Investment Strategies for Flexible Resources

Jan A. Van Mieghem
Kellogg Graduate School of Management at Northwestern University

November 10, 1997 (Extended technical report; an abbridged version appeared in *Management Science* 44(8)1071-1078, August 1998)

Abstract

This article studies optimal investment in flexible manufacturing capacity as a function of product prices (margins), investment costs and multivariate demand uncertainty. We consider a two-product firm that has the option to invest in product-dedicated resources and/or in a flexible resource that can produce either product, but has to make its investment decision before demands are observed. The flexible resource provides the firm with a hedge against demand uncertainty, but at a higher investment cost than the dedicated resources. Our analysis highlights the important role of price (margin) and cost mix differentials, which, in addition to the correlation between product demands, significantly affect the investment decision and the value of flexibility. Contrary to the intuition also prevalent in the academic literature, we show that it can be advantageous to invest in flexible resources even with perfectly positively correlated product demands.

Key Words: Flexibility, technology, strategy, capacity investment, prices, operational hedging, multi-dimensional newsvendor model.

1 Introduction

"The preserving of flexibility when faced with uncertainty" is no longer a neglected aspect of behavior under risk, as it was according to Jones and Ostroy [16] in 1984. Yet our understanding of flexibility still is based mainly on intuition that may be incomplete. Following the lead of Fine and Freund [11], we study optimal investment in flexible manufacturing resources and consider a firm that faces uncertain demands for its two products. The firm has the option to invest in product-dedicated resources and in a flexible resource that is able to produce either product, but has to make its investment decision before demand is observed. The flexible resource provides the firm with a hedge against demand uncertainty but at the expense of a higher investment cost than the dedicated resources.

The results advanced in this article highlight the important role of price (or, more precisely, unit contribution margin) and cost mix differentials, which, in addition to the correlation between product demands, significantly affect the investment decision in flexible technology and the value of flexibility. The ability of product-flexible technology to deal with changes or uncertainty in demand mix (that is, relative proportions of product quantities demanded) is well understood. This has led to a belief that flexible capacity provides no additional value when product demands are perfectly positively correlated. Fine and Freund [11, p. 459] offer the argumentation behind this belief:

"Because the demands for the two products move in lockstep, flexible capacity can only be useful if it can produce one product more cheaply than the dedicated capacity can. There will never be an opportunity to take advantage of the flexibility characteristic of the flexible capacity ... when the two products' demands are perfectly positively correlated." We will show that this belief is incomplete, if not false. Indeed, in addition to its adaptability to demand mix changes, product-flexible technology provides another opportunity for revenue improvement through its ability to exploit differentials in price (margin) mix. Product flexibility generates an option to produce and sell more of highly profitable products at the expense of less profitable products. More importantly, this option can remain valuable even with perfectly positively correlated product demand (i.e., when demand mix is constant and known with certainty).

In this article we will use the term "flexibility" in a natural manner to describe the capability of a resource to produce different products and, more generally, of a manufacturing process to produce identical products utilizing different resources. (The term "resource" is used to mean any long-lived human or physical asset that affects the production capabilities of a firm but is not consumed in the course of production.) For our purpose, there is no need to add another general definition of flexibility to the plethora already available in the literature (e.g., see the extensive, technology-oriented surveys by Gupta and Goyal [12] and Sethi and Sethi [23], or Carlsson [4] for an economics perspective).

Research on flexibility seems to follow five major paradigms. Technology focused studies of flexibility tend to use detailed (often at the expense of analytical intractability) stochastic programming or queueing models, e.g., Benjaafar [2], Burstein [3], Kulatilaka [17], Laengle, Griffin and Griffin [18] and Fine and Freund [11]. Following Stigler [24], economics oriented studies initially viewed flexibility as a generator of economies of *scale* in the presence of demand uncertainty and related it inversely to the curvature of the firm's total cost curve, see Carlsson [4] for extensive references. Choice set theory and information economics extend the notion of flexibility and relate it to the size of the choice set (a more flexible initial action preserves more choices for actions in the second period, e.g., Jones and Ostroy [16]) and the precision of available information (a more flexible position is preferred with an increase in uncertainty, e.g., Vives [25]). Following Milgrom and Roberts [20], flexibility recently has been identified with economies of *scope* in the production of differentiated goods. See, for example, Eaton and Schmitt [8], Röller and Tombak [22], Milgrom and Shannon [21], Athey and Schmutzler [1], and the related general theory of flexibility by de Groote [6] to which we will refer in Section 5. Finally, flexibility also relates to research on irreversible investments and real options: an irreversible investment reduces the firm's flexibility while, as will be shown in this article, an investment in flexible resources may create valuable production options that can be exercised when uncertainty is resolved. See, for example, Dixit and Pindyck [7] and He and Pindyck [14].

