Appendix with Explicit Results to: Price vs. Production Postponement: Capacity and Competition

Jan A. Van Mieghem
Kellogg Graduate School of Management, Northwestern University

Maqbool Dada
Krannert Graduate School of Management, Purdue University

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Here we first summarize the explicit solutions to our pricing-capacity and quantity-capacity models for both the monopoly and duopoly when uncertainty is uniformly distributed. Then follow the proofs and other technical extensions. (For clarity, we use the subscript M for monopoly and D for duopoly, as needed.) The main paper appeared in Management Science Vol. 45, No. 12, December 1999 pp. 1631-1649.

1 Explicit Solutions for the Uniform Distribution

1.1 Monopoly Price-Capacity Decisions

Lemma 1 *If ε is uniformly distributed with mean ε_0 and standard deviation σ, then the optimal capacity-constrained monopoly price $p_M(K)$ for a monopolist depends on the coefficient of variation $\frac{\sigma}{\varepsilon_0}$ as follows:*

1. *If $0 \leq \frac{\sigma}{\varepsilon_0} \leq \frac{1}{3\sqrt{3}}$ (low variability), then*

$$p_M(K) = \begin{cases} \frac{1}{3}(\varepsilon_0 - \sqrt{3}\sigma - K)\left(2 + \sqrt{1 + \frac{12\sqrt{3}\sigma K}{\left(\varepsilon_0 - \sqrt{3}\sigma - K\right)^2}}\right) & \text{if } K \leq \frac{1}{2}\varepsilon_0 + \sqrt{3}\sigma, \\ \frac{1}{2}\varepsilon_0 & \text{otherwise.} \end{cases}$$

2. *If $\frac{1}{3\sqrt{3}} < \frac{\sigma}{\varepsilon_0} \leq \frac{1}{2\sqrt{3}}$ (medium variability)*

$$p_M(K) = \begin{cases} \frac{1}{2}\varepsilon_0 + \frac{\sqrt{3}}{2}\sigma - \frac{1}{4}K & \text{if } K \leq 2\left(3\sqrt{3}\sigma - \varepsilon_0\right), \\ \frac{1}{3}(\varepsilon_0 - \sqrt{3}\sigma - K)\left(2 + \sqrt{1 + \frac{12\sqrt{3}\sigma K}{\left(\varepsilon_0 - \sqrt{3}\sigma - K\right)^2}}\right) & \text{if } 2\left(3\sqrt{3}\sigma - \varepsilon_0\right) < K \leq \frac{1}{2}\varepsilon_0 + \sqrt{3}\sigma, \\ \frac{1}{2}\varepsilon_0 & \text{otherwise.} \end{cases}$$

3. *If $\frac{1}{2\sqrt{3}} < \frac{\sigma}{\varepsilon_0} \leq \frac{1}{\sqrt{3}}$ (high variability):*

$$p_M(K) = \begin{cases} \frac{1}{2}\varepsilon_0 + \frac{\sqrt{3}}{2}\sigma - \frac{1}{4}K & \text{if } K \leq \frac{2}{3}\left(\varepsilon_0 + \sqrt{3}\sigma\right), \\ \frac{1}{3}\varepsilon_0 + \frac{\sqrt{3}}{3}\sigma & \text{otherwise.} \end{cases}$$

Lemma 2 *If ε is uniformly distributed with mean ε_0 and standard deviation σ, then the optimal price-capacity strategy for a monopolist depends on the coefficient of variation $\frac{\sigma}{\varepsilon_0}$ and the investment cost c as follows:*

1. *If $0 \leq \frac{\sigma}{\varepsilon_0} \leq \frac{1}{3\sqrt{3}}$ (low variability): if $c \leq \bar{c}_1 = \varepsilon_0 - \sqrt{3}\sigma$, then*

$$p = \text{the unique root of } 4p^3 - 2\left(\varepsilon_0 + c\right)p^2 + 2\sqrt{3}\sigma c^2 \text{ with } \frac{\varepsilon_0}{2} \leq p \leq \varepsilon_0 - \sqrt{3}\sigma, \quad (1)$$

$$K = \varepsilon_0 - p + \sqrt{3}\sigma\left(1 - \frac{2c}{p}\right). \quad (2)$$

If $c > \varepsilon_0 - \sqrt{3}\sigma$, then $K = 0$ and p is arbitrary.

2. *If $\frac{1}{3\sqrt{3}} < \frac{\sigma}{\varepsilon_0} \leq \frac{1}{2\sqrt{3}}$ (medium variability): if $c \leq c_2 = \frac{\varepsilon_0^2 - 3\sqrt{3}\sigma\varepsilon_0 + 6\sigma^2}{\sqrt{3}\sigma}$, then K and p are determined by (1)-(2). If $c_2 < c < \bar{c}_2 = \frac{\left(\varepsilon_0 + \sqrt{3}\sigma\right)^2}{8\sqrt{3}\sigma}$, then*

$$p = \frac{1}{6}\left(\varepsilon_0 + \sqrt{3}\sigma + \sqrt{\left(\varepsilon_0 + \sqrt{3}\sigma\right)^2 + 24\sqrt{3}\sigma c}\right), \tag{3}$$

$$K = 2\left(\varepsilon_0 + \sqrt{3}\sigma - 2p\right). \tag{4}$$

If $c \geq \bar{c}_2$, then $K = 0$ and p is arbitrary.

3. *If $\frac{1}{2\sqrt{3}} < \frac{\sigma}{\varepsilon_0} \leq \frac{1}{\sqrt{3}}$ (high variability): if $c \leq \bar{c}_2$, then K and p are determined by (3)-(4). If $c \geq \bar{c}_2$, then $K = 0$ and p is arbitrary.*

1.2 Monopoly Quantity-Capacity Decisions

Lemma 3 *If ε is uniformly distributed with mean ε_0 and standard deviation σ, then the optimal capacity-constrained monopoly quantity $q_M(K)$ for a monopolist depends on the coefficient of variation $\frac{\sigma}{\varepsilon_0}$ as follows:*

1. *If $0 \leq \frac{\sigma}{\varepsilon_0} \leq \frac{1}{2\sqrt{3}}$ (low & medium variability), then*

$$q_M(K) = \min\left(\frac{\varepsilon_0}{2}, K\right),$$

2. *If $\frac{1}{2\sqrt{3}} < \frac{\sigma}{\varepsilon_0} \leq \frac{1}{3\sqrt{3}}$ (high variability)*

$$q_M(K) = \min\left(\frac{\varepsilon_0 + \sqrt{3}\sigma}{3}, K\right).$$

Lemma 4 *If ε is uniformly distributed with mean ε_0 and standard deviation σ, then the optimal quantity-capacity strategy for a monopolist depends on the coefficient of variation $\frac{\sigma}{\varepsilon_0}$ and the investment cost c as follows: if $c \geq \varepsilon_0$, then $K = 0$, otherwise:*

1. *If $\frac{\sigma}{\varepsilon_0} \leq \frac{1}{2\sqrt{3}}$ (low & medium variability), then*

$$K = \frac{1}{2}\left(\varepsilon_0 - c\right). \tag{5}$$

2. *If $\frac{1}{2\sqrt{3}} < \frac{\sigma}{\varepsilon_0} \leq \frac{1}{\sqrt{3}}$ (high variability): if $c < 2\sqrt{3}\sigma - \varepsilon_0$, then*

$$K = \frac{1}{3}\left(2\left(\varepsilon_0 + \sqrt{3}\sigma\right) - \sqrt{\left(\varepsilon_0 + \sqrt{3}\sigma\right)^2 + 12\sqrt{3}c\sigma}\right), \tag{6}$$

otherwise $K = \frac{1}{2}\left(\varepsilon_0 - c\right)$.

