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Undescribable Events

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We develop a model of undescribable events. Examples of events that are well understood by economic agents but are prohibitively difficult to describe in advance abound in real life. This notion has also pervaded a substantial amount of economic literature. Undescribable events in our model are understood by economic agents—their consequences and probabilities are known—but are such that every finite description of such events necessarily leaves out relevant features that have a non-negligible impact on the parties' expected utilities. We illustrate our results using a simple coinsurance problem as a backdrop. When the only uncertainty faced by the two agents is an undescribable event the optimal finite coinsurance contract is no contract at all.

1. INTRODUCTION

1.1. *Motivation*

In the well-known case of *Jacobellis v. Ohio*,[1] Supreme Court Justice Potter Stewart argued that only "hard-core" pornography could be banned, but conceded

> I shall not today attempt to further define the kind of materials I understand to be embraced within the shorthand definition; and perhaps I could never succeed in doing so.

Stewart had said:

> But I know it when I see it. (Woodward and Armstrong, 1979, p. 94)

The describability problem faced by Justice Potter Stewart exemplifies well the type of circumstances we focus on in this paper: we seldom observe exhaustive *ex ante* rules, even though formulating such rules carries potentially enormous benefits.[2]

Consider the following familiar example. Academic institutions routinely decide whether to grant tenure to junior faculty members. An *ex ante* contingent tenure rule would spell out in advance a detailed set of conditions under which tenure would be granted as a function of a candidate's performance. Formulating such rule would entail considerable gains, such as reducing

1. *Jacobellis v. Ohio*, 378 U.S. 184 (1964).
2. These comprise not only the savings on litigation and court costs, but include also the benefits from the disincentives to "manipulative" behaviour, which non-exhaustive definitions may generate. The analysis of the strategic impact of the lack of exhaustive *ex ante* rules, while clearly important, is beyond the scope of our analysis here.

uncertainty, cutting down on the effort and resources spent in committee work and reducing the potential for allegations of inequity, bias, etc. Despite this, to our knowledge no research-oriented department in the U.S. has set forth such rule. Instead, decisions are usually made using a lengthy case-by-case process that often suffers from the drawbacks mentioned above. The thesis of this paper is that complete contingent contracting on something like the tenure decision is difficult because the underlying event, "the candidate has a tenurable vita", is inherently hard to describe *ex ante* in its full details.

What we call "undescribability" is a pervasive force in economic interactions. This is often noted in the context of contracting (although the quote form Justice Stewart illustrates that the issue extends well beyond contracting narrowly construed). In their seminal paper, Grossman and Hart note

> A basic assumption of the model is that the production decisions are sufficiently complex that they cannot be specified completely in an initial contract between the firms. We have in mind a situation in which it is prohibitively difficult to think about and describe unambiguously in advance how all the potentially relevant aspects of the production allocation should be chosen as a function of the many states of the world. (Grossman and Hart, 1986, p. 696)

This paper introduces a model that admits complex events. What would such events look like? In the tenure example, it is easy to write simple, clear-cut rules like "grant tenure if and only if the candidate publishes x or more papers in journal y". The problem is that such a rule is too coarse to capture the subtle ways in which membership in the event "the candidate has a tenurable vita" is determined as a function of observable characteristics of a candidate's record. The probability that the candidate will get tenure may even be 1 if enough papers are published. However, for some values of x and y in the sentence above the probability of promotion will be neither 0 nor 1.[3]

In our model *all ex ante* descriptions are feasible, provided only they are *finite* in a well-defined sense. Roughly speaking, an event is complex if any feasible *ex ante* description will leave a "positive measure" of exceptions. In a benchmark extreme we consider below, these exceptions are so pervasive that no contract is written in equilibrium. In intermediate cases, the best contract the parties can come up with may be of value, but still falls short of fully delineating the boundaries of the event to be contracted on.

Our framework can be viewed as a limit of large finite environments with slight imperfections (*e.g.* small contracting cost).[4] An advantage of the abstract limit model we consider is that complexity takes the very sharp form of a discontinuity between the *ex ante* problem of finding a finite description of the relevant events and the *ex post* evidence available to the decision-makers. While Justice Stewart conceded that the problem of formulating an *ex ante* definition of pornography is difficult, he was equally emphatic in asserting that when facing a specific event, "I know it when I see it". In the tenure example, this amounts to saying that *ex post*, when facing a specific case, a tenure committee "knows" a tenurable vitae "when they see it".[5] In this paper we develop a model in which the contracting parties find it impossible to convey their will to the enforcement agent (the court) by means of a finite description (contract) even though the event they would like to condition on is common knowledge among them.

3. We return to this example extensively in Section 8.2 below.

4. We do not report a formal version of this claim here in the interest of brevity.

5. For the time being, we ignore the issue of a possible *ex post* implementation mechanism (Moore and Repullo, 1988; Maskin and Tirole, 1999). We revisit this question in Section 9 below when we discuss some related literature.

1.2. *Desiderata*

We proceed with a list of critical requirements that we seek for our model of undescribable events.

1. *Expected utility.* We want a model in which the consequences and probabilities of the relevant events are understood by the parties, and hence all appropriate expected utility calculations can be carried out.
2. *Language based.* We want to take seriously the notion that we can distinguish between physical states and their description in *ex ante* agreements.[6]

   To capture this requirement, we work with a model in which physical states of nature can be described by means of a *language* in which a countable infinity of elementary statements are possible. Each elementary statement represents a particular feature that can be either present or not in a given state of nature (the sky can be either "blue" or "not blue").

   So, with little loss of generality, we take each physical state of nature $s \in \mathcal{S}$ to be fully described by an infinite list of elementary statements $\{s^1, \dots, s^i, \dots\}$ that determine which features are present in the state. Each feature s^i can either be present ($s^i = 1$) or not ($s^i = 0$) in each state so that $\mathcal{S}$ is a subset of $\mathcal{C} = \{0, 1\}^{\mathbb{N}}$, the set of all infinite sequences of 0's and 1's.[7]
3. *Rich language.* We model events that are undescribable because they are too complex and *not* because the contracting parties are endowed with a language that is simply too coarse relative to the environment they face. Since we want to rule out coarse languages, as a minimal requirement we will insist that the parties can write *ex ante* contracts that vary across any two states s' and s''.
4. *Finitely describable events.* The set of statements that can be included in an *ex ante* contract must embody the notion that there are in fact some events that are "[...] prohibitively difficult to [...] describe unambiguously in advance" (Grossman and Hart, 1986, p. 696). Given that we require that our model be language-based in the sense above, there is a completely natural way to model this notion. We will assume that only *finite* statements about the constituent features of a set of states can be included in the contract that the parties draw up.

1.3. *Modelling choices*

The complexity-based approach we advocate markedly departs from alternative plausible explanations based on irrationality, contract-writing costs, and non-verifiability that have been explored in earlier works. An obvious question is what a complexity-based explanation adds to these other approaches. Although we narrowly focus on the role of complexity to the exclusion of other considerations, this should be understood as an expository device to make our point as sharply as possible. A satisfactory explanation of undescribability is likely to involve a combination of factors. We briefly discuss below some of these factors.

6. Of course, this does not preclude, as will be the case in our model below that a "full description" of a state of nature will identify the actual state uniquely.

7. We limit the set of elementary statements to be at most countably infinite, in keeping with the view that in any logical endeavour a "statement" must be a finite string of symbols drawn from an alphabet that is itself at most countably infinite. Of course, depending on the cardinality of $\mathcal{S}$ a finite set of elementary statements could suffice to pin down a state uniquely. In this case $\mathcal{S}$ would have to be a finite set. The actual assumption embodied in our statement above is that a countable infinity of elementary statements is, in fact, always sufficient to uniquely identify a state s. This implies that the cardinality of $\mathcal{S}$ is *at most* $2^{\aleph_0}$.

This is also an appropriate point to observe that our "ambient space" $\mathcal{C}$ can be thought of as the Cantor set. To think of our set-up in this way lends it considerable more generality and may equip some readers with a useful way to picture the structure of the model. For reasons of space we do not pursue this interpretation any further in our analysis below. We are, nevertheless, grateful to an anonymous referee for raising this point.

