful coexistence is impossible, and each player chooses H with probability one. This represents an extreme case of the Schelling's "reciprocal fear of surprise attack". (Conversely, if $\bar{h}_i \leq 0$ for $i \in \{1, 2\}$, so there are no dominant-strategy hawks, then the unique equilibrium is for each player to choose D, regardless of type.

The arguments of Milgrom and Roberts [13] can be adapted to show that the unique equilibrium can be obtained by iterated deletion of (interim) dominated strategies.

3.3 The uniform independent case

Suppose η_1 and η_2 are independently drawn from a uniform distribution on $[\underline{\eta}, \bar{\eta}]$, where $\bar{\eta} - \underline{\eta} > d - c$. In this case, $F(s|t) = \left(s - \underline{\eta}\right) / \left(\bar{\eta} - \underline{\eta}\right)$ so

$$F_1(s|t) + F_2(s|t) = \frac{1}{\bar{\eta} - \underline{\eta}} < \frac{1}{d - c}.$$

Theorem 3 implies there is a unique BNE. Thus, with a uniform distribution, there is a unique equilibrium, as long as the support is big enough.

To simplify the calculations, suppose $k_2 = k \geq 0 = k_1$. Thus, player 2 is ex ante (weakly) more hostile than player 1. If only one kind of dominant strategy type exists, say only dominant strategy hawks, then the unique equilibrium is for both players to choose H regardless of type. For a more interesting interior equilibrium, assume $\bar{\eta} > 0$ and $\underline{\eta} + k < c - d$. This guarantees that the support of each player's types includes dominant strategy types of both kinds. We solve (7) to get $h_1^* = A + \left(\bar{\eta} - \underline{\eta}\right) B$ and $h_2^* = A + (d - c)B$, where

$$A \equiv -\frac{(d - c)\bar{\eta}}{\left(\bar{\eta} - \underline{\eta}\right) - (d - c)}$$

and

$$B \equiv -\frac{(d - c)k}{\left(\bar{\eta} - \underline{\eta}\right)^2 - (d - c)^2}$$

It can be verified that $\bar{\eta} > 0$ and $\underline{\eta} + k < c - d$ guarantee $h_i^* \in \left(\underline{h}_i, \bar{h}_i\right)$. If $k > 0$ then $h_1^* < h_2^*$, since $\bar{\eta} - \underline{\eta} > d - c$ and $B < 0$. Thus, if it should happen that both players' true types lie in the interval (h_1^*, h_2^*), then player 1 chooses H (because $h_1 > h_1^*$) but player 2 chooses D (because $h_2 < h_2^*$). In this sense, the ex ante less hostile player becomes ex post the most aggressive one - because he feels more threatened by the opponent.

3.4 Discussion

Suppose the players are symmetric: $k_1 = k_2 = 0$. Now the best response curves must intersect at the 45 degree line, and the unique equilibrium is a symmetric cut-off equilibrium, $h_1^* = h_2^* = h^*$. The symmetric cut off point is the unique solution in $[\underline{h}_i, \bar{h}_i]$ to the equation

$$h^* + (d - c)(1 - F(h^*|h^*)) = 0 \tag{8}$$

The intuition behind Theorem 3 may be brought out by a standard "stability" argument. Starting at the symmetric cutoff equilibrium h^*, suppose both simultaneously reduce their cut-off by ε (so a few more types use H). Then, consider type $h^* - \varepsilon$. If type $h^* - \varepsilon$ now prefers D, the initial equilibrium is stable, and this is what we want to verify. In fact there are two opposing effects. First, at the original cut-off h^*, type $h^* - \varepsilon$ strictly preferred D, so there is reason to believe he still prefers D. However, the opponent has now become more hostile. At the initial equilibrium, cutoff type h^* thought that the opponent would choose H with probability $1 - F(h^*|h^*)$. But after the perturbation, the new cutoff type $h^* - \varepsilon$ thinks that the opponent will choose H with probability $1 - F(h^* - \varepsilon|h^* - \varepsilon)$. If

$$F_1(h^*|h^*) + F_2(h^*|h^*) < \frac{1}{d - c} \tag{9}$$

then the first effect dominates, and type $h^* - \varepsilon$ will strictly prefer D after the perturbation, as required by stability. But (9) follows from (6). As usual, the stability condition guarantees intuitive comparative statics. By (8) and (9), an increase in $d - c$ will lead to more aggressive behavior (a reduction in equilibrium h^*).

If types are independent, then $F_2 \equiv 0$ and (6) simply requires that the density of the random variable η_i is sufficiently spread out, i.e., that there is "enough uncertainty" about types. Now suppose types are affiliated. This impacts the stability of the equilibrium via the expression $F_1(h^*|h^*) + F_2(h^*|h^*)$. There are two contradictory effects. On the one hand, affiliation causes type h^* to think the opponent is likely to be similar to himself, so $F_1(h^*|h^*)$ is large. This effect makes uniqueness less likely. On the other hand, affiliation causes $F_2(h^*|h^*)$ to be negative. This effect makes uniqueness more likely. While the first effect is easy to understand, in terms of concentrating the density of types in a smaller area, the second effect is more subtle. Intuitively, in the above stability argument, affiliation causes type $h^* - \varepsilon$ to be less pessimistic about the opponent's hostility than type h^*, making him more likely to prefer D. That is, the best response curves are more likely to have the slopes that guarantee stability and uniqueness.

4 Strategic Substitutes

Actions are strategic substitutes when $d < c$. In this case, we need to make sufficient assumptions to guarantee that (4) holds, otherwise a cutoff equilibrium may not exist.

4.1 Existence and uniqueness

Our main result for the case of strategic substitutes is the following.

Theorem 4 *Suppose $d < c$ and for all $x, s, t \in (\underline{\eta}, \bar{\eta})$,*

$$F_1(s|t) - F_2(s|t) < \frac{1}{c-d} \tag{10}$$

and

$$F_1(s|x) - F_2(x|t) < \frac{1}{c-d}. \tag{11}$$

There is a unique BNE. This BNE is a cutoff equilibrium.

