Coordinating Investment, Production and Subcontracting

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Abstract

We value the option of subcontracting to improve financial performance and system coordination by analyzing a competitive stochastic investment game with recourse. The manufacturer and subcontractor decide separately on their capacity investment levels. Then demand uncertainty is resolved and both parties have the option to subcontract when deciding on their production and sales. We analyze and present outsourcing conditions for three contract types: (1) *price-only contracts* where an ex-ante transfer price is set for each unit supplied by the subcontractor; (2) *incomplete contracts,* where both parties negotiate over the subcontracting transfer; and (3) *state-dependent* price-only and incomplete contracts for which we show an equivalence result.

While subcontracting with these three contract types can coordinate *production* decisions in the supply system, only state-dependent contracts can eliminate all decentralization costs and coordinate capacity *investment* decisions. The minimally sufficient price-only contract that coordinates our supply chain specifies transfer prices for a small number (6 in our model) of contingent scenarios. Our game-theoretic model allows the analysis of the role of transfer prices and of the bargaining power of buyer and supplier. We find that sometimes firms may be better off leaving some contract parameters unspecified ex-ante and agreeing to negotiate ex-post. Also, a price-focused strategy for managing subcontractors can backfire because a lower transfer price may decrease the manufacturer's profit. Finally, as with financial options, the option value of subcontracting increases as markets are more volatile or more negatively correlated.

Key Words: Supply chain, real investments, capacity, coordination, production, subcontracting, outsourcing, supply contracts.

1 Introduction

We present analytic models to study subcontracting and outsourcing, two prevalent business practices across many industries. While the word *subcontracting* has been used for nearly two centuries, *outsourcing* first appeared in the English language only as recently as 1982 [2]. Both terms refer to the practice of one company (the subcontractor or supplier) providing a service or good for another (the contractor, buyer or manufacturer). Subcontracting typically refers to the situation where the contractor "procures an item or service that is normally capable of economic production in the contractor's own facilities and that requires the contractor to make specifications available to the subcontractor [7]." Outsourcing refers to the special case where the contractor has no in-house production capability and is dependent on the subcontractor for the entire product volume.

We value the option of subcontracting and outsourcing to improve financial performance and system coordination by analyzing a two-stage, two-player, two-market stochastic game. In stage one, the manufacturer and subcontractor decide separately on their investment levels. Then demand uncertainty is resolved and both parties have the option to subcontract when deciding on their production levels in stage two, constrained by their earlier investment decisions. Subcontracting is viewed as a trade of the supplier's product for the manufacturer's money. Section 2 first analyzes two scenarios (the centralized firm vs. two independent firms without any subcontracting) for performance reference. In Section 3 we study *price-only contracts* where an ex-ante transfer price is set for each unit supplied by the subcontractor. We characterize the sub-game perfect investment strategy and formulate an outsourcing threshold condition in terms of the manufacturer's investment cost. A higher transfer price may increase the manufacturer's profit. This suggests that a price-focused strategy for managing subcontractors can backfire on the manufacturer. While a lower price allows cheap supply, it does not guarantee its availability. Our model confirms that optimal manufacturer and supplier capacity levels are imperfect substitutes with respect to capacity costs and contribution margins. We also show that manufacturers will indeed subcontract more when the level of market uncertainty (risk) increases and when markets are more negatively correlated. Similar to financial options, this increases the option value of subcontracting (real assets). In Section 4 we study two other contract types. One uses the *incomplete contracting approach* where no explicit contracts can be made and both parties negotiate over the subcontracting transfer. This allows us to analyze the role of the "bargaining power" of the contractor on outsourcing decisions and system performance improvement, which may be greater than with price-only contracts. The latter suggests that sometimes firms may be better off leaving some contract parameters unspecified ex-ante and agreeing to negotiate after demand is observed. Our third contract type consists of *state-dependent price-only and incomplete contracts* for which we show an equivalence result. While subcontracting with these three contract types can coordinate production decisions in the supply system, only state-dependent contracts can eliminate all decentralization costs and coordinate capacity investment decisions. We present the minimally sufficient price-only contract that achieves coordination. Section 5 closes with a discussion of more complex contracts in the literature and suggestions for further work.

Many literatures discuss the costs and benefits of subcontracting. According to the strategy literature, subcontracting and outsourcing occur because a firm may find it less profitable or infeasible to have all required capabilities in house: "a firm should concentrate on its core competencies and strategically outsource other activities [19]" and "not one company builds an entire flight vehicle, not even the simplest light plane, because of the exceptional range of skills and facilities required [1]". Subcontracting and outsourcing may also be "an impetus and agent for change" and "may improve unduly militant or change-resisting" employee relations [4]. These benefits come at a cost by exposing the contractor to strategic risks, such as dependence on the subcontractor (with its inherent loss of control and associated hold-up risk) and vulnerability (e.g., lower barriers to entry and loss of competitive edge and confidentiality) [19]. The operations literature highlights the flexibility that subcontracting offers to production and capacity planning. Like demand and inventory management, subcontracting allows for short term capacity adjustments in the face of temporal demand variations. Subcontracting, however, has the distinguishing feature that it "requires agreement with a third party who may be a competing firm with conflicting interests [14]." (The implication being that any reasonable model of subcontracting must incorporate multiple decision makers.) From a financial perspective, the main reported benefits of subcontracting and outsourcing are lower operating costs and lower investment requirements for the contractor, and the spreading of risk between the two parties. Empirical studies report that cost efficiency is the prime motivation for outsourcing maintenance [4] and information systems [16]. It is also argued that contractors 'push the high risk' onto subcontractors by having them "carry a disproportionate share of market uncertainties [8]." The financial costs of subcontracting include decreased scale economies to the contractor [10] and the transaction costs resulting from the initiation and management of the contracting relationship [19]. Finally, an extensive economics literature discusses our topic when studying vertical integration but that literature generally ignores capacity considerations.

Few papers explicitly study an analytic model of subcontracting. Kamien and Li [14] present a multi-period, game theoretic aggregate planning model with given capacity constraints and show that the option of subcontracting results in production smoothing. Kamien, Li and Samet [15] study Bertrand price competition with subcontracting in a deterministic game with capacity constraints implicit in their convex cost structure. Hanson [11] develops and empirically tests a model of the optimal sharing of the ownership of a given, exogenously determined number of units of an asset between a manufacturer and a subcontractor. Tournas [20] captures asymmetries in in-house information in a principal-agent model and compares them with the bargaining cost of a captive outside contractor. Brown and Lee [5] propose a flexible reservation agreement in which a manufacturer may reserve supplier capacity in the form of options. Finally, there is significant literature on outsourcing in supply-chains. Cachon and Lariviere [6] give an overview of various contract types, which will be discussed in more detail in Section 5.

Our model is different in that the capacity investment levels of both the manufacturer and the subcontractor are decision variables. Our multi-variate, multi-dimensional competitive newsvendor formulation is an extension of univariate, one-dimensional supply models and of the univariate competitive newsvendor models of Li [17] and Lippman and McCardle [18]. The multi-variate demand distribution allows us to investigate the important role of market demand correlation and provides a graphical interpretation of the solution. Our multi-dimensional model allows us to study the impact of subcontracting on both players' in-house investment levels and on the buyer's outsourcing decision, which is pre-assumed in captive-buyer captive-supplier models. We show that the higher complexity of subcontracting (two capacity decisions) makes coordination more difficult compared to traditional outsourcing models (only supplier capacity) in supply chains; we explicitly distinguish coordination of ex-ante investment decisions from ex-post production and sales decisions coordination. Finally, we have chosen to make both models essentially single-period and to posit no information asymmetries between the two parties. Therefore we shall not discuss how subcontracting can smooth production plans over time, create or mitigate information asymmetry problems, or affect the long-run competitive position of the firms.

2 A Subcontracting Model

2.1 The Model

Consider a two-stage stochastic model of the investment decision process of two firms. In stage one market demands are uncertain and both firms must decide separately, yet simultaneously, on their capacity investment levels. At the beginning of stage two, market demands are observed and both firms must decide on their production levels to satisfy optimally market demands, constrained by their earlier investment decisions. At this stage, both firms have the option to engage in a trade. The subcontractor S can supply the manufacturer M a transfer quantity $x_t \geq 0$ in exchange for a payment $p_t x_t$, as shown in Figure 1. Before we explain the specifics of the supply contract in the next section, let us discuss model features, notation and two reference scenarios that are useful in evaluating the impact of subcontracting on firm performance.

In the first reference scenario both firms operate completely *independent* of each other and subcontracting is not an option (i.e., transfer quantity $x_t = 0$). Both firms *go solo* and each will sell to its own market. For simplicity, we will assume that both firms have exclusive access to their respective markets. Because the subcontractor lacks the assembly, marketing and sales clout of the manufacturer, she does not have direct access to market M. In practice, however, the manufacturer may have access to market S through wholly owned upstream subsidiaries that provide them and

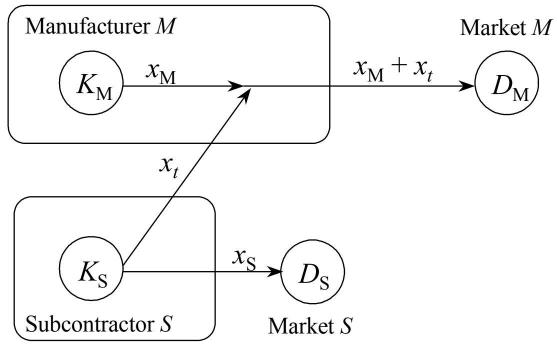

Figure 1: The Subcontracting Model.

others with parts or subsystems. General Motors, for example, owns Delphi Automotives that supplies GM and other auto assemblers with brake systems and other parts. At the same time, GM multisources some parts from outside, independent subcontractors. Thus, market M would represent the end market for cars and market S the intermediate market for parts. GM could compete in market S but we will abstract from such competition to highlight the subcontracting option. Also, notice that direct sourcing from market S instead of from the subcontractor is not an option for the manufacturer. This modeling assumption reflects the relationship-specific information typically present in subcontracting and it implies that we are not discussing the purchase of standardized, off-the-shelf products in commodity markets.

The second reference scenario represents the other extreme in which both firms are *integrated* and controlled by a single decision maker. In this *centralized* scenario the integrated firm will serve both markets. Subcontracting, then, is the intermediate scenario in which both firms are independently owned so that we have two decision makers, yet trading is possible. (Thus the subcontractor's technology is sufficiently flexible that it can produce the same product as the manufacturer's technology.)

Let $K_i \geq 0$ denote firm i's capacity investment level, where $i = M$ or S. Firm i is assumed to face a constant marginal investment cost $c_i > 0$, so that its capacity investment cost $c_i K_i$ is linear in the investment level. Production levels x_M and $x_S + x_t$ are linearly constrained by the capacity investment levels: $x_M \leq K_M$ and $x_S + x_t \leq K_S$. For simplicity, we assume that both firms make constant contribution margins p_i per unit sold in market i. Stronger, we will assume zero marginal production costs so that p_i represents the fixed sales price in market i. To avoid trivial solutions we assume that $c_i < p_i$. Let $D_i \geq 0$ denote the product demand in market i. Like Kamien and Li [14], we assume symmetric information in the sense that each firm has complete information about the other's cost and profit structure and investment level, and they share identical beliefs regarding future market demands. These beliefs can then be represented by a single, multi-variate probability measure P. For simplicity, we assume that market demands are finite with probability one and that P has a continuous density f on the sample space $\mathbb{R}^2_+$. The expectation operator will be denoted by E. We assume zero shortage costs and zero salvage values for both products and production assets[1]. Finally, both firms are assumed to be expected profit maximizers and the research question can thus be formulated in the two reference scenarios as follows.

[1]Relaxation of these assumptions to include convex investment costs, market-and-firm specific unit contribution margins p_{ij}, shortage costs and salvage values, and non-unit capacity consumption rates is relatively straightforward (as shown in [12]) at the expense of added notational complexity.

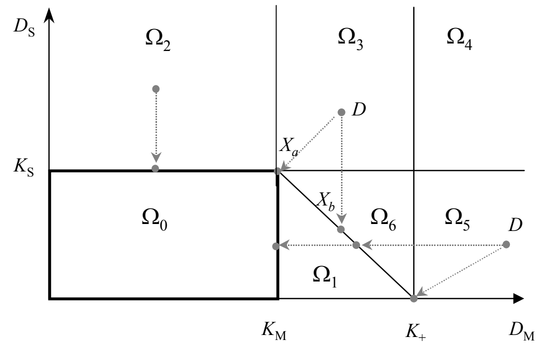

Figure 2: Production decisions and total market supply vector X, represented by arrows, depend on the demand D realization and the scenario.

2.2 Independents: Going Solo

When both firms do not subcontract, each firm decides on its production and sales decision x_i in stage two by maximizing its operating profit $\pi_i = p_i x_i$ subject to the capacity constraint $x_i \leq K_i$ and the demand constraint $x_i \leq D_i$. This 'product mix' linear program has optimal activity level $x_i^{solo} = \min(K_i, D_i)$ with profit $\pi_i^{solo} = p_i x_i^{solo}$. In stage 1, firm i chooses its optimal investment level K_i^{solo} so as to maximize its expected firm value, denoted by V_i, which is the expected operating profit minus investment costs:

$$K_i^{solo} = \arg\max_{K_i \geq 0} V_i^{solo}(K) \qquad \text{where} \qquad V_i^{solo}(K) = \mathrm{E}\pi_i^{solo}(K, D) - c_i K_i. \tag{1}$$

A critical fractile newsvendor solution is optimal: $K_i^{solo} = G_i(c_i/p_i)$, where G_i^{-1} is the tail distribution of D_i. To build some intuition for the solution technique that will be used below, let us summarize briefly how this familiar result can be derived using the multi-dimensional newsvendor model of Harrison and Van Mieghem [12]. It will be convenient to let capacity vector K partition the demand space $\mathbb{R}_+^2$ into 7 regions $\Omega_l(K), l = 0, 1, \cdots, 6$, as in Figure 2 (where we abbreviated the sum of the components of K by $K_+ = K_M + K_S$). The rectangular region $\Omega_0(K)$ is the capacity region of this two-firm supply system without subcontracting. Whenever D is outside this capacity region, some demand cannot be met and the optimal market supply $X = (x_M, x_S) \leq D$, represented by an arrow emanating from D, will be on the capacity frontier.