1.3 Capacity Decisions under Quantity Competition

Lemma 5 *If ε is uniformly distributed with mean ε_0 and standard deviation σ, then the optimal quantity-capacity strategy for a duopolist depends on the coefficient of variation $\frac{\sigma}{\varepsilon_0}$ and the investment cost c as follows: If $c \geq \bar{c} = \varepsilon_0$, then $q = K = 0$, otherwise $q = K$ where*

1. *If $0 \leq \frac{\sigma}{\varepsilon_0} \leq \frac{1}{3\sqrt{3}}$ (low variability):*

$$K_i = K_j = \frac{1}{3}(\varepsilon_0 - c).$$

2. If $\frac{1}{3\sqrt{3}} < \frac{\sigma}{\varepsilon_0} \le \frac{1}{\sqrt{3}}$ *(medium & high variability): if* $c \le \bar{c}_1 = \frac{-\varepsilon_0 + 3\sqrt{3}\sigma}{2}$*, then*

$$K_i = K_j = \frac{3}{8}\left(\varepsilon_0 + \sqrt{3}\sigma\right) - \frac{1}{8}\sqrt{\left(\varepsilon_0 + \sqrt{3}\sigma\right)^2 + 32c\sqrt{3}\sigma}. \tag{7}$$

Otherwise, $K_i = K_j = \frac{1}{3}(\varepsilon_0 - c)$.

Lemma 6 *If* ε *is uniformly distributed with mean* 1 *and standard deviation* σ*, then*

$$k_n = \begin{cases} \frac{n}{n+1}(1-c) & \text{if } \sqrt{3}\sigma \le \frac{1+nc}{n+1}, \\ \frac{1+\sqrt{3}\sigma}{n+2}\left(1 + n - \sqrt{1 + \frac{4\sqrt{3}n(n+2)c\sigma}{(1+\sqrt{3}\sigma)^2}}\right) & \text{elsewhere.} \end{cases} \tag{8}$$

and

$$k_n \nearrow k = \begin{cases} 1 - c & \text{if } \sqrt{3}\sigma \le \frac{1+nc}{n+1}, \\ 1 + \sqrt{3}\sigma - 2\sqrt{\sqrt{3}\sigma c} & \text{elsewhere.} \end{cases} \tag{9}$$

2 Proofs and Other Discussions

2.1 Scaling of the Demand curve

An arbitrary linear demand curve $p' = \varepsilon' - bD'$, where ε' has mean ε_0, reduces to $p = \varepsilon - D$ after scaling prices $p' = \varepsilon_0 p$ and quantities $q' = \frac{\varepsilon_0}{b} q$.

For the arbitrary demand curve, the unscaled marginal costs are $c' = \varepsilon_0 c$ so that unscaled revenues and costs are $\frac{\varepsilon_0^2}{b} pq$ and $\varepsilon_0^2 cK$, respectively.

2.2 Unimodality Assumption

The expected revenue function $\mathrm{E}\pi(p, K)$ for the deterministic, uniform (see Lemma's below) and exponential distribution (see paper) is indeed unimodal. If ε is normally distributed with mean 1 and standard deviation σ (and truncated to the left at zero, so that ε is non-negative) we have:

$$f(\varepsilon) = Ae^{-\frac{(\varepsilon-1)^2}{2\sigma^2}}, \qquad \text{for } \varepsilon \ge 0$$

and A is normalization constant:

$$A^{-1} = \int_0^\infty e^{-\frac{(\varepsilon-1)^2}{2\sigma^2}} d\varepsilon = \frac{\sqrt{2\pi}\sigma}{2}\left(1 + \mathrm{erf}\left(\frac{1}{\sqrt{2}\sigma}\right)\right).$$

Thus,

$$\mathrm{E}\pi(p, K) = Ap \int_p^{p+K} (\varepsilon - p)\, e^{-\frac{(\varepsilon-1)^2}{2\sigma^2}} d\varepsilon + ApK \int_{p+K}^\infty e^{-\frac{(\varepsilon-1)^2}{2\sigma^2}} d\varepsilon$$

and

$$\begin{aligned} \frac{1}{A}\frac{\partial}{\partial p}\mathrm{E}\pi &= \int_p^{p+K} (\varepsilon - 2p)\, e^{-\frac{(\varepsilon-1)^2}{2\sigma^2}} d\varepsilon + K \int_{p+K}^\infty e^{-\frac{(\varepsilon-1)^2}{2\sigma^2}} d\varepsilon, \\ \frac{1}{A}\frac{\partial^2}{\partial p^2}\mathrm{E}\pi &= p\left[e^{-\frac{(p-1)^2}{2\sigma^2}} - e^{-\frac{(p+K-1)^2}{2\sigma^2}}\right] - 2\int_p^{p+K} e^{-\frac{(\varepsilon-1)^2}{2\sigma^2}} d\varepsilon. \end{aligned}$$

Clearly, $\mathrm{E}\pi(\cdot, K)$ is positive for positive p except at $p = 0$ and $p = \infty$ where $\mathrm{E}\pi(0, K) = \mathrm{E}\pi(\infty, K) = 0$, and thus has at least one maximum. Also, for each positive K, $\mathrm{E}\pi(p, K)$ is concave increasing at $p = 0$ ($0 < \frac{\partial}{\partial p}\mathrm{E}\pi(0, K) < A\mathrm{E}\varepsilon = A$ and $\frac{\partial^2}{\partial p^2}\mathrm{E}\pi(0, K) < 0$) and convex decreasing near $p = \infty$. Because $\frac{\partial^2}{\partial p^2}\mathrm{E}\pi$ is the difference of two positive unimodal functions, it may have more than one root (inflection point) and

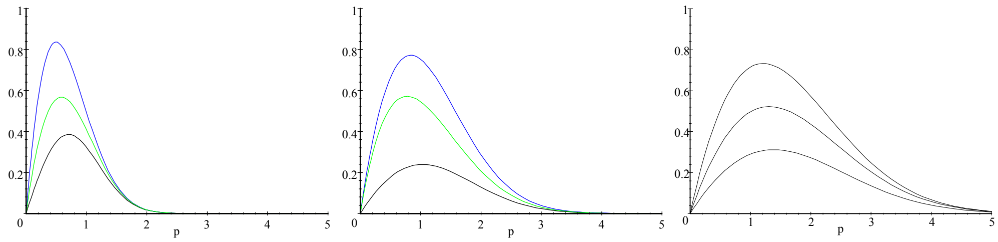

Figure 6: $\mathrm{E}\pi(\cdot, K)$ for the (truncated) normal distribution is also unimodal. The graphs (each one for $K = 0.5, 1$ and 2) show different variability levels: $\sigma = 0.5$ (left), 1 (middle) and 1.5 (right).

proving unimodality analytically is not straightforward. In our numerical simulations (see Figure 6), however, $\mathrm{E}\pi(\cdot, K)$ was always unimodal. Note that

$$\frac{1}{A}\frac{\partial}{\partial K}\mathrm{E}\pi = p\int_{p+K}^{\infty} e^{-\frac{(\varepsilon-1)^2}{2\sigma^2}} d\varepsilon > 0,$$
$$\frac{1}{A}\frac{\partial^2}{\partial K^2}\mathrm{E}\pi = -pe^{-\frac{(p+K-1)^2}{2\sigma^2}} < 0,$$

so that $\mathrm{E}\pi$ is concave increasing in K for each p.