1.4. *Irrationality*

It is easy to generate undescribable events by appealing to some form of irrationality. For instance, one may assume that the agents do not understand the model or that they are not sophisticated enough to incorporate probabilities in their decision processes. This is not what we do here. Agents in our model are rational: they understand the consequences and probabilities associated with a particular event; they can evaluate expected pay-offs from their actions, including the contracts they sign. The main restriction we impose on agents' abilities is that they are limited to *ex ante* contracts that are finite. We do not view this as a form of irrationality, but rather as a rational response to a formidably complex environment.

1.5. *Contracting costs*

One way to generate undescribable events is to introduce explicitly costs of writing finer and finer contracts. At an intuitive level, this explanation appears quite compelling.[8] Writing costs seem to be an undeniable feature of real world contracting problems. However, a critical trade-off arises in formally modelling this idea. Clearly, any sufficiently large cost function delivers *ex ante* contracts that are necessarily coarser than the first best (in the absence of such costs). But the *form* of undescribability (*i.e.* which events are describable and which are not) will depend on the assumed form of the cost function, an exercise which, at least to some degree, is necessarily arbitrary. The other side of the trade-off, of course, is that if one is able to trust the cost function that is postulated, a tighter characterization of equilibrium contracts will in general be available.

Using exclusively the restriction that contracts be finite has a further advantage, which manifests itself in two related ways.

First, unlike the results that can be obtained associating a higher cost to "longer sentences", our results below are immune to changes in the language that, for instance, recode two elementary statements into a single one. A finite statement in one language will correspond to a finite statement in the new one and vice versa. This immunity to recoding is relevant in a world in which languages evolve to capture more efficiently concepts that may once have been considered complex or difficult.

Second, one may want to consider the possibility that an undescribable event can itself become an elementary statement in the language; perhaps as a result of evolutionary pressure, or un-modelled "intuition". Although, depending on the context, this may or may not be plausible, our results are also immune to this phenomenon. Once the undescribable event is added to the language as an elementary statement, a new language is in place. If we now restrict contracts to finite sentences in the new language, a new set of undescribable events will arise.

1.6. *Observable but not verifiable events*

The contracting literature has identified another type of circumstance in which failure to condition on some events may arise. These are events that are "observable but not verifiable".[9] In this framework, whether the relevant event occurs is *observed* by (and is common knowledge among) the contracting parties. The problem is that whether the event has occurred cannot be observed by a third party. In particular, it cannot be *verified* by any third party that is charged with enforcing the terms of the contract (*e.g.* a court).

8. This approach was first proposed by Dye (1985) and was more recently pursued by Anderlini and Felli (1999) and Battigalli and Maggi (2002).

9. An exhaustive list of references here would be enormous and hence out of place. See, for instance, Holmström (1982) in which to our knowledge the term was first used in its current sense, the seminal paper by Hart and Moore (1988), and the survey by Tirole (1999).

Our approach is consistent with, and in fact complements, the observable but not verifiable story. In order for complete *ex ante* contracting to take place, two key ingredients are necessary. The parties need to describe at an *ex ante* stage their will to the court with full precision, and the court needs to be able to verify *ex post* in which category specified by the contract the actual state of the world falls. The observable but not verifiable approach takes away the court's ability to verify *ex post* what really took place. In this paper, we model the difficulties (impossibility) that the contracting parties face in describing their will to the court, leaving intact its ability to verify the realized state of the world *ex post*.

In a fully specified model of what courts do, it surely would have to be the case that their information structure is (at least to some extent) endogenous. If the ability to verify finer and finer events yields large potential gains from trade, then the appropriate resources will be invested to endow the court with the ability to do so. The model we develop here tells us that, even in the limit case in which the court can verify *all* events, the possibility is still open that the parties will lack the ability to describe them fully in their contractual agreement.

2. OVERVIEW

The plan of the rest of the paper is as follows. In Section 3 we set up the coinsurance problem that we use as a backdrop and derive the benchmark efficient allocation that the parties can achieve in the absence of any constraint. We then define the state space and the associated probability measure in Section 4. In Section 5 we proceed to give a formal definition of the notion of a finite contract. In Section 6 we piece together all these elements and proceed to evaluate the parties' expected utilities associated with any finite contract. Section 7 presents our first main result: the existence of undescribable events in our model. In Section 8 we return to our basic coinsurance problem and characterize the effects of undescribable events in this set-up. We show that when the only uncertainty that the parties face is an undescribable event the optimal finite contract is to specify no transfers at all: the no-contract outcome obtains. In Section 9 we return to some of the related papers we have mentioned above and expand our discussion of related literature. Section 10 outlines some extensions of our model and concludes the paper. All proofs are in the Appendix.[10]

3. THE CONTRACTING PROBLEM

For the sake of concreteness, throughout the paper, we work using a standard coinsurance problem as a backdrop. Two risk-averse agents, labelled $i = 1, 2$, face a risk-sharing problem. The uncertainty in the environment is captured by the realization of a state of nature, denoted by s; the set of all possible states of nature is denoted by $\mathcal{S}$. The preferences of agent i are represented by the state-dependent utility function $U_i : \mathbb{R} \times \mathcal{S} \to \mathbb{R}$. The agents' utilities depend on s according to whether s falls in a subset $\mathcal{Z}$ of the state space $\mathcal{S}$.

The two agents can agree to a state-contingent monetary transfer $t \in \mathbb{R}$, which by convention represents a payment from 2 to 1. Let $V : \mathbb{R} \to \mathbb{R}$ be a twice differentiable, strictly increasing, and concave function. We write the utility of $i = 1, 2$ in state s, if the transfer is t, as

$$U_1(t, s) = \begin{cases} V(1+t) & \text{if } s \in \mathcal{Z} \\ V(t) & \text{if } s \in \overline{\mathcal{Z}} \end{cases} \qquad U_2(t, s) = \begin{cases} V(-t) & \text{if } s \in \mathcal{Z} \\ V(1-t) & \text{if } s \in \overline{\mathcal{Z}} \end{cases} \tag{1}$$

where $\overline{\mathcal{Z}}$ denotes the complement of $\mathcal{Z}$ in $\mathcal{S}$.

10. In the numbering of equations, definitions, remarks, and so on, a prefix of "A" indicates that the relevant item is to be found in the Appendix.

Ex ante, 1 makes a take-it-or-leave-it offer of a contract $t\colon \mathcal{S} \to \mathbb{R}$ to 2, where $t(s)$ is the monetary transfer from 2 to 1 if state s is realized. Of course, 1's take-it-or-leave-it offer to 2 will have to satisfy a participation constraint for 2 which will be specified shortly.

The coinsurance problem we have just described is completely standard. Since in (1) we have specified the agents utilities so that complete insurance is in fact feasible, in the absence of any additional restrictions, the optimal contract t^* will involve only two levels of transfers $t_{\mathcal{Z}}$ and $t_{\overline{\mathcal{Z}}}$ with

$$t^*(s) = \begin{cases} t_{\mathcal{Z}} & \text{if } s \in \mathcal{Z} \\ t_{\overline{\mathcal{Z}}} & \text{if } s \in \overline{\mathcal{Z}} \end{cases} \tag{2}$$

and $1 + t_{\mathcal{Z}} = t_{\overline{\mathcal{Z}}}$ so that $\forall s \in \mathcal{S}$

$$U_1(t(s), s) = V(1 + t_{\mathcal{Z}}) = V(t_{\overline{\mathcal{Z}}}) \quad \text{and} \quad U_2(t(s), s) = V(-t_{\mathcal{Z}}) = V(1 - t_{\overline{\mathcal{Z}}}). \tag{3}$$

Agent 2's participation constraint can be easily specified if we define the probability $p = \Pr\{s \in \mathcal{Z}\}$ that s falls in $\mathcal{Z}$. In the absence of any agreed transfers 2's expected utility is $pV(0) + (1-p)V(1)$. Since 2 is the recipient of a take-it-or-leave-it offer, his participation constraint will bind. Therefore, in addition to (3) the optimal contract t^* is characterized by

$$pV(-t_{\mathcal{Z}}) + (1-p)V(1 - t_{\overline{\mathcal{Z}}}) = pV(0) + (1-p)V(1). \tag{4}$$

Clearly, equations (3) and (4) uniquely pin down the values of $t_{\mathcal{Z}}$ and $t_{\overline{\mathcal{Z}}}$, so that the characterization of the solution to our coinsurance problem in the standard case is complete.