Linear programming theory yields that the profit vector $\pi^{solo}(K, D)$ is unique and concave in K. Thus, the linear superposition $\mathrm{E}\pi_i^{solo}(K, D)$ and thus $V_i^{solo}(\cdot)$ are also concave so that the first order conditions of (1) are sufficient:

$$\frac{\partial}{\partial K_i} V_i^{solo} = -\nu_i^{solo} \text{ and } \nu_i^{solo} K_i^{solo} = 0,$$

where $\nu_i^{solo} \geq 0$ is the optimal Lagrange multiplier of the non-negativity constraint $K_i \geq 0$. Invoking [12], gradient and expectation can be interchanged to yield $\mathrm{E}\lambda_i(K^{solo}, D) = c_i - \nu_i^{solo}$,where λ_i is firm i's capacity shadow value: $\lambda_i = \frac{\partial \pi_i}{\partial K_i}$. The shadow value λ_i, which is the optimal dual variable of firm i's production linear program, equals a constant λ_i^l in each domain Ω_l of Figure 2. Thus, the expected marginal profit can be expressed as $\frac{\partial \pi_i}{\partial K_i} = \mathrm{E}\lambda_i = \sum_{l=1}^{6} \lambda_i^l P(\Omega_l(K))$. To simplify notation, define a 2×6 matrix Λ whose l-th column is the shadow vector in domain $\Omega_l : \Lambda_{il} = \lambda_i^l$. Similarly, define a 6×1 vector $\bar{P}(K)$ whose l-th coordinate is the probability of domain $\Omega_l : \bar{P}_l(K) = P(\Omega_l(K))$.

When both firms "go solo" the marginal vector is

$$\mathrm{E}\lambda = \Lambda^{solo}\bar{P}(K^{solo}) = \begin{bmatrix} p_M & 0 & p_M & p_M & p_M & p_M \\ 0 & p_S & p_S & p_S & 0 & 0 \end{bmatrix} \bar{P}(K^{solo}) = \begin{bmatrix} p_M P(D_M > K_M^{solo}) \\ p_S P(D_S > K_S^{solo}) \end{bmatrix}.$$

Because contribution margins exceed investment costs ($p_i > c_i$) both firms will invest ($\nu^{solo} = 0$) and the optimality equations directly yield the familiar newsvendor solutions.

2.3 Centralization

When both firms are controlled by one central decision maker, the optimal production and sales vector x in stage two maximizes system operating profit, subject to system capacity and demand constraints. Transfers x_t are possible and optimal activity levels x^{cen} and profit π^{cen} are the solution of the product mix linear program:

$$\begin{aligned} \pi^{cen} &= \max_{x\geq 0} p_M(x_M + x_t) + p_S x_S \\ \text{s.t. } x_M &\leq K_M, x_t + x_S \leq K_S, x_t + x_M \leq D_M, x_S \leq D_S. \end{aligned} \tag{2}$$

The optimal investment vector K^{cen} maximizes expected system value:

$$K^{cen} = \arg\max_{K\geq 0} V^{cen}(K) \qquad \text{where} \qquad V^{cen}(K) = \mathrm{E}\pi^{cen}(K, D) - c'K. \tag{3}$$

The option of transfers x_t enlarges the supply system's capacity region to $\Omega_0 \cup \Omega_1$, or Ω_{01} in short. Using this shorthand notation, if $D \in \Omega_{23456}$, demand exceeds supply and the optimal supply vector $X = (x_M + x_t, x_S)$ will be on the boundary of the capacity region Ω_{01}. The linear program (2) can be solved parametrically in terms of K and D (thereby directly manifesting the domains Ω_l defined earlier). If market M yields higher margins than market S, it gets priority in the capacity allocation decision yielding market supply vector X_b in Figure 2. Otherwise market S gets priority yielding vector X_a in Figure 2. As before, $\pi^{cen}(K, D)$ is concave and the shadow vector $\lambda(K, D)$ is constant in each domain so that the optimal capacity vector K^{cen} solves $\Lambda^{cen}\bar{P}(K^{cen}) = c - \nu^{cen}$ and $K^{cen\prime}\nu^{cen} = 0$, where

$$\Lambda^{cen} = \begin{bmatrix} 0 & 0 & \min(p) & p_M & p_M & \min(p) \\ 0 & p_S & p_S & \max(p) & p_M & \min(p) \end{bmatrix}. \tag{4}$$

If M-capacity is less expensive than S-capacity ($c_M < c_S$), it is profitable to invest in both types of capacity ($\nu^{cen} = 0$). Otherwise, it is optimal to supply both markets using only the cheaper S-capacity: $\nu_M^{cen} > 0$ and $K_M^{cen} = 0$. In the Appendix of [23] we show that V^{cen} is strict concave at K^{cen} so that the optimal investment vector is unique.

We now have completely characterized the optimal investment strategies in both reference scenarios. Clearly, system values under centralization V^{cen} (weakly) dominate those when both players go solo: $V^{cen} \geq V_+^{solo} = V_1^{solo} + V_2^{solo}$. The value gap $\Delta V^{solo} = V^{cen} - V_+^{solo}$ captures the costs of decentralization. In the remainder of this article, we will investigate how subcontracting can decrease the value gap and whether it can "coordinate" the supply network (that is, *eliminate* the value gap).

3 Subcontracting with Price-Only Contracts

A *price-only contract* specifies ex-ante to both parties the transfer (or "wholesale") price p_t that the manufacturer must pay for each unit supplied by the subcontractor. Because this simple contract does not specify a transfer quantity x_t or any other model variables, it cannot force a party to enter the subcontracting relationship. Using Cachon and Lariviere's [6] terminology, contract compliance is voluntary and both parties will enter the subcontracting relationship (or "trade") only if it benefits them. As before, both players must decide separately, yet simultaneously, on their capacity investments in stage 1 before uncertainty is resolved. The resulting capacity vector K is observable and becomes common information. After demand is observed, both parties make their individual production-sales decisions x in stage 2 where they have the option to subcontract. The manufacturer M can ask a supply x_t^M from the subcontractor S, who has the option to fill the request. That is, she offers a quantity $x_t^S \leq x_t^M$, which is accepted by M in exchange for a payment $p_t x_t$.

When making decisions, each player acts strategically and takes into account the other player's decisions. Any capacity vector K (or production vector x) with the property that no player can increase firm value by deviating unilaterally from K (or x) is a Nash equilibrium in pure strategies and is called simply an *optimal investment* (production) *vector*. Its resulting firm value (profit) vector is denoted by $V(K)$ (or $\pi(x)$). The analysis of our subcontracting model involves establishing and characterizing the existence of a Nash equilibrium in this two-player, two-stage stochastic game.

3.1 The Production-Subcontracting Subgame

As with any dynamic decision model, we start with stage 2 and solve the *production-subcontracting subgame* for any given pair (K, D). Both players decide sequentially on their production and transfer levels in order to maximize their own operating profit:

$$\begin{aligned} \max_{x_M, x_t, x_t^M \geq 0} \quad & p_M x_M + (p_M - p_t) x_t \\ \text{s.t.} \quad x_M &\leq K_M, \\ x_M + x_t &\leq D_M, \\ x_t &= \min(x_t^M, x_t^S), \end{aligned} \qquad \text{and} \qquad \begin{aligned} \max_{x_S, x_t^S \geq 0} \quad & p_S x_S + p_t x_t \\ \text{s.t.} \quad x_S + x_t^S &\leq K_S, \\ x_S &\leq D_S, \\ x_t &= \min(x_t^M, x_t^S). \end{aligned}$$

Incentives to subcontract depend on the transfer price p_t. First, M will only subcontract if $p_t < p_M$, otherwise the independent solo solution emerges. Thus, for the remainder of this article we will assume $p_t < p_M$ so that M will always prioritize his internal capacity and will ask S to fill the remaining demand: $x_M = \min(D_M, K_M)$ and $x_t^M = (D_M - K_M)^+$. Second, S has an incentive to fill M's demand if $p_t > p_S$, while she will prefer to fill her own market demand if $p_t < p_S$. Thus, we must distinguish between two cases:

1. High transfer price: $p_S < p_t < p_M$. S prefers filling M's request to the best of her capacity: $x_t^S = \min(K_S, x_t^M)$. The subcontracting transfer is $x_t = \min\left((D_M - K_M)^+, K_S\right)$, which materializes whenever M has excess demand, that is if $D \in \Omega_{13456}$. S will use any remaining capacity to fill her own market demand: $x_S = \min(D_S, K_S - x_t)$. (The resulting market supply vector in Figure 2 is X_b.)

2. Low transfer price: $p_t \leq \min(p)$. S prefers serving her own market: $x_S = \min(D_S, K_S)$. Any remaining capacity can fill M's demand: $x_t^S = \min(x_t^M, K_S - x_S)$. The subcontracting transfer is $x_t = \min\left((D_M - K_M)^+, (K_S - D_S)^+\right)$, and subcontracting will materialize when M has excess market demand *and* S has low market demand, that is if $D \in \Omega_{156}$. (The resulting market supply vector in Figure 2 is X_a.)

In both cases, the production vector $x(K, D)$ forms a unique Nash equilibrium because no player has an incentive to deviate unilaterally. At any transition point between the two cases (e.g., $p_S = p_t$), players are indifferent because they receive the same profit in either case, and a continuum of production vectors are Nash equilibria. This poses no problems, however, because

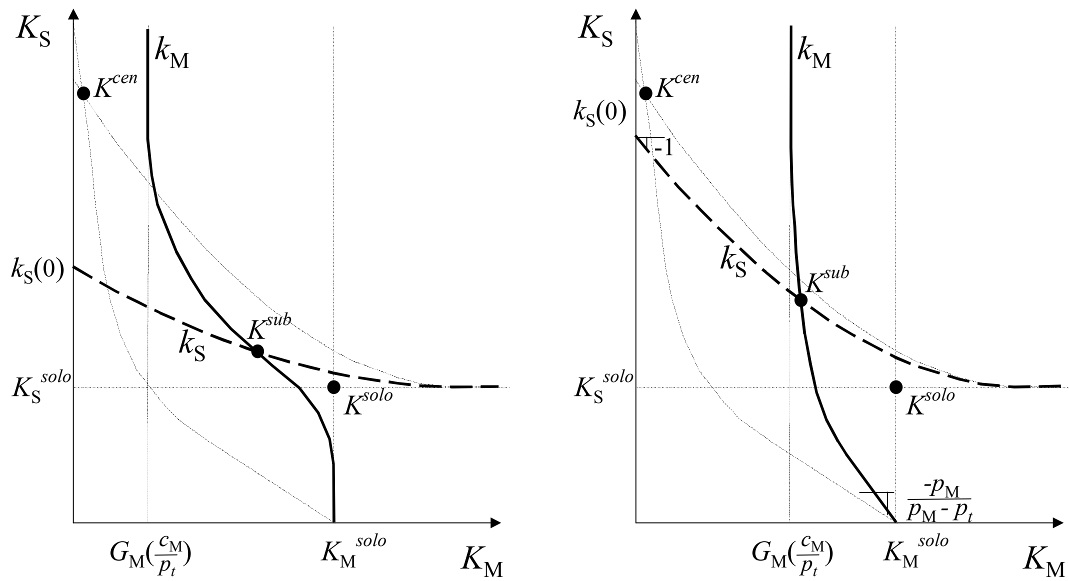

Figure 3: The intersection of the capacity reaction curves k_M (bold) and k_S (dashed bold) defines the optimal investment K^{sub} when subcontracting with low p_t (left) and high p_t (right).

linear programming theory yields that the associated profit vector $\pi(K, D)$ is unique and concave in K, which is all we need to solve the investment game in stage 1.

3.2 The Capacity Investment Game

To demonstrate the existence of a subgame perfect Nash equilibrium in pure investment strategies, we will show that the *capacity reaction curves* have a stable intersection point. Firm i's capacity reaction curve $k_i(\cdot)$ specifies its optimal investment level $K_i = k_i(K_j)$ given firm j has capacity K_j. It is defined pointwise as $k_i(K_j) = \arg\max_{K_i \geq 0} V_i(K)$. As before, $\mathrm{E}\pi_i(\cdot, D)$ and $V_i(\cdot)$ inherit concavity from $\pi_i(K, D)$ so that the first order conditions (FOC) are sufficient: $\Lambda^{sub}\bar{P}(K) = c - \nu$ and $\nu' K = 0$,where

$$\Lambda^{sub} = \begin{cases} \begin{bmatrix} p_t & 0 & p_t & p_M & p_M & p_t \\ 0 & p_S & p_S & p_t & p_t & p_S \end{bmatrix} & \text{if} \quad p_S \leq p_t < p_M, \\ \begin{bmatrix} p_t & 0 & p_M & p_M & p_M & p_M \\ 0 & p_S & p_S & p_S & p_t & p_t \end{bmatrix} & \text{if} \quad p_t < \min(p_S, p_M). \end{cases}$$

Firm $i's$ reaction curve is found by solving FOC$_i$ as a function of K_j. The Appendix of [23] shows that $-1 \leq \frac{dk_i}{dK_j} \leq 0$ and that axis crossings and asymptotes are as shown in Figure 3. Thus, the reaction curves have an intersection K^{sub} at which at least one reaction curve has a slope $\frac{dk_i}{dK_j} > -1$. Hence, K^{sub} is unique and stable (Nash).