In the appendix, we prove that all BNE must be cutoff equilibria. The inequality (10) implies

$$(c-d)F_2(x|t) > (c-d)F_1(s|x) - 1 \geq -1.$$

Therefore, (4) holds, so a cutoff equilibrium exists by Proposition 2. To complete the proof of Theorem 4, we only need to show that there can be at most one cutoff equilibrium.

It again suffices to show that the slope of the best response function β_i is less than one in absolute value (Vives [21]). Equation (5) implies

$$\begin{aligned} 1 + \beta_i'(x) &= 1 + \frac{(d-c)F_1(x - k_j|\beta_i(x) - k_i)}{1 - (d-c)F_2(x - k_j|\beta_i(x) - k_i)} \\ &= \frac{1 + (c-d)\left(F_2(x - k_j|\beta_i(x) - k_i) - F_1(x - k_j|\beta_i(x) - k_i)\right)}{\Psi_2^i(x, \beta_i(x))} > 0 \end{aligned}$$

The inequality is due to (4) and (11). Since we already know that $\beta_i'(x) < 0$ when $d < c$, we conclude that $-1 < \beta_i'(x) < 0$. Thus, the cutoff equilibrium is unique.

4.2 Diagrammatic exposition

Consider again the function $Q(\eta) \equiv F(\eta|\eta)$. By calculus,

$$\frac{\bar{\eta} - \underline{\eta}}{c-d} = \int_{\underline{\eta}}^{\bar{\eta}} \left(\frac{1}{c-d}\right) d\eta > \int_{\underline{\eta}}^{\bar{\eta}} \{F_1(\eta|\eta) - F_2(\eta|\eta\} \ d\eta \geq \int_{\underline{\eta}}^{\bar{\eta}} \{F_1(\eta|\eta) + F_2(\eta|\eta\} \ d\eta$$
$$= \int_{\underline{\eta}}^{\bar{\eta}} Q'(\eta) \ d\eta = Q(\bar{\eta}) - Q(\underline{\eta}) = 1$$

where the first inequality is due to (10) and the second is due to the fact that $F_2(\eta|\eta) \leq 0$. Therefore, for each $i \in \{1, 2\}$,

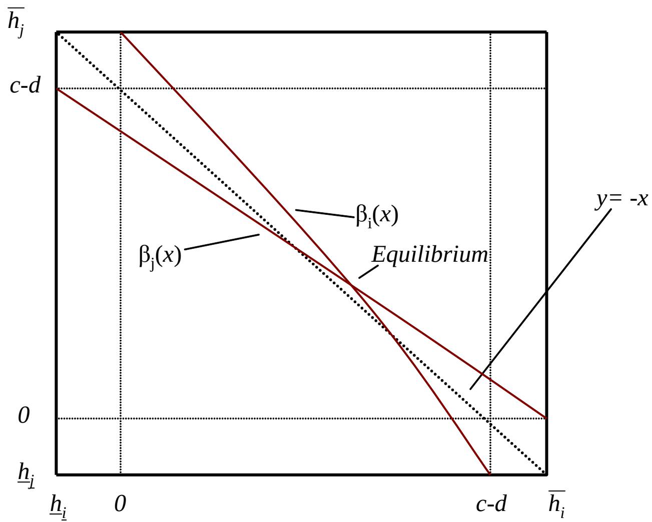
$$\bar{h}_i - \underline{h}_i = \bar{\eta} - \underline{\eta} > c - d.$$

This implies that either $\bar{h}_i > c - d$ or $\underline{h}_i < 0$ (or both). That is, the support includes dominant strategy types.

Suppose both players have dominant strategy hawks and doves means so that if player j uses cutoff x, $\beta_i(x)$ is interior. Then, we must have $\Psi^i(x, \beta_i(x)) = 0$ and so we can use (5) to study the best-response function β. Since $\Psi_1^i(x, y) > 0$, the best response functions are downward-sloping: $\beta_i'(x) < 0$. This implies $-1 < \beta_i'(x) < 0$ for all $x \in [\underline{h}_j, \bar{h}_j]$. This implies the two best-response functions cannot cross more than once, so there can be only one cutoff equilibrium:

Now assume that at least one player has dominant strategy doves or hawks but not both. Then, again either there is a unique equilibrium which is interior or there is a unique equilibrium with is a corner equilibrium. Suppose either $0 < \bar{h}_i < c - d$ or $c - d > \underline{h}_i > 0$ but not both. Then, both players have dominant strategy hawks or doves and opportunistic types. In that case, if (11) holds, there is a unique equilibrium and this equilibrium is interior.

Figure 3: Equilibrium for Strategic Substitutes



That is, unlike in the case of strategic complements, the presence of both dominant strategy types is not necessary for an interior equilibrium.

The incomplete information game with strategic substitutes is a submodular game. In the two player case, we can invert one player's strategy set to make the game supermodular. Hence, when there is a unique equilibrium, we can again invoke the result that a supermodular game with a unique equilibrium can be solved by iterated deletion of (interim) dominated strategies.

The sufficient condition for uniqueness with strategic substitutes contains one inequality that is symmetric with strategic complements (upto two negative signs). But it also contains a second condition that is used to prove the non-existence of non-cut-off equilibria. The issue of non-cutoff equilibria does not arise in a complete information setting where there is a full equivalence between two player supermodular and submodular games.

As in the case of strategic complements, it is easy to characterize the unique equilibrium. If the unique equilibrium is interior, then player i chooses the cut-off point $h_i^* \in \left(\underline{h}_i, \bar{h}_i\right)$ which solves

$$h_i^* + (d-c)\left(1 - F(h_j^* - k_j | h_i^* - k_i)\right) = 0. \tag{12}$$

In other cases, there are corner solutions.

The uniqueness conditions for the two classes of games, (11) and (6), can be compared. If types are independent, then $F(s|t)$ is independent of its second argument, so (6) and (11) both reduce to the condition

$$F_1(s|t) < \left|\frac{1}{d-c}\right|$$

for all $s, t \in (\underline{\eta}, \bar{\eta})$. If in addition η_1 and η_2 are independently drawn from a uniform distribution with support $[\underline{\eta}, \bar{\eta}]$, then the sufficient condition for uniqueness of equilibrium is

$$\bar{\eta} - \underline{\eta} > |d-c| .$$

However, if types are affiliated, so that $F_2(s|t) < 0$, then the uniqueness conditions for strategic substitutes are more stringent than those for strategic complements, because $F_2(s|t)$ enters with a *negative* sign in (11).