Starting with Hart and Moore (1988) a class of models that fall within the following broad sketch has become somewhat canonical in the incomplete contracting literature.[11] Two contracting parties, a buyer and a seller, have the opportunity to undertake an *ex ante* unobservable relationship-specific investment that affects the cost and/or value of the object (a "widget") of the potential exchange. Subsequently, the cost and value of the widget are realized, typically as a function of the realization of a state of nature as well as of the levels of relationship-specific investment. The presence of non-contractible variables in this set-up then gives rise to a hold-up problem, which in turn determines inefficient levels of *ex ante* investments. In particular the *ex ante* investments, the actual cost and value of the widget and the state of nature cannot be directly contracted on, even though it would be advantageous in principle to write an *ex ante* contract that conditions the sale price of the widget (and possibly whether the exchange is to take place or not) on these variables.

Our results below could be applied, virtually unchanged, to yield a model of the type we have just outlined in which one or more of the relevant variables cannot be profitably included in an *ex ante* contract because the relevant events are too complex. We use the coinsurance problem above as a backdrop purely for the sake of its simplicity.

4. STATES AND PROBABILITIES

4.1. *The state space*

We begin with a formal definition of our state space.

Definition 1 (*State Space*). The state space $\mathcal{S}$ is a countably infinite set $\{s_1, s_2, \ldots s_n, \ldots\}$. Each s_n is in turn an infinite sequence of the type $\{s_n^1, \ldots, s_n^i, \ldots\}$ with $s_n^i \in \{0, 1\}$ for every i and n.

11. What follows is not meant to be a summary description of the actual model analysed in Hart and Moore (1988), but merely a description of the main ingredients common to many contributions to this area of the literature. We also refer the reader to our discussion of related literature in Section 9 below.

Some comments are in order. As we mentioned above, each digit s_n^i of a given state s_n is interpreted as a feature of the state that can be either true (say $s_n^i = 1$) or false ($s_n^i = 0$).[12]

These features should be interpreted as elementary in a well-defined sense. They should not include sub-statements that preclude a feature from being true or false in conjunction with the truth or falsity of other elementary features. For instance, suppose that the physical possibilities limit the amount of rainfall (say in a 24-hour period) to be $r \in \{0, 1, 2, 3\}$ inches. The description of the state embodied in its features should specify the amount of rainfall. If we interpret, say feature 1 as "it rained at least 1 inch" and feature 2 as "it rained at least 2 inches" it is clear that any state in which feature 1 is false and feature 2 is true is not well defined. The elementary features, say features 1 and 2, describing rainfall should instead be interpreted as follows. Feature 1 is "the first digit of the binary expansion of the amount of rainfall in inches equals 1" and feature 2 is "the second digit of the binary expansion of the amount of rainfall in inches equals 1" so that all four combinations of true and false of the two features are logically consistent in principle.

The rainfall example takes us directly to a second concern.[13] Clearly, the cardinality of the set $\mathcal{C}$ of all possible sequences of the type $\{s_n^1, \ldots, s_n^i, \ldots\}$ is that of the continuum, $2^{\aleph_0}$. However, Definition 1 requires $\mathcal{S}$ to be a countable set.[14] A legitimate question to ask is then whether this imposes a hidden structure on the relationship between features. Some combinations of elementary features do not lead to a well-defined state in $\mathcal{S}$; some sentences consisting of elementary features are ruled out of the state space $\mathcal{S}$.

Recall, however, that our emphasis here is on what can be described by means of any *finite* sentence in the language provided by the elementary features. This is what finite contracts are limited to do. The set of *finite* sentences in our language is easily seen to be countable. As will be apparent below our results hold for a state space $\mathcal{S}$ in which *no* finite sentences are ruled out.[15] It is possible to go from the set $\mathcal{C}$ of all possible (finite and infinite) sentences in the language to our countable state space $\mathcal{S}$, with only infinite sentences being ruled out. Finally, nothing in our model forbids "correlation" between features in the structure of $\mathcal{S}$. This correlation could be sufficiently strong so that some finite sentences are in fact ruled out, but it is not necessary in any way.

4.2. *Probabilities*

The probability measure μ that we place over $\mathcal{S}$ is unconventional. We define the *density* of a set of states.

Definition 2 (*Density*). Given any $Q \subseteq \mathcal{S}$, let χ_Q denote the characteristic function of Q so that $\chi_Q(s_n) = 1$ if $s_n \in Q$ and $\chi_Q(s_n) = 0$ if $s_n \notin Q$. We define the density of Q to be

$$\mu(Q) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \chi_Q(s_n) \tag{5}$$

when the limit in (5) exists. The density is otherwise left undefined. We denote by $\mathcal{D}$ the collection of subsets of $\mathcal{S}$ that have a well-defined density.

12. It is easy to check that our analysis generalizes to a model in which each elementary feature can take a finite number of values instead of just two. The flavour of what we say also generalizes to the case in which the value of each feature lives in a well-behaved space (for instance a compact subset of a complete separable metric space). See also footnote 7 above.

13. We are grateful to an anonymous referee for raising this point.

14. As we discuss at some length in Section 10 below a state space $\mathcal{S}$ that is a "small" subset of $\mathcal{C}$ is a necessary ingredient of a model that delivers the results that we obtain here.

15. See footnote 20.

Two points should be noted. First, the density of a set $\mu(Q)$ is its "frequency" in the standard sense of the word. Thus, for instance, every finite set of states has a density of 0, the set of all "even numbered" states $\{s_2, s_4, s_6, \ldots\}$ has a density of $1/2$, and so on. Second, the density of a set (and whether or not it is well defined) depends on the *ordering* of the states $\{s_1, \ldots, s_n, \ldots\}$.[16] This ordering is taken as given and fixed throughout the paper.

We conclude this subsection with three observations that will become useful below.

First, given two sets Q' and Q'' that have well-defined densities and such that $\mu(Q') > 0$ and $\mu(Q' \cap Q'')$ is also well defined, we can define the conditional density $\mu(Q'' \mid Q')$ as $\mu(Q' \cap Q'')/\mu(Q')$.

Second, if we let Σ be the set of all subsets of $\mathcal{S}$, then there exists an extension to Σ of the density μ in Definition 2 above which is a finitely additive probability measure. In other words, there exists a finitely additive probability measure $\tilde{\mu}$ over $(\mathcal{S}, \Sigma)$ that for every set of states $B \subset \mathcal{S}$ satisfies $\tilde{\mu}(B) = \mu(B)$, whenever $\mu(B)$ is defined.[17]

Lastly, we note that while μ of Definition 2 (and hence $\tilde{\mu}$) obeys finite additivity it fails to be *countably additive*. Recall that for any singleton set $s_n \in \mathcal{S}$ we have that $\mu(s_n) = 0$. Clearly, this implies that $\sum_{n=1}^{\infty} \mu(s_n) = 0$. However, $\mathcal{S} = \bigcup_{n=1}^{\infty} s_n$, and $\mu(\mathcal{S}) = 1$. We return at some length to the specific role that the failure of countable additivity plays in our context in Sections 9 and 10 below.

Probabilities that obey finite additivity but fail to be countably additive are certainly not new in statistical and decision theory. For instance, the classic work by Savage (1954) only introduces countable additivity as a "special hypothesis" if and when needed.[18] The classical arguments to consider a postulate of countable additivity something that should be dispensed with if at all possible are well known (de Finetti, 1949; Dubins and Savage, 1976). We do not pursue them any further here.

5. FINITELY DEFINABLE SETS AND FINITE CONTRACTS

The set of *ex ante* contracts that our agents can draw up intuitively coincides with those agreements that can be embodied into *finite* statements in the language at their disposal. It is convenient to start our description of what a finite contract is by introducing the notion of a *finitely definable set*. Intuitively, these are subsets of $\mathcal{S}$ that can be defined referring only to a *finite* subset of their features.