3.3 Production versus Investment Coordination

Subcontracting with price-only contracts *can coordinate production decisions if* $p_S \leq p_t \leq p_M$ or $p_t \leq p_M \leq p_S$, because only then is the contingent production vector under subcontracting $x(K, D)$ equal to the production $x^{cen}(K, D)$ in the centralized scenario. (Incentive incompatibilities arise if $p_t < p_S < p_M$ and $D \in \Omega_{3456}$: under subcontracting the supplier will prioritize its own market

	c_M	c_S	p_M	p_S	p_t
K_M^{sub}	$-(\alpha_1+\alpha_3) \leq 0$	$\alpha_2 \geq 0$	$(\alpha_1+\alpha_3)P_{45} \geq 0$	$-\alpha_2 P_{236} \leq 0$	$(\alpha_1+\alpha_3)P_{136} - \alpha_2 P_{45}$
K_S^{sub}	$\alpha_1 \geq 0$	$-(\alpha_2+\alpha_4) \leq 0$	$-\alpha_1 P_{45} \leq 0$	$(\alpha_2+\alpha_4)P_{236} \geq 0$	$-\alpha_1 P_{136} + (\alpha_2+\alpha_4)P_{45}$
V_M^{sub}	$\beta_1 - K_M^{sub}$	$-\beta_2 \leq 0$	$Ex_{M+t}^{sub} - \beta_1 P_{45}$	$\beta_2 P_{236} \geq 0$	$\beta_5 \frac{\partial K_S^{sub}}{\partial p_t} - c_M \frac{\partial K_M}{\partial p_t} - Ex_t^{sub}$
V_S^{sub}	$\beta_3 \geq 0$	$-\beta_4 - K_S^{sub} \leq 0$	$-\beta_3 P_{45} \leq 0$	$Ex_S^{sub} + \beta_4 P_{236} \geq 0$	$-\beta_6 \frac{\partial K_M^{sub}}{\partial p_t} - c_S \frac{\partial K_S}{\partial p_t} + Ex_t^{sub}$

Table 1: Sensitivity of the optimal investment levels K^{sub} and value V^{sub}, where $\alpha, \beta \geq 0$. Table entries represent partial derivatives: $\alpha_1 = \frac{\partial K_S^{sub}}{\partial c_M}$ for example.

whereas the centralized system would prioritize market M.) This contract arrangement, however, *cannot coordinate ex-ante capacity investment decisions* or eliminate all decentralization costs as measured by the value gap $\Delta V = V^{cen} - V_+^{sub}$. Mathematically, the optimal centralized and subcontracting investment vectors in general differ as the unique solutions to $\Lambda^{cen}\bar{P}(K^{cen}) = c - \nu^{cen}$ and $\Lambda^{sub}\bar{P}(K^{sub}) = c - \nu^{sub}$, respectively, with $\Lambda^{cen} \neq \Lambda^{sub}$. Hence, $V_+^{sub} < V^{cen}$ because the value functions are strictly concave at the optimal investment vectors. Economically, our single-parameter price-only contract is unable to provide sufficient ex-ante incentives for both players to 'build' $K^{sub} = K^{cen}$. As in most realistic multi-player models, the first-best solution is not attained and decentralization comes at a cost.

These contracts do, however, mitigate decentralization costs and improve performance. Comparing the capacity reaction curves with the optimality curves that define the optimal centralized and solo investment (the thin lines in Figure 3) directly shows that subcontracting drives investment towards the centralized investment K^{cen}:

$$K_M^{cen} \leq K_M^{sub} \leq K_M^{solo} \qquad \text{and} \qquad K_S^{cen} \geq K_S^{sub} \geq K_S^{solo}.$$

This is what one expects: subcontracting allows the manufacturer to decrease his investment. The option of subcontracting means potentially more business for the supplier and thus warrants additional 'relationship-specific' investment. Next, we investigate how other model primitives impact the coordination improvement.

3.4 Sensitivity of the Investment-Subcontracting Strategies

The sensitivity of the optimal investment strategy with respect to changes in capacity costs c, contribution margins p, and transfer price p_t is summarized in Table 1. As expected, *optimal manufacturing and supplier capacity levels are imperfect substitutes* with respect to capacity costs c and margins p. Indeed, strategic decision making captured by our game-theoretic model makes one party's investment level and firm value dependent on the other party's cost and revenue structure. When the manufacturer faces higher investment costs, for example, he will decrease his investment level. The supplier anticipates that lower manufacturing capacity most likely will lead to higher supply requests x_t^M. This gives the supplier an incentive to increase her investment, reflecting the externalities in our model. The increase in K_S^{sub}, however, does not make up for the decrease in K_M^{sub} (because transfers are only made with a probability strictly less than one). A similar reasoning applies to a change in margin p, but it has a smaller impact than a cost change simply because the margin dependency is state-dependent. For example, an increase in p_M only warrants an increase in manufacturing capacity if demand is sufficiently large (e.g., $D \in \Omega_{45}$ if $p_t > p_S$). An increase in c_M, on the other hand, always justifies a decrease in manufacturing capacity, regardless of the demand realization. This result is in stark contrast to deterministic systems and one expects this sensitivity differential to increase in the amount of demand variability.

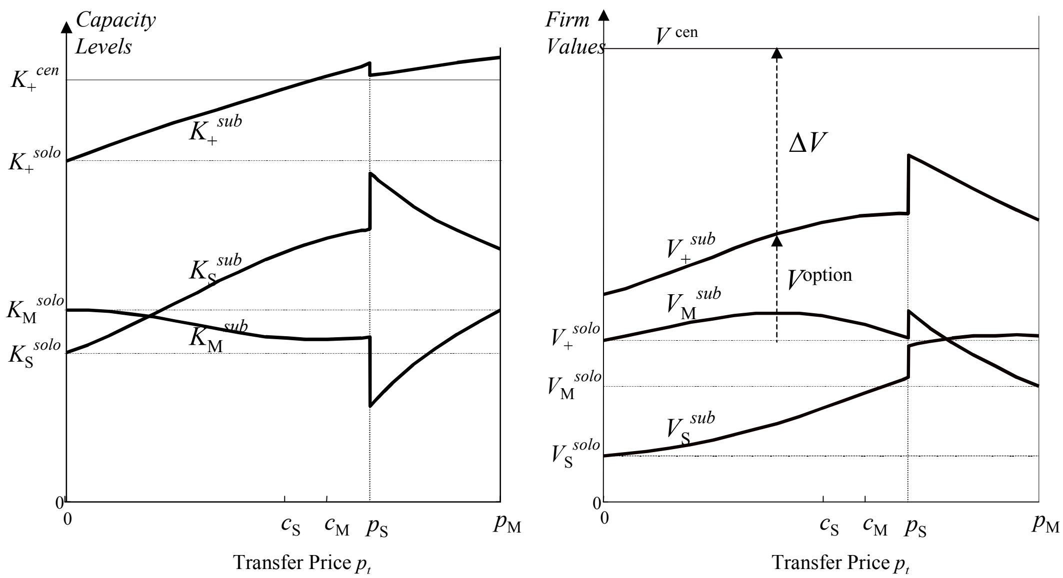

Figure 4: Capacity levels, the option value of subcontracting V^{option} and the decreased value gap ΔV as a function of the transfer price p_t when market demands are uniform but strongly negatively correlated.

More interestingly, while the supplier's value sensitivity directly reflects the externalities in the model, the manufacturer's value is a little more intricate. Clearly, an increase in supplier costs leads to a decrease in total system capacity, which impacts both parties' value negatively. An increase in manufacturing cost benefits the supplier who increases her capacity in anticipation of a larger total demand $x_t^M + D_S$. Recall that the centralized system would put all capacity with the supplier if $c_M > c_S$. Hence, anything that shifts capacity from the manufacturer to the supplier will tend to benefit the system. (The structure of the capacity reaction curves shows that K^{sub} moves toward K^{cen}—and thus V_+^{sub} moves toward V^{cen}—if c_M increases, even if $c_M \leq c_S$.) This effect can dominate to yield the unexpected result that the manufacturer's value *can be increasing in its investment cost.* The manufacturer enjoys spill-over benefits from increased supplier capacity that may outweigh his increased investment costs.

Similar effects can occur when increasing the transfer price. The table shows that this has a similar effect as a *simultaneous increase in margins* p_M and p_S. The absolute effect on investment levels and firm values is ambiguous. An increase in p_t makes subcontracting more expensive for the manufacturer relative to internal capacity investment. This is reflected by a rightward move of the manufacturer's reaction curve k_M in Figure 3. Increased transfer prices, however, give the supplier a higher incentive to increase her "relationship-specific" investment. Thus, while we expect K_M^{sub} to decrease and K_S^{sub} to increase, the supplier's reaction curve k_S can move upward more than k_M moves right so that K_M^{sub} increases and K_S^{sub} decreases. Figure 4 illustrates the intricate externalities that can occur in stochastic games. *Contract design*, or the choice of the optimal p_t, thus becomes very case specific and depends on the objective. (One can maximize manufacturer, supplier or system profits, or some combination, depending on how the transfer price is set. It may be the outcome of negotiation between the two partners, or it may be equal to an external reference price if another external supply market exists.) In all our numerical test problems, system profits were maximized at $p_t = p_S$ yielding a substantial improvement in the value gap ΔV, which is in agreement with economic theory stating that transfer prices should be set equal to outside opportunity costs. If the manufacturer sets the transfer price, however, he does not necessarily set it at p_S. Indeed, because of demand variability, a transfer price below p_S may yield optimal profits for the manufacturer. Figure 4 illustrates this possibility when market demands are strongly negatively correlated ($\rho = -0.9$). As argued earlier, the capacity levels are imperfect substitutes while Table 1 shows that total industry investment level K_+^{sub} is increasing in p_t. The figure also shows that in the context of our model *subcontracting may reduce or increase industry investment* compared to the solo or centralized setting. (While the figure shows that $K_+^{solo} < K_+^{sub}$, this is not true in general either.) Interestingly, similar to the c_M dependence described earlier, a higher transfer price may increase the manufacturer's profit. This suggests that *a price-focused strategy for managing subcontractors can backfire* on the manufacturer. While a lower price allows cheap supply, it does not guarantee its availability.

Finally, to study the effect of uncertainty on the optimal investment strategies, we consider a probability measure $P(\cdot \mid \gamma)$ with density $f(\cdot \mid \gamma)$ that is parameterized by an uncertainty measure γ such as an element of the mean demand vector or correlation matrix. Formally, the impact of changes in γ on the optimal investment strategy can be expressed as:

$$\frac{\partial}{\partial \gamma} K^{sub} = -|J|^{-1} \begin{bmatrix} \sum_{l=1}^{6} \left(J_{22} \Lambda_{1l}^{sub} - J_{21} \Lambda_{2l}^{sub} \right) P_l^{\gamma} \\ \sum_{l=1}^{6} \left(-J_{12} \Lambda_{1l}^{sub} + J_{11} \Lambda_{2l}^{sub} \right) P_l^{\gamma} \end{bmatrix},$$

where J is the Jacobian of the optimality equations $\Lambda^{sub} \bar{P}(K^{sub}) = c - \nu^{sub}$ and

$$P_l^{\gamma} = \frac{\partial}{\partial \gamma} P(\Omega_l(K^{sub}) \mid \gamma) = \int_{\Omega_l(K^{sub})} \frac{\partial}{\partial \gamma} f(z \mid \gamma) dz.$$

Although this expression is of limited practical value, it may be useful for estimating the sign of $\frac{\partial}{\partial \gamma} K^{sub}$. The appendix of [23] shows that $J_{22} \leq J_{21} \leq 0$ and $J_{11} \leq J_{12} \leq 0$. Thus, $\frac{\partial}{\partial \gamma} K_M^{sub}$ and $\frac{\partial}{\partial \gamma} K_S^{sub}$ may have *opposite signs* so that the optimal manufacturer and supplier investment levels would respond in opposite ways to changes in the demand distribution, akin to the substitution effect stated earlier. This effect is present for changes in the standard deviation or correlation of market demands in the example shown in Figure 5. For simplicity, we assumed identical mean and standard deviations[2] for D_M and D_S.

As shown in the left graphs of Figure 5, optimal investment levels are monotone in variability as measured by the standard deviation, but they can be increasing or decreasing. This is similar to the well-known effect in one-dimensional newsvendor models with symmetric demand distributions where optimal investment increases (decreases) in variability if the critical ratio $\frac{c}{p} > 0.5$ (< 0.5). More importantly, compared to the independent "solo" setting, an increase in market risk *decreases* the manufacturer's relative investment if there is a subcontracting option. This can be paraphrased by saying that *the manufacturer will subcontract more as market risk increases* and the subcontractor's response is to invest more[3]. The presence of demand uncertainty is a key driver in the option value of subcontracting, which is increasing in variability. Thus, similar to many financial options, more uncertainty is good for this real option. In absolute terms, however, more variability reduces firm values. The graph at the right in Figure 5 shows that *the manufacturer will subcontract less as market correlation increases*. Indeed, when market demands are positively correlated, the subcontracting option has less value so that the optimal fraction of capacity that is subcontracted decreases. In terms of our graphical solution technique of Figure 2, the triangular option region Ω_1 gets more probability mass as correlation becomes more negative.

[2]This example was generated numerically using a two-dimensional demand distribution parameterized by correlation and standard deviation in market demand. Explicit expressions for these distributions were first presented in [22, pp. 75-77].

[3]The subcontractor's optimal investment level seems to be less sensitive to risk, which may be explained by risk pooling: the supplier's effective demand pools over both markets and therefore is less variable.