2.3 Proof of Proposition 1

$\mathrm{E}\pi(p, K)$ is twice differentiable in p and

$$\frac{\partial V}{\partial p} = \int_p^{p+K} (\varepsilon - 2p) f(\varepsilon) d\varepsilon + \int_{p+K}^{\infty} K f(\varepsilon) d\varepsilon, \tag{10}$$
$$\frac{\partial^2 V}{\partial p^2} = pf(p) + 2F(p) - pf(p+K) - 2F(p+K). \tag{11}$$

We have assumed that the measure P is such that $\mathrm{E}\pi(p, K)$ is unimodal so that the first order condition is sufficient.

If $K > 0$, $\frac{\partial V}{\partial p} > 0$ if $p = 0$ so that $p_M(K)$ must be strictly positive and the first order condition is sufficient for the interior maximum. Implicit differentiation yields

$$\frac{\partial^2 V}{\partial p^2}\frac{dp_M}{dK} = -\frac{\partial^2 V}{\partial K \partial p} = -\left(1 - F(p_M + K) - p_M f(p_M + K)\right). \tag{12}$$

The following sample path argument shows that $\frac{dp_M}{dK} \leq 0$. Assume that $p_M(K)$ is unique and consider the three representative sample paths of $\pi(p, K, \varepsilon)$ (refer to Figure 1 in the paper). If K increases to $K + dK$, the revenue maximizing price for each sample path either remains the same (for low and medium values of ε : $\varepsilon < 2K$) or decreases (for high values of $\varepsilon > 2K$). Therefore, the unique maximum $p_M(K)$ of $\mathrm{E}\pi(p, K)$ (which is a convex superposition of the sample paths $\pi(p, K, \varepsilon)$) cannot increase when K increases to $K + dK$. ■

2.4 Proof of Proposition 2

The objective function

$$V = \int_p^{p+K} p(\varepsilon - p) f(\varepsilon) d\varepsilon + \int_{p+K}^{\infty} pK f(\varepsilon) d\varepsilon - cK \tag{13}$$

is differentiable with, in addition to (10), the necessary first order condition:

$$\frac{\partial V}{\partial K} = \int_{p+K}^{\infty} pf(\varepsilon) d\varepsilon - c = pP(\varepsilon \geq p + K) - c \tag{14}$$

For a boundary solution $p = 0, K > 0$ to be optimal we would need $c = 0$. Boundary solutions $p \geq 0, K = 0$ yield $V = 0$. Thus, any non-trivial solution is an interior solution of the first-order conditions which were assumed to be sufficient.

Implicit differentiation of $\frac{\partial V}{\partial p} = 0$ and $\frac{\partial V}{\partial K} = 0$ yields

$$\frac{\partial K}{\partial c} = \frac{\frac{\partial^2 V}{\partial p^2}}{\frac{\partial^2 V}{\partial K^2}\frac{\partial^2 V}{\partial p^2} - \left(\frac{\partial^2 V}{\partial p \partial K}\right)^2} \text{ and } \frac{\partial p}{\partial c} = -\frac{\frac{\partial^2 V}{\partial p \partial K}}{\frac{\partial^2 V}{\partial p^2}}. \tag{15}$$

Because there is an interior maximum, V is concave at that point (p, K) and thus $\frac{\partial K}{\partial c} < 0$. Also, if $\frac{\partial^2 V}{\partial p \partial K} > 0$, then $\frac{\partial p}{\partial c} > 0$. ■

2.5 Duality Result for Capacity-Constrained Price vs. Quantity-Setting

Proposition 6 (Duality) *The optimal capacity-constrained monopoly quantity is* $q_M(K) = \min(q^*, K)$, *where* q^* *is the unique solution to*

$$\int_{q^*}^{\infty} (\varepsilon - 2q^*) f(\varepsilon) d\varepsilon = 0. \tag{16}$$

Dual: The optimal capacity-constrained monopoly price is $p_M(K) = \max(p^*, p(K))$, *where* p^* *is the unique solution to*

$$\int_{p^*}^{\infty} (\varepsilon - 2p^*) f(\varepsilon) d\varepsilon = 0, \tag{17}$$

and $p(K)$ *solves the optimality equation of Proposition 1.*

Corollary 3 *If* ε *is bounded from above with probability one by* $\bar{\varepsilon}$, *then* $P(\Omega_2(p)) = 0$ *for all prices* p *such that* $p + K \geq \bar{\varepsilon}$ *and the capacity-constrained price is* $p_M = p^*$, *which is independent of capacity. If, in addition,* ε *is bounded from below with probability one by* $\underline{\varepsilon} \geq \frac{1}{2}$, *then* $P(\Omega_0(p)) = 0$ *and the capacity-constrained price* $p_M(K)$ *and the expected revenue function are independent of variability and equal to the deterministic solutions:* $\forall p \leq \underline{\varepsilon} \leq \bar{\varepsilon} \leq p + K$:

$$p_M(K) = \tfrac{1}{2} \text{ and } \mathrm{E}\pi_M^p(p, K) = p(1 - p). \tag{18}$$

The dual of $q_M(K) = \min(q^*, K)$ is $p_M(K) = \max(p^*, p(K))$, where $q^* = p^*$. In addition, the actual capacity-constrained price and quantity are related via the deterministic demand curve ($p = 1 - q$) *only if* variability is low and there is sufficient capacity: if ε has finite support $[\underline{\varepsilon}, \bar{\varepsilon}]$ with $\frac{1}{2} \leq \underline{\varepsilon} \leq \bar{\varepsilon} \leq \frac{1}{2} + K$, then $p_M(K) = q_M(K) = \frac{1}{2}$ independent of the level of moderate variability. Because their expected revenue functions differ, price-setting and quantity-setting in general yield different investment results. From the duality result, one could hope that in the special case of low variability, both would yield identical investment outcomes, but the next section will show that this is not true either.

Proof. We have that

$$q_M(K) = \arg\max_{0 \leq q \leq K} \mathrm{E}\pi_M(q, K) = \int_{q \leq K}^{\infty} (\varepsilon - q) q f(\varepsilon) d\varepsilon. \tag{19}$$

Given our assumption that f is such that $\mathrm{E}\pi_M(q, K)$ is unimodal concave-convex, the first order equations are sufficient. The unconstrained maximum q^* must satisfy $\int_{q^*}^{\infty} (\varepsilon - 2q^*) f(\varepsilon) d\varepsilon = 0$. If $q^* < K$, then $q_M(K) = q^*$, otherwise $\mathrm{E}\pi_M(q, K)$ is increasing over $[0, K]$ so that $q_M(K) = K$.

Dual: First note that for arbitrary large capacity K, clearly $p^* = p(K)$. Now, invoking Proposition 1 yields that $p(K)$ increases as K decreases, so that in general $p(K) \geq p^*$. ■

2.6 Proofs of Properties of Series k_n (Corollaries 1 and 2)

First consider the simpler case of k, which solves:

$$\int_{k}^{\infty} (\varepsilon - k) f(\varepsilon) d\varepsilon = c \tag{20}$$

if $c < 1$, and $k = 0$ if $c \geq 1$. The latter constraint guarantees a non-negative k. Indeed, the function

$$g(x) = \int_x^\infty (\varepsilon - x) f(\varepsilon) d\varepsilon \Rightarrow -1 \leq g^{'}(x) = -0 + \int_x^\infty (-1) f(\varepsilon) d\varepsilon = -\bar{F}(x) \leq 0,$$

is positive and decreasing (strictly over the domain of ε) and $g(0) = \mathrm{E}\varepsilon = 1$. Hence the solution (inverse function) $k(c)$ to $g(k) = c$ is unique, with $1 \geq g(x) \geq 1 - x$. Thus, $k(1) = 0$ and $k'(c) = \frac{1}{g'(k(c))} = \frac{-1}{\bar{F}(k(c))} \leq -1$, so that $k(c) \geq 1 - c$ and $k(c) - k(c + \Delta c) \geq \Delta c$.