This article is inspired and intimately related to the work by Fine and Freund [11]. However, our research method is different from their traditional discrete stochastic programming approach in that we use our "multi-dimensional newsvendor model," which was first presented in [13]. The resulting parsimonious descriptive model is amenable to analytic analysis and graphic interpretation, which allows our results to be easily taught and remembered. Its multi-dimensionality enriches the traditional newsvendor model by incorporating product, resource and demand differentiation through price and cost *vectors*, a technology *matrix* and a *multivariate* demand distribution. The intent of this article is to build new theory and intuition on the benefits of product flexibility to hedge against demand uncertainty by highlighting the role of price and cost mix differentials in addition to demand correlation. For our purposes here it suffices to analyze a one-period model. As such, our approach may be too highly stylized to serve as a practical decision support system which may need to consider more complex models for which one must resort to numerical methods, cf. [5, 19]. Alternately, multi-period extensions may be analyzed using our recent theory on quasi-reversible multi-dimensional investment under uncertainty [9, 13].

This article is organized as follows. The next section presents the model and section 3 discusses the optimal investment position using our multidimensional newsvendor solution. Section 4 shows that the optimal investment strategy has one of three distinct forms and analyzes the sensitivity of the optimal investment to changes in prices (margins) and investment costs. Finally, section 5 examines how the optimal investment depends on the multivariate demand distribution and presents closed-form solutions for perfectly correlated product demand, emphasizing the role of price differentials. (All proofs can be found in the Appendix.)

We conclude this introduction with some notational conventions. We will not distinguish in notation between scalars and vectors. All vectors are assumed to be column vectors, and primes denote transposes. Vector inequalities should be interpreted componentwise. As usual, E and ∇ denote the expectation and gradient-vector operators.

2 Modeling Investment in Flexible Resources

Consider a firm that has the option to invest in two product-dedicated resources and one flexible resource—respectively labeled resources 1, 2 and 3—to manufacture two products. First, the firm must decide on a non-negative vector of resource capacity levels $K \in \mathbb{R}^3_+$ for production, before the product demand vector $D \in \mathbb{R}^2_+$ is observed. After demand is observed, the firm chooses, constrained by its earlier resource investment, a vector $x = (y_1, y_2, z_1, z_2) \in \mathbb{R}^4_+$ of production quantities, where $y_j + z_j$ is the total production quantity of product j and y_j and z_j represent the quantities produced on the product-j-dedicated and flexible resource respectively. This multi-stage decision problem, also known as a stochastic program with recourse, is characteristic of *real option* models: first invest in capabilities, then receive some additional information, and finally exploit capabilities optimally contingent on the revealed information.

The firm's manufacturing process and production decisions are modeled as follows. Having chosen a capacity vector K and observed a demand vector D, the firm chooses its production vector x as the optimal solution of the following product mix problem so as to maximize operating profit:

$$\begin{aligned}
\max_{y,z\in\mathbb{R}^2_+} p_1(y_1+z_1) &+ p_2(y_2+z_2) && (1)\\
\text{subject to} \quad y_1 &\le K_1, && (2)\\
y_2 &\le K_2, && (3)\\
z_1+z_2 &\le K_3, && (4)\\
y_1+z_1 &\le D_1, && (5)\\
y_2+z_2 &\le D_2, && (6)
\end{aligned}$$

where $p \in \mathbb{R}^2_+$ is a *price* or *margin* vector whose jth component represents the *unit contribution margin* for product j (that is, sales price minus variable cost of production). The optimal objective value of the product mix problem (1)–(6) is the maximal *operating profit* and is denoted by $\pi(K,D) = (p_1,p_2,p_1,p_2)'x(K,D)$, where $x(K,D)$ is an associated optimal production vector. To keep the number of parameters manageable, we have implicitly made two assumptions in the product mix problem. First, product j variable production costs on the associated dedicated resource and on the flexible resource are identical. Second, because the contribution margins do not depend on the production quantities chosen, the firm is assumed to be a price taker in both the output and factor markets. Notice that although (4) seems to indicate that both products require an equal amount of the flexible resource to produce one unit, this is without any loss of generality since in this model it is just a matter of how units are defined.