While stag hunt captured the idea of Schelling's "reciprocal fear of surprise attack," chicken, a game with strategic substitutes captures a different logic of "escalating fear of conflict". Coordination types in chicken want to *mis*-coordinate with the opponent's action, particularly if he plays H. Coordination types with a low hostility level h are near indifferent between H and D if they are certain that the opponent plays D. But if there is positive probability that the opponent is a dominant strategy type, the "almost dominant strategy doves" strictly prefer to back off and play D. This in turn emboldens coordination types who are almost dominant strategy hawks to play H and the cycle continues. This escalation is more powerful if there is negative correlation between types and dovish coordination types with low h put high probability on hawkish coordination types with high h and vice-versa. But it is more natural to assume independence or positive correlation. In the latter case, the uniqueness condition for chicken is less likely to hold. We study independence and correlation in the context of the global games model in Section 5.

4.3 The uniform independent case

Suppose the idiosyncratic shocks η_1 and η_2 are independently drawn from a distribution which is uniform on $[\underline{\eta}, \bar{\eta}]$. The flatness condition requires "sufficient uncertainty": $\bar{\eta} - \underline{\eta} > c - d$. Then, we are assured that a unique equilibrium exists. Notice that the flatness condition implies that player 1 either must have dominant strategy hawks or dominant strategy doves with positive probability. We assume $c - d > \underline{\eta}$, otherwise we have a trivial equilibrium where both players play H whatever their type. We also assume $\bar{\eta} > 0$, otherwise there is a trivial equilibrium where player 1 always plays D and player 2 plays h iff $h_2 \geq 0$.

Let

$$\begin{aligned} h_1^* &= \frac{\left[\bar{\eta}\left(\bar{\eta} - \underline{\eta} - (c-d)\right) + k\left(\bar{\eta} - \underline{\eta}\right)\right](c-d)}{\left(\bar{\eta} - \underline{\eta}\right)^2 - (c-d)^2} > 0 \text{ and} \\ \tilde{h}_2 &= \frac{\left[\bar{\eta}\left(\bar{\eta} - \underline{\eta} - (c-d)\right) - k\right](c-d)}{\left(\bar{\eta} - \underline{\eta}\right)^2 - (c-d)^2} < c - d \end{aligned}$$

where these equations solve (12). Notice that as $k \geq 0$, $h_1^* \geq \tilde{h}_2$.

If $\tilde{h}_i \in [0, c-d]$ and $\tilde{h}_i \in [\underline{\eta} + k_i, \bar{\eta} + k_i]$, then the unique equilibrium is interior and has $h_i^* = \tilde{h}_i$. Therefore, for an interior equilibrium, we require

$$\begin{aligned} \frac{\bar{\eta}\left(\bar{\eta} - \underline{\eta} - (c-d)\right) + k\left(\bar{\eta} - \underline{\eta}\right)}{\left(\bar{\eta} - \underline{\eta}\right)^2 - (c-d)^2} &< 1 \text{ and} \\ \frac{\bar{\eta}\left(\bar{\eta} - \underline{\eta} - (c-d)\right) - k}{\left(\bar{\eta} - \underline{\eta}\right)^2 - (c-d)^2} &> 0. \end{aligned}$$

This implies

$$\begin{aligned} k &< \bar{\eta}\left(\bar{\eta} - \underline{\eta} - (c-d)\right) \text{ and} \\ k &< \frac{\left(c - d - \underline{\eta}\right)\left(\bar{\eta} - \underline{\eta} - (c-d)\right)}{\left(\bar{\eta} - \underline{\eta}\right)}. \end{aligned}$$

In other cases, there is a corner solution with either one or the other player always playing the same action.

5 Global games

Following Carlsson and van Damme [7], we assume the players' types are generated from an underlying parameter θ as follows. First, θ is drawn from a distribution H on $\Theta \equiv [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}$. The density h is strictly positive, continuously differentiable and bounded on Θ. Then, the idiosyncratic shock η_i is given by

$$\eta_i = \theta + \varepsilon E_i$$

where (E_1, E_2) is independent of θ and admits a continuous, symmetric density φ.[3] The support of E_i is $[-\frac{1}{2}, \frac{1}{2}]$. Player i knows his own draw η_i but not the opponent's draw η_j. Neither player can observe θ. We set $k_i = k_j = 0$ so $h_i = \eta_i$. Let $F^\varepsilon(\eta_j|\eta_i)$ and $f^\varepsilon(\eta_j|\eta_i)$ be the distribution and density of η_j conditional on player i's type η_i. Denote by φ^ε the joint density of $(\varepsilon E_i, \varepsilon E_j)$. Finally, ψ^ε is the density of $\varepsilon E_i - \varepsilon E_j$. Then, for $i, j = 1, 2, i \neq j$

$$f^\varepsilon(\eta_j|\eta_i) = \frac{\int h(\theta)\varphi^\varepsilon(\eta_i - \theta, \eta_j - \theta)d\theta}{\int\int h(\theta)\varphi^\varepsilon(\eta_i - \theta, \eta_j - \theta)d\eta_j d\theta}.$$

Lemma 5 *There is a constant $k(\eta_i)$ such that for all sufficiently small ε*

$$|F^\varepsilon(\eta_j|\eta_i) - \int_{y \leq \eta_j} \psi^\varepsilon(\eta_i - y)dy| \;\leq\; k(\eta_i)\varepsilon.$$

Proof. The proof follows the argument of Lemma 4.1 in Carlsson and van Damme [7] and is omitted. ■

5.1 Strategic Complements

We verify that (6), the sufficient condition for uniqueness with strategic complements is satisfied when ε is sufficiently small.

Notice that $F_1^\varepsilon(\eta_j|\eta_i) \geq 0$ for all $\eta_j, \eta_i \in [\underline{\eta}, \bar{\eta}]$. Also, when ε is small, $F_2^\varepsilon(\eta_j|\eta_i) \leq 0$ for all $\eta_j, \eta_i \in (\underline{\eta}, \eta_j)$ (see Van Zandt and Vives [20]).