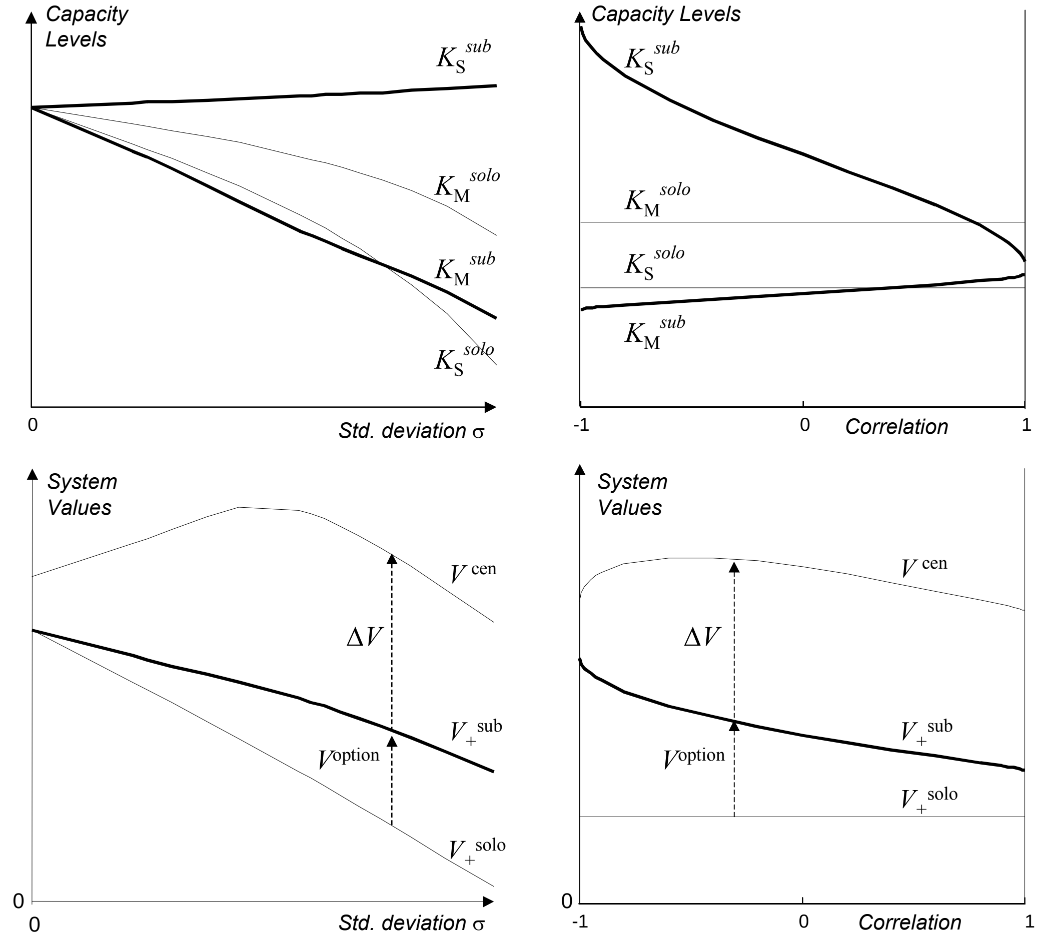

Figure 5: Optimal investment and the option value of subcontracting V^{option} as a function of standard deviation σ of demand assuming $\sigma_{D_M} = \sigma_{D_S}$ when market demands are uncorrelated (left) and of correlation when market demands are uniform (right).

3.5 Outsourcing or Complete Subcontracting: $K_M^{sub} = 0$

The structure of the capacity reaction curves shows that the optimal investment strategy has one of two distinct forms: either both firms invest or only the supplier invests. In the latter case, the manufacturer relies for all sourcing on the outside party. One can express an *outsourcing condition* in terms of a threshold $\bar{c}_M$ on the manufacturer's investment cost c_M as follows. Set $\bar{K} = (0, k_S(0))$ and define the threshold cost $\bar{c}_M = \Lambda_{1\cdot}^{sub} \bar{P}(\bar{K})$, where $\Lambda_{1\cdot}^{sub}$ is the first row of Λ^{sub}. Then the manufacturer should outsource if and only if his investment cost c_M exceeds the threshold cost $\bar{c}_M$.

Coordination of investment decisions would require that $\bar{c}_M = c_S$, because the centralized system puts all capacity at the supplier if $c_M > c_S$. Under price-only subcontracting, however, the threshold cost $\bar{c}_M$ depends not only on c_S, but also on the margins p, the "cost to subcontract" as expressed by the transfer price p_t, and the joint demand distribution P. Figure 6 illustrates that the 'outsourcing zone' of the strategy space is smaller than the outsourcing zone under centralization ($c_M \geq c_S$). This confirms that subcontracting with simple price contracts improves system performance as compared to the solo scenario (never outsourcing), yet it cannot eliminate the value gap ΔV in general.

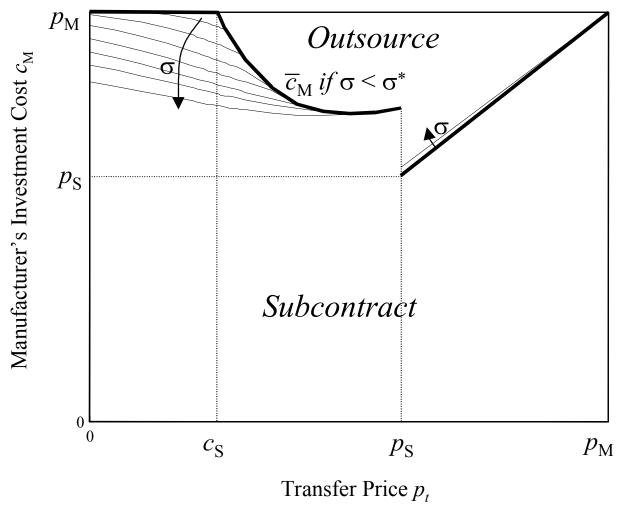

Figure 6: The threshold cost $\bar{c}_M$ partitions the strategy space. (Shown for increasing levels of uncertainty as measured by the standard deviation σ of demand assuming $\sigma_{D_M} = \sigma_{D_S}$)

In the Appendix of [23] we show that for low levels of demand uncertainty, the threshold level is *independent* of the demand distribution and

$$\bar{c}_M = \begin{cases} p_M & \text{if} \quad p_t < c_S, \\ p_t + \left(\frac{p_M}{p_t} - 1\right) c_S & \text{if} \quad c_S \leq p_t < p_S, \\ p_t & \text{if} \quad p_S \leq p_t < p_M. \end{cases} \tag{5}$$

Thus, $\bar{c}_M > c_S$ and with little demand uncertainty ($\sigma \leq \sigma^*$) and low transfer prices, no outsourcing will happen. Indeed, in this case M must still invest in in-house capacity because of two effects, both related to the low transfer price and low uncertainty. First, M cannot induce the supplier to fill his requests: with $p_t < (c_S <) p_S$, S will prioritize her own market. Second, little uncertainty and low transfer prices $p_t < c_S$ give the supplier insufficient incentive to invest in extra capacity to serve the manufacturer. For transfer prices higher than the supplier's capacity cost, outsourcing is possible because S now has an incentive to build extra capacity ($p_t > c_S$). For medium transfer prices, the threshold $\bar{c}_M$ is decreasing in p_t so that outsourcing becomes more likely with higher transfer prices p_t, reflecting a higher incentive for S. When the transfer price exceeds the supplier's margin, a discontinuous drop in $\bar{c}_M$ makes outsourcing even more likely: $p_t > p_S$ ensures M that its requests will now get priority by S. As the transfer price increases, however, subcontracting increasingly becomes more expensive for the manufacturer compared to in-house capacity so that M has less incentive to outsource.

When the level of demand uncertainty rises above a certain level ($\sigma > \sigma^*$), the threshold cost $\bar{c}_M$ will decrease for low to medium transfer prices ($p_t < p_S$) but increase for high transfer prices ($p_S < p_t < p_M$). Thus, for low to medium transfer prices, more uncertainty makes higher manufacturing requests more likely, creating a stronger incentive for the supplier to invest in extra capacity, which makes outsourcing more likely. For high transfer prices, on the other hand, more uncertainty increases the expected total transfer cost to the manufacturer who will prefer more in-house capacity, making outsourcing less likely.

4 Subcontracting with Other Contracts

4.1 Incomplete Contracts: Bargaining

In some situations, ex-ante contracts may be too expensive or impossible to specify or enforce. Start-up companies and companies in developing countries may find it too expensive to enforce execution of a contract [11], while investments by suppliers in quality, information sharing systems, responsiveness and innovation are often non-contractible. "Without the ability to specify contractually in advance the division of surplus from non-contractible investments, this surplus will be divided based on the ex-post bargaining power of the parties involved [3]." This *incomplete contracts approach* was first suggested by Grossman, Hart and Moore [9, 13] to study vertical integration. The negotiation on the surplus division can be cast as bilateral bargaining. Many bargaining games are possible (c.f. Kamien and Li [14, p. 1357]). Nash introduced a game that leads to splitting the surplus evenly. Rubinstein presents a sequential game in which player i gets fraction $\theta = \frac{1-\delta_i}{1-\delta_i\delta_j}$ of the surplus, where δ_i is the "impatience" or discount factor of player i, which is ex-ante observable. Whichever bilateral bargaining game is used, the manufacturer can ex-ante *expect* (but not contractually specify) to receive fraction θ of the surplus while the supplier will get fraction $\bar{\theta} = 1 - \theta$. One can also think of θ as the 'bargaining power' of the manufacturer.

The analysis is similar to before in that both firms have the option to engage in a trade at the beginning of stage two. The firms can decide jointly on production-sales decisions so that the resulting activity vector equals the vector chosen in the centralized scenario. Engaging in subcontracting thus yields a profit surplus $\Delta\pi(K, D) = \pi^{cen}(K, D) - \pi_+^{solo}(K, D) \geq 0$, and both parties thus have an incentive to implement the centralized production vector $x^{cen}(K, D)$ by engaging in the trade $x_t(K, D)$. Hence, production decisions are *always* coordinated with incomplete contracts (in contrast to price-only contracts where poor production decisions can occur if $p_t < p_S < p_M$).

Investment coordination, however, is *not* achieved. Indeed, the manufacturer's operating profit is $\pi_M^{solo} + \theta\Delta\pi$ while the supplier's is $\pi_S^{solo} + \bar{\theta}\Delta\pi$. Because $\frac{\partial}{\partial K_{j\neq i}}\pi_i^{solo} = 0$, the capacity reaction curves can be constructed again in terms of a shadow matrix:

$$
\begin{aligned}
\Lambda^{bar} &= \Lambda^{cen} + diag(\bar{\theta}, \theta)(\Lambda^{solo} - \Lambda^{cen}) \\
&= \begin{cases}
\begin{bmatrix} \bar{\theta}p_M & 0 & \bar{\theta}p_M + \theta p_S & p_M & p_M & \bar{\theta}p_M + \theta p_S \\ 0 & p_S & p_S & \bar{\theta}p_M + \theta p_S & \bar{\theta}p_M & \bar{\theta}p_S \end{bmatrix} & \text{if } p_M \geq p_S \\
\begin{bmatrix} \bar{\theta}p_M & 0 & p_M & p_M & p_M & p_M \\ 0 & p_S & p_S & p_S & \bar{\theta}p_M & \bar{\theta}p_M \end{bmatrix} & \text{if } p_M < p_S.
\end{cases}
\end{aligned}
$$

Both curves have a unique, stable intersection that defines the optimal investment vector K^{bar} that in general differs from K^{cen} because $\Lambda^{bar} \neq \Lambda^{cen}$. Thus, the division of the ex-post surplus gives the supplier an incentive to make a relationship-specific investment, yet insufficient[4] to implement K^{cen}. As shown in Figure 7, reduction in the value gap ΔV and the option value of subcontracting with incomplete contracts is maximal when surplus is divided not too unevenly (but it need not be a fair $50 - 50$ split). More importantly, incomplete contracts are not inferior to explicit price-only contracts. For example, comparing Figure 7 with corresponding Figures 4 and 6 shows that the option value can be larger and that outsourcing is more likely. The higher option value may reflect the fact that with incomplete contracts production coordination is always achieved. It also suggests that sometimes firms may be better off leaving some contract parameters unspecified ex-ante and agreeing to negotiate after demand is observed.

[4] Obviously, if *ex-ante* negotiations are allowed, both parties have an incentive to implement K^{cen} and investment coordination would be achieved.

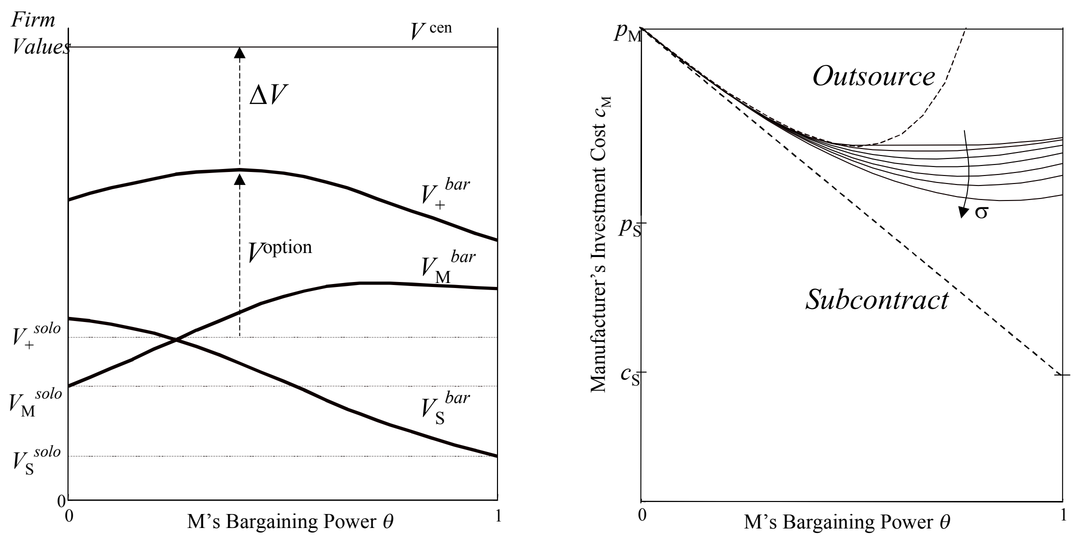

Figure 7: The option value of subcontracting with an incomplete contract and its outsourcing threshold (with dashed bounds) as a function of the manufacturer's bargaining power for the same model parameters as Figure 4.