Recall that for each state of nature s_n, $s_n^i \in \{0, 1\}$ indicates the value of the i-th digit. Define also

$$A(i, j) = \{s_n \in \mathcal{S} \quad \text{such that} \quad s_n^i = j\} \tag{6}$$

so that $A(i, j)$ is the set of states that have the i-th feature equal to $j \in \{0, 1\}$. These are the elementary statements of the underlying language to which we referred informally in Sections 1.2 and 4.1 above.

We are now ready to define the finitely definable subsets of $\mathcal{S}$. These are the sets that can be described in the language of our contracting parties.

Definition 3 (*Finitely Definable Sets*). Consider the algebra of subsets of $\mathcal{S}$ generated by the collection of sets of the type $A(i, j)$ defined in (6). Let this algebra be denoted by $\mathcal{A}$. We refer to any $A \in \mathcal{A}$ as a finitely definable set.

16. Since the probability of events in $\mathcal{D}$ is implicitly assumed to be common knowledge among the contracting parties, we are also implicitly assuming that the enumeration of states is the same for the two agents.

17. See, for example, Bhaskara Rao and Bhaskara Rao (1983, p. 41) for a proof.

18. See also the fundamental contributions in de Finetti (1937) and Koopman (1940*a*,*b*, 1941).

Elements of $\mathcal{A}$ can be obtained by complements and/or finite intersections and/or finite unions of the sets $A(i, j)$. Hence, every element of $\mathcal{A}$ can be defined by finitely many elementary statements about the features of the states of nature that it contains.

A suitable definition of a finite contract is now possible. The key feature of a finite contract is that it should specify a set of transfers that is conditional only on finitely definable sets. For simplicity we also restrict attention to contracts that specify a finite set of values for the actual transfer t. This is clearly without loss of generality in our simple coinsurance problem described in Section 3 above.

Definition 4 (*Finite Contracts*). A contract is finite if and only if the transfer rule $t(\cdot)$ that it prescribes is measurable with respect to $\mathcal{A}$ and takes finitely many values $\{t_1, \dots, t_M\}$. The set of finite contracts is denoted by $\mathcal{F}$.

While it is possible, as we do here, to take Definition 4 as a primitive that embodies the notion of a contract as a finite object, it is important to point out that this requirement can be supported in a different way (other than just taking Definition 4 at face value).

Anderlini and Felli (1994) put forward the idea that it is natural to consider contracts that yield a value for a sharing rule that is *computable* by a Turing machine as a function of the state of nature. The justification for this requirement is a claim that if a function is computable in a finite number of steps by any imaginable finite device then it must be computable by a Turing machine.[19] Obviously, any finite contract must be computable. It is also possible to show that the converse holds: requiring that contracts be finite exhausts the set of all computable contracts. For reasons of space, we omit any formal analysis of this topic.

6. EXPECTED UTILITIES

Recall that it is our goal to restrict attention to those cases in which the agents can base their choices on the expected utility that an *ex ante* contract yields. Since we want the agents to be able to contemplate all possible finite contracts, we need to ensure that all such contracts can be evaluated in this way. So far, there is nothing in our framework that guarantees that this is the case. This is because our Definition 2 above does not, by itself, guarantee that all finitely definable sets have a well-defined density. Proposition 1 below guarantees that this can indeed be done.

Before the formal result, it is convenient to state a formal definition.

Definition 5 (*Finitely Consistent State Space*). We say that a state space $\mathcal{S}$ as in Definition 1 is finitely consistent if it is the case that $\mathcal{A} \subseteq \mathcal{D}$ so that every $A \in \mathcal{A}$ has a well-defined density $\mu(A)$.

We are now ready for our first claim.

Proposition 1 (Existence). *There exists a state space $\mathcal{S}$ as in Definition* 1 *that is finitely consistent according to Definition* 5.[20]

The proof of the proposition is in the Appendix. One way to construct the $\mathcal{S}$ of Proposition 1 is to take the state space as a realization of countably many i.i.d. draws from a countably

19. This claim is known in the literature on computable functions as Church's thesis. See, for instance, Rogers (1967) or Cutland (1980).

20. Moreover, as is readily apparent from the proof of Proposition 1 in the Appendix, the finitely consistent $\mathcal{S}$ can be constructed so that $\mu(A) > 0$ for every $A \in \mathcal{A}$.

additive density $\hat{\mu}$ over $\mathcal{C} = \{0, 1\}^{\mathbb{N}}$ (the proof in the Appendix runs along these lines). It is then sufficient to observe that the law of large numbers guarantees that, with probability 1, the limit frequency of draws that fall into any finitely definable set A is in fact well defined and equal to its density $\hat{\mu}(A)$. The set of realizations of these i.i.d. draws that have the properties required by the statement of the proposition has probability 1 in the space of realizations of this process. It then follows that it must be non-empty. Hence, setting $\mathcal{S}$ to be equal to a "typical" realization of these i.i.d. draws as described is sufficient to prove the claim.

To evaluate the expected utility accruing to each party from any finite contract we will also need to refer to the conditional densities of certain events. This is an easy task if we restrict attention to finitely definable sets. The following remark is stated without proof since it is a direct consequence of the fact that, by assumption, since $\mathcal{A}$ is an algebra, the intersection of two finitely definable sets is itself finitely definable.

Remark 1 (Well-Defined Conditional Densities). Let $\mathcal{S}$ be a finitely consistent state space. Let A' and A'' be two finitely definable sets with $\mu(A') > 0$. Then the conditional density $\mu(A'' \mid A')$ is well defined.

Of course, to compute the expected utility of finite contracts, the parties must be able to compute more than the frequencies of finitely definable sets. They need to compute the density of the intersection of $\mathcal{Z}$ with any finitely definable set.

Our next definition makes precise what it means for a set of states to meet this requirement.

Definition 6 (*Well-Defined Conditional Frequencies*). A set $\mathcal{Z}$ has well-defined conditional frequencies if

$$\mathcal{Z} \cap A \in \mathcal{D} \quad \forall A \in \mathcal{A}.$$

In other words, $\mathcal{Z}$ has a well-defined density, conditional on any finitely definable set that has positive measure under μ.

The fact that undescribable events with well-defined conditional frequencies can arise in this model is the subject of our next section. For the time being, we remark that the expected utilities from any finite contract are well defined.

Definition 7 (*Expected Utilities*). Consider the coinsurance problem described in Section 3. Let μ be the density of Definition 2, and let $\mathcal{S}$ be a finitely consistent state space. Assume further that $\mathcal{Z}$ has well-defined conditional frequencies in the sense of Definition 6. Let also any finite contract $t : \mathcal{S} \to \{t_1, \dots, t_M\}$ be given. Then the expected utility to agent 1 from contract t is defined as

$$\mathrm{EU}_1(t) = \sum_{i=1}^{M} V(1 + t_i)\mu[t^{-1}(t_i) \cap \mathcal{Z}] + \sum_{i=1}^{M} V(t_i)\mu[t^{-1}(t_i) \cap \overline{\mathcal{Z}}] \tag{7}$$

while 2's expected utility is

$$\mathrm{EU}_2(t) = \sum_{i=1}^{M} V(-t_i)\mu[t^{-1}(t_i) \cap \mathcal{Z}] + \sum_{i=1}^{M} V(1 - t_i)\mu[t^{-1}(t_i) \cap \overline{\mathcal{Z}}]. \tag{8}$$

In this paper, we restrict attention to contracts that are measurable with respect to $\mathcal{A}$ and to contracting problems in which $\mathcal{Z}$ has well-defined conditional frequencies in the sense of

Definition 6. This ensures that expected utility has the simple form above. To deal with more general settings, an elaborate theory of integration with respect to finitely additive probabilities is available. [21]

7. UNDESCRIBABLE EVENTS

We begin with an abstract definition that captures the idea that in the model we have set up it is possible that a set $\mathcal{Z}$ may look the same if we consider its restriction over any finitely definable set, but at the same time may have a characteristic function that varies finely with the state of nature. It will be precisely this type of variability that finite contracts cannot capture and hence gives rise to undescribable events.

