Econometrica, Vol. 67, No. 4 (July, 1999), 919–920

DECOMPOSITION AND CHARACTERIZATION OF RISK WITH A CONTINUUM OF RANDOM VARIABLES: CORRIGENDUM

BY NABIL I. AL-NAJJAR[1]

THE ASSERTION THAT (i) implies (iv) in the Main Theorem in Al-Najjar (1995) is incorrect as stated. The following adds a simple technical regularity condition that restores the result. None of the remaining results is affected.

Notation and definitions follow those in Al-Najjar (1995).[2] Consider the conditions:

(i) *Weak measurability*: $(t,s) \mapsto (\mathbf{f}_t \mid \mathbf{f}_s)$ has measurable sections.[3]

(iv) *Approximation by finite samples*: There exists $f \in L_2$ such that $\{t^\infty \in T^\infty : \lim_{n\to\infty} \| f(t^n) - f \| = 0\}$ has τ^∞-probability 1.

Condition (iv) says that risk in the continuum economy can be consistently estimated from risks in large, randomly drawn samples of agents. Al-Najjar (1995) incorrectly asserted that (i) implies (iv). The problem is that (i) does not guarantee the measurability of the set $\{t^\infty : \lim \| f(t^n) - f \| = 0\}$.

However, (iv) is fully restored with a minor regularity condition strengthening the measurability of the sections in (i) to full measurability:

(i′) $(t,s) \mapsto (\mathbf{f}_t \mid \mathbf{f}_s)$ is measurable.

This condition, referred to in Al-Najjar (1998) as *measurability of the covariance structure*, is satisfied in virtually every economic application of interest (e.g., when $\mathbf{f}$ is i.i.d., exchangeable, or has a strict factor structure with exchangeable residuals).

PROOF OF (i) ⇒ (iv): Let $(\mathbf{g},\mathbf{h})$ be the decomposition of $\mathbf{f}$ in part (ii) of the Theorem into an aggregate component, $\mathbf{g}$, and idiosyncratic component, $\mathbf{h}$. As in the paper, it suffices to show that $\mathbf{h}$ satisfies (iv). From the definition of decomposition, (i′) implies that the real-valued function $(t,s) \mapsto (\mathbf{h}_t \mid \mathbf{h}_s)$ is measurable—this is where (i′) is needed. Fubini's Theorem and the fact that $\mathbf{h}$ is idiosyncratic imply that $A = \{(t,s) : \mathrm{cov}(\mathbf{h}_t, \mathbf{h}_s) = 0\}$ is measurable and has measure one.

[1] I am grateful to M. Ali Khan and Yeneng Sun for kindly pointing out the error. I also thank a co-editor, a referee, Dean Foster, Ehud Kalai, and Greg Greiff for their comments.

[2] For the reader's convenience, I recall the relevant definitions. A *process* is a function $\mathbf{f}: T \to L_2$ on the Lebesgue space $(T = [0,1], \mathscr{T}, \tau)$, where $\mathbf{f}_t$ represents agent t's random shock. Here, $L_2 = L_2(\Omega, \Sigma, P)$, where Ω is the probability space representing uncertainty about agents' shocks. The mapping $(t,s) \mapsto (\mathbf{f}_t \mid \mathbf{f}_s = E\mathbf{f}_t E\mathbf{f}_s + \mathrm{cov}(\mathbf{f}_t, \mathbf{f}_2)$ represents the inner product between the shocks of agents t and s. We also consider random draws of agent names, $t^\infty = (t_1, \ldots)$, represented by the probability space $(T^\infty, \mathscr{T}^\infty, \tau^\infty)$. Write $f(t^n) = (1/n)\Sigma_{l=1}^n \mathbf{f}_{t_n}$ for the average of the random shocks of the first n agents in a given random draw.

[3] This is equivalent to the weak measurability condition used in Al-Najjar (1995): for every $x \in L_2$, $t \mapsto (x \mid \mathbf{f}_t)$ is measurable. I have chosen to use (i) in this note to further clarify the issue to the reader. To prove equivalence, note first that the latter condition clearly implies (i). To prove the converse, define the closed linear space $H = \mathrm{span}\{\mathbf{f}_t : t \in [0,1]\}$. Any $x \in L_2$ can be uniquely written as $x = x_H + (x - x_H)$, where $x_H \in H$ is the orthogonal projection of x on H. Since $x - x_H$ is orthogonal to every element of H, $(x \mid \mathbf{f}_t) = (x_H \mid \mathbf{f}_t)$ for every t. But $x_H \in H$ means that there are sequences of agent names $\{t_1, \ldots\}$ and weights $\{\alpha_1, \ldots,\}$ such that $x_H = \Sigma_{k=1}^\infty \alpha_k \mathbf{f}_{t_k}$. Continuity of the inner product means that the sequence of functions $(\Sigma_{k=1}^K \alpha_k \mathbf{f}_{t_k} \mid \mathbf{f}_t) = \Sigma_{k=1}^K \alpha_k (\mathbf{f}_{t_k} \mid \mathbf{f}_t)$ converges pointwise (in t) to $(x_H \mid \mathbf{f}_t)$, as $K \to \infty$. Since each $(\mathbf{f}_{t_k} \mid \mathbf{f}_t)$ is measurable by Condition (i), the limit $(x_H \mid \mathbf{f}_t)$ must also be measurable.

The remainder of the proof is exactly as stated in the paper: "with τ^{∞}-probability 1, any sequence will consist of orthogonal random variables. The result now follows from the fact that the range of **h** is bounded" (p. 1221). *Q.E.D.*

To clarify the technical nature of the problem, take a process **f** such that for every agent s, $E(s)=0$ and $\mathbf{f}_s$ is independent of the shock of all other agents except for a set of measure zero A_s. This means that for *any* initial draw s, with probability one a second draw t is such that the shocks of agents s and t are independent. One may be tempted to conclude that the shocks of a randomly drawn *pair* of agents (t,s) will be independent with probability one. But, the set for which this holds, $A = \cup_{s \in [0,1]} A_s$, is an uncountable union of the sections A_s, so the measurability of A does not follow from that of the sections. Condition (i′) simply strengthens the measurability of the sections A_s to the full measurability of A. This technical problem is similar to what causes Fubini's theorem to fail for functions with measurable sections, but not jointly measurable. Khan and Sun (1996) found an example in this spirit based on the one that had appeared in Dobric (1987).

Dept. of Managerial Economics and Decision Sciences, J. L. Kellogg Graduate School of Management, Northwestern University, 2001 Sheridan Rd., Evanston, IL 60208, U.S.A.; al-najjar@nwn.edu

Manuscript received May, 1997; final revision received October, 1998.

REFERENCES

AL-NAJJAR, N. I. (1995): "Decomposition and Characterization of Risk with a Continuum of Random Variables," *Econometrica*, 63, 1195–1224.

——— (1998): "Factor Analysis and Arbitrage Pricing in Large Asset Economies," *Journal of Economic Theory*, 78, 231–262.

DOBRIC, V. (1987): "The Law of Large Numbers, Examples and Counterexamples," *Mathematica Scandinavica*, 60, 273–291.

KHAN, M. A., AND Y. SUN (1996): "On the Decomposition and Characterization of Risk with a Continuum of Random Variables," CORE Discussion Paper No. 9738.