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The efficiency of competitive mechanisms under private information

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Abstract

We consider the efficiency properties of exchange economies where privately informed traders behave strategically. Specifically, a *competitive mechanism* is any mapping of traders' reports about their types to an equilibrium price vector and allocation of the reported economy. In our model, some traders may have non-vanishing impact on prices and allocations regardless of the size of the economy. Although truthful reporting by all traders cannot be achieved, we show that, given any desired level of approximation, there is $\bar{N}$ such that *any* Bayesian–Nash equilibrium of *any* competitive mechanism of *any* private information economy with $\bar{N}$ or more traders leads, with high probability, to prices and allocations that are close to a competitive equilibrium of the true economy. In particular, allocations are approximately efficient. A key assumption is that there is small probability that traders behave non-strategically.

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1. Introduction

A cornerstone of competitive analysis is the assumption that traders in large markets ignore their influence on prices. This assumption has far-reaching implications on how prices are determined and whether market outcomes are efficient. Yet despite its central role, price-taking behavior is

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difficult to support with formal strategic foundation. A stark illustration is seen by considering a continuum economy with complete-information. Although individuals have no impact on the economy's fundamentals, if there are multiple competitive equilibria, individual traders may misreport their preferences if they believed doing so would lead to a more favorable equilibrium selection. But if price-taking behavior is problematic in idealized complete-information continuum economies, finite markets with incomplete information are likely to present an even more formidable challenge.

In light of the above observations, this paper focuses on the efficiency properties of competitive markets rather than on whether traders behave as price-takers. We consider exchange economies where privately informed traders behave strategically. A *competitive mechanism* is any mapping from traders' reports to a competitive equilibrium of the reported economy. A probability distribution on traders' types and a competitive mechanism define a Bayesian game where: players report types, the competitive mechanism selects an equilibrium for the reported economy, and payoffs are the utilities traders derive from the implied allocation.

Our main result is that, given any desired degree of approximation, there is $\bar{N}$ such that in any economy with $\bar{N}$ or more traders and *any* prior distribution on traders' types, *any* Bayesian–Nash equilibrium of *any* competitive mechanism for this economy yields an allocation that is approximately efficient. Since this result applies uniformly to all finite economies with $\bar{N}$ or more traders, no replica structure is invoked, nor is it necessary to make assumptions about the existence of a well defined limit economy.

Our results may be viewed as helping assess competitive mechanisms as a device to implement efficient outcomes. From the perspective of a designer, any mechanism that equates supply and demand does well, and robustly so across Bayesian–Nash equilibria, type distributions, endowments, and so on. A limitation of our approach is that we restrict attention to a family of direct mechanisms, and as such we expect agents to report their demand functions—possibly strategically.

We rely on two substantive assumptions: first, the set of possible types is finite. This ensures that a trader whose gain from misrepresenting his type is small will actually report the truth. [1] Second, we assume that there is positive probability of each trader being non-strategic. Although our main focus is on traders who report strategically, the presence of a small fraction of non-strategic traders significantly alters the behavior of strategic ones by making them report truthfully.

Looking for results that hold for any prior type distribution and any Bayesian–Nash equilibrium of any competitive mechanism reflects our concern that the efficiency of competitive markets should depend as little as possible on the fine details of the modeling choices. This concern for "robustness" is, of course, not new. There is an increasingly acute awareness among many economists that robustness should be a major concern in areas ranging from decision theory to macroeconomics and mechanism design. We believe that a concern for robustness is particularly relevant here: the competitive correspondence provides little more than consistency requirements (namely that that supply equals demand) on the outcome of the complex and unmodeled processes of market interactions. [2] Lacking a theory of how actual market outcomes arise, we find

[1] This assumption appears in a number of papers in the literature, including Gul and Postlewaite [5] and McLean and Postlewaite [11]. See Jackson and Manelli [8] for discussion.

[2] This is in the spirit of the idea, appearing in Kalai [9], that equilibria of a simultaneous move game might be thought of as representations of the outcomes of some complex underlying extensive game elaborations of the simultaneous game. Kalai's analysis is inapplicable in our context, however, because he focuses on anonymous mechanisms where the influence of every player on the final outcome disappears in the limit by assumption.

little compelling justifications to restrict the way markets map individual reports into prices and consumption bundles beyond the consistency requirements of a competitive equilibrium. This is why we do not make the common assumption of restricting attention to anonymous and continuous competitive mechanisms. It would be indeed disturbing if the efficiency of competitive institutions hinged on prices chosen anonymously or on a specific equilibrium being played.

The key conceptual tool underlying our analysis is the notion of influence we introduced in Al-Najjar and Smorodinsky [1]. Roughly, suppose we are given a random variable z that depends on the actions of all agents (e.g., the competitive price of a commodity). The influence of an agent on z is the maximum impact he can have on the expectation of z taking as given the stochastic actions of the other agents. A key observation is that an agent's incentive to mis-represent cannot exceed his influence, regardless of which equilibrium is being played, which beliefs other players holds, ..., etc. The analysis in Al-Najjar and Smorodinsky [1] provides bounds on average influence that are uniform over *all* random variables z provided only that the environment is slightly noisy. The key source of noise in our setting is the behavior of non-strategic types. While the rough influence argument sketched above is far from directly applicable here, we show that it can be extended to derive powerful dominance-like bounds on individuals' mis-representations in *any* equilibrium. By contrast, to show the existence of *one* efficient Bayesian–Nash equilibrium requires only the lining up of incentives around a cleverly chosen putative equilibrium profile.

A natural question is whether simplifying assumptions like the anonymity or continuity of the competitive mechanism, or the introduction of a replica structure, can simplify the proofs of our results or make it possible to obtain stronger ones. We comment on these points briefly; for more detailed discussion, see the working paper version of this article (available from the authors). First, since agents in our model are not ex ante identical (for instance, they can differ in their endowments), there is no reason to expect them to have the same influence. Thus anonymity has no bite, except in degenerate cases when all agents are ex ante symmetric. Second, continuity also has no bite since in the finite-player context of our model any selection from the competitive correspondence is, trivially, continuous. Finally, a replica structure is of no help in proving that efficiency obtains at *all* Bayesian–Nash equilibria. The reason is that a replica structure ensures symmetry in fundamentals but not in equilibrium behavior. There is no reason to believe that handling asymmetric play is substantively easier in replica economies compared to general economies. An important advantage of our methodology is that it sidesteps these issues entirely.

Turning to the literature, the vulnerability of competitive mechanisms to strategic interactions was first pointed out by Hurwicz [7]. One of the earliest formal studies of this problem appeared in Roberts and Postlewaite [14]. They consider sequences of exchange economies with complete information and show that the incentive to mis-report vanishes as the economy gets large. [3]

There is a large literature that asks a related question: "Is it possible to design an incentive compatible mechanism that yields an efficient allocation?" Examples of papers that pursue this line of enquiry include Gul and Postlewaite [5], Postlewaite and Schmeidler [12] and McLean and Postlewaite [11]. Our paper, by contrast, studies the efficiency properties of a particular institution, namely the competitive system of price determination and allows for the possibility that some individuals may retain a large impact on market outcomes, even in the limit. This should be compared with Gul and Postlewaite [5], the paper most closely related to ours. They consider

[3] Roberts and Postlewaite [14] define incentive compatibility relative to the competitive correspondence, a definition that differs (and is weaker than) the current standard where incentive compatibility is defined relative to mechanisms that select a particular outcome for any report profile.

the asymptotic efficiency of a Walrasian-like direct mechanism in a replica economy. [4] In their mechanism individuals' report their types and receive (slightly distorted) competitive allocations relative to a competitive equilibrium of the reported economy. The competitive equilibrium they use is anonymous (depends only on the frequencies of various types) and their result is that there is at least one Bayesian–Nash equilibrium where traders report truthfully and efficiency is achieved. [5]

The paper may also be compared to the literature on efficiency of large auctions, including Gresik and Satterthwaite [4], Rustichini et al. [15], Swinkels [16], Reny and Perry [13], Cripps and Swinkels [2], to name a few. In these papers, the goal is to examine the properties of a specific auction institution rather than design an optimal auction form. Competitive mechanisms may be viewed as natural extensions of double-auctions to divisible, multi-good environments. Compared to exchange economies, the environments studied in the large auction literature are restrictive. Assumptions like single good; discrete number of units of the good, symmetric distribution of valuations, and so on, are common. On the other hand, the large auctions framework provides a significant advantage, namely that strategic interactions underlying the price formation process can be explicitly modeled.