Definition 8 (*Undescribability*). We say that an event $\mathcal{Z} \subseteq \mathcal{S}$ is undescribable if $\mu(\mathcal{Z}) \in (0,1)$ and

$$\mu(\mathcal{Z}|A) = \mu(\mathcal{Z}) \quad \forall A \in \mathcal{A} \text{ with } \mu(A) > 0. \tag{9}$$

So, $\mathcal{Z}$ is undescribable if its density is defined and is neither 0 nor 1, and its conditional density is defined and is the same, conditional on any finitely definable set that has positive measure under μ.[22]

In other words, if $\mathcal{Z}$ is undescribable knowing that s belongs to any finitely definable subset of $\mathcal{S}$ does not help us to predict better whether it belongs to $\mathcal{Z}$ or to its complement: any "finite" piece of information about s has no value to this end. It should be noted at this point that the possibility that Definition 8 may have a non-trivial content is a feature of the model we have set up, which *does not* hold in say a standard model with a continuum of states with $\mathcal{Z}$ a measurable set. In fact, it is clear that in this case if $\mathcal{Z}$ satisfies (9) then it must have measure either 0 or 1.[23] This is not the case in our model.

Proposition 2 (Undescribable Events). *There exists a finitely consistent state space $\mathcal{S}$ such that, for every $p \in (0,1)$, there is an undescribable event $\mathcal{Z} \subset \mathcal{S}$ with $\mu(\mathcal{Z}) = p$.*

The formal proof of Proposition 2 is in the Appendix. Here we only sketch the argument for the case $p = 1/2$. Let $\mathcal{S}$ be a finitely consistent state space. We can then construct $\mathcal{Z}$ in the following way. For each given state of nature $s_n \in \mathcal{S}$, we set $s_n \in \mathcal{Z}$ and $s_n \in \overline{\mathcal{Z}}$ with equal probability and with i.i.d. draws across all the states s_n. The law of large numbers again guarantees that we can take $\mathcal{Z}$ to be a "typical" realization of this process to prove the claim. In fact, in any such typical realization, the law of large numbers ensures that the event $\mathcal{Z}$ has a density that is well defined and is equal to 1/2 conditional on any finitely definable subset of states. In other words, the event $\mathcal{Z}$ is undescribable.

The proofs of Propositions 1 and 2 reported in the Appendix both rely on the law of large numbers. An advantage of these arguments is that the resulting state spaces and undescribable

21. Dunford and Schwartz (1958) is a classic textbook, which provides a unified treatment of integration for both finite and countably additive measures. A more specialized treatment can be found in Bhaskara Rao and Bhaskara Rao (1983).

22. Notice that if $\mathcal{Z}$ is undescribable, it follows from our definition that it has well-defined conditional frequencies according to Definition 6. We are only interested in undescribable events that have well-defined conditional frequencies because we seek a definition of such events that allows all relevant expected utility calculations to be carried out—see the first of our Desiderata in Section 1.2 above.

23. Observe that if $\mathcal{Z} \neq \varnothing$ and we are allowed to take $A \subseteq \mathcal{Z}$, using (9) we trivially get that $1 = \mu(\mathcal{Z}|A) = \mu(\mathcal{Z})$. We return to the lack of undescribable events in the standard set-up in Section 10.2 below.

events are not knife-edge constructs, but probability-1 realizations of the stochastic processes used. However, it is legitimate at this point to ask whether there are constructive arguments that can be used instead.

The answer to the question is affirmative. In fact, the following construction proves both Propositions 1 and 2. We outline it for the case in which $\mu(A(i,j)) = 1/2$ for every i and every j, and $\mu(\mathcal{Z}) = 1/2$.

Start with the states being identified by their labels, the positive integers. Now assign all odd numbered states $(s_1, s_3, s_5, s_7, s_9, s_{11}, \ldots)$ to $\mathcal{Z}$ and all even-numbered states $(s_2, s_4, s_6, s_8, s_{10}, s_{12}, \ldots)$ to $\overline{\mathcal{Z}}$. The features of each state in $\mathcal{Z}$ can be thought of as forming a column of an infinite matrix $M(\mathcal{Z})$ with columns labelled $s_1, s_3, s_5, s_7, s_9, s_{11}, \ldots$. Similarly the features of each state in $\overline{\mathcal{Z}}$ can be thought of as forming a column of an infinite matrix $M(\overline{\mathcal{Z}})$ with columns labelled $s_2, s_4, s_6, s_8, s_{10}, s_{12}, \ldots$.

Now assign the values of the features of states in $\mathcal{Z}$ proceeding by row of the matrix $M(\mathcal{Z})$ as follows. In the first row assign 0 to the first element (so that $s_1^1 = 0$), assign 1 to the second element (so that $s_3^1 = 1$) and continue *ad infinitum* alternating each time a 0 and a 1. In the second row assign a value of 0 to the first two elements (so that $s_1^2 = s_3^2 = 0$), the a value of 1 to the next two elements (so that $s_5^2 = s_7^2 = 0$) and continue *ad infinitum* alternating two 0's and two 1's along the row. In the third row assign a value of 0 to the first four elements (so that $s_1^3 = s_3^3 = s_5^3 = s_7^3 = 0$) and a value of 1 to the next four elements (so that $s_9^3 = s_{11}^3 = s_{13}^3 = s_{15}^3 = 1$) and continue *ad infinitum* alternating four 0's and four 1's along the row. In the m-th row of $M(\mathcal{Z})$ assign a value of 0 to the first 2^{m-1} elements, a value of 0 to the next 2^{m-1} elements and so on *ad infinitum* to fill the entire matrix.

To assign the values of the features of states in $\overline{\mathcal{Z}}$, fill out the entries of the matrix $M(\overline{\mathcal{Z}})$ by reversing the assignment for $M(\mathcal{Z})$: in every position of $M(\overline{\mathcal{Z}})$ place a 1 if a 0 appears in the corresponding position in $M(\mathcal{Z})$ and a 0 if a 1 appears in the corresponding position in $M(\mathcal{Z})$. A schematic representation of the construction we have outlined is as follows.

$$
\overbrace{\begin{array}{ccccccc}
s_1 & s_3 & s_5 & s_7 & s_9 & s_{11} & \cdots \\
\hline
0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
0 & 0 & 1 & 1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1 & 1 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}}^{M(\mathcal{Z})}
\qquad
\overbrace{\begin{array}{ccccccc}
s_2 & s_4 & s_6 & s_8 & s_{10} & s_{12} & \cdots \\
\hline
1 & 0 & 1 & 0 & 1 & 0 & \cdots \\
1 & 1 & 0 & 0 & 1 & 1 & \cdots \\
1 & 1 & 1 & 1 & 0 & 0 & \cdots \\
1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}}^{M(\overline{\mathcal{Z}})}
$$

It then becomes easy to verify that all the requirements of Propositions 1 and 2 are satisfied with $\mu(A(i,j)) = 1/2$ for every i and every j, and $\mu(\mathcal{Z}) = 1/2$.

While the constructive argument we have outlined clears one concern, it raises a new one that is worth addressing at this point. It may be tempting to argue that the set $\mathcal{Z}$ that we constructed above is not complex and that it is (obviously) describable—after all, the construction we outlined is its description. We believe this to be misleading, however. To appreciate this point, let $\mathbb{N}$ be the set of natural numbers and fix any countable state space $\mathcal{S}$. Call a function $e\colon \mathbb{N} \to S$ an enumeration if it is one-to-one and onto (thus, under e, we are labelling a state $s \in \mathcal{S}$ as the $e^{-1}(s)$-th state). Given any infinite subset $\mathcal{Z} \subset \mathcal{S}$, it is obviously possible to find an enumeration $e_{\mathcal{Z}}$ under which $\mathcal{Z}$ has a simple description. For instance, one can easily find an $e_{\mathcal{Z}}$ under which $\mathcal{Z}$ corresponds to (*i.e.* $e_{\mathcal{Z}}^{-1}(\mathcal{Z})$ is) the set of even integers. Obviously $\mathcal{Z}$ is simple to describe, but only *given the enumeration* $e_{\mathcal{Z}}$.

To use the labels in $\mathbb{N}$ given by e to identify a set, the description of a set must therefore include a full specification of the enumeration needed to give it a simple representation (*e.g.* as the set of even integers), an infinite object by itself. The contracting agents in our model are endowed with a given language: the set of all possible finite sentences combining a fixed set of elementary features. This language is the only available vehicle to convey their will to the court. Thus, permutations of the integer labels of the states are meaningless to our contracting agents in identifying any particular set of states in an *ex ante* contract. The set $\mathcal{Z}$ constructed above is not describable in the language determined by the elementary features.