Assuming that the firm starts with no initial resources, it incurs an *investment cost* $C(K)$ if it chooses a capacity vector K. For simplicity we assume that investment costs are linear,

$$C(K) = c'K, \tag{7}$$

where $c \in \mathbb{R}^3_+$ is a vector of *marginal investment costs*, but the results presented below directly generalize to any convex function C. In order for the model to be realistic and to yield interesting results, we assume that flexible capacity is more expensive than dedicated, yet sufficiently inexpensive to be a viable alternative: $0 < c_1, c_2 < c_3 < c_1 + c_2$. Also, it should be economically justified to produce both products, *i.e.*, $c_1 < p_1$ and $c_2 < p_2$, where the more profitable product is given label 1, so that $p_1 \ge p_2 > 0$, and the *price* (or *margin*) *differential* will be denoted by $\Delta p = p_1 - p_2 \ge 0$. Finally, demand uncertainty is represented by a probability measure P over the demand space $\mathbb{R}^2_+$. For simplicity we assume that D is a continuous random vector that is finite with probability one and that has a joint probability density function g which is positive over its support. The firm seeks a strategy of investment and production that maximizes

$$V(K) = \mathrm{E}\pi(K,D) - C(K), \tag{8}$$

the expected value of operating profits minus resource investment costs. We denote the maximal value of $V(\cdot)$ by V^* and call any maximizer of $V(\cdot)$ an optimal investment vector.

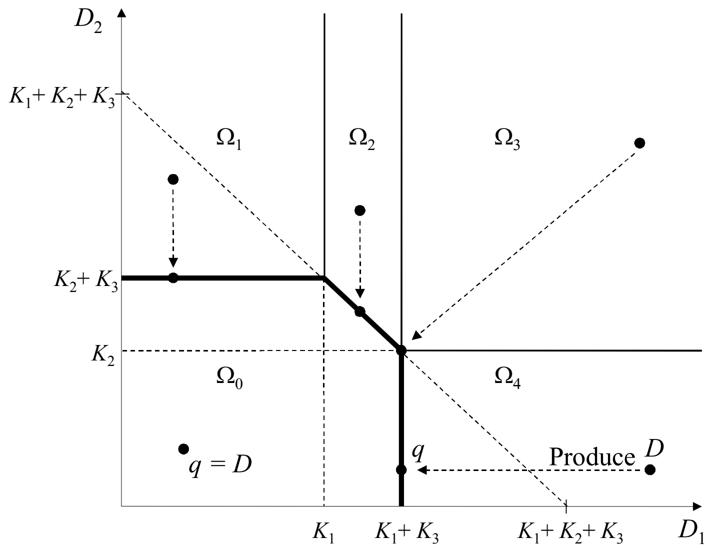

Figure 1: The total production quantities (product sales) $q = (y_1 + z_1, y_2 + z_2)$ and shadow prices λ depend on the capacity K and the demand D.

It is straightforward to incorporate demand shortage penalties and capacity salvage values into the model as follows. Assume product j carries a shortage penalty cost $c_{\mathrm{P},j} \geq 0$ for each unit of demand that is not satisfied (i.e., for each unit of $D_j > y_j + z_j$) and a unit of resource i has a salvage value $c_{\mathrm{S},i} < c_i$ at the end of the period. Then all results presented in this article remain valid if we inflate unit contribution margins p to $p + c_{\mathrm{P}}$, deflate marginal investment costs c to $c - c_{\mathrm{S}}$ and decrease the operating profit $\pi(K, D)$ by $c'_{\mathrm{P}} D$ (which also decreases $V(K)$ by the constant E $c'_{\mathrm{P}} D$).