2. The model

2.1. Complete information economies

An *exchange economy with complete information* with N agents, generically denoted by E^N, is a pair (θ^N, w^N) where $\theta^N = (\theta_1, \ldots, \theta_N) \in \Theta^N$ is a vector of types, and $w^N = (w_1, \ldots, w_N)$ is a vector of initial endowments, with each w_n belonging to the commodity space R^l_{++}. [6] Here, types are utility functions, so agent n of type θ_n has utility function U_{θ_n}. [7]

2.1.1. Endowments

We shall assume that, with large enough N, each trader owns a negligible portion of the economy's aggregate endowments. Formally, there exist $\xi^+ > \xi^- > 0$ such that the endowment of every trader in every economy belongs to the cube

$$W \equiv [\xi^-(1, \ldots, 1), \xi^+(1, \ldots, 1)].$$

We use $W^N \subset R^l_{++}$ to denote the set of initial endowment vectors for economies of size N. The set W is assumed fixed throughout the paper.

2.1.2. Utilities

There is a finite set of possible utility functions $\{U_\theta; \theta \in \Theta\}$, where Θ is a finite index set with cardinality $|\Theta|$. Each utility function

$$U_\theta : R^l_{++} \to R$$

[4] A key way in which their results are stronger is that they allow (a limited form of) common values.

[5] This is referred to as "weak implementation." See Gul and Postlewaite [5, Remark 4, pp. 1254–5] for a discussion of this point.

[6] *Notation*: We use superscripts to denote the size of the economy, and subscripts to index the agents in a given economy.

[7] The reference to the number of goods, l, is suppressed, as it is held fixed throughout the analysis.

is assumed to be continuous and corresponds to a preference that is strictly monotone, strictly convex and satisfies the boundary condition.[8] The set of utility functions Θ is assumed fixed throughout the paper.[9] Let $\mathring{\Delta}$ denote the subset of the unit simplex in R^l consisting of strictly positive prices. Our assumptions on the utility functions imply that the demand function

$$D_\theta : \mathring{\Delta} \times R^l_{++} \to R^l_{++}$$

corresponding to U_θ is well-defined, single valued, and continuous.

2.1.3. Sorting condition

We will require the following condition: for every $x \in R^l_{++}$, $\theta,\ \theta' \in \Theta$, for any pair of hyperplanes $H,\ H'$ in R^l such that H (resp. H') supports U_θ (resp. $U_{\theta'}$) at x, we have $H \neq H'$.

We call this a sorting condition because, given any (p, w), a trader of type θ demands a bundle that is distinct from that of a trader of a different type.

2.2. Competitive mechanisms and efficiency

Let $\mathcal{E}^N = \Theta^N \times W^N$ denote the space of complete information exchange economies with N agents. A *competitive equilibrium* for $E^N = (\theta^N, w^N) \in \mathcal{E}^N$ is a pair $(p, x^N) \in \mathring{\Delta} \times (R^l_{++})^N$ such that

1. Excess demand is non-positive: $\sum_{n=1}^N (x_n - w_n) \leqslant 0$.
2. Each agent maximizes utility: $x_n = D_{\theta_n}(p, w_n)$.

For $\eta > 0$, call (p, x^N) an *η-competitive equilibrium* if excess demand is non-positive, agents exhaust their budgets, and all but a fraction η of agents maximize utility. Formally[10]:

1. $\sum_{n=1}^N (x_n - w_n) \leqslant 0$.
2. $p \cdot x_n = p \cdot w_n$ for all n;
3. $\dfrac{\#\{n : |x_n - D_{\theta_n}(p, w_n)| \neq 0\}}{N} < \eta.$[11]

The *competitive equilibrium correspondence* is denoted[12]:

$$CE : \mathcal{E}^N \rightarrow\!\!\rightarrow \mathring{\Delta} \times (R^l_+)^N, \tag{1}$$

with generic value $(p, x^N) \in CE(E^N)$, where $p \in \mathring{\Delta}$ is an equilibrium price vector and $x^N = (x_1, \ldots, x_N) \in (R^l_+)^N$ a corresponding equilibrium allocation. For $\eta > 0$, let CE_η denote the η-competitive equilibrium correspondence.

[8] These are the preferences that correspond to $\P_{sc}$ in Mas-Colell [10, p. 168]. A preference $\succcurlyeq$ satisfies the boundary condition if for every bundle x, the weakly preferred set $\{y : y \succcurlyeq x\}$ is closed in R^l. See Mas-Colell [10, p. 68].

[9] Throughout the paper the cardinal measurement of utility is critical, so we do *not* reduce utility functions to their underlying preferences.

[10] The formal definition below is motivated by the conditions found in Hildenbrand [6, Proposition 7, p. 163]. Hildenbrand also requires that $|x_n - D_{\theta_n}(p, w_n)| < \eta$ for all n, a condition that need not hold in our main theorems.

[11] For $x \in R^k,\ k = 1, 2 \ldots$, we use $|x|$ to denote the max norm of x. For subsets $A,\ B$ of R^k, we abuse notation and write $|A - B|$ to denote the Hausdorff distance between A and B relative the max norm.

[12] Obviously CE should be indexed by N. We suppress this, however, since N will be clear from the context.

A function

$$\sigma : \mathcal{E}^N \to \mathring{\Delta} \times (R^l_+)^N \tag{2}$$

is a *competitive mechanism* if it is a selection of the competitive correspondence CE. We shall use the notation $\sigma_p(E^N)$ and $\sigma_{x^N}(E^N)$ to denote the price vector and the allocation implied by σ. Allowing *any* selection of the competitive correspondence reflects our ignorance of the detailed processes through which competitive markets operate.

Given $E^N = (\theta^N, w^N)$ and $\eta > 0$, a feasible allocation x^N is (ex post) η-*efficient* if there is no feasible allocation $\check{x}^N$ such that

$$U_{\theta_n}(\check{x}_n) > U_{\theta_n}(x_n) + \eta, \quad n = 1, \ldots, N. \tag{3}$$

Call x^N *efficient* if it is 0-efficient. [13]

Given an economy (θ^N, w^N), $Eff(\theta^N, w^N)$ and $Eff_\eta(\theta^N, w^N)$ will denote the set of all efficient and η-efficient allocations respectively.

2.3. Strategic vs. non-strategic types

Our primary interest is the behavior of strategically minded, privately informed traders. Our analysis will require, however, the consideration of the possibility that, with small probability, these traders may be non-strategic. By itself, the behavior of non-strategic traders is of little interest, since they truthfully report their private information. But their presence can exert significant influence on the behavior of strategic traders, as we shall see below.

To make this formal, a *non-strategic type* for trader n is one who always reports his true utility function. Let $\bar{\Theta} = \Theta$ denote the set of possible utility functions of a non-strategic type, with generic element denoted $\bar{\theta}_n$. A *strategic type*, on the other hand, is one who is free to mis-report his utility. We use $\hat{\Theta} = \Theta$ to denote the set of possible utility functions of a strategic trader, with generic element denoted $\hat{\theta}_n$. Finally, for each n, there is a 0–1 random variable, χ_n, that determines whether trader n is strategic ($\chi_n = 0$) or not ($\chi_n = 1$).

Thus, the type space is the product $\hat{\Theta} \times \bar{\Theta} \times \{0, 1\}$. Let Ψ^N denote the space of all distributions over the space of type profiles $[\hat{\Theta} \times \bar{\Theta} \times \{0, 1\}]^N$, with generic element ψ^N.

We will focus on a particular subset $\Psi^N(\varepsilon_c, \varepsilon_\chi) \subset \Psi^N$, where $\varepsilon_c,\ \varepsilon_\chi \in (0, 1]$. For a $\psi^N \in \Psi^N$, let $\psi^N_s,\ \psi^N_c,\ \psi^N_\chi$ denote its marginals on $\hat{\Theta}^N,\ \bar{\Theta}^N$ and $\{0, 1\}^N$, respectively. Then ψ^N belongs to $\Psi^N(\varepsilon_c, \varepsilon_\chi)$ if:

1. ψ^N_c and ψ^N_χ have independent marginals across n.
2. For all n, $\psi^N_c(\bar{\theta}_n = \theta) \geqslant \varepsilon_c \ \ \forall \theta \in \bar{\Theta}$.
3. For all n, $\psi^N_\chi(\chi_n = 1) \geqslant \varepsilon_\chi$.
4. Finally, we assign each trader his *true type*, θ_n, as follows: $\theta_n = \hat{\theta}_n$ if $\chi_n = 0$ and $\theta_n = \bar{\theta}_n$ if $\chi_n = 1$.