8. COINSURANCE WITH UNDESCRIBABLE EVENTS

8.1. *No coinsurance*

The possibility that the contract t^* in the coinsurance problem described in Section 3 above may prescribe transfers conditional on an undescribable event has far-reaching consequences on what the contracting parties can achieve by means of a finite contract.

In this case, any finite contract will be unable to capture any of the fine variability of t^*. As a consequence, the agents will choose a trivial contract that prescribes a transfer of $t = 0$ in every possible state. This is, of course, the same as saying that *no contract* will be drawn up.

Consider the coinsurance problem described in Section 3. For a given $\mathcal{S}$, μ, and $\mathcal{Z}$, let t^{**} be the optimal *finite* coinsurance contract, if it exists. In other words, if it is well defined let t^{**} be the solution to

$$\begin{array}{lll} \max\limits_{t} & \mathrm{EU}_1(t) & \\ \text{s.t.} & \mathrm{EU}_2(t) & \geq \ \mu(\mathcal{Z})V(0)+\mu(\overline{\mathcal{Z}})V(1) \\ & t \in \mathcal{F} & \end{array} \tag{10}$$

where $\mathrm{EU}_i(t)$ are the parties' expected utilities as in Definition 7 above.

Proposition 3 (Optimal No Coinsurance). *Consider the coinsurance problem described in Section* 3. *Then there exist an* $\mathcal{S}$ *and* μ *and* $\mathcal{Z}$ *with* $\mu(\mathcal{Z}) \in (0,1)$ *with the following properties.*

1. *The set* $\mathcal{Z}$ *has well-defined conditional frequencies.*
2. *The optimal finite contract* t^{**} *that solves problem* (10) *exists and is unique, up to a set of states of* μ*-measure 0.*
3. *The optimal finite contract* t^{**} *prescribes no transfer between the agents in every state of nature. In other words,* $t^{**}(s) = 0$ *for every* $s \in \mathcal{S}$*, up to a set of states of* μ*-measure* 0.

Once again the formal proof of Proposition 3 is in the Appendix. Intuitively, Proposition 3 is a fairly direct consequence of Propositions 1 and 2 coupled with the concavity of the agents' preferences.

Start with a finitely consistent state space $\mathcal{S}$. Recall now that in the coinsurance problem described in Section 3 the parties are able to achieve full insurance by agreeing on a transfer contingent on the event $\mathcal{Z}$. We now choose the event $\mathcal{Z}$ to be undescribable as in Proposition 2. Let $p_{\mathcal{Z}}$ and $p_{\overline{\mathcal{Z}}}$ be the densities of $\mathcal{Z}$ and $\overline{\mathcal{Z}}$, respectively, conditional on any $A \in \mathcal{A}$.

Since $\mathcal{Z}$ is undescribable, any attempt by the parties to condition on a finite set of characteristics (the only feasible *ex ante* description available to them) will leave them with a set of states of which only a fraction $p_{\mathcal{Z}}$ actually belongs to $\mathcal{Z}$. This is true whatever finitely definable subset of $\mathcal{S}$ the parties decide to condition their contract on. The fact that the parties are risk averse now implies that the optimal finite contract should specify the same transfer from 2 to 1 contingent on *any* finitely definable subset of $\mathcal{S}$. Any transfer function that varies across two finitely definable sets of states will be strictly dominated (in terms of the parties expected utility) by a constant transfer that coincides with the average of the transfer function we started from.

The optimal contract t^{**} is now obtained from the observation that the only constant (across all states) transfers from 2 to 1 that are compatible with 2's participation constraint are non-positive. Since 1's expected utility is increasing in the constant transfer from 2, the optimal finite contract must clearly prescribe a transfer of 0 in all states.

The allocation entailed by the optimal finite contract coincides with the no-contract outcome. Clearly, the fact that the two parties to the contract are strictly risk averse implies that party 1's expected utility associated with the no-contract outcome is bounded away from the full-insurance contract t^* described in Section 3.

The extreme prediction that the parties will choose an allocation equivalent to no-contract at all of course derives from the particular event $\mathcal{Z}$. An example of a less extreme situation is our next concern.

8.2. *Some coinsurance*

By way of motivation, consider again our "tenurable vitae" example in the Introduction to the paper. Clearly, the extreme situation described above in which the null contract obtains is not fully suited to this case.

For simplicity, imagine a case in which there is one journal that matters at all. For simplicity again, assume that the candidate can only possibly publish either 0 or 1 paper in this journal by the time the case is reviewed and that the number of papers published in this journal is described by the value of the first feature of $s \in \mathcal{S}$. Clearly, knowing whether the candidate has a journal publication (knowing the first feature of s) is *not* irrelevant to whether the candidate has a tenurable vitae.

In the spirit of our remarks in the "Introduction" to the paper, we now imagine that although the value of the first feature is important in the tenure decision, it does not seal the case either way. The event that the candidate has a tenurable vitae is too complex. Some vitae with no journal publications are tenurable while some are not. Some vitae with a journal publication are tenurable while some are not. Knowing that the candidate has a journal publication makes it more likely that she deserves tenure. The example that follows is designed to show that our set-up can easily capture intermediate situations of this kind.[24]

We partition the state space $\mathcal{S}$ into two subsets: $A_0 = A(1,0)$ in which $\mathcal{Z}$ is more likely and $A_1 = A(1,1)$ in which it is less likely.[25] In fact, with a simple adaptation of the argument used to prove Proposition 2, we can pick two numbers $\underline{p} < \overline{p}$ in $(0,1)$ and ensure that $\mathcal{Z}$ is such that for any $A \in \mathcal{A}$

$$\mu(\mathcal{Z}|A) = \begin{cases} \underline{p} & \text{if } A \subseteq A_0 \\ \overline{p} & \text{if } A \subseteq A_1. \end{cases} \tag{11}$$

A simple adaptation of the argument used to prove Proposition 3 is now sufficient to show that the optimal finite contract t^{**} in this case must satisfy

$$t^{**}(s) = \begin{cases} t_0 & \text{if } s \in A_0 \\ t_1 & \text{if } s \in A_1 \end{cases} \tag{12}$$

with $t_1 < t_0$.

24. The example is deliberately kept as simple as possible. It is immediate to see how it generalizes to the case of a finite set of relevant conditioning events. As we outline in Section 10 below, much more far-reaching generalizations are possible, but are beyond the scope of this paper.

25. Note that, using the definition of $A(i, j)$ given in (6), A_0 is the set of states in which the first feature is equal to 0, while A_1 is the set of states in which it is equal to 1.

Intuitively, the two agents face two sources of uncertainty. These are the two events $s \in A_0$ as opposed to $s \in A_1$ and $s \in \mathcal{Z}$ as opposed to $s \in \overline{\mathcal{Z}}$. The first event is insurable since A_0 and A_1 are finitely definable sets. Because of the undescribability of $\mathcal{Z}$ relative to A_0 and A_1, the second cannot be usefully captured by a finite contract, except for the fact that it is *correlated* with the first. The optimal coinsurance contract between the agents then exhibits partial insurance against the event $s \in A_0$ as opposed to $s \in A_1$.

9. RELATED LITERATURE

As we have discussed already, the intuitive notion of an event that is impossible to include in an *ex ante* contract because it is too complex has been extensively used in the incomplete contracting literature.[26] Here, we restrict ourselves to some related papers that we have not discussed before.

Anderlini and Felli (1994) and Al-Najjar (1999) are two existing contributions that are closely related to the results presented here. In Anderlini and Felli (1994), the contracting parties are restricted to *ex ante* agreements that are finite in a sense that is analogous to the one we postulate in this paper. However, crucially, in Anderlini and Felli (1994), there is a continuum of states of nature. One of the results reported there is the so-called "approximation result": in a model with a continuum of states, under general conditions of continuity, the restriction that only finitely many of the constituent features of a state of nature can be included in any *ex ante* agreement has a negligible impact on the parties' expected utilities.