Notice that

$$\int_{y' \leq \eta_j + \Delta} \psi^\varepsilon(\eta_i + \Delta - y')dy' = \int_{y \leq \eta_j} \psi^\varepsilon(\eta_i - y)dy$$

as ψ^ε depends only on the difference between E_1 and E_2. Therefore, from Lemma 5, for a given Δ, there is constant $k(\eta_i, \Delta)$ such that for all ε sufficiently small,

$$|F^\varepsilon(\eta_j|\eta_i) - F^\varepsilon(\eta_j + \Delta|\eta_i + \Delta)| \;\leq\; k(\eta_i, \Delta)\varepsilon$$

[3] We assume symmetry to make this approach consistent with our basic model.

or

$$\frac{|F^{\varepsilon}(\eta_j|\eta_i) - F^{\varepsilon}(\eta_j + \Delta|\eta_i + \Delta)|}{\Delta} \leq k'(\eta_i, \Delta)\varepsilon$$

where $k'(\eta_i, \Delta) = k(\eta_i, \Delta)/\Delta$. As this inequality must hold for small Δ, for all ε sufficiently small,

$$F_1^{\varepsilon}(\eta_j|\eta_i) + F_2^{\varepsilon}(\eta_j|\eta_i) \leq \varepsilon.$$

And hence the global games information structure satisfies our sufficient condition for uniqueness (6). The interested reader can refer to Carlsson and van Damme [7] and notice that this is a different approach to proving uniqueness.

5.2 Strategic Substitutes

We show that our sufficient condition identified in Theorem 4 is *not* satisfied by the global games information structure. Hence, in this case, we present an alternative theory of uniqueness generated by payoff uncertainty. To simplify, assume all distributions are uniform. If player i draws $\eta_i \in [\underline{\theta} + \varepsilon, \bar{\theta} - \varepsilon]$, then his posterior beliefs about θ are given by a uniform distribution on $[\eta_i - \varepsilon, \eta_i + \varepsilon]$. Player i's beliefs about η_j are given by a symmetric, triangular distribution around η_i with support $[\eta_i - 2\varepsilon, \eta_i + 2\varepsilon]$. If $s, t \in [\underline{\theta} + \varepsilon, \bar{\theta} - \varepsilon]$ then

$$F(t|s) = \begin{cases} 1 & \text{if} \quad t \geq s + 2\varepsilon \\ 1 - \frac{1}{2}\left(1 - \frac{t-s}{2\varepsilon}\right)^2 & \text{if} \quad s \leq t \leq s + 2\varepsilon \\ \frac{1}{2}\left(1 - \frac{s-t}{2\varepsilon}\right)^2 & \text{if} \quad s - 2\varepsilon \leq t \leq s \\ 0 & \text{if} \quad t \leq s - 2\varepsilon \end{cases}$$

which implies

$$F(t|s) + F(s|t) = 1. \tag{13}$$

We also compute

$$F_1(t|s) = \begin{cases} 0 & \text{if} \quad t > s + 2\varepsilon \\ \frac{1}{4\varepsilon^2}(s - t + 2\varepsilon) & \text{if} \quad s \leq t \leq s + 2\varepsilon \\ \frac{1}{4\varepsilon^2}(t - s + 2\varepsilon) & \text{if} \quad s - 2\varepsilon \leq t \leq s \\ 0 & \text{if} \quad t \leq s - 2\varepsilon \end{cases}$$

Notice that

$$F_1(t|s) = F_1(s|t). \tag{14}$$

Differentiating (13) with respect to s yields

$$F_2(t|s) + F_1(s|t) = 0.^4 \tag{15}$$

For $|s-t| < 2\varepsilon$ we have, using (14) and (15),

$$F_1(s|t) - F_2(s|t) = 2F_1(s|t) = \frac{1}{2\varepsilon^2}\left(|s-t| + 2\varepsilon\right)$$

which reaches a maximum $2/\varepsilon$ when $|s-t| = 2\varepsilon$. Also,

$$F_1(s|x) - F_2(x|t) = F_1(s|x) + F_1(t|x)$$

by (15), which also reaches a maximum $2/\varepsilon$. By Theorem 4, there is a unique BNE if $\varepsilon > 2(c-d)$. Thus, "large idiosyncratic uncertainty" guarantees uniqueness. In fact the game can be solved by iterated deletion of (interim) dominated strategies.

Proposition 6 *If $c > d$ then there is a unique BNE for any $\varepsilon > 2(c-d)$.*

When uniqueness obtains, it is a stark consequence of the dynamic triggered by "deterrence by fear." The chance of conflict with dominant strategy hawks triggers dovish behavior among the coordination types. When $\varepsilon > 2(c-d)$, type 0 of player i puts positive probability on player j being a dominant strategy hawk. This implies that type 0 and types just higher play D. Similarly, type $c-d$ of player i puts positive probability on player j being a dominant strategy dove. This implies type $c-d$ and types just below play H. This process of iterated deletion of dominated strategies identifies a unique equilibrium.

Our sufficient conditions for uniqueness imply ε has to be big enough. What happens if ε is small ? Then, if there is not much asymmetry between the two players, multiple equilibria exist. Specifically, if $k_2 = k > 0 = k_1$, then multiple equilibria exist for k small.

Proposition 7 *Suppose $c > d$, and $k_2 = k > 0 = k_1$. If $\varepsilon < (c-d)/2$, then multiple equilibria exist for $k > 0$ sufficiently small, specifically, whenever $k < 2\varepsilon\left(1 - \frac{2\varepsilon}{c-d}\right)$..*

[4] Combining (14) and (15) yields $F_1(t|s) + F_2(t|s) = 0$ so (6) certainly holds. Therefore, for the case of strategic complements, we have: If $c < d$ then there is a unique BNE for any $\varepsilon > 0$.

The proof is in the appendix.