3 The Multi-dimensional Newsvendor Solution

Like most multi-stage decision problems, our model is analyzed backward by first solving for the optimal contingent production decisions $x(K, D)$, and the associated three-vector $\lambda(K, D)$ of optimal dual variables, or *shadow prices*, of the capacity constraints (2)–(4) in the product mix problem. Parametric analysis of that linear program (*i.e.*, using the Simplex method) leads us to partition the demand space $\mathbb{R}^2_+$ given a capacity vector $K \in \mathbb{R}^3_+$ into five domains as shown in Figure 1, where the thick-lined $\Omega_0(K)$ is the firm's production capacity region. This allows us to express the optimal contingent primal and dual variables: $x = (D_1, K_2, 0, K_3)'$ and $\lambda = (0, p_2, p_2)'$ if $D \in \Omega_1(K)$, $x = (K_1, K_2, D_1 - K_1, K_3 - D_1 + K_1)'$ and $\lambda = (p_2, p_2, p_2)'$ if $D \in \Omega_2(K)$, $x = (K_1, K_2, K_3, 0)'$ and $\lambda = (p_1, p_2, p_1)'$ if $D \in \Omega_3(K)$, $x = (K_1, D_2, K_3, 0)'$ and $\lambda = (p_1, 0, p_1)'$ if $D \in \Omega_4(K)$; and $\lambda = 0$ and any vector x of production quantities satisfying $y_1 + z_1 = D_1$ and $y_2 + z_2 = D_2$ is optimal if $D \in \Omega_0(K)$.

From basic linear programming theory we know that $\pi(\cdot, \cdot)$, and thus $V(\cdot)$, is concave so that the Kuhn-Tucker first-order conditions are necessary and sufficient to maximize $V(\cdot)$. In [13], we show that differentiation and expectation can be interchanged so that $\nabla \mathrm{E}\pi(\cdot, D) = \mathrm{E}\lambda(\cdot, D)$. Finally, because the shadow price vector is constant in each domain $\Omega_i(K)$, we can express the optimality equations in terms of the dual variables as follows:

Proposition 1 *An investment vector $K^* \in \mathbb{R}^3_+$ is optimal if and only if there exists a $\nu \in \mathbb{R}^3_+$ such that*

$$\begin{pmatrix} 0 \\ p_2 \\ p_2 \end{pmatrix} P(\Omega_1(K^*)) + \begin{pmatrix} p_2 \\ p_2 \\ p_2 \end{pmatrix} P(\Omega_2(K^*)) + \begin{pmatrix} p_1 \\ p_2 \\ p_1 \end{pmatrix} P(\Omega_3(K^*)) + \begin{pmatrix} p_1 \\ 0 \\ p_1 \end{pmatrix} P(\Omega_4(K^*)) = c - \nu, \tag{9}$$

$$\nu' K^* = 0, \tag{10}$$

It is readily shown that the optimal investment level K^* is unique. Proposition 1 greatly enhances the intuitive content of the model by providing a solution technique with a graphical interpretation. The optimal investment is found by superimposing the multivariate demand distribution onto Figure 1 and adjusting the thick lines of the feasible region (these are determined by K) such that the probabilities of the four domains $\Omega_1, \cdots, \Omega_4$ offset the marginal investment cost c as in the optimality equation (9). Generalizing the language of the familiar one-dimensional newsvendor model, one can say that it is optimal to invest up to a critical "fractile" of the multivariate demand distribution, thereby balancing "overage costs" with "underage costs."

4 How Optimal Investment depends on Costs and Prices

Proposition 2 will highlight the role of investment costs by showing that the optimal investment strategy must take one of the following forms: (i) invest in dedicated resources only, (ii) do not invest in dedicated capacity for product 2 but invest in dedicated capacity for product 1 as well as in flexible capacity, or (iii) invest in all three types of capacity. No other combinations of investment can be optimal. Before we can explicitly write out the firm's optimal investment policy in terms of the model primitives, we must first define two non-negative threshold values $\underline{c}_3$ and $\bar{c}_3$, as follows. Strategy (i) corresponds to a boundary solution of Proposition 1 of the form $\bar{K} = (\bar{K}_1, \bar{K}_2, 0)$ which is the unique solution to