Note that Nature in this model first independently draws a utility-type for each of the strategic and non-strategic versions of a player, then flips an independent coin to determine whether this

[13] Under our assumption that preferences are monotonic, this coincides with the more common definition that requires Eq. (3) to hold with weak inequality for all n and strict inequality for at least one n.

player is strategic or not. An alternative representation of private information is that Nature first flips a coin to determine whether the player is strategic or not, then choose independently his utility-type. Note that the two formulations induce identical distributions on true types, as defined in item 4 above. The one we chose is more convenient in the proofs and allows us to be completely general about the distribution ψ_s^N of the strategic types.

2.4. Private information economies

A *private information economy*, generically denoted (ψ^N, w^N), consists of:

1. *Endowment profile*: $w^N \in W^N$;
2. *Type distribution*: Type profiles are drawn according to a probability distribution ψ^N on $[\hat{\Theta} \times \bar{\Theta} \times \{0,1\}]^N$;
3. *Information structure:* $\psi^N,\ w^N$ are common knowledge; each trader n is informed of his own type realization.

Note that we impose no conditions on the prior probability distribution, ψ^N, and any correlation is allowed. In particular, the distribution over strategic types, ψ_s^N, in a private information economy may be degenerate.

2.5. The market game

A competitive mechanism σ and a private information economy (ψ^N, w^N), give rise to a game of incomplete information $\Gamma(\sigma, \psi^N, w^N)$ as follows:

1. Types are drawn according to ψ^N.
2. A strategy for trader n is a reporting function $\tilde{\theta}_n : \hat{\Theta} \times \bar{\Theta} \times \{0,1\} \to \Delta(\tilde{\Theta})$, where $\tilde{\Theta} = \Theta$ denotes the set of possible reports, with generic element $\tilde{\theta}$. We require that $\tilde{\theta}_n(\hat{\theta}_n, \bar{\theta}, 1) = \bar{\theta}_n$ for all $\bar{\theta}_n$ (that is, non-strategic types report truthfully). We write $\tilde{\theta}_n$ instead of $\tilde{\theta}_n(\hat{\theta}_n, \bar{\theta}, \chi_n)$ for simplicity. A strategy profile is denoted $\tilde{\theta}^N = (\tilde{\theta}_1, \ldots, \tilde{\theta}_N)$.
3. Given a vector of reports $\tilde{\theta}^N$, the competitive mechanism picks a competitive equilibrium $\sigma(\tilde{\theta}^N, w^N)$. Player n's payoff is $U_{\theta_n}(\sigma_{x_n}(\tilde{\theta}^N, w^N))$.
4. A type distribution ψ^N, a strategy profile $\tilde{\theta}^N$, and a competitive mechanism σ give rise to a probability distribution P on vectors of prices, types, reports, and allocations:

$$\mathring{\Delta} \times [\hat{\Theta} \times \bar{\Theta} \times \{0,1\} \times \Theta \times \tilde{\Theta} \times (R^l_{++})]^N.$$

This distribution is used, among other things, by traders to compute their expected payoffs.

We finally note that in this market game, (interim) individual rationality is automatically satisfied since non-strategic traders always report their true types, and strategic traders always have the option to do so.

3. Main results and their interpretation

3.1. Main results

Our first result concerns the relationship between Bayesian–Nash equilibrium outcomes and the competitive equilibria of the true economy:

Theorem 1. *For any $\eta > 0$, there is $\bar{\varepsilon} > 0$ such that for any $0 < \varepsilon < \bar{\varepsilon}$ and any pair $(\varepsilon_c, \varepsilon_\chi)$, satisfying $\varepsilon_c \cdot \varepsilon_\chi \geqslant \varepsilon$, there exists $\bar{N}$ such that for any private information economy (ψ^N, w^N), with $\psi^N \in \Psi^N(\varepsilon_c, \varepsilon_\chi)$ and $N > \bar{N}$, any competitive mechanism σ and any Bayesian–Nash equilibrium of $\Gamma(\sigma, \psi^N, w^N)$*

$$P\left\{\sigma(\tilde{\theta}^N, w^N) \in CE_\eta(\theta^N, w^N)\right\} > 1 - \eta. \tag{4}$$

The strength of the theorem is in the order of quantifiers: under our assumptions, once ε and $\bar{N}$ are chosen (as a function of the model's primitives and the desired degree of approximation η), the conclusion that prices and allocations are close to a competitive equilibrium of the true economy holds uniformly over all Bayes–Nash equilibria of all competitive mechanisms of private information economies with $N > \bar{N}$ traders.

The next theorem is our main result on efficiency:

Theorem 2. *For any $\eta > 0$, there is $\bar{\varepsilon} > 0$ such that for any $0 < \varepsilon < \bar{\varepsilon}$ and any pair $(\varepsilon_c, \varepsilon_\chi)$, satisfying $\varepsilon_c \cdot \varepsilon_\chi \geqslant \varepsilon$, there exists $\bar{N}$ such that for any private information economy (ψ^N, w^N), with $\psi^N \in \Psi^N(\varepsilon_c, \varepsilon_\chi)$ and $N > \bar{N}$, any competitive mechanism σ and any Bayesian–Nash equilibrium of $\Gamma(\sigma, \psi^N, w^N)$*

$$P\left\{\sigma_{x^N}(\tilde{\theta}^N, w^N) \in Eff_\eta(\theta^N, w^N)\right\} > 1 - \eta. \tag{5}$$

The interpretation of this result is similar. The remainder of this section provides intuition for the proofs and discusses some of the complications that arise. We also discuss alternative modeling assumptions.

3.2. Intuition

A basic intuition underlying all papers in this literature is that the influence of individual traders on market outcomes decreases as the economy becomes large. However, since we consider all mechanisms, including non-anonymous ones, it is possible that a non-vanishing fraction of traders retain significant influence regardless of the size of the economy. A key assumption to deal with this problem is that each trader is non-strategic with positive probability. This ensures that there is strategic uncertainty about the opponents' reports and about the outcome of the mechanism that is bounded away from zero uniformly in the number of traders and the strategies they play. Using the terminology we introduced in Al-Najjar and Smorodinsky [1], define the "influence" of a trader on an abstract outcome function as the maximum change in the expected outcome that can be caused by changes in this trader's actions. See also Fudenberg et al. [3] and Swinkels [16] for related concepts and results.

Earlier work provides a general bound on the average influence players possess in noisy environments (Lemma A.2). Although a useful start, this result is not sufficient to deal with the complications arising in our setting of exchange economies and incomplete information. A significant part of our argument is a build up to Lemma A.7 which provides bounds on traders' ability to influence prices. Roughly, the idea is to divide the simplex of prices into a finite collection of subsets, each of which is small enough that the payoff of all traders changes very little within each subset. Lemma A.7 shows that, uniformly in N, competitive mechanisms, equilibria and type distributions, all but a decreasing fraction of traders cannot shift the price vector from one component of the partition to another. As the number of traders increases, the fraction of potentially influential traders becomes small.

The next step is to show that traders with small influence report their types truthfully (Lemma A.9). Here the assumption of a finite type space plays an important role. With a *continuum* of types, it is still possible to show that the magnitude of a misrepresentation of traders with small influence is small. The difficulty is that there is no guarantee that a large number of small misrepresentations will not add up to a large distortion. With a discrete type space, any misrepresentation will lead to a discrete decrease in a trader's payoff. Unless this trader has a large offsetting influence on prices, he would strictly prefer to report truthfully.

In summary, the argument so far is that uniformly in N, competitive mechanisms, equilibria and type distributions, all but a vanishing fraction of traders report their types truthfully. There is little we can say about the behavior and impact of the influential traders. However, as N grows large, the economy with a decreasing fraction of misrepresentations becomes increasingly close in distribution to the true economy. Since the set of equilibria of a competitive mechanism depends only on the distribution of characteristics, a competitive equilibrium of an economy with a decreasing fraction of misreports is an approximate equilibrium of the true economy when N is large. Finally, a competitive equilibrium of an economy with a decreasing fraction of misreports is efficient for that economy, and so must be approximately efficient for the true economy.

3.3. The probabilistic nature of the results

The statements in Theorems 1 and 2 are probabilistic in nature. Our techniques do not allow us to strengthen them to deterministic ones. The following example elaborates on this point. It relies on facts of encoding theory to construct distributions in which, in every realization, the *noisy* reports of players $-n$ can be used to uniquely recover the type of player n.