The previous two propositions show that a unique equilibrium exists for sufficiently large ε, but multiple equilibria can exist for small ε, if k is also small. Fixing k, however, and taking $\varepsilon \to 0$ must produce a unique equilibrium, because this is the "global games" case. Indeed, if ε is small then the types are highly correlated. In this case, the process of iterated elimination of interim dominated strategies selects a unique outcome, where all opportunistic types of player 1 "fold" and play D, and all opportunistic types of player 2 play H. The process begins with the aggressive opportunistic types of player 1 backing off as they put high probability on player 2 being a dominant strategy hawk and the process continues. This is the reverse of the logic underlying Proposition 6. Finally, since (D, H) is the risk-dominant outcome for the complete information game with $\varepsilon = 0$ and $k > 0$, this result agrees with the conclusion of Carlsson and van Damme [7].

Proposition 8 *Suppose $c > d$, and $k_2 = k > 0 = k_1$. Moreover, suppose $k + \underline{\theta} + \varepsilon < 0$ and $\bar{\theta} - \varepsilon > c - d$ (so dominant strategy hawks and doves exist for each player). If ε is sufficiently small, specifically $\varepsilon < k/2$, then there is a unique BNE, where player 1 plays H iff $h_1 \geq c - d$ and player 2 plays H iff $h_2 \geq 0$.*

The proof is in the Appendix

To summarize, deterrence by fear leads to a unique equilibrium in two cases. The chance of conflict with dominant strategy hawks triggers dovish behavior among the opportunistic types. For this to occur, either types must be independent and the prior diffuse or it must be common knowledge that one player is inherently more aggressive than the other. This in turn persuades opportunistic types to play H, as they put positive probability of their opponent playing D, and the process continues.

6 Conclusion

We derive sufficient conditions for uniqueness of equilibrium as "small slope" conditions on appropriately defined best-response functions. They are related to quite standard conditions for uniqueness of equilibrium from the Industrial Organization literature. The connection can be explored further. Even when there are multiple equilibria, the equilibrium set is well-behaved in games with strategic complementarities such as some models of Bertrand competition with product differentiation. It may be possible to use techniques from the monotone comparative statics literature to study games with payoff uncertainty with multiple equilibria. This is an interesting topic for future research.

Finally, we note that while we can unify uniqueness conditions for games with independent types and highly correlated types as in global games, there are differences between the two models. In the global game model, the uncertainty serves as an equilibrium selection device for the underlying complete information model and the game with "small noise" is studied. The way that uncertainty is introduced into the model can lead to a quite different selection (see Weinstein and Yildiz [22]). In the model with independent types, the game actually studied is a game with incomplete information and "large noise". Adding complex higher order beliefs to this model does not affect the Bayesian equilibrium as there is only one such underlying equilibrium to begin and it is reached via iterated deleted of dominated strategies. In this sense the conflict game with independent types has stronger robustness propeorties that its global game equivalent.

7 Appendix

7.1 Proof of non-existence of non-cutoff equilibria

To complete the proofs of Theorem 3 (strategic complements) and Theorem 4 (strategic substitutes), we will show that, under the hypotheses of these theorems, any BNE must be a cutoff equilibrium. Since the proofs for strategic complements and substitutes are in part parallel, it is convenient to combine them. Recall that the hypotheses of either theorem imply that (4) holds.

It suffices to show that *one* player use a cutoff strategy, because (4) guarantees that the best response to a cutoff strategy must be a cutoff strategy. Notice that a constant strategy (always D or always H) is a special case of a cutoff strategy, so if either player uses a constant strategy we are done. From now on, assume neither player uses a constant strategy: some types play D and some types play H.

For $i \in \{1, 2\}$, define

$$x_i \equiv \inf \{h_i : \sigma_i(h_i) = H\} \tag{16}$$

and

$$y_i \equiv \sup \{h_i : \sigma_i(h_i) = D\} \tag{17}$$

By definition, $x_i \leq y_i$. Notice that if $x_i = y_i$ then player i uses a cutoff strategy. We will show that there is always some player i such that $x_i = y_i$.

Recall that $\delta_j(h_i)$ denotes the probability that player $j \neq i$ plays D, conditional on player i's type being h_i. By definition of x_i, player i's type x_i weakly prefers H, so (2) implies

$$x_i + (1 - \delta_j(x_i))(d - c) \geq 0 \tag{18}$$

Similarly, player i's type y_i weakly prefers D, so

$$y_i + (1 - \delta_j(y_i))(d - c) \leq 0 \tag{19}$$

By definition, $\sigma_j(h_j) = D$ for all $h_j < x_j$ and $\sigma_j(h_j) = H$ for all $h_j > y_j$. Therefore, for $j \neq i$,

$$F(x_j - k_j | x_i - k_i) \leq \delta_j(x_i) \leq F(y_j - k_j | x_i - k_i) \tag{20}$$

and

$$F(x_j - k_j | y_i - k_i) \leq \delta_j(y_i) \leq F(y_j - k_j | y_i - k_i). \tag{21}$$

From now on we consider the two cases separately.

Strategic complements. Without loss of generality, assume

$$y_1 - k_1 \geq y_2 - k_2. \tag{22}$$

We will show that $y_1 = x_1$ so player 1 uses a cutoff strategy..

Since $c < d$, (18) and the first inequality of (20) (with $i = 1$) imply

$$x_1 + (1 - F(x_2 - k_2 | x_1 - k_1))(d - c) \geq 0 \tag{23}$$

Similarly, (19) and the first inequality of (21) imply

$$y_1 + (1 - F(y_2 - k_2 | y_1 - k_1))(d - c) \leq 0 \tag{24}$$

Combining (23) and (24) yields

$$F(y_2 - k_2 | y_1 - k_1) - F(x_2 - k_2 | x_1 - k_1) \geq \frac{1}{d - c}(y_1 - x_1) \tag{25}$$

By (22), the inequality (25) implies

$$F(y_1 - k_1|y_1 - k_1) - F(x_1 - k_1|x_1 - k_1) \geq \frac{1}{d-c}(y_1 - x_1). \tag{26}$$

By the mean value theorem, there is $z \in (\underline{\eta}, \bar{\eta})$ such that

$$(F_1(z|z) + F_2(z|z))(y_1 - x_1) = F(y_1 - k_1|y_1 - k_1) - F(x_1 - k_1|x_1 - k_1)$$

and substituting this in (26) yields

$$(F_1(z|z) + F_2(z|z))(y_1 - x_1) \geq \frac{1}{d-c}(y_1 - x_1). \tag{27}$$

But we know that $y_1 \geq x_1$, and the hypothesis of Theorem 3 says that

$$F_1(z|z) + F_2(z|z) < \frac{1}{d-c}$$

Therefore, (27) implies $y_1 = x_1$. Therefore, the proof of Theorem 3 is complete.