$$p_1 P(\Omega_3(\bar{K})) + p_1 P(\Omega_4(\bar{K})) = p_1 P(D_1 > \bar{K}_1) = c_1, \tag{11}$$

$$p_2 P(\Omega_1(\bar{K})) + p_2 P(\Omega_3(\bar{K})) = p_2 P(D_2 > \bar{K}_2) = c_2, \tag{12}$$

while the third optimality equation reduces to $c_3 > \bar{c}_3$ where

$$\bar{c}_3 = c_1 + c_2 - p_2 P(\Omega_3(\bar{K})). \tag{13}$$

Strategy (ii) is a boundary solution of the form $\underline{K} = (\underline{K}_1, 0, \underline{K}_3)$ which is the unique solution to

$$p_2 P(\Omega_2(\underline{K})) + p_1 P(\Omega_3(\underline{K})) = c_1, \tag{14}$$

$$p_2 P(\Omega_1(\underline{K})) + p_2 P(\Omega_2(\underline{K})) + p_1 P(\Omega_3(\underline{K})) = c_3. \tag{15}$$

while the second optimality equation reduces to $c_3 < \bar{c}_3$ where

$$\underline{c}_3 = c_2 + P(\Omega_3(\underline{K}))\Delta p. \tag{16}$$

Conditions (12) and (14) yield that $0 \leq p_2 P(\Omega_3(\bar{K})) \leq c_2$ and $p_1 P(\Omega_3(\underline{K})) \leq c_1$, respectively. Because $\Delta p = p_1 - p_2 \geq 0$, it follows that $c_1 \leq \bar{c}_3 \leq c_1 + c_2$ and $c_2 \leq \underline{c}_3 \leq \bar{c}_3$. We can now state the main result:

Proposition 2 *The optimal investment strategy has one of three distinct forms, depending on the marginal cost of flexibility c_3:*

(i) If $c_3 > \bar{c}_3$, it is optimal to invest only in dedicated resources and $K^* = \bar{K}$.

(ii) If $c_3 < \underline{c}_3$, it is optimal to invest only in the product-1 dedicated resource and the flexible resource and $K^* = \underline{K}$. This requires a positive price differential $\Delta p > 0$.

(iii) Otherwise, it is optimal to invest in all three resources and K^* solves (9) with $\nu = 0$.

Because $\max(c_1, c_2) \leq \bar{c}_3 \leq c_1 + c_3$, there are values for c_3 that make strategy (*i*) and/or (*iii*) optimal (both are possible if $\max(c_1, c_2) < \bar{c}_3 < c_1 + c_3$). However, a positive price differential $\Delta p > 0$ is necessary, but not sufficient, for strategy (*ii*) to be optimal: because we cannot guarantee that $\underline{c}_3 \geq c_1$, the problem parameters p, c_1, c_2 and P may be such that strategy (*ii*) is never optimal for any value of c_3. Besides understanding the pivotal role of the marginal cost of the flexible resource, it is also interesting to see how the optimal investment level changes as the entire marginal cost vector c changes:

Proposition 3 *The optimal value V^* is a non-increasing convex function of the marginal capacity costs c with gradient $\nabla_c V^* = -K^* \leq 0$, and the optimal investment vector K^* has cost sensitivity terms*

$$\nabla_c K^{*\prime} = \begin{pmatrix} -(\alpha_1 + \alpha_2 + \alpha_4) & -\alpha_1 & \alpha_1 + \alpha_2 \\ -\alpha_1 & -(\alpha_1 + \alpha_3 + \alpha_5) & \alpha_1 + \alpha_3 \\ \alpha_1 + \alpha_2 & \alpha_1 + \alpha_3 & -(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_6) \end{pmatrix}, \tag{17}$$

where $\alpha(K^*) \in \mathbb{R}^6_+$ and $\alpha > 0$ under optimal strategy (iii); $\alpha_2, \alpha_4, \alpha_6 > 0$ and all other $\alpha_j = 0$ under strategy (ii); and $\alpha_4, \alpha_5 > 0$ and all other $\alpha_j = 0$ under strategy (i).