Let us highlight the role of the bargaining power θ, because the sensitivity of the investment strategy to other parameters is similar to that under price-only contracts. As earlier, we can express an outsourcing condition in terms of a threshold $\bar{c}_M$ on the manufacturer's investment cost c_M. The appendix of [23] derives the following bounds on the outsourcing threshold:

$$\theta c_S \min\left(1, \frac{p_M}{p_S}\right) + \bar{\theta} p_M \leq \bar{c}_M \leq \min\left(p_M, \bar{\theta} p_M + \frac{\theta}{\bar{\theta}} c_S\right).$$

The threshold is decreasing (almost linearly) for small θ, which implies that *outsourcing is more likely for more powerful manufacturers.* The argument, however, cannot be generalized to very powerful manufacturers ($\theta \to 1$): the threshold may be *increasing* near $\theta = 1$ as shown in Figure 7. Indeed, if θ is near 1, outsourcing is less likely because the subcontractor receives less ex-post surplus and has less ex-ante incentive to make a relation-specific investment. Similarly, if bargaining power is very small, most surplus goes to the supplier. As with price-only contracts, $\bar{c}_M \neq c_S$, and the outsourcing zone under this contract is again smaller than the zone under centralization: mere supplier cost advantage of the subcontractor is *not* sufficient for the manufacturer to outsource because the surplus division incentive is insufficient for the subcontractor to implement the centralized capacity level.

4.2 State-dependent Price-Only Contracts

A *state-dependent price-only contract* specifies an ex-ante transfer price $p_t(K, D)$ for each possible contingent state vector. (Such a contract requires that capacity levels are not only observable by the two firms as assumed earlier, but also verifiable by a third party.) Not only can these contracts improve performance because of their increased degrees of freedom, optimal *state-dependent price-only contract design can coordinate investment decisions and eliminate all decentralization costs.* Indeed, it is directly verified that the sufficient condition $\Lambda^{sub} = \Lambda^{cen}$ is satisfied if $p_M \leq p_S$ with $p_t(K, D) = 0$ for $D \in \Omega_1(K)$ and $p_t = p_M$ in Ω_{56}. Such a contract achieves investment coordination (and production coordination because $p_t \leq p_M \leq p_S$) by aligning incentives: subcontracting is costless in Ω_1 (equal to S's marginal opportunity cost when going solo) giving M the correct incentive to reduce its investment to K_M^{cen} and rely on subcontracting, while a transfer price in Ω_{56} equal to M's marginal opportunity profit p_M gives S the right incentive to increase its investment to K_S^{cen}. Similarly, if $p_S \leq p_M$ coordination calls for $p_t = 0$ in Ω_1, $p_t = p_S$ in Ω_{36} and $p_t = p_M$ in Ω_{45}. Higher incentives are now necessary for S to prioritize market M above its own market (as the centralized system would do): transfer prices must at least equal its own margin in Ω_{36} and be higher in Ω_{45} to induce production and investment coordination.

It is surprising that our model setup allows us to characterize these necessary and sufficient conditions for coordination this easily. In addition, notice that these sufficient state-dependent contracts are actually simpler than their name suggests: one only must specify the transfer prices under six scenarios Ω_i in our model and not for each state D.

State-dependent price-only contracts can be related to incomplete contracts as follows. The execution of the inter-firm transfer $x_t^{bar}(K^{bar}, D)$ and the surplus division is implemented by specifying the quantity $x_t(K, D)$ to be provided by the subcontractor and the unit transfer price p_t^{bar} to be paid by the manufacturer. This transfer price is defined implicitly in the bargaining model in that it guarantees the correct division of surplus: $\pi_S^{bar} = p_S x_S^{cen} + p_t^{bar} x_t^{cen}$ (recall that $x^{bar} = x^{cen}$ and $x_M^{cen} = x_M^{solo}$) and rearranging terms yields

$$p_t^{bar} x_t^{cen} = \bar{\theta} p_M x_t^{cen} + \theta p_S (x_S^{solo} - x_S^{cen}). \tag{6}$$

This transfer payment $p_t^{bar} x_t^{cen}$ is the composition of two terms: $p_M x_t^{cen}$ is the gross surplus derived from subcontracting while $p_S(x_S^{solo} - x_S^{cen})$ is the subcontractor's opportunity cost or the profit forgone by subcontracting. The gross surplus is received by the manufacturer who pays the share $\bar{\theta} p_M x_t^{cen}$ to the subcontractor. The subcontractor bears the opportunity cost and is compensated by the manufacturer for the share $\theta p_S(x_S^{solo} - x_S^{cen})$.

Solving (6) in each domain Ω_i yields the state-dependent transfer prices: If $p_M < p_S$, $p_t^{bar} = \bar{\theta} p_M$ in Ω_{156} (its value in Ω_{0234} is irrelevant because no transfer occurs then). This $p_t^{bar}(K, D)$ contract yields production coordination because $p_t^{bar} = \bar{\theta} p_M \leq p_M < p_S$. Thus, if $p_M < p_S$, this state-dependent price-only contract is equivalent to the incomplete contract with parameter θ: it implements identical centralized production decisions and the particular choice of $p_t(K, D)$ guarantees that expected operating profits equal those under the bargaining model and hence their investment vectors are identical. If $p_M > p_S$, however, the existence of an equivalent state-dependent price-only contract is not guaranteed in general. Solving (6) yields $p_t^{bar} = \bar{\theta} p_M$ in Ω_1, $p_t^{bar} = \bar{\theta} p_M + \theta p_S$ in Ω_{34}, $p_t^{bar} = \bar{\theta} p_M + \theta p_S \frac{D_S}{K_S}$ in Ω_5 and $p_t^{bar} = \bar{\theta} p_M + \theta p_S \frac{D_+ - K_+}{D_M - K_M}$ in Ω_6. In general, such a price-only contract does not guarantee production coordination. If, however, M has limited bargaining power so that $\bar{\theta} p_M \geq p_S$, then $p_S \leq p_t(K, D) \leq p_M$ and production coordination is guaranteed so that the p_t contract yields the same investment vector as the incomplete contract.

4.3 State-dependent Incomplete Contracts

A state-dependent incomplete contract is an incomplete contract with *state-dependent* surplus division (bargaining) parameter $\theta(K, D)$. Given their equivalence with state-dependent price-only contracts if supplier margins are higher ($p_S \geq p_M$), it is not surprising that these contracts also coordinate the supply system. Indeed, if $p_M \leq p_S$, the sufficient condition for investment coordination $\Lambda^{bar} = \Lambda^{cen}$ is satisfied with $\theta = 1$ (M receives all surplus) in Ω_1, any constant θ in Ω_{234} and $\theta = 0$ (S receives all surplus) in Ω_{56}. (This $\theta(K, D)$ is also found by requiring that the equivalent p_t^{bar} is coordinating: $p_t^{bar} = \bar{\theta} p_M = 0$ in Ω_1 and $p_t^{bar} = \bar{\theta} p_M = 1$ in Ω_{56}.)

Similarly, with $p_M > p_S$, equality of Λ^{bar} and Λ^{cen} requires $\theta = 1$ in Ω_{13}, any constant θ in Ω_2, $\theta = 0$ in Ω_{45}, but no constant θ in Ω_6 exists to equalize $\Lambda_{\cdot 6}^{bar}$ and $\Lambda_{\cdot 6}^{cen}$. Hence, we must look for a variable function $\theta(K,D)$ over Ω_6. As before, to achieve investment coordination this $\theta(K,D)$ must satisfy the sufficient FOC equality $\mathrm{E}\lambda^{bar} = \mathrm{E}\lambda^{cen} (= \Lambda^{cen}\bar{P}(K))$. If θ varies over a domain Ω_i, however, the marginal profit vector in that domain must be expanded to $\lambda^{bar,i} = \Lambda^{bar}_{\cdot i} + (\frac{\partial\theta}{\partial K_M}, \frac{-\partial\theta}{\partial K_S})'\Delta\Pi_i$, where $\Delta\Pi_i = \Delta\pi$ in domain Ω_i. Hence, the FOC become a system of partial differential equations: with $i = 6$, $\theta(K,D)$ must satisfy

$$\begin{bmatrix} \bar{\theta}p_M + \theta p_S \\ \bar{\theta}p_S \end{bmatrix} + \Delta\pi(K,D) \begin{bmatrix} \frac{\partial\theta}{\partial K_M} \\ -\frac{\partial\theta}{\partial K_S} \end{bmatrix} = \begin{bmatrix} p_S \\ p_S \end{bmatrix}, \tag{7}$$

where $\Delta\pi(K,D) = (p_M - p_S)(D_M - K_M) + p_S(K_S - D_S)$. Luckily, a valid solution $0 \leq \theta(K,D) \leq 1$ is inspired by the equivalent $p_t^{bar} = \bar{\theta}p_M + \theta p_S \frac{D_+ - K_+}{D_M - K_M}$ in Ω_6. Recall that a state-dependent price-only contract requires $p_t = p_S$ in Ω_6 to induce investment coordination. Solving $p_t^{bar} = p_S$ for θ yields

$$\theta(K,D) = \frac{(p_M - p_S)(D_M - K_M)}{(p_M - p_S)(D_M - K_M) + p_S(K_S - D_S)} = \frac{(p_M - p_S)(D_M - K_M)}{\Delta\pi(K,D)} \text{ in } \Omega_6, \tag{8}$$

which indeed satisfies (7). Hence, this incomplete contract with truly state-dependent $\theta(K,D)$ coordinates the supply system if $p_M > p_S$.

5 Discussion and Extensions

In addition to the three contracts studied here, many other contract structures can be used to regulate subcontracting by adding more parameters to the contract specification. Cachon and Lariviere [6] give an overview of more sophisticated contracts used in the literature, which typically also specify some conditions on the transfer quantity x_t or on the manufacturer's liability of the supplier's excess capacity. Cachon and Lariviere show that these more advanced contracts can, but do not necessarily, improve system coordination and highlight the role of the information structure and the verifiability (and thus enforcement) of the players' actions. In the presence of information asymmetries, complex contracts provide for a powerful signaling device that can improve performance. Tsay [21] has shown that some price-quantity contracts also improve system coordination. While we analyzed only simple contracts, we believe that many of the characteristics of more complex outsourcing contracts will carry over to our subcontracting model.

Other extensions such as the inclusion of specific transaction costs and merging costs are relatively straightforward. We have assumed that the initiation and management of the subcontracting relationship was costless. A positive cost is directly incorporated so that both parties would enter into the relationship only if the ex-post surplus exceeds the transaction cost. Similarly, one can include merging costs, which would explain why both parties do not always choose to merge into a single, centralized organization. Another variation is to make both firms more equal 'partners' by dropping the non-negativity constraint on x_t to allow for bi-directional transfers. (This also yields a two-location inventory model with transfers between profit centers.)

Allowing for demand-dependent sales prices (and thus margins) by incorporating downward sloping demand curves (our firms are assumed to be price takers) would yield a duopoly model more in-line with traditional economics. This generalization to incorporate tactical pricing decisions, however, comes at considerable cost. One not only loses the connection to the traditional newsvendor model and its intuitive, graphical interpretation, but the competitive pricing decision under uncertainty greatly increases the complexity of the analysis[5]. Allowing for non-exclusive

[5]Allowing for inter-firm subcontracting transfers would amount to putting yet another layer of complexity on the competitive investment-pricing model that we studied in [24].

market access is an easier extension that, we believe, will not change the qualitative insights obtained here. Finally, the time-horizon can be extended to a multi-period setting to study the effect of predictable temporal demand variations, such as over a product life cycle (stochastic temporal variations most likely will lead to a production smoothing effect as studied by Kamien and Li [14]).

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6 Appendix

All first order conditions (FOC) for optimality are of the form $\mathrm{E}\lambda = \Lambda\overline{P} = c$,where the 2×6 matrix Λ is function only of p_t, p_M and p_S, while the vector $\overline{P}$ is function only of K (and of parameters in the probability distribution). The structure of the FOCs (or capacity reaction curves) and uniqueness of an optimal solution will be established using partial derivatives which are found by implicitly differentiating one or both FOC. Let x represent a cost or margin parameter of interest. Total differentiation of the FOC yields:

$$\frac{d}{dx}\mathrm{E}\lambda = \frac{d}{dx}\Lambda\overline{P} = \left(\frac{\partial}{\partial x}\Lambda\right)\overline{P} + J\frac{\partial}{\partial x}K,$$

where J is the Jacobian matrix of the FOC: $J_{ij} = \frac{\partial \mathrm{E}\lambda_i}{\partial K_j} = \frac{\partial V_i}{\partial K_j \partial K_i}$, which can be calculated explicitly:

$$J = \Lambda \left[\begin{array}{cc} \frac{\partial}{\partial K_M}\overline{P} & \frac{\partial}{\partial K_S}\overline{P} \end{array}\right] = \Lambda\left(\nabla_K \overline{P}'\right)',$$

where the 2×6 matrix $\nabla_K \overline{P}'$ can be expressed in terms of the line integrals L_{ij} of the probability density $f(\cdot)$ over the boundary between domains Ω_i and Ω_j and $L_{ij,kl} = L_{ij} + L_{kl}$:

$$\nabla_K \overline{P}' = \left[\begin{array}{cccccc} L_{16} - L_{01} & L_{23} & L_{34} - L_{23} & -L_{34} & -L_{56} & L_{56} - L_{16} \\ L_{16} & -L_{02} & L_{34} - L_{36} & -L_{34} - L_{45} & L_{45} - L_{56} & L_{56} + L_{36} - L_{16} \end{array}\right].$$

For example: $L_{23} = \int_{K_S}^{\infty} f(K_M, D_S)dD_S$. Thus, all effort is reduced to showing that $J = \Lambda\left(\nabla_K \overline{P}'\right)'$ is invertible which then yields

$$\frac{\partial}{\partial x}K = J^{-1}\frac{dc}{dx} - J^{-1}\left(\frac{\partial}{\partial x}\Lambda\right)\overline{P}. \tag{9}$$

Thus, letting $x = c_i$ we directly have that

$$\left[\begin{array}{cc} \frac{\partial}{\partial c_M}K & \frac{\partial}{\partial c_S}K \end{array}\right] = J^{-1}, \tag{10}$$

and the slope of $k_i(\cdot)$, the FOC for K_i given K_j, follows from totally differentiating the i'th FOC: $\frac{\partial}{\partial K_i}\mathrm{E}\lambda_i \frac{dk_i}{dK_j} + \frac{\partial}{\partial K_j}\mathrm{E}\lambda_i = 0$ or

$$\frac{dk_i}{dK_j} = -\frac{\frac{\partial^2}{\partial K_j \partial K_i}V_i}{\frac{\partial^2}{\partial K_i^2}V_i} = -\frac{J_{ij}}{J_{ii}}.$$