Strategic substitutes. Assume without loss of generality that

$$x_1 - k_1 \leq x_2 - k_2 \tag{28}$$

Since $c > d$, (18) and the second inequality of (20) imply

$$x_i + (1 - F(y_j - k_j|x_i - k_i))(d - c) \geq 0 \tag{29}$$

By the same reasoning,

$$y_i + (1 - F(x_j - k_j|y_i - k_i))(d - c) \leq 0 \tag{30}$$

Set $i = 1$ in (29) and (30) and combine the two inequalities to get

$$F(y_2 - k_2|x_1 - k_1) - F(x_2 - k_2|y_1 - k_1) \geq \frac{1}{c-d}(y_1 - x_1). \tag{31}$$

Set $i = 1$ in (29) and $i = 2$ in (30) and combine the two inequalities to get

$$F(y_2 - k_2|x_1 - k_1) - F(x_1 - k_1|y_2 - k_2) \geq \frac{1}{c-d}(y_2 - x_1). \tag{32}$$

Set $i = 2$ in (29) and $i = 1$ in (30) and combine the two inequalities to get

$$F(y_1 - k_1|x_2 - k_2) - F(x_2 - k_2|y_1 - k_1) \geq \frac{1}{c-d}(y_1 - x_2). \qquad (33)$$

Now we need to consider several cases.

Case A: $y_1 - k_1 \geq y_2 - k_2$. Using this inequality and (28), the inequality (31) implies

$$F(y_1 - k_1|x_1 - k_1) - F(x_1 - k_1|y_1 - k_1) \geq \frac{1}{c-d}(y_1 - x_1) \qquad (34)$$

Applying the mean value theorem twice,

$$\begin{aligned}
& F(y_1 - k_1|x_1 - k_1) - F(x_1 - k_1|y_1 - k_1) \\
= \; & [F(y_1 - k_1|x_1 - k_1) - F(x_1 - k_1|x_1 - k_1)] + [F(x_1 - k_1|x_1 - k_1) - F(x_1 - k_1|y_1 - k_1)] \\
= \; & F_1(t|x_1 - k_1)(y_1 - x_1) + F_2(x_1 - k_1|s)(x_1 - y_1) \\
= \; & (F_1(t|x_1 - k_1) - F_2(x_1 - k_1|s))(y_1 - x_1)
\end{aligned}$$

for some $s, t \in \left(\underline{\eta}, \bar{\eta}\right)$. Substituting this in (34) yields

$$(F_1(t|x_1 - k_1) - F_2(x_1 - k_1|s))(y_1 - x_1) \geq \frac{1}{c-d}(y_1 - x_1)$$

But we know that $y_1 \geq x_1$, so the hypothesis of Theorem 4 implies $x_1 = y_1$.

Case B: $y_1 - k_1 < y_2 - k_2$. In this case, the inequality (33) implies

$$F(y_2 - k_2|x_2 - k_2) - F(x_2 - k_2|y_2 - k_2) \geq \frac{1}{c-d}(y_1 - x_2). \qquad (35)$$

Case B has three subcases.

Sub-case B1: $y_1 \geq y_2$. Here, (35) implies

$$F(y_2 - k_2|x_2 - k_2) - F(x_2 - k_2|y_2 - k_2) \geq \frac{1}{c-d}(y_2 - x_2).$$

This is symmetric with inequality (34), but for player 2 instead of player 1. Applying the mean-value theorem therefore yields $x_2 = y_2$.

Sub-case B2: $y_2 > y_1$ and $k_2 \geq k_1$. Here, from (32) we obtain

$$F(y_2 - k_1|x_1 - k_1) - F(x_1 - k_1|y_2 - k_1) \geq \frac{1}{c-d}(y_2 - x_1)$$

This is symmetric with inequality (34), but with y_2 replacing y_1. Applying the mean-value theorem therefore yields $x_1 = y_2$. But this contradicts $y_2 > y_1 \geq x_1$, so sub-case B2 is impossible.

Sub-case B3: $y_2 > y_1$ and $k_2 < k_1$. Here, from (33), we obtain

$$F(y_1 - k_1 | x_2 - k_1) - F(x_2 - k_1 | y_1 - k_1) \geq \frac{1}{c-d} (y_1 - x_2)$$

This is symmetric with inequality (34), but with x_2 replacing x_1. Applying the mean-value theorem therefore yields $x_2 = y_1$. Substituting this into (33), we obtain

$$F(y_1 - k_1 | y_1 - k_2) - F(y_1 - k_2 | y_1 - k_1) \geq 0. \tag{36}$$

However, recall that $y > x$ implies $F(y|x) - F(x|y) > 0$. Since $y_1 - k_2 > y_1 - k_1$, the inequality (36) cannot hold, so sub-case B3 is impossible.

Thus, in both case A and case B, some player uses a cutoff strategy. Therefore, the proof of Theorem 4 is complete.

7.2 Proof of Proposition 7

Suppose $k < 2\varepsilon \left(1 - \frac{2\varepsilon}{c-d}\right)$. Notice this implies $2\varepsilon < c - d$ to guarantee that $k > 0$. Here types are not highly correlated, and the process of iterated elimination of dominated strategies cannot achieve anything after the first "round". Consider type $h_1 = c - d$. In the second "round", he cannot rule out the possibility that player 2 will choose D when $h_2 < c - d$. Moreover, the event that $h_2 = \eta_2 + k < c - d$ has positive probability when $\eta_1 = h_2 = c - d$ and $k < 2\varepsilon$. Therefore, we cannot eliminate H for type $h_1 = c - d$. Player 1's type $h_1 = 0$ cannot eliminate H, because H has not been eliminated for $h_2 > 0$. Since neither of the "boundary" opportunistic types can eliminate H, no opportunistic type at all can eliminate H. Clearly, they cannot eliminate D either. Thus, no opportunistic type of player 1 can eliminate any action in round 2. A similar argument applies to player 2.