Example. Consider an economy of N agents, where each agent can be one of three types. Formally, $\Theta = \{A, B, C\}$. Assume that whenever there are approximately $\frac{N}{2}$ agents of type A and approximately $\frac{N}{2}$ agents of type B then the economy possesses (at least) two equilibria prices, denoted P_{bad} and P_{good}. In addition, assume that the price vector P_{bad} is less desired to any agent of type C than P_{good}.

We now construct the set of possible type vector realizations. For that let us choose an arbitrary set of vectors, denoted by $\theta^N(n) \in \Theta^N$, $n = 1, \ldots, N$, with the following properties [14]:

- $|\{j : \theta_j^N(n) \neq \theta_j^N(m)\}| \geqslant \frac{N}{10} \quad \forall m \neq n.$
- $\frac{N}{2} \leqslant |\{j : \theta_j^N(n) = A\}| \leqslant \frac{N+1}{2}.$
- $\theta_n^N(n) = C.$

The interesting property of this set of vectors is that it is enough that most of the population reports truthfully (over 95%) for the mechanism to fully reconstruct the true vector, and in particular learn the value (identity) of n.

[14] The number of indices where two vectors do not coincide is known as the Hamming distance between two vectors. We adopt a technique from coding theory and show, in Lemma A.14, that for large enough values of N, it is possible to construct such N distinct vectors.

The probability distribution by which type vectors are chosen is as follows: (a) The vector $(C, C, C, \ldots, C)$ is assigned some small probability $\alpha > 0$. (b) The residual probability, $1 - \alpha$, is distributed evenly among the N vectors, $\theta^N(n)$, $n = 1, \ldots, N$.

Consider the following mechanism for choosing prices. If the reported economy is similar to one of the vectors $\theta^N(n) \in \Theta^N$ then if agent n reports B or C the chosen price is P_{bad} and otherwise it is P_{good}. Recall that agent n in $\theta^N(n)$ is in fact of type C.

Let us consider the analysis done by agent n who learns that he is of type C. This agent, assigns low a posteriori probability to the type vector $(C, C, C, \ldots, C)$ and high a posteriori probability to $\theta^N(n)$. No matter what he will report the resulting vector of reports will be quite similar to $\theta^N(n)$ in which case, due to the structure of the mechanism, he is pivotal and should report his type as A. As a result, if the vector $(C, C, \ldots, C)$ is actually realized, then all agents that are strategic will report A and so the reported vector is quite far from the true vector and we lose our efficiency result.

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Appendix A.

The main argument behind our results is that most traders in a large economy report truthfully:

Proposition 3. *Given any* $\delta > 0$, *there exists* $\bar{\varepsilon} > 0$ *such that for any* $0 < \varepsilon < \bar{\varepsilon}$ *and any pair* $(\varepsilon_c, \varepsilon_\chi)$, *satisfying* $\varepsilon_c \cdot \varepsilon_\chi \geqslant \varepsilon$, *there exists integers* $\bar{N}$, J, M *such that for any private information economy* (ψ^N, w^N), *with* $\psi^N \in \Psi^N(\varepsilon_c, \varepsilon_\chi)$ *and* $N > \bar{N}$, *any competitive mechanism* σ *and any Bayesian–Nash equilibrium of* $\Gamma(\sigma, \psi^N, w^N)$

$$P\left\{\#\{n : \tilde{\theta}_n \neq \theta_n\} \geqslant J \cdot M \cdot |\Theta| \cdot N^{3/4}\right\} < \delta.$$

The key feature of this proposition is the order of the quantifiers: the integers J and M do not depend on the size of the economy N, the competitive mechanism σ used, or the Bayesian–Nash equilibrium played (however, the identity of the traders who may not report the true type may depend on these factors).

Two ideas underlie the proposition: first, in a large economy, most traders will not be "pivotal" in determining equilibrium prices, a conclusion we reach in Lemma A.7. Here, we use the concept and results on influence from Al-Najjar and Smorodinsky [1]. See also Fudenberg et al. [3] who establish related results in the case where traders' types are independent and Swinkels [16] who develops similar notions for proving efficiency in large private value auctions. Second, non-pivotal traders strictly prefer to report truthfully, which is the conclusion of Lemma A.9.

A.1. An analysis of pivotalness

Throughout this section, we will fix $\varepsilon_c,\ \varepsilon_\chi,\ \varepsilon > 0$ such that $\varepsilon_c \cdot \varepsilon_\chi \geqslant \varepsilon$. Define the influence of the strategic type of trader n on a function $F : \tilde{\Theta}^N \to [0, 1]$, given a distribution $\psi^N \in \Psi^N(\varepsilon_c, \varepsilon_\chi)$

and a reporting strategy profile, $\tilde{\theta}^N$ (which need not be in equilibrium):

$$V_n(F; \hat{\theta}_n) = \max_{\tilde{\theta}_n} E(F|\hat{\theta}_n, \tilde{\theta}_n) - \min_{\tilde{\theta}_n} E(F|\hat{\theta}_n, \tilde{\theta}_n).$$

Here, expectations are taken with respect to the distribution on $[\hat{\Theta} \times \bar{\Theta} \times \{0, 1\} \times \Theta \times \tilde{\Theta}]^N$ determined by ψ^N and $\tilde{\theta}^N$. Informally, this is the maximal impact of trader n's report on F, conditional on knowing he is strategic and that his strategic type (and therefore also his true type) is $\hat{\theta}_n$.

The key result on influence we need is:

Lemma A.1. *There exists an integer K, such that for every N, $\psi^N \in \Psi^N(\varepsilon_c, \varepsilon_\chi)$, reporting strategy $\tilde{\theta}^N$ and function $F : \tilde{\Theta}^N \to [0, 1]$:*

$$\sum_{n=1}^{N} \sum_{\hat{\theta}_n \in \hat{\Theta}} \psi_s^N(\hat{\theta}_n) \cdot V_n(F; \hat{\theta}_n) < K\sqrt{N}.$$

To prove this result, we need two intermediate concepts of influence. The first is

$$V_n(F, \hat{\theta}^N) = \max_{\tilde{\theta}_n} E(F|\hat{\theta}^N, \tilde{\theta}_n) - \min_{\tilde{\theta}_n} E(F|\hat{\theta}^N, \tilde{\theta}_n).$$

This is the influence of the strategic type of trader n on F given a strategic type profile $\hat{\theta}^N$. (Note the difference between $V_n(F, \hat{\theta}_n)$ and $V_n(F, \hat{\theta}^N)$; the later calculates influence under the assumption that the entire vector of strategic types, $\hat{\theta}^N$, is known.) From Al-Najjar and Smorodinsky [1] we have the following result:

Lemma A.2. *There exists an integer K, such that for every N and function $F : \tilde{\Theta}^N \to [0, 1]$:*

$$\sum_{n=1}^{N} V_n(F; \hat{\theta}^N) < K\sqrt{N}.$$

Proof. Note that, conditional on knowing the vector of strategic types, traders' reports are independent. Furthermore, due to the existence of non-strategic types, any type will be reported with probability at least ε. Consequently Theorem 2 in Al-Najjar and Smorodinsky [1] applies, which is the desired inequality. □

To prove Lemma A.1, define $\tilde{\theta}_n^+$ and $\tilde{\theta}_n^-$ to be the pair of types such that

$$V_n(F; \hat{\theta}_n) = E(F|\hat{\theta}_n, \tilde{\theta}_n^+) - E(F|\hat{\theta}_n, \tilde{\theta}_n^-).$$

Similarly, define $\check{\theta}_n^+ = \check{\theta}_n^+(\hat{\theta}^N)$ and $\check{\theta}_n^- = \check{\theta}_n^-(\hat{\theta}^N)$ to be the pair of types such that

$$V_n(F; \hat{\theta}^N) = E(F|\hat{\theta}^N, \check{\theta}_n^+) - E(F|\hat{\theta}^N, \check{\theta}_n^-).$$

Proof of Lemma A.1.

$$\begin{aligned}
V_n(F;\hat{\theta}_n) &= E(F|\hat{\theta}_n,\tilde{\theta}_n^+) - E(F|\hat{\theta}_n,\tilde{\theta}_n^-)\\
&= \sum_{\hat{\theta}^N\in\hat{\Theta}^N} \psi_s^N(\hat{\theta}^N|\hat{\theta}_n)\,[E(F|\hat{\theta}^N,\tilde{\theta}_n^+) - E(F|\hat{\theta}^N,\tilde{\theta}_n^-)]\\
&\leqslant \sum_{\hat{\theta}^N\in\hat{\Theta}^N} \psi_s^N(\hat{\theta}^N|\hat{\theta}_n)\,[E(F|\hat{\theta}^N,\tilde{\theta}_n=\check{\theta}_n^+(\theta^N)) - E(F|\hat{\theta}^N,\tilde{\theta}_n=\check{\theta}_n^-(\theta^N))]\\
&= \sum_{\hat{\theta}^N\in\hat{\Theta}^N} \psi_s^N(\hat{\theta}^N|\hat{\theta}_n)\,V_n(F;\hat{\theta}^N).
\end{aligned}$$

