Choice under Aggregate Uncertainty

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Aggregation

- The utility of a decision maker depends on outcome profiles:

$$s = (x_1, \ldots, x_n)$$

$$S = X_1 \times \cdots \times X_n$$

- An attractive representation is via linear aggregators:

$$V(x_1, \ldots, x_n) = \sum_{i=1}^{n} v_i(x_i)$$

- Incorporate uncertainty (??): given $P \in \Delta(S)$, EU is:

$$\int_S \sum_{i=1}^{n} v(x_i)\, dP(s) \equiv \int_S V(s)\, dP(s)$$

Example 1: Dixit-Stiglitz CES Aggregator

- Representative consumer derives utility from consuming a variety of products
- The Dixit-Stiglitz aggregator is:

$$\left(\sum_{i=1}^{n} q_i^{\rho}\right)^{\frac{1}{\rho}}, \quad 0 < \rho < 1$$

  - $q_i \geq 0$: units of variety i
  - ρ measures taste for variety
  - *Is ρ cardinally meaningful? Risk neutrality to income gambles?*

Example 2: Utilitarian Social Welfare Functions

- A social planner's utility is an additive function of the utilities of members of society

$$\sum_{i=1}^{n} v_i(x_i)$$

  - v_i: utility function of individual i
  - x_i: his consumption bundle

Example 3 : Games Against Multiple Opponents

- Player's payoff depends on actions of n opponents
- Linear dependence expressed as:

$$\sum_{i=1}^{n} v_i(a_i)$$

  - $v_i(a_i)$: payoff impact of individual i's action on utility

Aggregation and Aggregate Uncertainty

Aggregate uncertainty is irrelevant with linear aggregators:

$$\begin{aligned} \int_S V(s)\, dP(s) &\equiv \int_S \left(\sum_{i=1}^{n} v(x_i) \right) dP(s) \\ &\equiv \sum_{i=1}^{n} \left(\int_S v(x_i)\, dP(s) \right) \\ &\equiv \sum_{i=1}^{n} E_{p_i(x_i)} v(x_i) \end{aligned}$$

$p_i(x_i)$: marginal distribution of P on the i's coordinate

Only the marginals of P matter; correlation is irrelevant

.. but aggregate uncertainty does (should) matter!

- Society responds differently to aggregate vs. idiosyncratic risks with the same marginals
  - Terrorism vs. idiosyncratic traffic deaths
- Idiosyncratic vs. aggregate strategic uncertainty in games
- Robson (1996): Evolutionary reasons why *Nature* may design utility to distinguish between the two
- Halevy and Feltkamp (2005) model of uncertainty aversion

Motivation: Many paradoxes motivating non-Bayesian decision criteria are consequences of using additive aggregators and insensitivity to aggregate uncertainty

Mathematical Structure

- $S = X_1 \times \cdots \times X_n$
- $n \geq 3$
- X_i is connected, complete, separable metric space
- $\Delta(S)$ Borel probability measures on profiles
- p_i is the marginal of $P \in \Delta(S)$ on X_i
- P is *independent* if

$$P = p_1 \times \cdots \times p_n$$

Expected Utility

Preference relation $\succsim$ on the set of social lotteries $\Delta(S)$

Expected Utility (EU)

$\succsim$ has a representation:

$$\int_S U(s)\, dP(s),$$

for a cardinally unique and continuous U.

Aggregation

New medical procedure with outcomes: recovery x, or death y

Three profiles:

$$\bar{x} = (x, x, \ldots, x)$$
$$\bar{y} = (y, y, \ldots, y)$$

$$s = (\underbrace{x, \ldots, x}_{n/2}, \underbrace{y, \ldots, y}_{n/2})$$

Obviously

$$U(\bar{x}) \quad > \quad U(\bar{y})$$

$U(s)$??

Aggregation

- Fix a non-empty subset $I \subset \{1, \ldots, n\}$
- s_I is profile s restricted to I
- *Conditional preference:* h is preferred to h' given s means

$$(h_I, s_{I^c}) \succsim (h'_I, s_{I^c})$$

Sure Thing Principle (STP)

For all profiles s, s', h, h',

$$(h_I, s_{I^c}) \succsim (h'_I, s_{I^c}) \iff (h_I, s'_{I^c}) \succsim (h'_I, s'_{I^c})$$

Theorem

$\succsim$ *satisfies* EU *and* STP

$$\Longleftrightarrow$$

$\succsim$ *has an* aggregative utility *representation:*

$$\mathcal{U}(P) = \int_S u\left(\frac{1}{n}\sum_{i=1}^{n} v_i(s_i)\right)\, dP(s)$$

- *v_i's are continuous and non-constant functions* $v_i : X_i \to \mathbb{R}, i = 1, \ldots, n$
- *u is increasing, continuous function* $u : \text{range}\left(\frac{1}{n}\sum_{i=1}^{n} v_i\right) \to \mathbb{R}$.