6.1 Centralized Reference Scenario

The optimal solution K^{cen} is at the intersection of the two FOC curves. We have that

$$J^{cen} = \begin{cases} \left[\begin{array}{cc} -(p_M - p_S)L_{34,56} - p_S L_{16,23} & -(p_M - p_S)L_{34,56} - p_S L_{16} \\ -(p_M - p_S)L_{34,56} - p_S L_{16} & -(p_M - p_S)L_{34,56} - p_S L_{02,16} \end{array}\right] & \text{if } p_S \leq p_M, \\ \left[\begin{array}{cc} -p_M L_{16,23} & -p_M L_{16} \\ -p_M L_{16} & -p_S L_{02} - (p_S - p_M)L_{45,36} - p_M L_{16} \end{array}\right] & \text{if } p_M < p_S. \end{cases}$$

All entries in J are nonpositive with $J_{11} \leq J_{12} \leq 0$ and $J_{22} \leq J_{21} \leq 0$ so that $|J| \geq 0$ and

$$\begin{aligned} |J| &= \begin{cases} (p_M - p_S)p_S L_{34,56}L_{02,23} + p_S^2(L_{16}L_{02} + L_{23}L_{02} + L_{23}L_{16}) & \text{if } p_S \leq p_M, \\ p_M p_S L_{16,23}L_{02} + p_M(p_S - p_M)L_{16,23}L_{45,36} + p_M^2 L_{23}L_{16} & \text{if } p_M < p_S. \end{cases} . \\ -1 &\leq \frac{dk_i}{dK_j} = -\frac{J_{ij}}{J_{ii}} \leq 0. \end{aligned}$$

Clearly, if $c_M > c_S$, it is optimal to invest only in S-capacity: $\nu_M > 0$ so that $k_M^{cent}(\cdot) = 0$ and $k_S^{cent}(\cdot) = K_S^{cent}$. If $p_S \leq p_M$, $p_S P_{236} + p_M P_{45} = c_S$. Because $P_{02} = 0$, either L_{16} and/or $L_{34,56}$ are positive so that FOC_S is strict concave at K_S^{cen} ($J_{22} < 0$), ergo uniqueness. If $p_M < p_S$, $p_S P_{234} + p_M P_{56} = c_S$ and either $L_{34,36}$ and/or L_{16} are positive, again showing uniqueness.

Otherwise, if $c_M < c_S$, we invest in both capacities and at least one of the terms in $|J|$ is positive so that V^{cen} is strict concave at the unique optimal K^{cen}. We can compute some points of the centralized curves:

- If $K_S = 0$ and $c_M < c_S$, then $P_{01356} = 0, L_{23} = L_{34}, L_{16,56} = 0$ and $p_M P_4 = p_M P(D_M > K_M^{cent}) = c_M$. Thus,

$$k_M^{cent}(0) = K_M^{solo} \text{ and } \frac{dk_M}{dK_S} = \begin{cases} -\frac{p_M - p_S}{p_M} > -1 & \text{if } p_S \le p_M, \\ 0 & \text{if } p_S > p_M. \end{cases}.$$

- If $K_S \to \infty$ and $c_M < c_S$, then $P_{23456} = 0$ so that $L_{34,56,16} = 0$ and $\mathrm{E}\lambda_M = 0 < c_M$ so that

$$k_M(\infty) = 0 \text{ and } \frac{dk_M(\infty)}{dK_S} = 0,$$

  a situation that remains if K_S decreases as long as $P_{01}((0, K_S)) = 1$. Clearly, this minimal K_S increases in correlation and variability.

- If $K_M = 0$, then $P_{02} = 0$ and $L_{02} = 0$. Thus,

$$\begin{cases} \frac{dk_S^{cent}(0)}{dK_M} = -1 & \text{if } p_S \le p_M, \\ -1 \le \frac{dk_S^{cent}(0)}{dK_M} \le 0 & \text{if } p_S > p_M. \end{cases}.$$

- If $K_M \to \infty$, then $P_{13456} = 0$ so that $L_{34,56,16} = 0$ and $p_S P_2 = c_S$. Thus,

$$k_S^{cent}(\infty) = G_S(\frac{c_S}{p_S}) = K_S^{solo} \text{ and } \frac{dk_S^{cent}(\infty)}{dK_M} = 0,$$

  a situation that remains if K_M decreases as long as $P_{456}((K_M, k_S(\infty)) = 0$. Clearly, this minimal K_M increases in variability.

6.2 Subcontracting with Price-Only Contracts

The Jacobian becomes

$$J = \begin{cases} \begin{bmatrix} -(p_M - p_t)L_{34,56} - p_t L_{01,23} & -(p_M - p_t)L_{34,56} \\ -(p_t - p_S)L_{34,56} - p_S L_{16} & -(p_t - p_S)L_{34,56} - p_S L_{02,16} \end{bmatrix} & \text{if } p_S \le p_t \le p_M, \\ \begin{bmatrix} -p_t L_{01} - p_M L_{23} - (p_M - p_t)L_{16} & -(p_M - p_t)L_{16} \\ -p_t L_{16} & -p_S L_{02} - (p_S - p_t)L_{45,36} - p_t L_{16} \end{bmatrix} & \text{if } p_t < \min(p). \end{cases}.$$

6.2.1 Uniqueness of the solution K^{sub}

All entries in J are nonpositive with $J_{11} \le J_{12} \le 0$ and $J_{22} \le J_{21} \le 0$ so that $|J| \ge 0$ and

$$\begin{aligned} |J| &= \begin{cases} (p_M - p_t)p_S L_{34,56}L_{02} + (p_t - p_S)p_t L_{01,23}L_{34,56} + p_t p_S L_{01,23}L_{02,16} & \text{if } p_S \le p_t \le p_M, \\ (p_M - p_t)p_S L_{02}L_{16} + (p_M - p_t)(p_S - p_t)L_{16}L_{45,36} + p_t(p_S - p_t)L_{01}L_{45,36} & \\ \quad + p_M(p_S - p_t)L_{23}L_{45,36} + p_t p_S L_{01}L_{02} + p_t^2 L_{01}L_{16} + p_M p_S L_{23}L_{02} + p_M p_t L_{23}L_{16} & \text{if } p_t < \min(p). \end{cases}. \\ -1 &\le \frac{dk_i}{dK_j} = -\frac{J_{ij}}{J_{ii}} \le 0. \end{aligned}$$

Existence of an intersection follows from the relative position of axis crossings and asymptotes:

- If $K_S = 0$, then $P_{01356} = 0, L_{23} = L_{34}, L_{01,16,56} = 0$ and $p_M P_4 = p_M P(D_M > K_M) = c_M$. Thus,

$$k_M(0) = G_M(\frac{c_M}{p_M}) = K_M^{solo} \text{ and } \frac{dk_M(0)}{dK_S} = \begin{cases} -\frac{p_M - p_t}{p_M} > -1 & \text{if } p_S \le p_t \le p_M, \\ 0 & \text{if } p_t < \min(p). \end{cases}.$$

  ($\frac{dk_M(0)}{dK_S}$ remains 0 as K_S increases with low p_t until P_1 becomes positive. Clearly, this maximal K_S decreases in correlation and variability.)

- If $K_S \to \infty$, then $P_{23456} = 0$ so that $L_{34,56,16} = 0$ and $\mathrm{E}\lambda_M = p_t P_1 \leq p_t$. Thus, if $p_t < c_M$, we have $k_M(\infty) = 0$, else
$$k_M(\infty) = G_M(\frac{c_M}{p_t}) < K_M^{solo} \text{ and } \frac{dk_M(\infty)}{dK_S} = 0,$$
a situation that remains if K_S decreases as long as $P_{01}\left((k_M(\infty), K_S)\right) = 1$. Clearly, this minimal K_S increases in correlation and variability. Note that $k_M(\cdot)$ is continuous in p_t for $p_S < p_t < p_M$, except at $p_t = c_M$ if D_M is bounded from below by a positive number with probability one (the demand density is zero at $D_M = 0$).
- If $K_M = 0$, then $P_{02} = 0$ so that $L_{02} = 0$. For high p_t we have that $p_S P_{36} + p_t P_{45} = c_S$ and because $p_S < p_t$, we have that
$$\frac{dk_S(0)}{dK_M} = -1.$$
With small variability, we have that $k_S(0) \simeq D_+$ (exact: $P(D_+ > k_S(0)) = \frac{c_S}{p_S}$. Indeed, if $K_S \ll (\gg) D_+$, we would have that $P_{3456} = 1(0)$, which cannot satisfy FOC$_S$.) For low p_t we have that $p_S P_{34} + p_t P_{56} = c_S$. If $p_t < c_S$, then $P_{34} > 0$. If D has low variability in the sense that $P(\Omega_1(K = (0, K_S^{solo}))) = 0$, then $k_2(\cdot)$ is discontinuous at $p_t = c_2$ and we have that
$$\begin{aligned} k_2(0) &\approx D_+ \text{ (exactly: } P_{56} = \frac{c_S}{p_t}) \text{ and } \frac{dk_S(0)}{dK_M} = -1 \text{ IF } p_t > c_S, \\ k_S(\cdot) &= K_S^{solo} \approx D_S \text{ and thus } \frac{dk_S(0)}{dK_M} = 0 \text{ IF } p_t < c_S. \end{aligned}$$
If D has high variability, $0 \leq \frac{dk_S(0)}{dK_M} \leq -1$.
- If $K_M \to \infty$, then $P_{13456} = 0$ so that $L_{34,56,16} = 0$ and $p_S P_2 = c_S$. Thus,
$$k_S(\infty) = G_M(\frac{c_S}{p_S}) = K_S^{solo} \text{ and } \frac{dk_S(\infty)}{dK_M} = 0,$$
a situation that remains if K_M decreases as long as $P_{456}\left((K_M, k_S(\infty)\right) = 0$. Clearly, this minimal K_M increases in variability.

Uniqueness of K^{sub} follows from $-1 < \frac{dk_M}{dK_S}$ at intersection (assume high p_t, low p_t is similar)

- If $p_M \geq p_t > c_M : 0 < P_{13645} < 1$ and because P is a continuous measure we have that $L_{23,01} > 0$ so that V_M is strict concave at the optimal K_M and thus the reaction curve $k_M(\cdot)$ is unique. Moreover
$$-1 < \frac{dk_M}{dK_S} \leq 0 \text{ (and } \frac{dk_M}{dK_S} = 0 \text{ if } P_{45} = 0).$$
- If $p_M > c_M \geq p_t : 0 < P_{45} < 1$ so that $L_{34,56} > 0$. Again the reaction curve $k_M(\cdot)$ is unique but now, as long as $k_M > 0$:
$$-1 \leq \frac{dk_M}{dK_S} < 0 \text{ (and } \frac{dk_M}{dK_S} = -1 \text{ if } P_{012} = 0).$$
At the intersection K^{sub} we have that $-1 < \frac{dk_M}{dK_S}$ which shows uniqueness (indeed $P_{012} = 0$ would imply $P_{3456} = 1$, which cannot be a solution to FOC$_S : p_S P_{36} + p_t P_{45} \geq \min(p_S, p_t) = p_S > c_S$).

Similarly for firm 2's reaction curves ($p_S P_{236} + p_t P_{45} = c_S$), if follows that

- Because $p_t > p_S > c_S : 0 < P_{236}, P_{45} < 1$ and thus $0 < P_{01} < 1$ and $L_{02,16} > 0$ so that V_S is strict concave at the optimal K_S and thus the reaction curve $k_S(\cdot)$ is unique. Moreover
$$-1 \leq \frac{dk_S}{dK_M} \leq 0 \text{ (and } \frac{dk_S}{dK_M} = -1 \text{ if } P_2 = 0 \text{ and } \frac{dk_S}{dK_M} = 0 \text{ if } P_{13456} = 0).$$

Given that the two reaction curves are unique with $-1 \leq \frac{dk_i}{dK_j} \leq 0$, and the relative axis crossings are as given higher together with $-1 < \frac{dk_M}{dK_S}$ at any solution to the FOC, it follows that hey have a unique intersection which is a stable, and thus Nash, equilibrium.