The series k_n involves the more complicated series of functions:

$$g_n(x) = \int_x^\infty (\varepsilon - \frac{n+1}{n} x) f(\varepsilon) d\varepsilon, \tag{21}$$

with

$$\begin{aligned} g_n(0) &= \mathrm{E}\varepsilon = 1, \\ g_n^{'}(x) &= \frac{1}{n} x f(x) + \int_x^\infty (-\frac{n+1}{n}) f(\varepsilon) d\varepsilon = \frac{1}{n} x f(x) - \frac{n+1}{n} \bar{F}(x) \geq -\frac{n+1}{n}. \end{aligned}$$

Again, $k_n(1) = 0$ and $g_n^{'}(x) \geq -\frac{n+1}{n}$, so that $g_n(x) \geq 1 - \frac{n+1}{n} x$ and $k_n \geq \frac{n}{n+1}(1-c)$. Because $g_n'(0) = -\frac{n+1}{n}$, g_n is initially decreasing with a minimum at x_n^*, where

$$\frac{x_n^* f(x_n^*)}{\bar{F}(x_n^*)} = n + 1,$$

at which

$$\begin{aligned} g_n(x_n^*) &= \int_{x_n^*}^\infty \varepsilon f(\varepsilon) d\varepsilon - \frac{n+1}{n} x_n^* \bar{F}(x_n^*) \\ &= \int_{x_n^*}^\infty \varepsilon f(\varepsilon) d\varepsilon - \frac{1}{n} x_n^{*2} f(x_n^*). \end{aligned}$$

Although many distributions yield a unique minimum (from which uniqueness of k_n would follow), there may be distributions with multiple extrema. Hence, we define k_n as the smallest positive root to $g_n(x) = c \leq 1$. Such a root always exists by Weierstrass' theorem because g_n is continuous with $g_n(0) = 1, g_n(\infty) = 0$ (because ε is finite with probability one so that F is a real distribution with $x\overline{F}(x) \to 0$ as $x \longrightarrow \infty$). Because k_n is the smallest positive root, we have that g_n is decreasing at k_n so $-\frac{n+1}{n} \leq g_n'(k_n(c)) \leq 0$ so that $k_n^{'}(c) = \frac{1}{g_n'(k_n(c))} \leq -\frac{n}{n+1}$.

Because $g_n(x)$ is increasing in n we have that the series k_n is increasing.

Last, because the integrand of g_n is negative for $\varepsilon < \frac{n+1}{n} k_n$, we have

$$g_n(x) < g\left(\frac{n+1}{n} x\right) = \int_{\frac{n+1}{n} x}^\infty (\varepsilon - \frac{n+1}{n} x) f(\varepsilon) d\varepsilon. \tag{22}$$

Because the function $g(x) = \int_x^\infty (\varepsilon - x) f(\varepsilon) d\varepsilon$ is monotone decreasing (see above) and $g(k) = c$ per definition, we have that $\frac{n+1}{n} k_n \leq k$. ■

Examples: If ε is exponentially distributed with mean 1, we have that

$$g_n(x) = \int_x^\infty (\varepsilon - \frac{n+1}{n} x) e^{-\varepsilon} d\varepsilon = e^{-x}(1 - \frac{x}{n}),$$

which is convex-concave with indeed a unique minimum at

$$x_n^* = n + 1$$

where $g_n(x_n^*) = -\frac{1}{n} e^{-(n+1)} < 0$, so that $g(x) = c$ has a unique solution.

If ε is uniformly distributed over $[a,b]$, setting $\mu = \frac{1}{b-a}$

$$\begin{aligned} k_n &= \frac{n}{n+1}(1-c) && \text{if } k_n \le a, \\ k_n &= \frac{1}{(n+2)}\left(b + bn - \sqrt{b^2 + 2nc(n+2)/\mu}\right) && \text{elsewhere,} \end{aligned}$$

because

$$\int_{k_n}^{b} (\varepsilon - \frac{n+1}{n}k_n)\mu d\varepsilon = \frac{1}{2}\mu \frac{b^2 n - 2bk_n n - 2bk_n + k_n^2 n + 2k_n^2}{n}.$$

Hence:

$$\begin{aligned} k &= 1-c && \text{if } k_n \le a, \\ k &= b - \sqrt{2c/\mu} && \text{elsewhere.} \end{aligned}$$

2.7 Proof of Proposition 3

Proof. Any choice $K > q^*$, implies that $\frac{d}{dK}\mathrm{E}\pi_M(q^*, K) = 0$ and thus $\frac{d}{dK}V_M(q^*, K) = -c$, which cannot be optimal at positive cost. Thus, it must be that $K = q^*$ and necessary optimality equation is $\int_K^\infty (\varepsilon - 2K)f(\varepsilon)d\varepsilon = c$, which has only has a solution if $c < \bar{c}$, where $\bar{c} = \frac{d}{dK}\mathrm{E}\pi_M(K,K)|_{K=0} = \int_K^\infty (\varepsilon - 2K)f(\varepsilon)d\varepsilon|_{K=0} = \mathrm{E}\varepsilon = \varepsilon_0$. ■

2.8 Proof of Proposition 4

Proof. First assume that K_i is large such that

$$\max_{0 \le q_i \le K_i} \int_{q_i+q_j}^{\infty} (\varepsilon - q_i - q_j) q_i f(\varepsilon) d\varepsilon \tag{23}$$

has an interior optimum $q_i(q_j)$ which satisfies

$$\int_{q_i(q_j)+q_j}^{\infty} (\varepsilon - 2q_i(q_j) - q_j) f(\varepsilon) d\varepsilon = 0 \text{ and } q_i f(q_+) - 2\bar{F}(q_+) < 0, \tag{24}$$

where $1 - F(x) = \bar{F}(x) \ge 0$. It follows directly that the unconstrained intersection of the reaction curves, if it exists, is symmetric:

$$\int_{q_+}^{\infty} (\varepsilon - q_+) f(\varepsilon) d\varepsilon = q_i \overset{q_i = q_D^* = \frac{1}{2}q_+}{\Leftrightarrow} \int_{2q_D^*}^{\infty} (\varepsilon - 3q_D^*) f(\varepsilon) d\varepsilon = 0.$$

Clearly, from Proposition 3 we know that $q_i(q_j = 0) = q_M(K_i) = q_M^*$ if K_i is large ($K_i \ge q_M^*$). The implicit function theorem yields

$$\frac{\partial q_i(q_j)}{\partial q_j} = -\frac{\bar{F}(q_+) - q_i f(q_+)}{2\bar{F}(q_+) - q_i f(q_+)}, \tag{25}$$

so that if $f(q_+)/\bar{F}(q_+) \le q_i^{-1}$, we have that $-1 \le \frac{\partial q_i(q_j)}{\partial q_j} \le 0$ with $q_i(0) = q_M^*$. If K_j is also large with interior optimum, it's reaction curve is decreasing with slope ≤ -1 and it intersects the $q_i = 0$ axis at q_M^*. Thus, the two reaction curves have exactly one intersection and that equilibrium is symmetric and is denoted by (q_D^*, q_D^*) where $q_D^* \le q_M^*$.