Let $h^* = \frac{(c-d)(2\varepsilon - k)^2}{8\varepsilon^2}$. Notice that

$$h^* + k - 2\varepsilon = \frac{(c-d)(2\varepsilon - k)^2}{8\varepsilon^2} - (2\varepsilon - k) = \left(\frac{(c-d)(2\varepsilon - k)}{8\varepsilon^2} - 1\right)(2\varepsilon - k) > 0$$

as long as

$$\frac{(c-d)(2\varepsilon - k)}{8\varepsilon^2} > 1$$

and

$$h^* + k + 2\varepsilon = \frac{(c-d)(2\varepsilon - k)^2}{8\varepsilon^2} + k + 2\varepsilon < c - d$$

as long as

$$\frac{(c-d)(2\varepsilon - k)}{8\varepsilon^2} < \frac{c - d - (k + 2\varepsilon)}{2\varepsilon - k}$$

Players' strategies are as follows: player 1 plays D iff $h_1 \leq h^*$; player 2 plays D iff $h_2 \leq 0$ or $h_2 \in [h^*, c-d]$.

Consider player 1 first. For player 1 of type h^*, the probability that player 2 plays H is $F(h^*|h^* - k) = \frac{(2\varepsilon - k)^2}{8\varepsilon^2}$ and he is indifferent between H and D. Higher types are more aggressive and assess a lower probability that player 2 plays H. These types strictly prefer to play H and, by a symmetric argument, lower types prefer to play D.

We must also show player 2's strategy is at a best-response. Assume $k < 2\varepsilon\left(1 - \frac{2\varepsilon}{c-d}\right)$. For player 2, if he is a dominant strategy type, the specified strategy is clearly optimal. For $h_2 \in [h^* + k - 2\varepsilon, h^*]$, $\Pr\{h_1 < h^*|h_2\} = 1 - \frac{1}{8\varepsilon^2}\left((h_2 - k + 2\varepsilon) - h^*\right)^2 = \delta_1(h_2)$. Substituting this into (39), the net gain from playing H rather than D becomes

$$h_2 + \frac{(d-c)}{8\varepsilon^2}\left(h_2 - h^* - k + 2\varepsilon\right)^2. \tag{37}$$

This is quadratic in h_2 and equals zero when $h_2 = h^*$. It reaches a maximum at

$$\hat{h} = h^* + k - 2\varepsilon + \frac{4\varepsilon^2}{c-d}$$

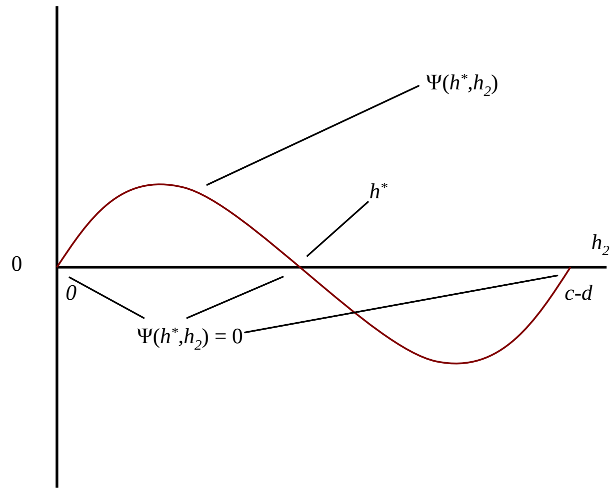
which is interior to the interval $[h^* + k - 2\varepsilon, h^*]$ as long as $k < 2\varepsilon\left(1 - \frac{2\varepsilon}{c-d}\right)$. In fact, (37) is clearly strictly positive for $h_2 \in [h^* + k - 2\varepsilon, h^*)$. For $h_2 \in [0, h^* + k - 2\varepsilon]$, player 2 knows his opponent plays D and then it is optimal to play H as (39) is equal to $h_2 \geq 0$. There is a similar argument for $h_2 \in (h^*, c-d]$ and so the entire $\Psi^2(h^*, h_2)$ picture is:

There is another equilibrium with the roles of players 1 and 2 reversed.[5] There is also an equilibrium where player 2 plays H when $h_2 \geq 0$ and player 1 plays H when $h_1 \geq c - d$.

[5] Let $h^* = (c-d)\left(1 - \frac{(2\varepsilon-k)^2}{8\varepsilon^2}\right)$. Notice that

$$h^* - k - 2\varepsilon = (c-d)\left(1 - \frac{(2\varepsilon - k)^2}{8\varepsilon^2}\right) - 2\varepsilon - k > 0$$

Figure 4: $\Psi^2(h^*,h_2)$



Now suppose $2\varepsilon > c - d$. Then, the above argument fails as $\hat{h} > h^*$.

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as long as

$$(c-d)\left(1 - \frac{(2\varepsilon - k)^2}{8\varepsilon^2}\right) > k + 2\varepsilon$$

and

$$h^* - k + 2\varepsilon = (c-d)\left(1 - \frac{(2\varepsilon - k)^2}{8\varepsilon^2}\right) + 2\varepsilon - k < c - d$$

as long as

$$(c-d)\left(1 - \frac{(2\varepsilon - k)^2}{8\varepsilon^2}\right) < c - d - (2\varepsilon - k)$$

Players' strategies are as follows: player 2 plays D iff $h_2 \leq h^*$; player 1 plays D iff $h_1 \leq 0$ or $h_1 \in [h^*, c-d]$.

Consider player 2 first. For player 2 of type h^*, the probability that player 1 plays H is $F(h^*|h^* - k) = 1 - \frac{(2\varepsilon - k)^2}{8\varepsilon^2}$ and he is indifferent between H and D. Higher types are more aggressive and assess a lower probability that player 1 plays H. These types strictly prefer to play H and, by a symmetric argument, lower types prefer to play D.