6.2.2 Sensitivity of K^{sub}

The intersection point K is the unique solution to the FOC and it follows from the FOC that one will never invest to cover all demand with probability 1. In other words, if $K > 0$, then $0 < P_{01} < 1$ and at least one of the terms in $\det(J)$ is positive so that $|J| > 0$ and J is invertible:

$$\begin{aligned} J^{-1} &= \begin{cases} |J|^{-1}\begin{bmatrix} -(p_t - p_S)L_{34,56} - p_S L_{02,16} & (p_M - p_t)L_{34,56} \\ (p_t - p_S)L_{34,56} + p_S L_{16} & -(p_M - p_t)L_{34,56} - p_t L_{01,23} \end{bmatrix} & \text{if } p_S \le p_t \le p_M, \\ |J|^{-1}\begin{bmatrix} -p_S L_{02} - (p_S - p_t)L_{45,36} - p_t L_{16} & (p_M - p_t)L_{16} \\ p_t L_{16} & -p_t L_{01} - p_M L_{23} - (p_M - p_t)L_{16} \end{bmatrix} & \text{if } p_t < \min(p). \end{cases} \\ &= -\begin{bmatrix} \alpha_1 + \alpha_3 & -\alpha_2 \\ -\alpha_1 & \alpha_2 + \alpha_4 \end{bmatrix}. \end{aligned}$$

Because $\frac{\partial K_i}{\partial c_j} = J_{ij}$, we have that both capacities are imperfect substitutes w.r.t. the marginal cost vector. Partials w.r.t. the margins are

$$\begin{aligned} \frac{\partial}{\partial p_M}K &= \begin{cases} -J^{-1}\begin{bmatrix} 0&0&0&1&1&0 \\ 0&0&0&0&0&0 \end{bmatrix} P = \begin{bmatrix} \alpha_1+\alpha_3 \\ -\alpha_1 \end{bmatrix} P_{45} & \text{if } p_S \le p_t \le p_M, \\ -J^{-1}\begin{bmatrix} 0&0&1&1&1&1 \\ 0&0&0&0&0&0 \end{bmatrix} P = \begin{bmatrix} \alpha_1+\alpha_3 \\ -\alpha_1 \end{bmatrix} P_{3456} & \text{if } p_t < \min(p). \end{cases} \\ \frac{\partial}{\partial p_S}K &= \begin{cases} -J^{-1}\begin{bmatrix} 0&0&0&0&0&0 \\ 0&1&1&0&0&1 \end{bmatrix} P = \begin{bmatrix} -\alpha_2 \\ \alpha_2+\alpha_4 \end{bmatrix} P_{236} & \text{if } p_S \le p_t \le p_M, \\ -J^{-1}\begin{bmatrix} 0&0&0&0&0&0 \\ 0&1&1&1&0&0 \end{bmatrix} P = \begin{bmatrix} -\alpha_2 \\ \alpha_2+\alpha_4 \end{bmatrix} P_{234} & \text{if } p_t < \min(p). \end{cases} \\ \frac{\partial}{\partial p_t}K &= \begin{cases} -J^{-1}\begin{bmatrix} 1&0&1&0&0&1 \\ 0&0&0&1&1&0 \end{bmatrix} P = \begin{bmatrix} (\alpha_1+\alpha_3)P_{136} - \alpha_2 P_{45} \\ -\alpha_1 P_{136} + (\alpha_2+\alpha_4)P_{45} \end{bmatrix} & \text{if } p_S \le p_t \le p_M, \\ -J^{-1}\begin{bmatrix} 1&0&0&0&0&0 \\ 0&0&0&0&1&1 \end{bmatrix} P = \begin{bmatrix} (\alpha_1+\alpha_3)P_1 - \alpha_2 P_{56} \\ -\alpha_1 P_1 + (\alpha_2+\alpha_4)P_{56} \end{bmatrix} & \text{if } p_t < \min(p). \end{cases} \end{aligned}$$

While $\frac{\partial}{\partial p_t}K$ cannot be signed in general, we do have that $\frac{\partial}{\partial p_t}K_+ > 0$.

6.2.3 Sensitivity of V^{sub}

We have that $\frac{dV_i}{dx} = \frac{\partial V_i}{\partial K_M}\frac{\partial K_M}{\partial x} + \frac{\partial V_i}{\partial K_S}\frac{\partial K_S}{\partial x} + \frac{\partial V_i}{\partial x}$, where $\frac{\partial V_i}{\partial K_i} = 0$ under optimal investment. The cross-partial $\frac{\partial V_i}{\partial K_j} = \frac{\partial}{\partial K_j}\mathrm{E}\pi_i = \mathrm{E}\frac{\partial}{\partial K_j}\pi_i = \mathrm{E}\lambda_{i,j}$ can be computed as before by the weighted average of the constant $\lambda^j_{i,j}$ in each domain l:

$$\begin{aligned} \frac{\partial V_M^{sub}}{\partial K_S} &= \mathrm{E}\lambda_{1,2} = (p_M - p_t)P_{45} \ge 0 \\ \frac{\partial V_S^{sub}}{\partial K_M} &= \mathrm{E}\lambda_{2,1} = \begin{cases} -p_t P_1 - (p_t - p_S)P_{36} \le 0 & \text{if } p_S \le p_t \le p_M, \\ -p_t P_1 \le 0 & \text{if } p_t < \min(p). \end{cases} \end{aligned}$$

Denoting $\mathrm{E}\lambda_{1,2} = \beta_5 \ge 0$ and $\mathrm{E}\lambda_{2,1} = -\beta_6 \le 0$, we get

$$\begin{aligned} \frac{\partial V_M^{sub}}{\partial c_M} &= \alpha_1\beta_5 - K_1^{sub} = \beta_1 - K_M^{sub}, & \frac{\partial V_S^{sub}}{\partial c_M} &= \beta_6(\alpha_1+\alpha_3) = \beta_3 \ge 0, \\ \frac{\partial V_M^{sub}}{\partial c_S} &= -(\alpha_2+\alpha_4)\beta_5 = -\beta_2 \le 0, & \frac{\partial V_S^{sub}}{\partial c_S} &= -\beta_6\alpha_2 - K_S^{sub} = -\beta_4 - K_S^{sub} \le 0. \end{aligned}$$

As expected $\frac{\partial V_M}{\partial c_S}, \frac{\partial V_S}{\partial c_S}$ are negative and $\frac{\partial V_S}{\partial c_M}$ is positive, while $\frac{\partial V_M}{\partial c_M}$ cannot be signed in general. For price sensitivity consider high transfer prices (the other case is similar but replace the P_{45} by P_{3456}, P_{236} by P_{234}, P_{136} by P_1, and P_{45} by P_{56}):

$$\begin{aligned} \frac{\partial V_M^{sub}}{\partial p_M} &= -\beta_5\alpha_1 P_{45} + \mathrm{E}x_{1+t} = -\beta_1 P_{45} + \mathrm{E}x_{1+t}, & \frac{\partial V_S^{sub}}{\partial p_M} &= -\beta_6(\alpha_1+\alpha_3)P_{45} = -\beta_3 P_{45} \le 0, \\ \frac{\partial V_M^{sub}}{\partial p_S} &= \beta_5(\alpha_2+\alpha_4)P_{236} = \beta_2 P_{236} \ge 0, & \frac{\partial V_S^{sub}}{\partial p_S} &= \beta_6\alpha_2 P_{236} + \mathrm{E}x_S = \beta_4 P_{236} + \mathrm{E}x_S \ge 0. \\ \frac{\partial V_M^{sub}}{\partial p_t} &= \beta_5\frac{\partial K_S^{sub}}{\partial p_t} - c_M\frac{\partial K_M^{sub}}{\partial p_t} - \mathrm{E}x_t, & \frac{\partial V_S^{sub}}{\partial p_t} &= -\beta_6\frac{\partial K_M^{sub}}{\partial p_t} - c_S\frac{\partial K_S^{sub}}{\partial p_t} + \mathrm{E}x_t. \end{aligned}$$

6.3 Outsourcing Conditions

First note that from the structure of the manufacturer's reaction curve it follows that the threshold cost $\bar{c}_M \geq p_t$, because a necessary condition for outsourcing is that $c_M \geq p_t$ so that $G_M(\frac{c_M}{p_t}) = 0$.

6.3.1 Low transfer price: $p_t < \min(p)$

Because $K_M = 0$, we have that $P_{02} = 0$ and $P_{13456} = 1$ and the optimality equations yield

$$\begin{aligned} p_t P_1 + p_M P_{3456} &= \bar{c}_M, \\ p_S P_{34} + p_t P_{56} &= c_S, \end{aligned}$$

so that $\frac{c_S}{p_S} \leq P_{3456} \leq \frac{c_S}{p_t}$

$$p_t + \frac{p_M - p_t}{p_S} c_S \leq \bar{c}_M = p_t + (p_M - p_t) P_{3456} \leq p_t + \frac{p_M - p_t}{p_t} c_S.$$

Also,

$$\bar{c}_M = c_S + p_t P_1 + (p_M - p_S) P_{34} + (p_M - p_t) P_{56} \geq^{\text{if } p_M \geq p_S} c_S.$$

Notice that with low levels of uncertainty, one either has

$$\begin{aligned} p_t &< c_S : \bar{K}_S = k_S(0) \simeq D_S \text{ (exactly: } p_S P_{34} = c_S), P_{3456} = 1 \Rightarrow \bar{c}_M = p_M. \\ c_S &< p_t : \bar{K}_S = k_S(0) \simeq D_+ \text{ (exactly: } p_t P_6 = c_S), P_{16} = 1, P_{02345} = 0 \Rightarrow \bar{c}_M = p_t + \frac{p_M}{p_t} c_S - c_S. \end{aligned}$$

As uncertainty increases, $\bar{c}_M$ will *decrease*. Indeed, if $p_t < c_S$, increasing uncertainty will decrease P_{3456} from 1 and increase P_1, but P_1 has lower coefficient $p_t < p_M$ in the definition of $\bar{c}_M$. If $p_t > c_S$, increasing uncertainty will decrease P_6 and increase P_{534}. From FOC 2 we see that P_6 will decrease more than P_{345} will increase ($p_S > p_t$); thus P_1 will also increase, but again less than the decrease in P_6, so that $\bar{c}_M$ will decrease because $p_t < p_M$.

6.3.2 High transfer price: $p_S < p_t < p_M$

Because $K_M = 0$, we have that $P_{02} = 0$ and $P_{13456} = 1$ and the optimality equations yield

$$\begin{aligned} p_t P_{136} + p_M P_{45} &= \bar{c}_M, \\ p_S P_{36} + p_t P_{45} &= c_S, \end{aligned}$$

so that $0 \leq P_{45} \leq \frac{c_S}{p_t}$

$$p_t \leq \bar{c}_M = p_t + (p_M - p_t) P_{45} \leq p_t + \frac{p_M - p_t}{p_t} c_S.$$

Again, with limited levels of uncertainty, one can only have ($p_t > p_S > c_S$):

$$\bar{K}_S = k_S(0) \simeq D_+ \text{ (exact: } p_S P_6 = c_S), P_{16} = 1, P_{02345} = 0 \Rightarrow \bar{c}_M = p_t.$$

As uncertainty increases, P_{16} will decrease from 1 and P_{345} will grow, leading to an *increase* in $\bar{c}_M$ because $p_t < p_M$. Finally, notice that $\bar{c}_M$ is discontinuous at $p_t = p_S$.

6.4 Incomplete Contracts (Bargaining)

6.4.1 FOC

With incomplete contracting the players' revenue functions are:

$$\begin{aligned} \pi_M^b &= \pi_M^{solo} + \theta \Delta\pi = \theta \pi^{cen} + \bar{\theta} \pi_M^{solo} - \theta \pi_S^{solo}, \\ \pi_S^b &= \pi_S^{solo} + \bar{\theta} \Delta\pi = \bar{\theta} \pi^{cen} - \bar{\theta} \pi_M^{solo} + \theta \pi_S^{solo}. \end{aligned}$$

Because $\frac{\partial}{\partial K_{j\neq i}}\pi_i^{solo}=0$, the capacity reaction curves can be constructed in terms of our primitive shadow matrices

$$
\begin{aligned}
\Lambda^{bar} &= diag(\bar\theta,\theta)\Lambda^{solo} + diag(\theta,\bar\theta)\Lambda^{cen} \\
&= \Lambda^{solo} + diag(\theta,\bar\theta)(\Lambda^{cen}-\Lambda^{solo}) = \Lambda^{cen} + diag(\bar\theta,\theta)(\Lambda^{solo}-\Lambda^{cen}) \\
&= \begin{cases}
\begin{bmatrix} \overline{\theta}p_M & 0 & \overline{\theta}p_M+\theta p_S & p_M & p_M & \overline{\theta}p_M+\theta p_S \\ 0 & p_S & p_S & \overline{\theta}p_M+\theta p_S & \overline{\theta}p_M & \overline{\theta}p_S \end{bmatrix} & \text{if } p_M > p_S \\
\begin{bmatrix} \overline{\theta}p_M & 0 & p_M & p_M & p_M & p_M \\ 0 & p_S & p_S & p_S & \overline{\theta}p_M & \overline{\theta}p_M \end{bmatrix} & \text{elsewhere.}
\end{cases}
\end{aligned}
$$

6.4.2 Outsourcing

Case 1: $p_M > p_S$. Because $K_M=0$, we have that $P_{02}=0$ and $P_{13456}=1$ and the optimality equations yield:

$$
\begin{aligned}
\overline{\theta}p_M P_1 + (\overline{\theta}p_M+\theta p_S)P_{36} + p_M P_{45} &= \bar c_M, \\
p_S P_3 + (\overline{\theta}p_M+\theta p_S)P_4 + \overline{\theta}p_M P_5 + \overline{\theta}p_S P_6 &= c_S,
\end{aligned}
$$

so that

$$
\begin{aligned}
\bar c_M &= \overline{\theta}p_M(1-P_{3456}) + (\overline{\theta}p_M+\theta p_S)P_{36} + p_M P_{45} \\
&= \overline{\theta}p_M + \theta p_S P_{36} + \theta p_M P_{45} \\
&= \overline{\theta}p_M + \theta\left(c_S - (\overline{\theta}p_M+\theta p_S)P_4 - \overline{\theta}p_M P_5 + \theta p_S P_6\right) + \theta p_M P_{45} \\
&= \overline{\theta}p_M + \theta\left(c_S - (\overline{\theta}p_M+\theta p_S)P_4 - \overline{\theta}p_M P_5 + p_M P_{45} + \theta p_S P_6\right) \\
&= \overline{\theta}p_M + \theta\left(c_S - (-\theta p_M+\theta p_S)P_4 + \theta p_M P_5 + \theta p_S P_6\right) \\
&= \overline{\theta}p_M + \theta c_S + \theta^2\left((p_M-p_S)P_4 + p_M P_5 + p_S P_6\right)
\end{aligned}
$$

$$
\overline{\theta}p_M + \theta c_S \le \bar c_M = \overline{\theta}p_M + \theta c_S + \theta^2\left((p_M-p_S)P_4 + p_M P_5 + p_S P_6\right) \le \overline{\theta}p_M + \theta c_S + \frac{\theta^2}{\overline{\theta}}c_S = \overline{\theta}p_M + \frac{\theta}{\overline{\overline{\theta}}}c_S.
$$

Case 2: $p_M \le p_S$. With $K_M=0$, we have that $P_{02}=0$ and $P_{13456}=1$ and the optimality equations yield:

$$
\begin{aligned}
(1-\theta)p_M P_1 + p_M P_{3456} &= \bar c_M, \\
p_S P_{34} + (1-\theta)p_M P_{56} &= c_S,
\end{aligned}
$$

If follows directly from FOC_1 :

$$
(1-\theta)p_M < \bar c_M < p_M
$$

and from FOC_2 : $(p_S > p_M)$

$$
(1-\theta)p_M P_{3456} < p_M P_{34} + (1-\theta)p_M P_{56} < c_S < p_S P_{34} + p_M P_{56} < p_S P_{3456}
$$

so that

$$
\frac{c_S}{p_S} < P_{3456} < \frac{c_S}{(1-\theta)p_M}
$$

$$
\begin{aligned}
\bar c_M &= (1-\theta)p_M(1-P_{3456}) + p_M P_{3456} \\
&= (1-\theta)p_M + \theta p_M P_{3456}
\end{aligned}
$$

And hence:

$$
(1-\theta)p_M + \theta\frac{p_M}{p_S}c_S < \bar c_M = (1-\theta)p_M + \theta p_M P_{3456} < (1-\theta)p_M + \frac{\theta}{1-\theta}c_S.
$$

6.4.3 State-Dependent Price-Only Contracts and Coordination

It is directly verified that the sufficient condition for coordination $\Lambda^{sub} = \Lambda^{cen}$ is satisfied if $p_M \leq p_S$ with $p_t(K, D) = 0$ for $D \in \Omega_1(K)$ and $p_t = p_M$ in Ω_{56}. Similarly, if $p_S \leq p_M$ coordination calls for $p_t = 0$ in Ω_1, $p_t = p_S$ in Ω_{36} and $p_t = p_M$ in Ω_{45}.