Averaging over the possible types of trader n:

$$\begin{aligned}
\sum_{\hat{\theta}_n\in\hat{\Theta}} \psi_s^N(\hat{\theta}_n)\,V_n(F;\hat{\theta}_n) &\leqslant \sum_{\hat{\theta}_n\in\hat{\Theta}} \psi_s^N(\hat{\theta}_n) \sum_{\hat{\theta}^N\in\hat{\Theta}^N} \psi_s^N(\hat{\theta}^N|\hat{\theta}_n)\,V_n(F;\hat{\theta}^N)\\
&= \sum_{\hat{\theta}^N\in\hat{\Theta}^N} V_n(F;\hat{\theta}^N) \sum_{\hat{\theta}_n\in\hat{\Theta}} \psi_s^N(\hat{\theta}^N|\hat{\theta}_n)\psi_s^N(\hat{\theta}_n)\\
&= \sum_{\hat{\theta}^N\in\hat{\Theta}^N} \psi_s^N(\hat{\theta}^N)\,V_n(F;\hat{\theta}^N).
\end{aligned}$$

Summing over n, we obtain:

$$\begin{aligned}
\sum_{n=1}^{N}\sum_{\hat{\theta}_n\in\hat{\Theta}} \psi_s^N(\hat{\theta}_n)\,V_n(F;\hat{\theta}_n) &\leqslant \sum_{n=1}^{N}\sum_{\hat{\theta}^N\in\hat{\Theta}^N} \psi_s^N(\hat{\theta}^N)\,V_n(F;\hat{\theta}^N)\\
&= \sum_{\hat{\theta}^N\in\hat{\Theta}^N} \psi_s^N(\hat{\theta}^N)\sum_{n=1}^{N} V_n(F;\hat{\theta}^N).
\end{aligned}$$

We apply Lemma A.2 to obtain:

$$\sum_{n=1}^{N}\sum_{\hat{\theta}_n\in\hat{\Theta}} \psi_s^N(\hat{\theta}_n)\,V_n(F;\hat{\theta}_n) \leqslant \sum_{\hat{\theta}^N\in\hat{\Theta}^N} \psi_s^N(\hat{\theta}^N)\,K\sqrt{N} = K\sqrt{N}. \qquad \square$$

The following observation follows from the Markov inequality:

Lemma A.3. *Let Q be an arbitrary probability distribution on Θ^N and $G_n:\Theta\to\{0,1\}$, $n=1,\ldots,N$, an arbitrary set of functions. Then for every $L\in[0,N]$,*

$$LQ\left\{\sum_{n=1}^{N} G_n(\theta_n)\geqslant L\right\} \leqslant \sum_{n=1}^{N} Q\big\{G_n(\theta_n)=1\big\}.$$

Proof. The Markov inequality states that for every non-negative random variable, X, and any number $L \geqslant 0$ we have $L \cdot Prob(X \geqslant L) \leqslant E(X)$. Now set $X = \sum_n G_n(\theta_n)$ and use the linearity of the expectation to get the desired result. [15] □

Lemma A.4. *For every* α, $\check{\delta} > 0$ *there exists* $\bar{N}$ *such that for any* $N > \bar{N}$, *and function* $F : \tilde{\Theta}^N \to [0, 1]$

$$\psi_s^n \left\{ \#\{n : V_n(F; \hat{\theta}_n) > \alpha\} > N^{\frac{3}{4}} \right\} < \check{\delta}.$$

That is, there is low probability of drawing a type profile with more than $N^{\frac{3}{4}}$ traders with large influence on an arbitrary function F.

Proof. We begin by applying Lemma A.3 to the functions: $G_n = \chi_{\{V_n(F;\hat{\theta}_n)>\alpha\}}$, the indicator function of the set $\{\hat{\theta}_n : V_n(F; \hat{\theta}_n) > \alpha\}$, to obtain:

$$N^{\frac{3}{4}} \psi_s^N \left\{ \#\{n : V_n(F; \hat{\theta}_n) > \alpha\} > N^{\frac{3}{4}} \right\} \leqslant \sum_{n=1}^{N} \psi_s^N \left\{ V_n(F; \hat{\theta}_n) > \alpha \right\}. \tag{6}$$

Observe that for any trader n and any strategic type $\hat{\theta}_n$ it is always the case that $\alpha\, \chi_{\{V_n(F;\hat{\theta}_n)>\alpha\}} \leqslant V_n(F; \hat{\theta}_n)$. Therefore

$$\begin{aligned}
\alpha \cdot \sum_{n=1}^{N} \psi_s^N \left\{ V_n(F; \hat{\theta}_n) > \alpha \right\} &= \alpha \sum_{n=1}^{N} \sum_{\hat{\theta}^N \in \hat{\Theta}^N} \psi_s^N(\hat{\theta}^N)\, [\chi_{\{V_n(F;\hat{\theta}_n)>\alpha\}}] \\
&= \sum_{n=1}^{N} \sum_{\hat{\theta}^N \in \hat{\Theta}^N} \psi_s^N(\hat{\theta}^N)\, [\alpha \chi_{\{V_n(F;\hat{\theta}_n)>\alpha\}}] \\
&\leqslant \sum_{n=1}^{N} \sum_{\hat{\theta}^N \in \hat{\Theta}^N} \psi_s^N(\hat{\theta}^N)\, V_n(F; \hat{\theta}_n) \\
&= \sum_{n=1}^{N} \sum_{\hat{\theta}_n \in \hat{\Theta}} \psi_s^N(\hat{\theta}_n)\, V_n(F; \hat{\theta}_n).
\end{aligned}$$

Combining this fact with Eq. (6) and Lemma A.1, we conclude that

$$\psi_s^N \left\{ \#\{n : V_n(F; \hat{\theta}_n) > \alpha\} > N^{\frac{3}{4}} \right\} \leqslant \frac{K\sqrt{N}}{\alpha N^{\frac{3}{4}}}.$$

The proof is obtained by taking N large enough such that $\frac{K\sqrt{N}}{\alpha N^{\frac{3}{4}}}$ is smaller than $\check{\delta}$. □

[15] The proof of the Markov inequality is straightforward: if X is positive random variable then: $E(X) = \sum_i x_i\, P(X = x_i) \geqslant \sum_{i:x_i \geqslant L} x_i\, P(X = x_i) \geqslant \sum_{i:x_i \geqslant L} L\, P(X = x_i) = LP(X \geqslant L)$.

A.2. Results from general equilibrium theory

In this subsection, and in this subsection only, we allow economies with a continuum of agents, so we assume $N \in \{1, 2, \ldots\} \cup \infty$. Here, E^N with $N = \infty$ will refer to an economy with a continuum of agents, in the sense of Mas-Colell [10].

Let $\mathcal{E} = \bigcup_{N=1}^{\infty} \mathcal{E}^N \cup \mathcal{E}^\infty$ denote the space of complete information economies of all cardinalities. Endow $\mathcal{E}$ with the notion of convergence defined in Mas-Colell [10, pp. 222–223]. Under this notion of convergence, a sequence of economies $\{E_k^{N_k}\}$, with cardinalities N_k, converge to an economy E^N (with possibly $N = \infty$) if:

1. $N_k \to N$, where $N = \infty$ if $\{N_k\}$ is unbounded.
2. The distribution on characteristics in economy $E_k^{N_k}$, denoted μ_k, converges weakly to the distribution μ of E^N.
3. The support of μ_k converges to μ in the Hausdorff metric.

Let $f : \overset{\circ}{\Delta} \times \mathcal{E} \to \mathbb{R}^l$ denote the excess demand function:

$$f(p, (\theta^N, w^N)) = \sum_{n=1}^{N} [D_{\theta_n}(p, w_n) - w_n].$$

Proposition 5.8.3 in Mas-Colell [10, p. 224] shows that f is continuous and satisfies the boundary condition. That is, for any sequence of economies $\{E_k\}$ in $\mathcal{E}$ (with possibly varying cardinalities) and a corresponding sequence of prices $\{p_k\}$ such that $E_k \to E$, $p_k \to p$, and p belongs to the boundary of the simplex, then $f(p_k, E_k) \to \infty$. Furthermore, Proposition 5.8.1 in Mas-Colell [10, p. 223] shows that $\mathcal{E}$ is metrizable and separable and, under our assumption that the space of characteristics is compact, is itself a compact space.