If K_i is small ($K_i < q_M^*$), the response function $q_i(q_j)$ is constant at $q_i = K_i$ for small q_j. After a certain value of q_j, the optimum of (23) becomes the interior point $q_i(q_j)$ from before. Thus, if $K \le (q_D^*, q_D^*)$, the unique equilibrium is $q = K$ and if $K \ge (q_D^*, q_D^*)$, the unique equilibrium remains (q_D^*, q_D^*). Thus, the only remaining case is that $K_i < q_D^*$, while $K_j > q_D^*$ (or its symmetric counterpart). Let q_c denote the unique intersection of firm j's (unrestricted) reaction curve $q_j(q_i)$ with $q_i = K_i$: $q_c = q_j(K_i)$. It directly follows that $q_D^* \le q_c \le q_M^*$. Now: if $K_j \in (q_D^*, q_c]$, the unique equilibrium is $q = K$; otherwise if $K_j > q_c$, the unique equilibrium is $q = (K_i, q_c)$. This also shows that if a firm has excess capacity ($\forall K_i >$ the unique equilibrium $q_i(K)$) we have that $\frac{\partial}{\partial K_i} q(K) = 0$.

In conclusion: there is a unique pure strategy equilibrium which cannot be larger than (q_M^*, q_M^*), and thus, because in that case $q_i(q_j)$ and $q_j(q_i) \leq q_M^*$, a sufficient condition is that

$$\forall x, y \in [0, q_M^*] : \frac{f(x+y)}{\bar{F}(x+y)} \leq \frac{1}{x}. \tag{26}$$

(The argument can be relaxed by requiring $f(q_+)/\bar{F}(q_+) < \frac{3}{2} q_i^{-1}$ so that $-1 \leq \frac{\partial q_i(q_j)}{\partial q_j} < 1$.) ■

2.9 Proof of Proposition 5

Proof. First consider the duopoly with $n = 2$. Any choice $K_i > q_i(K)$, implies that $\frac{d}{dK_i} \mathrm{E}\pi_i(q(K), K) = 0$ and thus $\frac{d}{dK_i} V_i(q(K), K) = -c$, which cannot be optimal at positive cost. Thus, it must be that $K = q(K)$ and the capacity reaction curves become:

$$\max_{0 \leq K_i} V_i(K) = \int_{K_+}^{\infty} (\varepsilon - K_+) K_i f(\varepsilon) d\varepsilon - cK_i. \tag{27}$$

Clearly if $\frac{\partial}{\partial K_i} V_i|_{K_i=0} = \int_{K_j}^{\infty} (\varepsilon - K_j) f(\varepsilon) d\varepsilon - c < 0$, firm i will not invest. Thus, if $\int_0^{\infty} \varepsilon f(\varepsilon) d\varepsilon = \varepsilon_0 \leq c$, no firm will invest and a unique trivial equilibrium follows: $q = K = 0$. If $c < \varepsilon_0$, firm $i's$ reaction curves becomes:

$$K_i \bar{F}(K_+) = \int_{K_+}^{\infty} (\varepsilon - K_+) f(\varepsilon) d\varepsilon - c \text{ and } K_i f(K_+) - 2\bar{F}(K_+) < 0. \tag{28}$$

A similar argument as in the preceding proof shows that there is a unique intersection of the reaction curves if $\forall x, y \in [0, q^*] : \frac{f(x+y)}{\bar{F}(x+y)} \leq \frac{1}{x}$. Moreover, this intersection is symmetric: $K_i = K_j = \frac{1}{2} K_+$ and because $\frac{f(x+y)}{\bar{F}(x+y)} \leq \frac{1}{x} < 2\frac{1}{x}$ the sufficient optimality condition is satisfied. The extension to $n > 2$ follows directly for a symmetric equilibrium (there may exist additional equilibria). ■

2.10 Lemma 1: Capacity-constrained monopoly pricing under uniform uncertainty

Depending on the location of (p, K) relative to the domain $[a, b]$ of ε we distinguish the six possible cases for $\mathrm{E}\pi_M(K, p)$. For ease of notation let $a = \varepsilon_0 - \sqrt{3}\sigma, b = \varepsilon_0 + \sqrt{3}\sigma$ and $\mu = (b - a)^{-1}$ if $b > a$.

Case 1: $a < p < p + K < b$: We have

$$\mathrm{E}\pi_M(p, K) = p\left(\frac{K}{2}\right) K\mu + pK(b - p - K)\mu, \tag{29}$$

$$\mathrm{E}\pi_M(p, K) = \frac{1}{2} pK\mu\left(-2p - K + 2b\right). \tag{30}$$

Capacity constrained monopoly pricing has sufficient interior optimality condition:

$$\frac{\partial V_M}{\partial p} = \frac{1}{2} K\mu\left(-K - 4p + 2b\right) = 0 \text{ and } \frac{\partial^2 V_M}{\partial p^2} = -2K\mu < 0. \tag{31}$$

Thus the capacity-constrained monopoly price is

$$p_M(K) = \frac{2b - K}{4}, \tag{32}$$

with corresponding revenue function

$$\mathrm{E}\pi_M(p_M(K), K) = \frac{\mu}{16} K\left(2b - K\right)^2. \tag{33}$$

This case is optimal for any K such that $a < p < p + K < b$ or:

$$a < \frac{b}{2} - \frac{K}{4} \text{ and } \frac{3K}{2} < b \Leftrightarrow K < \min(2(b-2a), \frac{2}{3}b) \tag{34}$$

This requires $b > 2a$ (medium and high variability). Note that $2(b-2a) < \frac{2}{3}b$ iff $b < 3a$ (medium variability).

Case 2: $p < a < p + K < b$: We have

$$\begin{aligned} \mathrm{E}\pi_M &= p(a - p + \frac{p+K-a}{2})\,(p+K-a)\,\mu + pK(b-p-K)\mu, & (35)\\ \mathrm{E}\pi_M(p,K) &= -\frac{1}{2}p\mu\left(p^2 - 2ap + 2pK - 2Kb + a^2 + K^2\right). & (36)\end{aligned}$$

The sufficient conditions for an optimal (p, K) interior in $\{(p, K) : 0 < p < a < p + K < b\}$ are

$$\begin{aligned} \frac{\partial \mathrm{E}\pi_M}{\partial p} &= -\frac{1}{2}\mu\left(3p^2 - 4ap + 4pK - 2Kb + a^2 + K^2\right) = 0,\\ \frac{\partial^2 \mathrm{E}\pi_M}{\partial p^2} &= -\mu\,(3p - 2a + 2K) < 0.\end{aligned}$$

Capacity constrained monopoly pricing has necessary interior optimality condition $\frac{\partial V_M}{\partial p} = 0$ or

$$p_M(K) = \frac{2}{3}(a-K) + \frac{1}{3}\sqrt{\left((a-K)^2 + 6K(b-a)\right)}, \tag{37}$$

(the other root is not a maximum) with corresponding revenue function

$$\mathrm{E}\pi_M(p_M(K), K) = \frac{\mu}{27}\left[(a-K)\left(18(b-a)K - (a-K)^2\right) + \left(\sqrt{(a-K)^2 + 6K(b-a)}\right)^3\right]. \tag{38}$$

This case is optimal for any level K such that $p < a < p + K < b$ or:

$$\begin{aligned} \frac{2}{3}(a-K) + \frac{1}{3}\sqrt{a^2 - 8aK + K^2 + 6Kb} &< a & (39)\\ \Leftrightarrow \sqrt{a^2 - 8aK + K^2 + 6Kb} &< (a+2K) & (40)\\ \Leftrightarrow a^2 - 8aK + K^2 + 6Kb - (a+2K)^2 &< 0 & (41)\\ \Leftrightarrow a^2 - 8aK + K^2 + 6Kb - (a+2K)^2 &< 0 & (42)\\ \Leftrightarrow 2(b-2a) &< K, & (43)\end{aligned}$$

and

$$\begin{aligned} a &< \frac{1}{3}(2a+K) + \frac{1}{3}\sqrt{(a-K)^2 + 6K(b-a)} < b & (44)\\ \Leftrightarrow &\quad (a-K) < \sqrt{(a-K)^2 + 6K(b-a)} < 3b - (2a+K) & (45)\\ \Leftrightarrow &\quad a^2 - 8aK + K^2 + 6Kb < (3b - (2a+K))^2 \text{ and } 0 < 3b - (2a+K) & (46)\\ \Leftrightarrow &\quad 4K < 3b - a \text{ and } K < 3b - 2a. & (47)\end{aligned}$$