The restriction to finite agreements clearly precludes the agents from writing some possible *ex ante* contracts.[27] Intuitively, the reason why the impact of this restriction is, in fact, negligible lies in the requirement that the parties must be able to compute the expected utilities that an *ex ante* agreement generates. If an *ex ante* agreement yields well-defined expected utilities to the contracting parties, then it must yield them utility levels that are integrable (in the standard sense) as a function of the state of nature. Since a function that is integrable can always be approximated by a sequence of step functions, it is now enough to notice that (a sufficiently rich set of) step functions can be viewed as finite *ex ante* agreements. In the model studied in Anderlini and Felli (1994), a "rich language" is sufficient to generate the approximation result that instead fails to hold in this paper.

Intuitively, the difference between the two environments can be traced to the cardinality of the state space (countable vs. continuous) and the nature of the associated probability measure (finitely additive frequencies in this paper, standard probability measures over the interval $[0, 1]$ in Anderlini and Felli (1994)).[28]

Using techniques similar to the ones used here, Al-Najjar (1999) addresses the question of whether competitive differences between agents get washed out by imitation. In a model with a continuum of states it is possible to show that the performance of a successful agent can be replicated asymptotically as more and more data become available: a version of the approximation result described above holds in this case. However, in a complex environment, imitation does not

26. This line of research has proved extremely fertile. Among other things, it has afforded important insights concerning the boundaries of a firm (Grossman and Hart, 1986), the allocation of ownership rights over physical assets (Hart and Moore, 1990), the allocation of authority (Aghion and Tirole, 1997), and power (Rajan and Zingales, 1998) in organizations.

27. A simple counting argument suffices to prove this point. It is easy to see that in the world of Anderlini and Felli (1994) there are countably many possible finite *ex ante* contracts, while there are uncountably many possible *ex ante* agreements.

28. We return to the approximation result of Anderlini and Felli (1994) in Section 10.2.

eliminate all competitive advantages, even in the limit when an arbitrarily large amount of data becomes available.[29]

The approach taken in Al-Najjar, Casadesus-Masanell and Ozdenoren (2003) to modelling complexity and undescribability is conceptually and technically different from what we do here. As in the present paper, the overarching idea is the discontinuity between the difficult task of formulating *ex ante* rules for dealing with all instances of a decision problem vs. the relative ease of picking the right action once a particular instance of the problem is realized. In their paper, the decision-maker's inability to think through all instances in advance is modelled as a subjective probabilistic model reflecting his *ex ante* cognitive uncertainty about what will be revealed *ex post*.

Finally, we view this paper as orthogonal to the debate on the role of message games in incomplete contracting models. In particular, a number of authors have argued that message games can, in fact, substitute for complete *ex ante* contracting.[30] The contracting parties play an *ex post* message game in which their private information is revealed in equilibrium. This enables them to make the contractual outcome depend on events that the *ex ante* contract neglects. As we have stressed already, our contribution here is to model undescribable events that cannot feature in an *ex ante* contract. If these are present, then the type of message game that is appropriate to the environment at hand will be the parties only hope to condition on the events that they cannot specify directly in their *ex ante* agreement.

10. CONCLUSIONS AND FURTHER RESULTS

We have shown that it is possible to construct a contracting environment in which some events have the following properties. Their probabilities and consequences are understood by all concerned, and all agents involved use this information to calculate expected utilities arising from any possible finite *ex ante* contract. Yet, these events are undescribable in the sense that any attempt to capture them by means of a finite *ex ante* agreement must fail.

The contracting parties cannot describe these events to any degree that will improve their expected utilities relative to an agreement that ignores them altogether. This is so notwithstanding the fact that the contracting parties' language can in fact distinguish between any two states.

10.1. *Intermediate cases*

As our example in Section 8.2 shows, it is possible to envision intermediate cases in which, say, knowing that the first feature of a state is 0 tells us something about its membership of $\mathcal{Z}$, but it is still the case that $\mathcal{Z}$ cannot be approximated by a finite contract.

In an earlier version of this paper (Al-Najjar *et al.*, 2002) we develop formally a batch of results that deal with a much larger class of these intermediate cases. What follows is a brief sketch.

It is possible to characterize tightly what the optimal finite contract looks like in the general case in which the conditional density of $\mathcal{Z}$ is not equal across all finitely definable sets in the algebra $\mathcal{A}$. Applying a theorem by Kolmogorov,[31] we can identify the unique countably additive measure on the set $\mathcal{C} = \{0, 1\}^{\mathbb{N}}$ that agrees with the conditional density of $\mathcal{Z}$, $\mu(\mathcal{Z}|\cdot)$, on every A

29. In a paper subsequent to the present one Al-Najjar and Casadesus-Masanell (2002) apply a similar construction to the one used here to a principal–agent model. The incomplete agency contract that results may leave a role for trust and for agent discretion in their agreement.

30. See Hart and Moore (1999), Maskin and Tirole (1999), Segal (1999), Tirole (1999), Maskin (2002), and Reiche (2006) among others.

31. See, for instance, Billingsley (1995, theorem 2.3) or Doob (1994, theorem V.6)

in $\mathcal{A}$.[32] Using this measure and keeping fixed the parties' utility functions we can then define an auxiliary contracting problem on the state space $\mathcal{C}$.

Since the ingredients of the auxiliary contracting problem are all standard it can be solved using familiar techniques. It is then possible to show that the solution to the auxiliary problem fully characterizes the optimal finite contract in the general case.

Hence, the optimal finite contract is not null in the general case. It captures the variability of the conditional density of $\mathcal{Z}$ that can be embodied in its unique countably additive translation to $\mathcal{C}$ that we have mentioned above.

All other variability in the characteristic function of $\mathcal{Z}$ cannot be captured at all by any finite contract. Hence, it can be safely ignored in the characterization of the optimal finite contract that the parties will sign.

10.2. *Necessity*

Our last concern is to revisit the unconventional features of our model of undescribable events. There are two that stand out. First, the measure μ that we place on $\mathcal{S}$ is finitely additive but, as we remarked above, fails to be *countably additive*. Second, our state space $\mathcal{S}$ is a "small" (countable) subset of the set $\mathcal{C} = \{0, 1\}^{\mathbb{N}}$ of all logically conceivable states.

In an earlier version of this paper (Al-Najjar *et al.*, 2003) we demonstrate that both unconventional features of our model are necessary ingredients of a model that delivers our results.

Broadly speaking, in Al-Najjar *et al.* (2003) we show that given any state space $\mathcal{S}$ equipped with a countably additive measure the approximation result of Anderlini and Felli (1994) must hold. That is, any set in the σ-algebra on which the countably additive measure is defined can be approximated more and more closely by a sequence of sets in the algebra (not the σ-algebra) on which the measure is defined. Hence, no undescribable events in the sense of Definition 8 can be found in this case. Although the details of the argument are relatively involved, the claim is a reasonably direct consequence of the definition of countable additivity and of Carathéodory's Extension Theorem.[33]

In a nutshell, in Al-Najjar *et al.* (2003) we also show that if a state space $\mathcal{S} \subseteq \mathcal{C} = \{0, 1\}^{\mathbb{N}}$ equipped with a measure μ admits an undescribable event as in Definition 8 then $\mathcal{S}$ must be a small subset of $\mathcal{C}$ in a well-defined sense.[34] The argument behind this claim utilizes again the theorem by Kolmogorov we cited above.[35] Very roughly speaking this guarantees that if $\mathcal{S}$ were to contain a whole "cylinder" of $\mathcal{C}$ then any measure placed on $\mathcal{S}$ that is finitely additive would also have to satisfy countable additivity on the same subset of $\mathcal{C}$. This, in turn, would contradict the result described in the paragraph above.

We view the necessity claims we have just outlined as indicating that the results in this paper can in fact be read in two ways. The first is to conclude that it is indeed possible to model formally the notion of an undescribable event. The second is that the model we used to this end, complete with its unconventional features, is what it takes to get a formal hold of this notion. There is a sense in which a rejection of the unconventional ingredients that we use here is equivalent to saying that the formal notion of an event that is undescribable because it is too complex rather than because the parties do not have a sufficiently rich language is unattainable.