We will make use of some elementary results:

Lemma A.5. *There is a compact set $\Delta \subset \overset{\circ}{\Delta}$ such that every competitive equilibrium price vector for $E \in \mathcal{E}$ belongs to Δ.*

Proof. If this were not true, then there is a sequence of economies $\{E_k\}$ in $\mathcal{E}$ that has an accumulation point p with at least one price equal to zero. (Note that the E_k's have varying number of agents. We do not refer to N in this proof, however, to simplify notation.) Since the space of economies is compact, passing to subsequences if necessary, we may assume that $E_k \to E$ and $p_k \to p$. Since the excess demand function is both continuous and satisfies the boundary condition, $f(p, E) = \lim_{k\to\infty} f(p_k, E_k) = \infty$. A contradiction. □

Lemma A.6. *For any $\beta > 0$ there exists a finite set of endowments $\check{W} = \{w(1), \ldots, w(M)\} \subset W$ such that for any $w \in W$ there exists $\check{w} \in \check{W}$ with*

$$|E_\nu U_\theta(D_\theta(p, w)) - E_\nu U_\theta(D_\theta(p, \check{w}))| < \beta/3$$

uniformly over all distributions ν over Δ (*where Δ is the compact set provided by Lemma* A.5).

Proof. Fix θ. By our assumptions on the U_θ's and D_θ's, the function $\max_{p\in\Delta} |U_\theta(D_\theta(p, w)) - U_\theta(D_\theta(p, w'))|$ is continuous on the compact set $W \times W$, and hence uniformly continuous.

Thus, for every $\beta > 0$ there is $\varepsilon > 0$ such that $|w - w'| < \varepsilon$ implies that $|U_\theta(D_\theta(p, w)) - U_\theta(D_\theta(p, w'))| < \beta/3$ for all $p \in \Delta$. The desired result is obtained by choosing a set $\{w^1, \ldots, w^M\} \subset W$ so that ε-balls centered around $w(m)$, $m = 1, \ldots, M$ constitute a cover of W.

Finally, note that for all $p \in \Delta$

$$\begin{aligned}&|E_\nu U_\theta(D_\theta(p, w)) - E_\nu U_\theta(D_\theta(p, w(m)))|\\ &\quad\leqslant \int_\Delta |U_\theta(D_\theta(p, w)) - U_\theta(D_\theta(p, w(m)))| d\nu(p)\\ &\quad\leqslant \int_\Delta \beta/3\, d\nu(p) = \beta/3. \qquad \square\end{aligned}$$

A.3. Players' influence on prices

We use Lemma A.4 to bound traders' influence on prices in equilibrium. The arbitrary function F in that lemma will now be replaced by functions $F^{j,\theta_n,m}_{w^N}$ defined below. First, define

$$z = \max\left\{U_\theta(D_\theta(p, w)) : \theta \in \Theta,\, p \in \Delta,\, w \in W\right\}.$$

This is well defined since U_θ and D_θ are continuous and Θ, Δ, W are compact.

Given $\theta \in \Theta$, $j \in \{1, \ldots, J\}$ and $m \in \{1, \ldots, M\}$, define:

$$A^{j,\theta,m} = \left\{p \,:\, z\,\frac{j-1}{J} \leqslant U_\theta(D_\theta(p, w(m))) \leqslant z\,\frac{j}{J}\right\}.$$

That is, $A^{j,\theta,m}$ is the set of all prices that keep the utility of a trader of type θ and endowment $w(m)$ within the interval $z\left[\frac{j-1}{J}, \frac{j}{J}\right]$.

Also, given a vector of initial endowments, $w^N \in W^N$, let

$$B^{j,\theta,m}_{w^N} = \left\{\tilde{\theta}^N : \sigma_p(\tilde{\theta}^N, w^N) \in A^{j,\theta,m}\right\}.$$ [16]

In words, $B^{j,\theta,m}_{w^N}$ is the set of all type profiles which, when combined with w^N, result in an equilibrium price vector belonging to $A^{j,\theta,m}$. Let $F^{j,\theta,m}_{w^N} : \tilde{\Theta}^N \to \{0, 1\}$ be the indicator function of $B^{j,\theta,m}_{w^N}$. We are interested in the impact of the strategic type of trader n on $F^{j,\theta,m}_{w^N}$. Given $\tilde{\theta}', \tilde{\theta}'' \in \tilde{\Theta}$:

$$\begin{aligned}&P(B^{j,\theta,m}_{w^N} | \hat{\theta}_n = \theta', \tilde{\theta}_n = \theta') - P(B^{j,\theta,m}_{w^N} | \hat{\theta}_n = \theta', \tilde{\theta}_n = \theta'')\\ &\quad= E_P(F^{j,\theta,m}_{w^N} | \hat{\theta}_n = \theta', \tilde{\theta}_n = \theta') - E_P(F^{j,\theta,m}_{w^N} | \hat{\theta}_n = \theta', \tilde{\theta}_n = \theta'')\\ &\quad\leqslant V_n(F^{j,\theta,m}_{w^N}; \hat{\theta}_n = \theta').\end{aligned}$$

[16] Note that we are using two values of endowment in this definition: the mechanism σ uses true endowment, approximate endowment $w(m)$ is used in $A^{j,\theta,m}$.

Lemma A.7. *For every $\alpha > 0$ and $\delta > 0$, there exists $\bar{N}$ such that for any $N > \bar{N}$, private information economy $(\psi^N, w^N) \in \Psi^N(\varepsilon_c, \varepsilon_\chi)$:*

$$P\left\{ \#\left\{n : \exists(j,\theta,m) \;\; V_n(F^{j,\theta,m}_{w^N}; \hat{\theta}_n) > \alpha\right\} > J \cdot M \cdot |\Theta| \cdot N^{\frac{3}{4}} \right\} \leqslant \delta.$$

That is, with high probability, the number of traders who may have significant influence over the price vector is bounded by $N^{\frac{3}{4}}$ times a constant that does not depend on N.

Proof. Apply Lemma A.4 with $\check{\delta} = \frac{\delta}{J \cdot M \cdot |\Theta|}$ to conclude:

$$\begin{aligned}
&P\left\{ \#\left\{n : \exists(j,\theta,m) V_n(F^{j,\theta,m}_{w^N}; \hat{\theta}_n) > \alpha\right\} > J \cdot M \cdot |\Theta| \cdot N^{\frac{3}{4}} \right\} \\
&\leqslant P\left\{ \exists(j,\theta,m) : \; \#\left\{n \mid V_n(F^{j,\theta,m}_{w^N}; \hat{\theta}_n\} > \alpha\right\} > N^{\frac{3}{4}} \right\} \\
&\leqslant P\Big(\bigcup_{j,\theta,m} \left\{ \#\{n \mid V_n(F^{j,\theta,m}_{w^N}; \hat{\theta}_n\} > \alpha\} > N^{\frac{3}{4}} \right\} \Big) \\
&\leqslant \sum_{j,\theta,m} P\left\{ \#\{n \mid V_n(F^{j,\theta,m}_{w^N}; \hat{\theta}_n\} > \alpha\} > N^{\frac{3}{4}} \right\} \\
&\leqslant J \cdot M \cdot |\Theta| \cdot \frac{\delta}{J \cdot M \cdot |\Theta|} = \delta. \qquad \square
\end{aligned}$$

A.4. Small influence leads to truth-telling

We show that a utility maximizing trader with small influence on the price vector reports his true type as his unique best response. First, we need the following lemma:

Lemma A.8. *Under the sorting condition, there is $\beta > 0$ such that for every $\theta, \theta' \in \Theta$, $w \in W$, $p \in \Delta$, $U_\theta(D_\theta(p,w)) - U_\theta(D_{\theta'}(p,w)) > \beta$.*

Proof. The sorting condition implies that $D_\theta(p,w) \neq D_{\theta'}(p,w)$ for any $p \in \Delta$. Utility maximization and strict convexity in turn imply that $U_\theta(D_\theta(p,w)) - U_\theta(D_{\theta'}(p,w)) > 0$ for any p and w. The conclusion of the lemma now follows from the facts that $(p,w) \mapsto U_\theta(D_\theta(p,w)) - U_{\theta'}(D_{\theta'}(p,w))$ is a strictly positive, continuous function on the compact domain $\Delta \times W$ and thus must have a strictly positive minimum. $\square$

Recall that $\sigma_{x_n}(E^N)$ denotes the projection of $\sigma(E^N)$ on x_n, i.e., the consumption bundle of agent n implied by the mechanism σ when applied to the economy E^N.