Thus we have

$$2(b-2a) < K < \min\left(3b - 2a, \frac{3b-a}{4}\right), \tag{48}$$

and because $3b - 2a > \frac{3b-a}{4}$ and we must have $2(b-2a) < \frac{3b-a}{4} \Leftrightarrow b < 3a$. This, this case requires

$$b < 3a \text{ (low \& medium variability) and } 2(b-2a) < K < \frac{3b-a}{4}.$$

Case 3: $p+K<a$: We have $\mathrm{E}\pi_M = pK$. Thus the capacity-constrained monopoly price is

$$p_M(K) = a - K. \tag{49}$$

with corresponding revenue function

$$\mathrm{E}\pi_M(p_M(K),K) = (a-K)K.$$

This case is suboptimal as it is a boundary solution. ($p+K = a-(a-c)/2 < a \Leftrightarrow c < a$)

Case 4: $p<a<b<p+K$: In this case (**??**) becomes $\mathrm{E}\pi_M = p(a-p+\frac{b-a}{2}) = p(\frac{a+b}{2}-p)$ with optimal response:

$$p_M(K) = \frac{a+b}{4}, \tag{50}$$

with corresponding revenue function

$$\mathrm{E}\pi_M(p_M(K),K) = \left(\frac{a+b}{4}\right)^2.$$

This case is optimal for any level K such that $p<a<b<p+K$ or

$$b<3a \text{ and } \frac{3b-a}{4}<K. \tag{51}$$

Case 5: $a<p<b<p+K$: We have $\mathrm{E}\pi_M = \frac{\mu}{2}p(b-p)^2$, with optimal response:

$$p_M(K) = \frac{b}{3}, \tag{52}$$

and corresponding revenue function

$$\mathrm{E}\pi_M(p_M(K),K) = \frac{2}{27}\mu b^3.$$

This case is optimal for any level K such that $a<p<b<p+K$ or

$$3a<b \text{ and } \frac{2b}{3}<K. \tag{53}$$

Case 6: $a<b<p<p+K$: We have $\mathrm{E}\pi_M = 0$, so that any price $p>b$ cannot be optimal for a positive K. ■

2.11 Lemma 2: Monopoly capacity investment with price-setting (uniform uncertainty)

We build on the results of lemma 1. There are three scenarios, depending on the level of variability. In all scenarios, $\mathrm{E}\pi_M(K,p_M(K))$ is strictly concave increasing in K up to a (scenario-dependent) level after which it becomes constant. Let $\bar{c} = \frac{d}{dK}\mathrm{E}\pi_M(K,p_M(K))|_{K=0}$. Thus, for any positive cost $c<\bar{c}$, there is a unique optimal capacity level K. Now, depending on the scenario, we may have to break up the interval $(0,\bar{c})$ to derive the optimal K. For the low variability scenario we have that only case 2 (from lemma 1) is needed and

$$\begin{aligned}&\frac{d}{dK}\mathrm{E}\pi_M(K,p_M(K))\\ =\;&\frac{\mu}{9}\left[K^2-12Kb+10aK-5a^2+6ab-\sqrt{(a^2-8aK+K^2+6Kb)}\,(4a-K-3b)\right].\end{aligned}$$

It is easily verified that $\bar{c}_{\mathrm{M,low}} = \frac{d}{dK}\mathrm{E}\pi_M(K,p_M(K))|_{K=0} = a$ and $\frac{d}{dK}\mathrm{E}\pi_M(K,p_M(K))|_{K=\frac{3b-a}{4}} = 0$.

For the medium variability scenario we have that for $K<2(b-2a)$ case 1 is valid:

$$\frac{d}{dK}\mathrm{E}\pi_M(K,p_M(K)) = \frac{1}{16}\mu\,(2b-K)\,(2b-3K)\,,$$

so that $\bar{c}_{\mathrm{M,medium}} = \frac{1}{4}\mu b^2$. At $K = 2(b-2a)$ case 2 becomes valid, at that point the derivative of $\mathrm{E}\pi_M(K, p_M(K))$ is $\mu a\,(3a-b)$. Thus, for the medium variability scenario, we have that there is a unique optimal capacity level K, which is determined by case 2 if $0 < c < \mu a\,(3a-b)$, and by case 1 if $\mu a\,(3a-b) \leq c < \bar{c}_{\mathrm{M,medium}} = \frac{1}{4}\mu b^2$.

Finally, for the high variability scenario we have that only case 1 applies so that we have $\bar{c}_{\mathrm{M,high}} = \bar{c}_{\mathrm{M,medium}} = \frac{1}{4}\mu b^2$.

Thus, to get the explicit expressions for the optimal K, we only need to consider cases 1 and 2:

Case 1: $a < p < p + K < b$: We have that

$$V_M(K, p_M(K)) = \frac{\mu}{16} K\,(2b-K)^2 - cK, \tag{54}$$

The optimal capacity level satisfies

$$\frac{\partial}{\partial K} V_M(K, p_M(K)) = 0 \Leftrightarrow \mu\,(2b-K)\,(2b-3K) = 16c \tag{55}$$

with unique positive solution $K = \frac{2}{3}\left(2b + \sqrt{b^2 + 12c/\mu}\right)$ and thus $p = \frac{1}{6}\left(b + \sqrt{b^2 + 12c/\mu}\right)$ and corresponding objective value:

$$\begin{aligned} V_i^*(c) &= \mu p(b-2p)(-2p-2b+4p+2b) - c2(b-2p), && (56)\\ &= 2\left(p^2\mu - c\right)(b-2p)\,. && (57) \end{aligned}$$

Conditions $a < p, p+K = 2b-3p < b$ and $K = 2(b-2p) \geq 0$ require $\max(a, \frac{b}{3}) < p < \frac{b}{2}$. Thus: either $\frac{b}{2} > a > \frac{b}{3}$ (medium variability) and

$$a < \frac{1}{6}\left(b + \sqrt{b^2 + 12c/\mu}\right) \leq \frac{b}{2} \Leftrightarrow c_2 = a\mu\,(3a-b) < c \leq \bar{c}_{\mathrm{medium}} = \frac{1}{4}\mu b^2,$$

or $\frac{b}{3} > a$ (high variability) and

$$\frac{b}{3} < \frac{1}{6\mu}\left(\mu b + \sqrt{(\mu^2 b^2 + 12c\mu)}\right) \leq \frac{b}{2} \Leftrightarrow 0 < c \leq \bar{c}_{\mathrm{high}} = \frac{1}{4}\mu b^2.$$

Case 2: $p < a < p + K < b$: The optimality equation for this case $\frac{\partial}{\partial K}\mathrm{E}\pi_M(K, p_M(K)) = c$ is a third-order polynomial with a unique solution satisfying $p < a < p + K < b$ and

$$4K^3 + (2c + a - 11b)\,K^2 + 2\left(2a^2 + 3b^2 + 10ca - 12cb - ab\right)K + (a-c)\left(a^2 - 9ca - 3ab + 9cb\right) = 0.$$

However, an easier (but equivalent) expression is obtained by switching the order of optimization (which is allowed, given the uniqueness of the optimum in the zone $p < a < p + K < b$). We have that

$$V_M = -\frac{1}{2} p\mu\left(p^2 - 2ap + 2pK - 2Kb + a^2 + K^2\right) - cK. \tag{58}$$