The v_i's are unique up to common positive affine transformation, and u is cardinally unique given the v_i's.

Sketch of the Proof

- Debreu (1960)'s aggregation theorem says that there exist continuous, cardinally unique $v_1, \ldots, v_n$ such that

$$V(s) = \sum_{i=1}^{n} v_i(x_i)$$

represents $\succsim$ restricted to S

- By identifying s with the dirac measure δ_s that puts unit mass on s, we have

$$S \subset \Delta(S)$$

- $U, V : S \to \mathbb{R}$ represent identical ordinal ranking

- There exists a strictly increasing function u

$$U(s) = u(V(s))$$

Sensitivity to Aggregate Uncertainty

Indifference to Aggregate Uncertainty

$\succsim$ is *indifferent to aggregate uncertainty* if

$$P \sim Q$$

whenever P, Q have equal marginals:

$$p_i = q_i, \; \forall i.$$

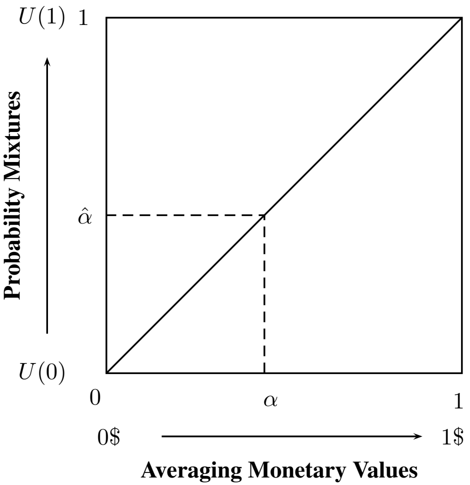
Theorem

$\succsim$ is indifferent to aggregate uncertainty if and only if u is affine.

Motivation in terms of risk aversion:

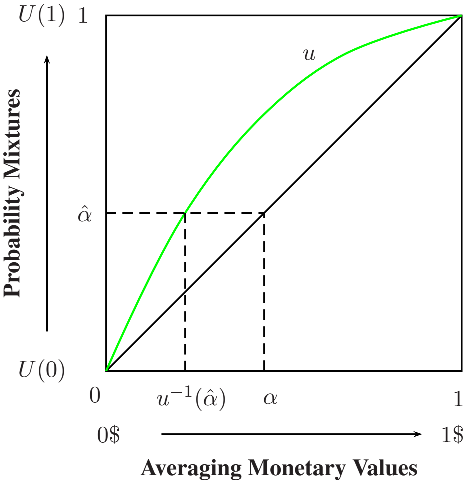
$\hat{\alpha}$:
lottery over prizes 1$, 0$

α:
average of two dollar amounts 1$, 0$



Risk aversion:

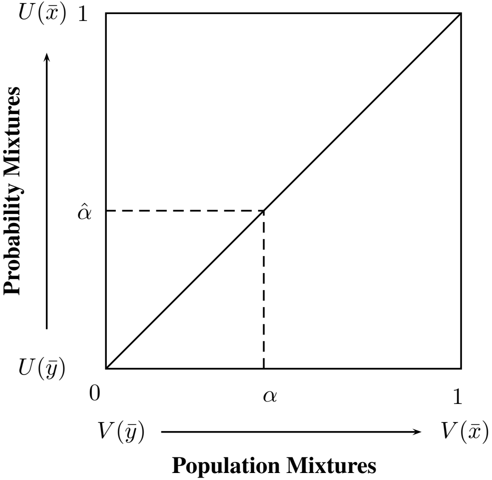
The two mixture operations are not equivalent



Probability mixture:
α% chance everyone gets *x*
$1-\alpha$% chance everyone gets *y*

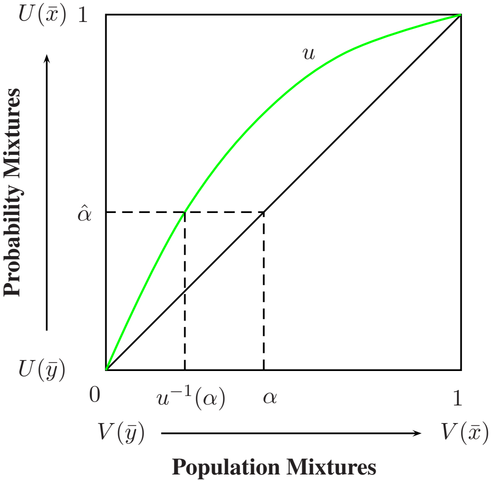
Population mixture:

α% get *x*
$1-\alpha$% get *y*



Hedging via fractional profiles:

