Discussion of “Consensus Expectations and Conventions”
Golub and Morris (2015)

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This Discussion

(1) My own (non-technical) understanding of consensus expectations.

(2) A decomposition result

(3) “Souped-up” consensus expectations
Consensus Expectations: Definition

- **First-order expectations:**
  \[ X^i(1) = \mathbb{E}^i[Y|t^i] \]

- **Iterated average expectations:**
  \( i \)'s expectation of the “average” expectation in the society
  \[ X^i(k + 1) = \mathbb{E} \left[ \sum_{j=1}^{n} \gamma^{ij} X^j(k) | t^i \right] \]

- **Consensus expectations:**
  (the average expectation of the average expectation of .... everybody)
  \[ c^* = \lim_{k \to \infty} X^i(k) \]
Example: “Hot Potato Game”

- A security traded among $n$ different traders
- Trader $i$ runs into trader $j$ with probability $\gamma_{ij}$ and “dumps” the asset on her.
- Traders uncertain about the fundamental value of the asset $\theta$ and the valuations of others.
- With probability $1 - \rho$ the world ends you’re stuck with the hot potato
- The asset price reflects
  - asset’s fundamental value $\theta$;
  - $i$’s expectation of the average valuation of her counterparties;
  - $i$’s expectation of the average expectation of her counterparties’
    average expectations of their own counterparties;
- Consensus expectations: equilibrium asset price in the highly speculative game as $\rho \to 1$. 
Consensus Expectations

- **Main result**: CE is a deterministic object and does not depend on agent index $i$

$$c^* = \sum_{i=1}^{n} \sum_{k=1}^{K^i} p^i_k E^i \left[ \theta \left| t_k^i \right. \right]$$

- Consensus expectations is a convex combination of agents’ first-order expectations.

- Extremely simple characterization.

- **Caution**: weights $p^i_k$ are extremely complicated objects and depend on
  - the extent of network externalities: $\gamma^{ij}$
  - agents’ expectations: $\pi^i(t^j_l | t^i_k)$
Consensus Expectations

- **Main result**: CE is a deterministic object and does not depend on agent index \( i \)

\[
c^* = \sum_{i=1}^{n} \sum_{k=1}^{K^i} p_{ik}^i \mathbb{E}^i \left[ \theta | t^i_k \right]
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Consensus Expectations

\[ c^* = \sum_{i=1}^{n} \sum_{k=1}^{K} p_k^i E^i \left[ \theta \mid t_k^i \right] \]

- Define \( B_{kl}^{ij} = \pi^i(t_l^j \mid t_k^i) \)

\[
B = \begin{bmatrix}
\gamma^{11} B^{11} & \cdots & \gamma^{1n} B^{1n} \\
\vdots & \ddots & \vdots \\
\gamma^{n1} B^{n1} & \cdots & \gamma^{nn} B^{nn}
\end{bmatrix}
\]

- weights in consensus expectations:

\[ p' B = p' \]

- \( p_k^i \): information & interaction centrality

- captures not only whether people care about you, but also what they think about you.
Whose Expectations Matter?

$$B = \begin{bmatrix}
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\end{bmatrix}$$

- If trader $i$ never faces trader $j$ it doesn’t matter what he thinks of her!
- What matters is what $j$’s direct counterparties think of her!
- Even though $i$ is certain that $j$ would eventually obtain the security.
Two possible states $G$ and $B$ with returns 1 and 0, *ex ante* equally likely.

Each trader can be of two types ($g$ or $b$)

- $\mathbb{P}(\theta = G|t_i = g) = p$
- $\mathbb{P}(\theta = G|t_i = b) = 1 - p$

\[
\begin{align*}
\mathbb{P}(t_{i+1} = g|t_i = g) &\approx 1 & \mathbb{P}(t_{i+1} = g|t_i = b) &= \frac{1}{2} \\
\mathbb{P}(t_{i-1} = g|t_i = g) &= \frac{1}{2} & \mathbb{P}(t_{i+1} = g|t_i = b) &\approx 0
\end{align*}
\]

**equilibrium price** = $p$
An Example from the Paper: Cyclic Optimism

- Two possible states \( G \) and \( B \) with returns 1 and 0, \textit{ex ante} equally likely.
- Each trader can be of two types (\( g \) or \( b \))
- \( P(\theta = G|t_i = g) = p \)
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equilibrium price = \( 1 - p \)
Network and Information Interaction

$$B = \begin{bmatrix} \gamma^{11}B^{11} & \cdots & \gamma^{1n}B^{1n} \\ \vdots & \ddots & \vdots \\ \gamma^{n1}B^{n1} & \cdots & \gamma^{nn}B^{nn} \end{bmatrix}$$

- Consensus expectations is determined by the eigenvector of $B$.
- Expectations and network interactions are not necessarily “separable”.
Comment: A Decomposition Result

• Suppose agents have common type sets: \( T_i = T \)

• Agents hold “symmetric expectations”:

\[
B_{kl}^{ij} = \pi^i(t^i_j|t^i_k) = \hat{\pi}(t_l|t_k) = \hat{B}_{kl}
\]

• In this case \( B = \Gamma \otimes \hat{B} \)

**Theorem**

*If \( \gamma^{ii} = 0 \) and agents hold symmetric expectations, then*

\[
p^i_k = \text{network centrality}_i \cdot \hat{\pi}(t_k).
\]

• The interaction network and beliefs no longer interact.

• Complete information and CPA-T would be special cases.

• The real bite of the results is when agents hold asymmetric expectations.
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- The real bite of the results is when agents hold asymmetric expectations.
Comment: Souped Up Consensus Expectations

- Consensus Expectations:

\[ X^i(k + 1) = \mathbb{E} \left[ \sum_{j=1}^{n} \gamma^{ij} X^j(k) | t^i \right] \]

- But what if agents care about a potentially different average expectation at different levels?

\[ \hat{X}^i(k + 1) = \mathbb{E} \left[ \sum_{j=1}^{n} \gamma^{ij}(k) \hat{X}^j(k) | t^i \right] \]

**Theorem**

Under suitable connectivity assumptions on \( \{\Gamma(k)\}_{k=1}^{\infty} \) and beliefs, “Souped-up” consensus expectations

\[ \lim_{k \to \infty} \hat{X}^i(k) \]

is always a deterministic object and independent of the index of the player \( i \).
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Summary

- A very interesting paper, formalizing and characterizing a new concept
- Many applications (coordination games, relaxing the common prior assumption, equilibrium robustness)
- Meta-Theorem 1: network interactions and incomplete information interact with one another.
- Meta-Theorem 2: at some level, network interactions and incomplete information are the same object (see Stephen’s other paper).
- (Almost) all infinite regress of average expectations lead to a consensus expectation!