We must also show player 1's strategy is at a best-response. Assume $k < 2\varepsilon\left(1 - \frac{2\varepsilon}{c-d}\right)$. For player 1, if he is a dominant strategy type, the specified strategy is clearly optimal. For $h_1 \in [h^*, h^* + 2\varepsilon - k]$, $\Pr\{h_2 < h^*|h_1\} = \frac{1}{8\varepsilon^2}\left(h^* - k - (h_1 - 2\varepsilon)\right)^2 = \delta_2(h_1)$. Substituting

Indeed, Proposition 6 shows that a unique equilibrium exists in this case. Large ε approximates independence and is the polar opposite of the "global games" conclusion.

7.3 Proof of Proposition 8

We can use the triangular distribution to determine the beliefs of opportunistic types.

Conditional on θ, player 1's type $h_1 = \eta_1$ is uniformly distributed on $[\theta - \varepsilon, \theta + \varepsilon]$, while player 2's type $h_2 = k + \eta_2$ is uniformly distributed on $[k+\theta-\varepsilon, k+\theta+\varepsilon]$. A strategy $\sigma_i : [\underline{h}, \bar{h}] \to \{H, D\}$ for player i is a measurable function which specifies a choice $\sigma_i(h_i) \in \{H, D\}$ for each type h_i. If player i's type h_i thinks player j will choose D with probability $\delta_j(h_i)$, then his net gain from choosing H instead of D is

$$h_i + (d - c)(1 - \delta_j(h_i)). \tag{39}$$

Remark 9 *If $\eta_1 < -k - 2\varepsilon$, then player 1 knows that $h_2 = k + \eta_2 < 0$, so player 2 must be a dominant strategy dove. If $\eta_1 > c - d - k + 2\varepsilon$, then player 1 knows that $h_2 = k + \eta_2 > c - d$, so player 2 must be a dominant strategy hawk. If $\eta_2 < -2\varepsilon$, then player 2 knows that $h_1 = \eta_1 < 0$, so player 1 must be a dominant strategy dove. If $\eta_2 > c - d + 2\varepsilon$, then player 2 knows that $h_1 = \eta_1 > c - d$, so player 1 must be a dominant strategy hawk.*

Consider the process of eliminating (interim) dominated strategies. In the first "round" of elimination, D is eliminated for dominant strategy hawks

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this into (39), the net gain from playing H rather than D becomes

$$h_1 + (d - c)\left(1 - \frac{1}{8\varepsilon^2}\left(h^* - k - (h_1 - 2\varepsilon)\right)^2\right). \tag{38}$$

This is quadratic in h_1 and equals zero when $h_1 = h^*$. It reaches a minimum at

$$\hat{h} = h^* - k + 2\varepsilon - \frac{4\varepsilon^2}{c - d}$$

which is interior to the interval $[h^*, h^* + 2\varepsilon - k]$ as long as $k < 2\varepsilon\left(1 - \frac{2\varepsilon}{c-d}\right)$. In fact, (38) is clearly strictly negative for $h_1 \in (h^*, h^* + 2\varepsilon - k]$. For $h_1 \in (h^* + 2\varepsilon - k, c - d]$, player 1 knows his opponent plays H and then it is optimal to play D as (39) is negative. There is a similar argument for $h_1 \in [0, h^*)$.

($h_i \geq c-d$) and H for dominant strategy doves. Now consider the second round.

Suppose $\varepsilon < k/2$. This is the case of highly correlated types. When $c-d-k+2\varepsilon < \eta_1 < c-d$, player 1 knows that player 2 is a dominant strategy hawk (by remark 9). Hence, H can be eliminated for player 1. Indeed, even if η_1 is slightly below $c-d-k+2\varepsilon$, H can be eliminated, because player 2 is highly likely to be a dominant strategy hawk. Let η_1' be the largest η_1 such that H cannot be eliminated for player 1's type η_1 in round 2. Notice that $\eta_1' < c-d-k+2\varepsilon$. Now if $h_2 = \eta_2 + k$ is slightly below $c-d$, then player 2 knows that player 1 has a positive probability of having a type between η_1' and $c-d$. Such types of player 1 had H removed in round 2 of the elimination of interim dominated strategies. Therefore, in round 3, D must be eliminated for types of player 2 slightly below $c-d$. Let η_2' be the largest η_2 such that D cannot be eliminated for player 2 in round 3. In round 4, player 1's types slightly below η_1' will be able to remove D, etc.

We claim that this process must eventually eliminate H for all $h_1 \in (0, c-d)$, and D for all $h_2 \in (0, c-d)$. If this were not true, then the process cannot proceed below some $h_1^* > 0$ and $h_2^* > 0$. Now, $h_2^* > h_1^* + k - 2\varepsilon$, otherwise type h_1^* knows that $h_2 \geq h_2^*$, and all such types have eliminated D, but then H must be eliminated for types slightly below h_1^*.

Consider player 2's type h_2^*, with private component $\eta_2^* = h_2^* - k$. He knows that $h_1 = \eta_1 \leq h_2^* - k + 2\varepsilon < c-d-k+2\varepsilon \leq c-d$. Now H has been eliminated for all $h_1 \in (h_1^*, c-d)$, and according to type h_2^*, the probability that player 1's type lies in this interval is $1 - F(h_1^*|h_2^* - k)$. Therefore, if D cannot be eliminated for type h_2^*, it must be the case that type h_2^* weakly prefers D when the opponent uses H with probability at most $F(h_1^*|h_2^* - k)$. This implies

$$h_2^* + (d-c)F(h_1^*|h_2^* - k) \leq 0$$

By a similar argument, if H cannot be eliminated for type h_1^*, then type h_1^* must prefer H when the opponent uses H with probability at least $1 - F(h_2^* - k|h_1^*)$. That is,

$$h_1^* + (d-c)\left(1 - F(h_2^* - k|h_1^*)\right) \geq 0$$

Subtracting the first inequality from the second yields

$$h_1^* - h_2^* \geq (c-d)\left(1 - F(h_2^* - k|h_1^*) - F(h_1^*|h_2^* - k)\right) = 0$$

where the last equality uses 13. However, this contradicts $h_2^* > h_1^* + k - 2\varepsilon$.

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