The necessary conditions for an optimal (p, K) interior in $\{(p, K) : 0 < p < a < p + K < b\}$ are

$$\begin{aligned} \frac{\partial V_M}{\partial p} &= -\frac{1}{2}\mu\left(3p^2 - 4ap + 4pK - 2Kb + a^2 + K^2\right) = 0,\\ \frac{\partial V_M}{\partial K} &= -p\mu\,(p - b + K) - c = 0. \end{aligned}$$

The optimal capacity level is most easily solved for algebraically by first solving for K as a function of p using $\frac{\partial V}{\partial K} = 0$ such that $K = \frac{-\rho^2\mu + \rho\mu b - c}{\rho\mu}$. Then substitute into $\frac{\partial V}{\partial p} = 0$ and solve for p. The solution p is the unique root of $f(x) = -4\mu x^3 + \mu\,(a + b + 2c)\,x^2 - c^2$ in the interval $[0, a]$. [Necessary and sufficient condition for $f(x)$ to have a single root in $[0, a]$ is that $c^2 - 2\mu a^2 c + 4\mu a^3 - \mu(a+b)a^2 = (c - \alpha_1)(c - \alpha_2) \leq 0$. Because $\alpha_1 + \alpha_2 < 0$, we have a positive root α iff $\alpha_1\alpha_2 = 4\mu a^3 - \mu(a+b)a^2 > 0 \Leftrightarrow 3a > b$. One can verify that this single positive root is $\alpha = c_2$ so that there is a valid price for this case if $3a > b$ and $c < c_2$. Because $c_2 > a$ when $b < 2a$, the condition $c < a$ when $b < 2a$ guarantees the uniqueness of the root.] ■

2.12 Lemma 3: Capacity-constrained monopoly quantity setting

We have two cases:

Case 1: $K < a$: Because $q \leq K$, the revenue function becomes

$$\mathrm{E}\pi_M = \mathrm{E}(\varepsilon - q)q = (\varepsilon_0 - q)q,$$

so that the capacity-constrained monopoly quantity is $q_M(K) = \min\left(\frac{\varepsilon_0}{2}, K\right)$. (Note that if $b > 3a$ (high variability) we have that $\frac{\varepsilon_0}{2} > a$, so that $q_M(K) = K$.)

Case 2: $K \geq a$: If $q = \frac{\varepsilon_0}{2} < a \Leftrightarrow b < 3a$ (low & medium variability), the solution above holds. Otherwise $a \leq q \leq b$, (clearly, one will never set $q > b$), and the revenue function becomes

$$\mathrm{E}\pi_M = \mathrm{E}\left[(\varepsilon - q)q | a \leq q \leq b\right] P(a \leq q \leq b) = \int_q^b (\varepsilon - q)q\mu d\varepsilon = \frac{1}{2}q\mu\left(b - q\right)^2, \tag{59}$$

with a unique maximum at $q = \frac{b}{3}$ so that $q_M(K) = \min\left(\frac{b}{3}, K\right)$.

Conclusion:

1. If $b < 3a$ (low & medium variability):

$$q_M(K) = \min\left(\frac{\varepsilon_0}{2}, K\right), \tag{60}$$

2. If $b > 3a$ (high variability):

$$q_M(K) = \min\left(\frac{b}{3}, K\right). \tag{61}$$

2.13 Lemma 4: Monopoly capacity investment with quantity-setting (uniform uncertainty)

We have two cases:

Case 1: $b < 3a$ (low & medium variability). The value function becomes

$$V(K) = \begin{cases} K\left(\varepsilon_0 - K\right) - cK & \text{if} \quad K \leq \frac{\varepsilon_0}{2}, \\ \left(\frac{\varepsilon_0}{2}\right)^2 - cK & \text{if} \quad K > \frac{\varepsilon_0}{2}. \end{cases} \tag{62}$$

Clearly, $K > \frac{\varepsilon_0}{2}$ is suboptimal. Thus, $K \leq \frac{\varepsilon_0}{2}$, and the optimal capacity investment is

$$K = \frac{1}{2}\left(\varepsilon_0 - c\right) \Rightarrow V = \left(\frac{\varepsilon_0 - c}{2}\right)^2 \text{ for low \& medium variability,} \tag{63}$$

with conditions: $c < \varepsilon_0$ and $\frac{1}{2}\left(\varepsilon_0 - c\right) < \frac{\varepsilon_0}{2}$ (which is clearly satisfied with low&medium variability).

Case 2: $b > 3a$ (high variability). The value function becomes

$$V(K) = \begin{cases} K\left(\varepsilon_0 - K\right) - cK & \text{if} \quad K \leq a, \\ \frac{1}{2}K\mu\left(b - K\right)^2 - cK & \text{if} \quad a < K \leq \frac{b}{3}, \\ \frac{2}{27}b^3\mu - cK & \text{if} \quad K > \frac{b}{3}. \end{cases} \tag{64}$$

Clearly, $K > \frac{b}{3}$ is suboptimal. Depending on the cost c, V can have a maximum in the zone $[0, a]$ or in $[a, b/3]$. A maximum in $[0, a]$ is as in case 1 with conditions $c < \varepsilon_0$ and $K = \frac{1}{2}\left(\varepsilon_0 - c\right) < a \Leftrightarrow \frac{a+b}{2} - 2a = \frac{b-3a}{2} < c$. If $c < \frac{b-3a}{2}$, then the optimal K is in between a and $b/3$ and maximizes

$$V = \frac{1}{2}K\mu\left(b - K\right)^2 - cK, \tag{65}$$

with unique maximum:

$$K = \frac{1}{3}\left(2b - \sqrt{b^2 + 6c/\mu}\right), \tag{66}$$

with conditions $0 < a < \frac{1}{3}\left(2b - \sqrt{b^2 + 6c/\mu}\right) < b \Leftrightarrow 3a - 2b < -\sqrt{b^2 + 6c/\mu} < b$ and because we must have high variability this reduces to $c < \frac{1}{2}\left(b - 3a\right) = 2\sqrt{3}\sigma - \varepsilon_0$. ■

2.14 Lemma 5: Optimal duopoly capacity investment under quantity-setting with uniform uncertainty

2.14.1 The quantity-setting subgame

We now consider the case of two firms; initially each firm i and j sets capacity at K_i and K_j for a total industry capacity of $K_+ = K_i + K_j$. Given the capacity choices in stage 1, each firm sets production quantity $q_i \leqslant K_i$ and $q_j \leqslant K_i$ for a total industry production of $q_+ = q_i + q_j$. Hence, the expected profit for firm i is:

$$\mathrm{E}\pi_i = \int_{q_+}^{\infty} (\varepsilon - q_+)\, q_i f(\varepsilon) d\varepsilon.$$

Case 1: $q_+ < a$. The revenue function becomes $\mathrm{E}\pi_i = \mathrm{E}(\varepsilon - q_+)\, q_i = (\varepsilon_0 - q_+) q_i$ so that firm i's quantity reaction curve is

$$q_i(q_j|K_i) = \min\left(\frac{\varepsilon_0 - q_j}{2}, K_i\right) \text{ if } q_+ < a \Leftrightarrow \min\left(\frac{a+b+2q_j}{4}, K_i + q_j\right) < a.$$

An interior solution $\frac{\varepsilon_0 - q_j}{2}$ requires $\frac{\varepsilon_0}{2} = \frac{a+b}{4} < a \Leftrightarrow b < 3a$ (low & medium variability).