Aggregative utility with concave *u* implies willingness to substitute probability mixtures by population mixtures



Dixit-Stiglitz Revisited

- The standard Dixit-Stiglitz CES aggregator is often written

$$\left(\frac{1}{n}\sum q_i^{\rho}\right)^{\frac{1}{\rho}}$$

  Not clear whether the exponent $\frac{1}{\rho}$ has cardinal meaning

- Using our representation and the assumption that $v_i = v_j$ and u are all CES, we obtain

$$\mathcal{U}(P) = \int \left(\frac{1}{n}\sum q_i^{\rho}\right)^{\frac{\kappa}{\rho}} dP$$

Dixit-Stiglitz Revisited

$$\left(\frac{1}{n}\sum q_i^{\rho}\right)^{\frac{\kappa}{\rho}}$$

- ρ is the familiar elasticity of substitution
- Consider bundles: $\alpha q = (\alpha q_1, \ldots, \alpha q_n)$, $\alpha > 0$ then

$$\kappa = \frac{\alpha U''(\alpha q)}{U'(\alpha q)}$$

- κ is the induced relative risk aversion wrt to changes in consumption levels
- $\frac{\kappa}{\rho}$ has cardinal meaning of risk aversion relative to taste for variety

Portfolio Choice

- In portfolio theory returns are perfectly fungible

$$\mathcal{U}(P) = \int u\left(\frac{1}{n}\sum x_i\right)\, dP.$$

- This is aggregative utility where the v_i = the identity.
- More general form is

$$\mathcal{U}(P) = \int u\left(\frac{1}{n}\sum v_i(x_i)\right)\, dP.$$

Mental Accounting

Consider the general form:

$$u\left(\frac{1}{n}\sum v_i(x_i)\right)$$

The concavity of the v_i's may be interpreted as capturing:

> *"utility unrelated to consumption. [...] An investor may interpret a big loss on a stock as a sign that he is a second-rate investor, thus dealing his ego a painful blow, and he may feel humiliation in front of friends and family when word about the failed investment leaks out." (Barberis and Huang)*

Halevy-Feltkamp 2005

- Halevy-Feltkamp (RES 2005):
  "Bayesian Model of Uncertainty Aversion"
- Urn with known composition and an "ambiguous urn"
- Agent's payoff is a concave function of two draws from the same urn
  $$u(x_1 + x_2)$$
- They show that
  - Agent prefers risky urn (with known composition)
  - Agents have a strict preference to randomize when facing the ambiguous urn

Aggregation of idiosyncratic risk

Regularity Assumptions

For every n, the aggregative utility $\mathcal{U}_n$ satisfies:

(i) The range of v_i is contained in $[0, 1]$ for every i;

(ii) The range of u is contained in $[0, 1]$;

(iii) u is Lipschitz continuous, with Lipschitz constant K.

Aggregation of idiosyncratic risk

Compare aggregative utility of the *n*th problem:

$$\mathcal{U}_n(P) = \int_S u\left(\frac{1}{n}\sum_{i=1}^{n} v_i(s_i)\right) dP(s) \tag{1}$$

Next, imagine "moving P inside $u(\,\cdot\,)$":

$$\bar{\mathcal{U}}_n(P) = u\left(\frac{1}{n}\sum_{i=1}^{n} E_{p_i} v_i(s_i)\right) \tag{2}$$

As n increases, $u\left(\frac{1}{n}\sum v_i(s_i)\right)$ "concentrates" around $\bar{\mathcal{U}}_n(P)$

Aggregation of idiosyncratic risk

Use concentration inequalities to prove:

Theorem

$$\left|\mathcal{U}_n(P) - \bar{\mathcal{U}}_n(P)\right| < \epsilon + 2e^{-2n\left(\frac{\epsilon}{K}\right)^2}$$

For every

- *Independent P*
- *v_i's and u satisfying the regularity assumption*

The Conditionally i.i.d. Case

- $S_i = S_j$ for all i, j
- Regularity Assumption holds
- P^μ is conditionally i.i.d.
  1. Draw θ using μ
  2. Use θ to draw profile s i.i.d.

Theorem

$$\left| \int_\Theta \int_S u\left(\frac{1}{n}\sum_{i=1}^{n} v_i(s_i)\right) \, dP^\theta(s) \, d\mu(\theta) \right.$$

$$-$$

$$\left. \int_\Theta u\left(\frac{1}{n}\sum_{i=1}^{n} E_{p_i^\theta} v_i(s_i)\right) \, d\mu(\theta) \right|.$$

$$\longrightarrow 0$$

uniformly in μ

Compound lotteries reduce, of course, but idiosyncratic risk makes $\sum v_i(s_i)$ concentrate around its mean while aggregate risk does not