Lemma A.9. *There is $\alpha > 0$ small enough such that if $V_n(F^{j,\theta,m}_{w^N}; \hat{\theta}_n) < \alpha$ for all j, θ, m then for all $\theta', \theta'' \in \Theta$, $\theta' \neq \theta''$ and any initial endowment w^N,*

$$\begin{aligned}
&E_P\big(U_{\theta'}(\sigma_{x_n}(\tilde{\theta}^N, w^N)) \big| \, \hat{\theta}_n = \theta', \chi_n = 0, \tilde{\theta}_n = \theta'\big) \\
&\quad > E_P\big(U_{\theta'}(\sigma_{x_n}(\tilde{\theta}^N, w^N)) \big| \, \hat{\theta}_n = \theta', \chi_n = 0, \tilde{\theta}_n = \theta''\big).
\end{aligned}$$

Proof. By Lemma A.8 there exists $\beta > 0$ such that for every price vector $p \in \Delta$, every θ', θ'' and $w \in W$

$$U_{\theta'}(D_{\theta'}(p, w)) - U_{\theta'}(D_{\theta''}(p, w)) > \beta.$$

Based on the value of β we now choose J to be large enough so $\frac{1}{J} < \frac{\beta}{9}$ and choose α to be sufficiently small to satisfy $J \cdot \alpha < \frac{\beta}{9}$.

Suppose that $V_n(F^{j,\theta,m}_{w^N}; \hat{\theta}_n) < \alpha$ for all j, θ, m. Consider the LHS of the desired inequality. By Lemma A.6, there exists an endowment, $w(m)$, in the grid of endowments, such that the following holds (conditioning on $\chi_n = 0$ is assumed throughout but suppressed for notational simplicity):

$$\begin{aligned}
&E_P\big(U_{\theta'}(\sigma_{x_n}(\tilde{\theta}^N, w^N))\big|\hat{\theta}_n = \theta', \tilde{\theta}_n = \theta'\big)\\
&\quad= \sum_{\tilde{\theta}^N \in \tilde{\Theta}^N} U_{\theta'}(\sigma_{x_n}(\tilde{\theta}^N, w^N))\ P(\tilde{\theta}^N|\hat{\theta}_n = \theta', \tilde{\theta}_n = \theta')\\
&\quad\geqslant \sum_{\tilde{\theta}^N \in \tilde{\Theta}^N} U_{\theta'}(D_{\tilde{\theta}}(\sigma_p(\tilde{\theta}^N, w^N), w(m)))P(\tilde{\theta}^N|\hat{\theta}_n = \theta', \tilde{\theta}_n = \theta') - \frac{\beta}{3}\\
&\quad= \sum_j \sum_{\tilde{\theta}^N \in B^{j,\theta',m}_{w^N}} U_{\theta'}(D_{\theta'}(\sigma_p(\tilde{\theta}^N, w^N), w(m)))P(\tilde{\theta}^N|\hat{\theta}_n = \theta', \tilde{\theta}_n = \theta') - \frac{\beta}{3}\\
&\quad\geqslant \sum_j \sum_{\tilde{\theta}^N \in B^{j,\theta',m}_{w^N}} \frac{j}{J}\ P(\tilde{\theta}^N|\hat{\theta}_n = \theta', \tilde{\theta}_n = \theta') - \frac{\beta}{3}\\
&\quad= \sum_j \frac{j}{J} \cdot P(B^{j,\theta',m}_{w^N}\ |\hat{\theta}_n = \theta', \tilde{\theta}_n = \theta') - \frac{\beta}{3}.
\end{aligned}$$

Now consider the RHS of the inequality. For the same endowment, $w(m)$, the following holds:

$$\begin{aligned}
&E_P\big(U_{\theta'}(\sigma_{x_n}(\tilde{\theta}^N, w^N))\big|\hat{\theta}_n = \theta', \tilde{\theta}_n = \theta''\big)\\
&\quad= \sum_{\tilde{\theta}^N \in \tilde{\Theta}^N} U_{\theta'}(\sigma_{x_n}(\tilde{\theta}^N, w^N))\ P(\tilde{\theta}^N|\hat{\theta}_n = \theta', \tilde{\theta}_n = \theta'')\\
&\quad\leqslant \sum_{\tilde{\theta}^N \in \tilde{\Theta}^N} U_{\theta'}(D_{\theta''}(\sigma_p(\tilde{\theta}^N, w^N), w(m)))P(\tilde{\theta}^N|\hat{\theta}_n = \theta', \tilde{\theta}_n = \theta'') + \frac{\beta}{3}\\
&\quad\leqslant \sum_{\tilde{\theta}^N \in \tilde{\Theta}^N} U_{\theta'}(D_{\theta'}(\sigma_p(\tilde{\theta}^N, w^N), w(m)))P(\tilde{\theta}^N|\hat{\theta}_n = \theta', \tilde{\theta}_n = \theta'') - \beta + \frac{\beta}{3}\\
&\quad= \sum_j \sum_{\tilde{\theta}^N \in B^{j,\theta',m}_{w^N}} U_{\theta'}(D_{\theta'}(\sigma_p(\tilde{\theta}^N, w(m)))P(\tilde{\theta}^N|\hat{\theta}_n = \theta', \tilde{\theta}_n = \theta'') - \frac{2\beta}{3}\\
&\quad\leqslant \sum_j \sum_{\tilde{\theta}^N \in B^{j,\theta',m}_{w^N}} \frac{j+1}{J} P(\tilde{\theta}^N|\hat{\theta}_n = \theta', \tilde{\theta}_n = \theta'')\ - \frac{2\beta}{3}\\
&\quad= \sum_j \frac{j+1}{J} \cdot P(B^{j,\theta',m}_{w^N}|\hat{\theta}_n = \theta', \tilde{\theta}_n = \theta'') - \frac{2\beta}{3}.
\end{aligned}$$

Subtracting the RHS from the LHS we get:

$$\begin{aligned}
LHS - RHS &\geqslant \sum_j \frac{j}{J} P(B_{w^N}^{j,\theta',m} | \hat{\theta}_n = \theta', \tilde{\theta}_n = \theta') - \frac{\beta}{3} \\
&\quad - \sum_j \frac{j+1}{J} \cdot P(B_{w^N}^{j,\theta',m} | \hat{\theta}_n = \theta', \tilde{\theta}_n = \theta'') + \frac{2\beta}{3} \\
&= \sum_j \frac{j}{J} \big(P(B_{w^N}^{j,\theta',m} | \hat{\theta}_n = \theta', \tilde{\theta}_n = \theta') - P(B_{w^N}^{j,\theta',m} | \hat{\theta}_n = \theta', \tilde{\theta}_n = \theta'') \big) \\
&\quad - \sum_j \frac{1}{J} \cdot P(B_{w^N}^{j,\theta',m} | \hat{\theta}_n = \theta', \tilde{\theta}_n = \theta'') + \frac{\beta}{3} \\
&\geqslant - \sum_j V_n(F_{w^N}^{j,\theta',m}; \theta') - \frac{1}{J} + \frac{\beta}{3} \\
&\geqslant - \sum_j \alpha - \frac{1}{J} + \frac{\beta}{3} \\
&= -J \cdot \alpha - \frac{1}{J} + \frac{\beta}{3}.
\end{aligned}$$

By the choice of J and α LHS-RHS $\geqslant \frac{\beta}{9}$, and in particular positive. $\square$

Lemma A.10. *Given any $\delta > 0$, there exists $\bar{\varepsilon} > 0$ such that for any $0 < \varepsilon < \bar{\varepsilon}$ and any pair $(\varepsilon_c, \varepsilon_\chi)$, satisfying $\varepsilon_c \cdot \varepsilon_\chi \geqslant \varepsilon$, there exists integers $\bar{N}$, J, M such that for any private information economy (ψ^N, w^N), with $\psi^N \in \Psi^N(\varepsilon_c, \varepsilon_\chi)$ and $N > \bar{N}$, any competitive mechanism σ and any Bayesian–Nash equilibrium of $\Gamma(\sigma, \psi^N, w^N)$*

$$P\left\{ \#\{n : \chi_n = 0,\ \tilde{\theta}_n \neq \hat{\theta}_n\} \geqslant J \cdot M \cdot |\Theta| \cdot N^{3/4} \right\} < \delta.$$

Proof. Follows from Lemmas A.7 and A.9. $\square$

Proof of Proposition 3. By definition, $\tilde{\theta}_n = \bar{\theta}_n = \theta_n$ whenever $\chi_n = 1$.

$$\begin{aligned}
\#\{n : \tilde{\theta}_n \neq \theta_n\} &= \#\{n : \{\tilde{\theta}_n \neq \bar{\theta}_n \text{ and } \chi_n = 1\} \cup \{\tilde{\theta}_n \neq \hat{\theta}_n \text{ and } \chi_n = 0\}\} \\
&= \#\{n : \{\tilde{\theta}_n \neq \hat{\theta}_n \text{ and } \chi_n = 0\}\}.
\end{aligned}$$

The result now follows from Lemma A.10. $\square$

A.5. Proof of Theorem 1

The proof is straightforward: the requirements of non-positive excess demand and budget balance for all agents immediately follow from the fact that σ always selects a competitive equilibrium given any perturbed reported economy $(\tilde{\theta}^N, w^N)$. The only remaining requirement is that when N is large, with high probability, all but a vanishing fraction of agents choose bundles that are optimal relative to their true types. This however follows from Proposition 3.