The equivalent transfer price is defined implicitly in the bargaining model in that it guarantees the correct division of surplus $\pi_S^{bar} = p_S x_S^{cen} + p_t^{bar} x_t^{cen}$(recall that $x^{bar} = x^{cen}$). We know that

$$\begin{aligned}
\pi_S^{bar} &= \pi_S^{solo} + \bar{\theta}\Delta\pi \\
&= \bar{\theta}\pi^{cen} - \bar{\theta}\pi_M^{solo} + \theta\pi_S^{solo} \\
&= \bar{\theta}(p_M(x_M^{cen} + x_t^{cen}) + p_S x_S^{cen}) - \bar{\theta} p_M x_M^{solo} + \theta p_S x_S^{solo} \\
&= \bar{\theta}(p_M x_t^{cen} + p_S x_S^{cen}) + \theta p_S x_S^{solo} \qquad \text{[because } x_M^{cen} = x_M^{solo}\text{]}
\end{aligned}$$

so that

$$\begin{aligned}
p_t^{bar} x_t^{cen} &= (1 - \theta)(p_M x_t^{cen} + p_S x_S^{cen}) + \theta p_S x_S^{solo} - p_S x_S^{cen} \\
&= \bar{\theta} p_M x_t^{cen} + \theta p_S (x_S^{solo} - x_S^{cen}),
\end{aligned}$$

or

$$p_t^{bar} = \bar{\theta} p_M + \theta p_S \frac{x_S^{solo} - x_S^{cen}}{x_t^{cen}}.$$

Case 1: $p_M \leq p_S$. We know that no transfers happen in Ω_{0234} and in the other domains:

	x_t^{cen}	x_S^{solo}	x_S^{cen}	p_t^{bar}
Ω_1	$D_M - K_M$	D_S	D_S	$\bar{\theta} p_M$
Ω_{56}	$K_S - D_S$	D_S	D_S	$\bar{\theta} p_M$

Case 2: $p_M > p_S$. We know that no transfers happen in Ω_{02} and in the other domains:

	x_t^{cen}	x_S^{solo}	x_S^{cen}	p_t^{bar}
Ω_1	$D_M - K_M$	D_S	D_S	$\bar{\theta} p_M$
Ω_3	$D_M - K_M$	K_S	$K_+ - D_M$	$\bar{\theta} p_M + \theta p_S$
Ω_4	K_S	K_S	0	$\bar{\theta} p_M + \theta p_S$
Ω_5	K_S	D_S	0	$\bar{\theta} p_M + \theta p_S \frac{D_S}{K_S}$
Ω_6	$D_M - K_M$	D_S	$K_+ - D_M$	$\bar{\theta} p_M + \theta p_S \frac{D_+ - K_+}{D_M - K_M}$.

6.4.4 State-Dependent Bargaining and Coordination

Assume $\theta(K, D)$. Coordination would require that the FOCs for K_i^{bar} are identical with those of K_i^{cen}. Or, in each domain Ω_l, $\frac{\partial}{\partial K_i}\pi_i^{bar} = \Lambda_{il}^{cen}$. With incomplete contracting the players' revenue functions are:

$$\begin{aligned}
\pi_M^b &= \pi_M^{solo} + \theta\Delta\pi = \theta\pi^{cen} + \bar{\theta}\pi_M^{solo} - \theta\pi_S^{solo}, \\
\pi_S^b &= \pi_S^{solo} + \bar{\theta}\Delta\pi = \bar{\theta}\pi^{cen} - \bar{\theta}\pi_M^{solo} + \theta\pi_S^{solo}.
\end{aligned}$$

Because $\frac{\partial}{\partial K_{j\neq i}}\pi_i^{solo} = 0$, the FOCs can be constructed in terms of our primitive shadow matrices *and* the gradient of θ:

$$\nabla \mathrm{E}\pi^{bar} = \mathrm{E}\lambda^{bar} = \Lambda^{bar}\bar{P}(K) + \left(\frac{\partial\theta}{\partial K_M}, \frac{-\partial\theta}{\partial K_S}\right)' \mathrm{E}\Delta\pi(K, D)$$

Requiring identical first-order conditions $\mathrm{E}\lambda^{bar} = \Lambda^{cen}\bar{P}(K)$ now yields a system of partial differential equations:

$$\Lambda^{cen} = \Lambda^{bar} + (\theta_M, -\theta_S)' \Delta\Pi$$

where θ_i denotes $\frac{\partial}{\partial K_i}\theta$ and $\Delta\Pi$ is a 1×6 matrix with $\Delta\Pi_l = \mathrm{E}\Delta\pi(K, D)$ if $D \in \Omega_l$.

Lemma 1 *If $p_M \leq p_S$, state-dependent bargaining with $\theta = 1$ in Ω_1, any constant θ in Ω_{234} and $\theta = 0$ in Ω_{56} coordinates investment decisions.*

Proof: Using the results of Λ^{bar} with a fixed θ and calculating $\Delta\pi$ in each domain yields:

$$\begin{aligned}
\Lambda &= \begin{bmatrix} \overline{\theta} p_M & 0 & p_M & p_M & p_M & p_M \\ 0 & p_S & p_S & p_S & \overline{\theta} p_M & \overline{\theta} p_M \end{bmatrix} + \\
& \quad \begin{bmatrix} \theta_M \\ -\theta_S \end{bmatrix} \begin{bmatrix} p_M(D_M - K_M) & 0 & 0 & 0 & p_M(D_+ - K_+) & p_M(D_+ - K_+) \end{bmatrix}. \\
\Lambda^{cen} &= \begin{bmatrix} 0 & 0 & p_M & p_M & p_M & p_M \\ 0 & p_S & p_S & p_S & p_M & p_M \end{bmatrix}.
\end{aligned}$$

Coordination requires that $\Lambda = \Lambda^{cen}$. The appropriate solution for $\theta(K, D)$ is found directly by inspection: $\theta = 1$ in Ω_1, any constant θ in Ω_{234} and $\theta = 0$ in Ω_{56}. To show that this is the unique solution, we solve the system of partial differential equations in each domain:

- In domain Ω_1:

$$\begin{aligned}
(D_M - K_M)\frac{\partial}{\partial K_M}\theta - \theta &= -1, \\
\frac{\partial}{\partial K_S}\theta &= 0,
\end{aligned}$$

with unique solution:

$$\theta = \frac{C_1 - K_M}{D_M - K_M},$$

where C_1 is a function that does not depend on K. To satisfy the boundary conditions $0 \leq \theta \leq 1$ in Ω_1, we must have $C_1 = D_M$ and hence $\theta = 1$ so that the manufacturer gets all the surplus in Ω_1.

- In domains Ω_{234} we have equality. (There are no transfers; centralized production = solo production; hence any θ will do.

- In domains Ω_{56}:

$$\begin{aligned}
\frac{\partial}{\partial K_M}\theta &= 0, \\
(D_+ - K_+)\frac{\partial}{\partial K_S}\theta + \theta &= 0.
\end{aligned}$$

The unique solution of the second PDE is:

$$\theta = C_5(D_+ - K_+),$$

and the first PDE requires that $C_5 = 0$, so that $\theta = 0$ in Ω_5 and all surplus goes to the supplier.

Lemma 2 *If $p_M > p_S$, state-dependent bargaining with $\theta = 1$ in Ω_{13}, any constant θ in Ω_2, $\theta = 0$ in Ω_{45}, and $\theta = \frac{(p_M - p_S)(D_M - K_M)}{\Delta\pi(K,D)}$ in Ω_6 coordinates investment decisions.*

Proof: Calculation of $\Delta\pi$ in each domain: We have that $\Delta\pi = 0$ in domain Ω_{02} and:

$$\Delta\pi = \begin{cases}
p_M D_M + p_S D_S - p_M K_M - p_S D_S = p_M(D_M - K_M) & \text{in } \Omega_1 \\
p_M D_M + p_S(K_+ - D_M) - p_M K_M - p_S K_S = (p_M - p_S)(D_M - K_M) & \text{in } \Omega_3 \\
p_M K_+ - p_M K_M - p_S K_S = (p_M - p_S)K_S & \text{in } \Omega_4 \\
p_M K_+ - p_M K_M - p_S D_S = p_M K_S - p_S D_S & \text{in } \Omega_5 \\
p_M D_M + p_S(K_+ - D_M) - p_M K_M - p_S D_S = (p_M - p_S)(D_M - K_M) + p_S(K_S - D_S) & \text{in } \Omega_6
\end{cases}.$$

Combining with the results of Λ^{bar}

$$\begin{aligned}
\Lambda &= \begin{bmatrix} \bar{\theta}p_M & 0 & \bar{\theta}p_M+\theta p_S & p_M & p_M & \bar{\theta}p_M+\theta p_S \\ 0 & p_S & p_S & \bar{\theta}p_M+\theta p_S & \bar{\theta}p_M & \bar{\theta}p_S \end{bmatrix} + \\
& \begin{bmatrix} \theta_M \\ -\theta_S \end{bmatrix} \begin{bmatrix} p_M(D_M-K_M) \\ 0 \\ (p_M-p_S)(D_M-K_M) \\ (p_M-p_S)K_S \\ p_M K_S - p_S D_S \\ (p_M-p_S)(D_M-K_M)+p_S(K_S-D_S) \end{bmatrix}', \\
\Lambda^{cen} &= \begin{bmatrix} 0 & 0 & p_S & p_M & p_M & p_S \\ 0 & p_S & p_S & p_M & p_M & p_S \end{bmatrix}
\end{aligned}$$

By inspection we again see that we should have $\theta = 1$ in Ω_{13}, $\theta = 0$ in Ω_{45}. Domain Ω_6 is more difficult as no single constant θ yields a solution. Hence, we must look for a variable θ function. Recall that a state-dependent price-only contract requires $p_t = p_S$ in Ω_6 to induce investment coordination. Now use the equivalent p_t^{bar} :

$$p_t^{bar} = \bar{\theta}p_M + \theta p_S \frac{D_+ - K_+}{D_M - K_M} = p_S$$

which yields the following candidate soluation function

$$\theta(K, D) = \frac{(p_M - p_S)(D_M - K_M)}{(p_M - p_S)(D_M - K_M) + p_S(K_S - D_S)} \text{ in } \Omega_6.$$

We now verify that this θ function satisfies the sufficient FOC, which simplify to require in domain Ω_6:

$$\begin{aligned}
\begin{bmatrix} \bar{\theta}p_M + \theta p_S \\ \bar{\theta}p_S \end{bmatrix} + \Delta\pi(K,D) \begin{bmatrix} \theta_M \\ -\theta_S \end{bmatrix} &= \begin{bmatrix} p_S \\ p_S \end{bmatrix} \\
\Leftrightarrow ((p_M - p_S)(D_M - K_M) + p_S(K_S - D_S)) \begin{bmatrix} \frac{\partial\theta}{\partial K_M} \\ -\frac{\partial\theta}{\partial K_S} \end{bmatrix} &= \begin{bmatrix} (p_S - p_M)(1-\theta) \\ p_S\theta \end{bmatrix}
\end{aligned}$$

Test the FOC to verify correctness of the candidate solution $\theta = \frac{(p_M-p_S)(D_M-K_M)}{(p_M-p_S)(D_M-K_M)+p_S(K_S-D_S)}$:

$$\begin{aligned}
FOC_1 \quad &: \quad [(p_M - p_S)(D_M - K_M) + p_S(K_S - D_S)] \frac{\partial}{\partial K_M} \frac{(p_M - p_S)(D_M - K_M)}{(p_M - p_S)(D_M - K_M) + p_S(K_S - D_S)} \\
&= \quad (p_S - p_M) \frac{p_S(K_S - D_S)}{(p_M - p_S)(D_M - K_M) + p_S(K_S - D_S)} \\
&= \quad (p_S - p_M) \left[1 - \frac{(p_1 - p_2)(D_1 - x)}{(p_1 - p_2)(D_1 - x) + p_2(y - D_2)} \right] \\
&= \quad (p_S - p_M)(1 - \theta) \Rightarrow OK! \\
FOC_2 \quad &: \quad -[(p_M - p_S)(D_M - K_M) + p_S(K_S - D_S)] \frac{\partial}{\partial K_S} \frac{(p_M - p_S)(D_M - K_M)}{(p_M - p_S)(D_M - K_M) + p_S(K_S - D_S)} \\
&= \quad p_S \frac{(p_M - p_S)(D_M - K_M)}{(p_M - p_S)(D_M - K_M) + p_S(K_S - D_S)} \\
&= \quad p_S\theta \Rightarrow OK!
\end{aligned}$$

Conclusion: the candidate solution $\theta(K, D)$ satisfies the FOC and hence our state-dependent bargaining contract induces investment coordination.