32. In the extreme case addressed by Proposition 3 this would be the uniform measure on $\mathcal{C}$.

33. See, for instance, Royden (1988, ch. 12.2).

34. To be precise, there we show that $\mathcal{S}$ must have Lebesgue measure 0 (appropriately translated) in $\mathcal{C}$. Other notions of "smallness" may be considered as well. Depending on whether one takes the point of view of the contracting agents or that of the analyst, one notion may be more appropriate than others. In particular, we conjecture that $\mathcal{S}$ is of Baire first category in $\mathcal{C}$ whenever undescribable events obtain. We are grateful to an anonymous referee for raising this point.

35. See footnote 31 above.

APPENDIX

Proof of Proposition 1. Consider the set $\mathcal{C}$ of infinite sequences of 0's and 1's, $\mathcal{C} = \{0,1\}^{\mathbb{N}}$, with typical element c and let c^i be the i-th digit of the sequence c. Let also

$$\tilde{A}(i,j) = \{c \in \mathcal{C} \quad \text{such that} \ \ c^i = j\}. \tag{A.1}$$

Let H denote the set of all infinite sequences $\{c_1,\dots,c_n,\dots\}$ with $c_n \in \mathcal{C}$ for every n. Let $\{\tilde{c}_n\}_{n=1}^{\infty}$ be an infinite sequence of i.i.d. random variables with (countably additive) distribution $\tilde{\mu}$ over $\mathcal{C}$, and let P be the (product) probability distribution that this yields for H.

For any i and j now consider the event $M(i,j) \subset H$ such that $\lim_{N\to\infty}(1/N)\sum_{n=1}^{N}\chi_{\tilde{A}(i,j)}(c_n) = \tilde{\mu}(\tilde{A}(i,j))$. By the law of large numbers, $P(M(i,j)) = 1$ for every i and j.

Now define,

$$M = \bigcap_{\substack{i\in\mathbb{N} \\ j\in\{0,1\}}} M(i,j). \tag{A.2}$$

Clearly, since $P(M(i,j)) = 1$ for every i and j, and of course P is countably additive, we must also have $P(M) = 1$, and therefore $M \neq \varnothing$.

It is now sufficient to choose $\mathcal{S}$ to be equal to any element of M to prove the claim. ‖

Proof of Proposition 2. Fix any $p \in (0,1)$ as in the statement of the proposition. Assume that $\mathcal{S}$ is finitely consistent and that it has the property that any finitely definable set A contains a countable infinity of elements. This is clearly possible from the construction in the proof of Proposition 1.

Define a stochastic process $\{\tilde{h}_1,\dots,\tilde{h}_n,\dots\}$ where each random variable $\tilde{h}_n$ takes values in $\{0,1\}$. Let H denote the set of all realizations of this process, and let P be the probability distribution on H under which $\{\tilde{h}_1,\dots,\tilde{h}_n,\dots\}$ are i.i.d. random variables with distribution $(p, 1-p)$. Notice that a realization $h = \{h_1,\dots,h_n\dots\} \in H$ of this process can be taken to be a candidate for the characteristic function $\chi_{\mathcal{Z}}\colon \mathcal{S} \to \{0,1\}$. We now proceed to show that the claim can be proved by setting $\chi_{\mathcal{Z}}$ equal to any such realization of this process in a set of probability 1.

Let any $h \in H$ be given and let $A(h)$ be the set of states s_n such that $s_n \in A$ and $h_n = 1$. The law of large numbers holds for any $A \in \mathcal{A}$ in the following sense. There is a set $H_A \subset H$ with $P(H_A) = 1$ such that $h \in H_A$ implies that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{A(h)}(s_n) = p\,\mu(A). \tag{A.3}$$

Since $P(H_A) = 1$, clearly $Q = \cap_{A\in\mathcal{A}} H_A$ also has probability 1. Therefore, $Q \neq \varnothing$. Now select any element $h = \{h_1,\dots,h_n,\dots\}$ of Q, and set $\chi_{\mathcal{Z}}(s_n) = h_n$ for every n. This is our candidate $\chi_{\mathcal{Z}}$.

Since equation (A.3) holds for any $A \in \mathcal{A}$ it is obvious that $\mathcal{Z}$ is undescribable as in Definition 8. Again from the fact that equation (A.3) holds for any $A \in \mathcal{A}$, it is clear that $\mathcal{Z}$ has well-defined conditional frequencies as in Definition 6. Lastly, again from equation (A.3) it is immediate that for any $A \in \mathcal{A}$ with $\mu(A) > 0$ we must have that $\mu(\mathcal{Z}|A) = p$, as required. ‖

Lemma A1. *Consider problem* (10). *Let $\mathcal{Z}$ have well-defined conditional frequencies as in Definition* 6 *and be undescribable as in Definition* 8.

Let any finite contract $t(\cdot) \in \mathcal{F}$ that is feasible in problem (10) *be given, and $\{t_1,\dots,t_M\}$ be the range of $t(\cdot)$. Finally, for every $i = 1,\dots,M$, let T_i be the inverse image of t_i under $t(\cdot)$.*

Assume now that $t(\cdot)$ has the following property. There exist an $i \in \{1,\dots,M\}$ and a $j \in \{1,\dots,M\}$ such that $\mu(T_i) > 0$ and $\mu(T_j) > 0$. Then there exists another finite contract $t'(\cdot) \in \mathcal{F}$ that is constant *over $T_i \cup T_j$, which is also feasible in problem* (10) *and that yields a higher expected utility for agent* 1.

Proof. Let $t'(\cdot)$ be the same as $t(\cdot)$ for every $s_n \notin T_i \cup T_j$, and set

$$t'(s_n) = \frac{\mu(T_i)t_i + \mu(T_j)t_j}{\mu(T_i) + \mu(T_j)} \quad \forall s_n \in T_i \cup T_j. \tag{A.4}$$

The claim now follows directly by concavity of V, defining U_1 and U_2 as in (1). The rest of the details are omitted. ‖

Lemma A2. *Let $\mathcal{Z}$ have well-defined conditional frequencies* (*as in Definition* 6) *and be undescribable* (*as in Definition* 8). *Then an optimal finite contract t^{**} that solves problem* (10) *exists unique, up to a set of states of μ-measure* 0. *Moreover, $t^{**}(s_n) = 0$ for all $s_n \in \mathcal{S}$, up to a set of states of μ-measure* 0.

Proof. Let $\mathcal{Z}$ as in the statement of the lemma be given. Consider now the following maximization problem:

$$\begin{aligned} \max_{x} \quad & V(1+x)\,\mu(\mathcal{Z}) + V(x)\,\mu(\overline{\mathcal{Z}}) \\ \text{s.t.} \quad & V(-x)\,\mu(\mathcal{Z}) + V(1-x)\,\mu(\overline{\mathcal{Z}}) \geq V(0)\,\mu(\mathcal{Z}) + V(1)\,\mu(\overline{\mathcal{Z}}) \\ & x \in \mathbb{R}. \end{aligned} \tag{A.5}$$

The strict concavity of $V(\cdot)$ implies that problem (A.5) has a unique solution by completely standard arguments. Let this solution be denoted by $\tilde{x}$.

The expected utility $V(-x)\,\mu(\mathcal{Z}) + V(1-x)\,\mu(\overline{\mathcal{Z}})$ is monotonically decreasing in x. Therefore, the constraint in problem (A.5) is satisfied only when $x \leq 0$. Since the objective function in problem (A.5), $V(1+x)\,\mu(\mathcal{Z}) + V(x)\,\mu(\overline{\mathcal{Z}})$, is monotonically increasing in x we conclude that the unique solution of problem (A.5) is $\tilde{x} = 0$.

From Lemma A1 above it is immediate that a solution to problem (A.5) must yield a solution to problem (10). Therefore setting $t^{**}(s_n) = 0$ for every $s_n \in \mathcal{S}$ yields the unique (up to a set of μ-measure 0) solution to problem (10). ‖

Proof of Proposition 3. Let $\mathcal{S}$ be a finitely consistent state space. Using Proposition 2 we can now choose $\mathcal{Z}$ to have well-defined conditional frequencies, and be undescribable, with $\mu(\mathcal{Z}) \in (0, 1)$. The claim now follows directly from Lemma A2. ‖

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