A.6. Proof of Theorem 2

We first show the following:

Proposition 4. *For every $\eta > 0$ there is $\tau > 0$ such that: for any $E^N = (\theta^N, w^N)$, $\check{E}^N = (\check{\theta}^N, \check{w}^N) \in \mathcal{E}^N$ such that $w_n = \check{w}_n$ for every n, and $\frac{\#\{n:\theta_n \neq \check{\theta}_n\}}{N} < \tau$; if $x^N \in Eff_0(E^N)$ then $x^N \in Eff_\eta(\check{E}^N)$.*

That is, if E^N and $\check{E}^N$ are identical except that they may disagree about the types of no more than a fraction τ of agents, then any competitive equilibrium allocation for E^N is an η-efficient allocation for $\check{E}^N$. We first need some preliminary lemmas.

Lemma A.11. *Given δ we can find $\check{\delta}$ such that for all endowment vectors and type profiles, $|p - p'| < \check{\delta}$ implies that demands at these prices must be at most δ apart.*

Proof. This follows from the continuity of the demand functions, which in turn follows from our assumptions on the utility functions. □

Define

$$X = \{x = D_\theta(p, w) : \theta \in \Theta, p \in \Delta, w \in W\}$$

and let $\mathbf{1}$ be the unit vector in R^l (i.e.,the vectors with all entries equal to 1).

Lemma A.12. *For every $r > 0$ there is $q(r) > 0$ such that for every $x \in X$ and $\theta \in \Theta$:*

$$U_\theta(x) - \frac{r}{2} \leqslant U_\theta(x - q(r)\mathbf{1}).$$

Proof. If this were not true, then there is r, θ, x such that for every k

$$U_\theta(x) > U_\theta(x) - \frac{r}{2} > U_\theta\left(x - \frac{1}{k}\mathbf{1}\right).$$

This is impossible since U_θ is continuous on the compact set $X \times \Theta$ which makes it uniformly continuous. □

Lemma A.13. *There is $q^+ > 0$ such that for every $\theta \in \theta$ and x, $\check{x} \in X$, $U_\theta(\check{x} + q^+\mathbf{1}) > U_\theta(x)$.*

Proof. Obvious from the fact that Θ and X are compact. □

Proof of Proposition 4. Let x^N be as in the statement of the Proposition and assume, by way of contradiction, that $\check{x}^N$ is a feasible allocation with the property that $U_{\check{\theta}_n}(\check{x}_n) > U_{\check{\theta}_n}(x_n) + \eta$ for all n.

Define $I = \{n \in N : \theta_n = \check{\theta}_n\}$ and note that $\frac{\#I}{N} > 1 - \tau$. We form a new allocation $\dot{x}^N$ as follows:

$$\dot{x}_n = \begin{cases} \check{x}_n - q(\eta)\mathbf{1} & \text{if } n \in I, \\ \check{x}_n + \dfrac{\#I}{N - \#I}\, q(\eta)\,\mathbf{1} & \text{if } n \notin I, \end{cases}$$

where $q(\eta)$ is the multiple of $\mathbf{1}$ provided in Lemma A.12. That is, under the allocation $\dot{x}^N$ a quantity $q(\eta)$ is removed of each good from each member of I. The total collected, $[\#I \cdot q(\eta)]\mathbf{1}$, is distributed equally over the remaining $N - \#I$ agents.

Note that for every $n \in I$,

$$U_{\theta_n}(x_n) = U_{\check{\theta}_n}(x_n) < U_{\check{\theta}_n}(\check{x}_n) - \frac{\eta}{2} < U_{\check{\theta}_n}(\dot{x}_n) = U_{\theta_n}(\dot{x}_n), \tag{7}$$

so for these agents $\dot{x}^N$ dominates x^N in E^N.

On the other hand, we have $\frac{\#I}{N-\#I}\, q(\eta)\, \mathbf{1} > \frac{1-\tau}{\tau}\, q(\eta)\, \mathbf{1}$.

Fix η and choose τ small enough so that $\frac{1-\tau}{\tau}\, q(\eta) > q^+$, where q^+ is the number obtained in Lemma A.13. Then, for $n \notin I$

$$U_{\theta_n}(x_n) < U_{\theta_n}\left(\check{x}_n + \frac{1-\tau}{\tau}\, q(\eta)\, \mathbf{1}\right) \leqslant U_{\theta_n}(\dot{x}_n). \tag{8}$$

From Eqs. (7) and (8) we conclude that every agent strictly prefers $\dot{x}^N$ to x^N in E^N. This a contradiction with the assumption that x^N is efficient in E^N. □

Proof of Theorem 2. Fix $\eta > 0$. Use Proposition 4 to find τ with the properties asserted in that proposition. Given this value of $\tau > 0$, use Proposition 3 to find an $\bar{\varepsilon}$ that satisfies the requirements of the Theorem. □

A.7. Miscellaneous proofs

Lemma A.14. *There exists a set of N binary vectors in $\{0,1\}^N$, denoted $X(n)$, $n = 1, \ldots, N$, such that $|\{j : X_j(n) \neq X_j(m)\}| \geqslant \frac{N}{10}$ $\forall m \neq n$, and $|\{j : X_j(n) = 1\}| = \frac{N}{2}$.*

Proof. Assume N is large and is furthermore divisible by 10. Let $\mathcal{C} = \{X \in \{0,1\}^N : \sum_j X_j = \frac{N}{2}\}$. For each element in $X \in \mathcal{C}$ let us denote by $V(X) = \{Y \in \mathcal{C} : \sum_j |X_j - Y_j| \leqslant \frac{N}{10}\}$.

Let $X(1)$ be an arbitrary vector in $\mathcal{C}$. If we can choose $X(n)$, for $n = 2, \ldots, N$ such that $X(n) \in \mathcal{C} - \bigcup_{j=1}^{n-1} V(X(j))$, then we are done.

To show this it is sufficient to show that $\lim_{N\to\infty} \frac{|\mathcal{C}|}{|V(X)|} = \infty$. On the one hand, $|\mathcal{C}| = \frac{N!}{(\frac{N}{2})!(\frac{N}{2})!}$, which, by Stirling's formula, is approximately $\frac{2^N}{\sqrt{\pi N}}$.[17] On the other hand, note that $V(X) \subset \{Y \in \{0,1\}^N : \sum_j |X_j - Y_j| \leqslant \frac{N}{10}\}$. Therefore

$$|V(X)| \leqslant \left| \left\{ Y \in \{0,1\}^N : \sum_j |X_j - Y_j| \leqslant \frac{N}{10} \right\} \right| \leqslant 2^{\frac{N}{10}} \frac{N!}{\left(\frac{N}{10}\right)!\left(\frac{9N}{10}\right)!}.$$

Using Stirling's formula we deduce that $|V(X)| \leqslant \frac{1.5^N}{\sqrt{0.1\pi N}}$. Therefore

$$\lim_{N\to\infty} \frac{\mathcal{C}}{|V(X)|} \approx \lim_{N\to\infty} \frac{\frac{2^N}{\sqrt{\pi N}}}{\frac{1.5^N}{\sqrt{0.1\pi N}}} = \infty. \qquad \square$$

[17] Stirling's formula asserts that $\lim_{N\to\infty} \frac{N!}{\sqrt{2\pi N}(\frac{N}{e})^N} = 1$.

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