Case 2: $q_+ \geq a$. Clearly, one will never set $q_+ > b$, so that the revenue function becomes

$$\mathrm{E}\pi_i = \int_{q_i+q_j}^{b} (\varepsilon - q_i - q_j) q_i \mu d\varepsilon = \frac{1}{2} q_i \mu \left(b - q_i - q_j\right)^2,$$

with a unique maximum at $q_i = \frac{1}{3}(b - q_j)$ so that

$$q_i(q_j|K_i) = \min\left(\frac{1}{3}(b - q_j), K_i\right) \text{ if } q_+ \geq a \Leftrightarrow \min\left(\frac{1}{3}(b + 2q_j), K_i + q_j\right) \geq a. \tag{67}$$

Thus, firm i's reaction curve $q_i(q_j|K)$ is piecewise linear. If capacity K_i is sufficiently large, the reaction curve is strictly decreasing in q_j whenever $q_i > 0$ with decreasing slopes > -1 as shown in Figure 7. If capacity is low, the reaction curve is the same, but restricted to the rectangle $[0, K_i] \times [0, K_j]$. In any case, as discussed in the proof of Proposition 5, there is a unique equilibrium at the intersection of both firm's reaction curves.

Assuming capacity is sufficiently large, the equilibrium is $q = (q_D^*, q_D^*)$ depends on the level of variability:

$$q_D^* = \begin{cases} \frac{a+b}{6} & \text{if} \quad \frac{1}{2}(b-a) \leq \frac{1}{2}(3a - b) \Leftrightarrow b < 2a \text{ (low variability)}, \\ \frac{b}{4} & \text{otherwise} \quad \text{(medium \& high variability)}. \end{cases} \tag{68}$$

In general, denoting the unrestricted reaction curves by $q_i^U(\cdot)$, the equilibrium is

$$q(K) = \begin{cases} (K_i, K_j) & \text{if} \quad 0 \leq K_i, K_j < q_D^*, \\ \left(\min\left(K_i, q_i^U(K_j)\right), K_j\right) & \text{if} \quad 0 \leq K_j < q_D^* \leq K_i, \\ \left(K_i, \min\left(K_j, q_j^U(K_i)\right)\right) & \text{if} \quad 0 \leq K_i < q_D^* \leq K_j, \\ (q_D^*, q_D^*) & \text{if} \quad q_D^* \leq K_i, K_j. \end{cases} \tag{69}$$

2.14.2 The Capacity Stage (Full Game)

We know from Proposition 6 that in equilibrium $q = K$ and $K = \frac{1}{2}(K_+, K_+)$ where

$$\int_{K_+}^{\infty} \left(\varepsilon - \frac{3}{2} K_+\right) f(\varepsilon) d\varepsilon = c \tag{70}$$

for $c < \varepsilon_0$, and $q = K = 0$ for $c \geq \varepsilon_0$.

Case 1:$0 \leq K_+ \leq a$: $\varepsilon_0 - \frac{3}{2} K_+ = c \Leftrightarrow$

$$K_+ = \frac{2}{3}(\varepsilon_0 - c) \text{ and } V_+ = \mathrm{E}(\varepsilon - K_+)\, K_+ - cK_+ = \frac{2}{9}(c - \varepsilon_0)^2, \tag{71}$$

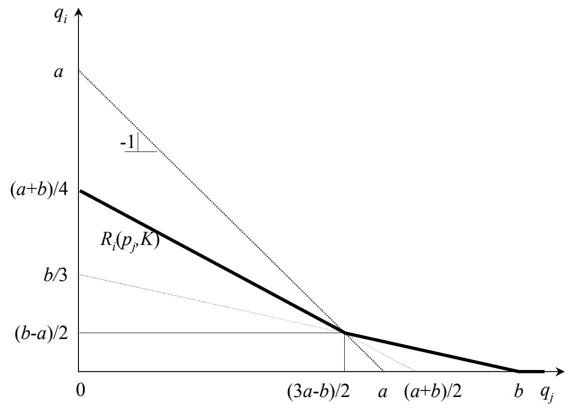

Figure 7: Quantity-reaction curve for firm i for uniform uncertainty (assuming $K_i > \frac{1}{4}(a+b)$).

with condition:

$$\frac{2}{3}(\varepsilon_0 - c) \leq a \Leftrightarrow c \geq \varepsilon_0 - \frac{3}{2}a. \tag{72}$$

With low variability ($b < 2a$), $\varepsilon_0 - \frac{3}{2}a < 0$, so that this holds for all $c < \varepsilon_0$. For medium and high variability this case requires $c \geq \varepsilon_0 - \frac{3}{2}a = \frac{b-2a}{2}$.

Case 2:$a < K_+$:

$$V_i = \int_{K_+}^{b} (\varepsilon - K_i - K_j) K_i \mu d\varepsilon - cK_i, \tag{73}$$

$$\frac{\partial}{\partial K_i} V_i = \int_{K_i+K_j}^{b} (\varepsilon - 2K_i - K_j) \mu \, d\varepsilon - c = 0, \tag{74}$$

$$\frac{\partial^2}{\partial K_i^2} V_i = \mu (3K_i + 2K_j - 2b) < 0. \tag{75}$$

Thus, with $K_i = K_j = \frac{1}{2}K_+$:

$$\int_{K_+}^{b} \left(\varepsilon - \frac{3}{2}K_+\right) \mu d\varepsilon = \frac{1}{2}\mu (b - K_+)(b - 2K_+) = c \text{ and } \mu\left(\frac{5}{2}K_+ - 2b\right) < 0 \text{ and } K_+ > a, \tag{76}$$

$$\Leftrightarrow \quad a < K_+ = \frac{1}{4}\left(3b \pm \sqrt{b^2 + 16c/\mu}\right) < \frac{4}{5}b, \tag{77}$$

so that only the negative root remains and

$$V_+(c) = \int_{K_+}^{b} (\varepsilon - K_+) K_+ \mu d\varepsilon - cK_+ = \frac{1}{2}K_+\mu (b - K_+)^2 - cK, \tag{78}$$

$$= \frac{1}{8}\left(3b - \sqrt{b^2 + 16c/\mu}\right) \mu \left(b - \frac{1}{4}\left(3b - \sqrt{b^2 + 16c/\mu}\right)\right)^2 - c\frac{1}{4}\left(3b - \sqrt{b^2 + 16c/\mu}\right), \tag{79}$$

$$= \frac{1}{128}\left(3b - \sqrt{b^2 + 16c/\mu}\right)\left[\mu\left(b + \sqrt{b^2 + 16c/\mu}\right)^2 - 32c\right], \tag{80}$$

with conditions:

$$a < \frac{1}{4}\left(3b - \sqrt{b^2 + 16c/\mu}\right) < \frac{4}{5}b, \tag{81}$$

$$4a - 3b < -\sqrt{b^2 + 16c/\mu} < \frac{1}{5}b. \tag{82}$$

Thus, we must have $b > \frac{4}{3}a$ and

$$b^2 + 16c/\mu < (3b - 4a)^2 \Leftrightarrow c < \frac{1}{2}(b - 2a). \tag{83}$$

This requires $b > 2a$ (medium & high variability). ■

2.14.3 Quantity competition under exponential uncertainty

With exponential uncertainty, the quantity-optimized revenue is

$$\mathrm{E}\pi_i(q_i(K), K) = \min(1, K_i)\exp(-\min(1, K_i) - \min(1, K_j)), \tag{84}$$

with capacity reaction curves

$$(1 - K_i)\exp(-K_i - \min(1, K_j)) = c \text{ and } K_i \leq 1. \tag{85}$$

The symmetric duopoly equilibrium $K = \frac{1}{2}(K_+, K_+)$ where K_+ solves for $c < \bar{c} = 1$:

$$\left(1 - \tfrac{1}{2}K_+\right)\exp(-K_+) = c. \tag{86}$$