Proposition 3 extends Lemma 2 and Theorem 2 of Fine and Freund [11] to our multi-dimensional newsvendor solution with continuous demand distribution and has a similar interpretation. It is not surprising that the optimal value V^* and the optimal investment level K_j^* of resource j do not increase as the marginal cost of resource j increases. The more interesting result is the *substitution effect* implicit in the off-diagonal terms of (17): as the marginal cost of a dedicated resource increases, the decrease in the optimal investment level of *both* dedicated resources is partially offset by an increase in the optimal level of the flexible resource, and vice versa for an increase in the marginal cost of the flexible resource. However, summing terms shows that the substitution is incomplete: $\frac{\partial}{\partial c_1}(K_1^* + K_2^* + K_3^*) \leq \frac{\partial}{\partial c_1}(K_1^* + K_3^*) \leq 0 \leq \frac{\partial}{\partial c_1}(K_2^* + K_3^*)$, and similar relations hold for c_2 and c_3. The impact of price (margin) changes is:

Proposition 4 *The optimal value V^* is an increasing convex function of the price (margin) vector p with gradient $\frac{\partial}{\partial p_i}V^* = \mathrm{E}(y_i + z_i) > 0$, and the optimal investment vector K^* has price sensitivity terms*

$$\nabla_p K^{*\prime} = \begin{pmatrix} \beta_2 & -\beta_1 & \beta_1 + \beta_3 \\ -\beta_5 + \beta_6 + \beta_7 & \beta_4 & \beta_5 - \beta_6 + \beta_8 - \beta_9 \end{pmatrix}, \tag{18}$$

where $\beta(K^*) \in \mathbb{R}_+^9$ and $\beta > 0$ with $\beta_4 + \beta_7 + \beta_8 > \beta_9$ under optimal strategy (iii); $\beta_2, \beta_3, \beta_5, \beta_7, \beta_8 > 0$ and all other $\beta_j = 0$ under strategy (ii); and $\beta_2, \beta_4 > 0$ and all other $\beta_j = 0$ under strategy (i).

Obviously, a higher price (margin) p_j, and thus a higher price differential $\Delta p|_{p_2 = cte}$, is beneficial and warrants a higher investment level K_j^* in the corresponding dedicated resource. More interesting is the impact of an increase in p_1, or equivalently in the price differential $\Delta p|_{p_2 = cte}$, on the flexibility investment. Such increase asks for a substitution of dedicated product 2 (the less profitable product) capacity into flexible capacity with a positive net effect: $\frac{\partial}{\partial \Delta p}(K_2^* + K_3^*)|_{p_2 = cte} = \beta_3 > 0$. Vice-versa for an increase in p_2 under strategy (ii), though the effect is less-pronounced under strategy (i), but total capacity is always increasing: $\frac{\partial}{\partial p_i}(K_1^* + K_2^* + K_3^*) > 0$. The impact of this substitution on price-mix exploitation will be explained in the next section.

5 How Optimal Investment Depends on Demand Uncertainty

The optimal investment level K^* and the threshold costs $\underline{c}_3$ and $\bar{c}_3$ depend not only on the contribution margins p and the marginal investment costs c but also on the entire demand distribution. Some insights follow directly from the graphical interpretation of the solution: a translation of the demand distribution by (δ_1, δ_2) is absorbed by a corresponding shift in the optimal investment level of the *dedicated* resources without affecting the optimal level of the flexible resource: K^* changes to $(K_1^* + \delta_1, K_2^* + \delta_2, K_3^*)$. Also, as the level of uncertainty in the demand distribution decreases, the optimal level of the dedicated resources tends to the mean demand while the optimal level of the flexible resource tends to zero. Indeed, the flexible resource provides us with a *real option* that can be exercised after uncertainty is realized. If there is no uncertainty this option has no value and one will invest in dedicated resources only: $K^* = (D_1, D_2, 0)$. Dedicated resources seem to serve as "base capacity," whereas the flexible resource serves as an optimal cost/benefit response to variability in demand. Because the optimal investment solution depends on the entire shape of the demand distribution—a familiar result of the newsvendor model—it cannot be expressed in terms of a few demand parameters, such as the mean demand or its variance, only. Thus it is difficult to draw any general conclusions on the impact of the demand distribution on the optimal investment strategy. However, the following sensitivity result may help to illustrate how the optimal investment level depends on a particular parameter θ of the demand distribution.

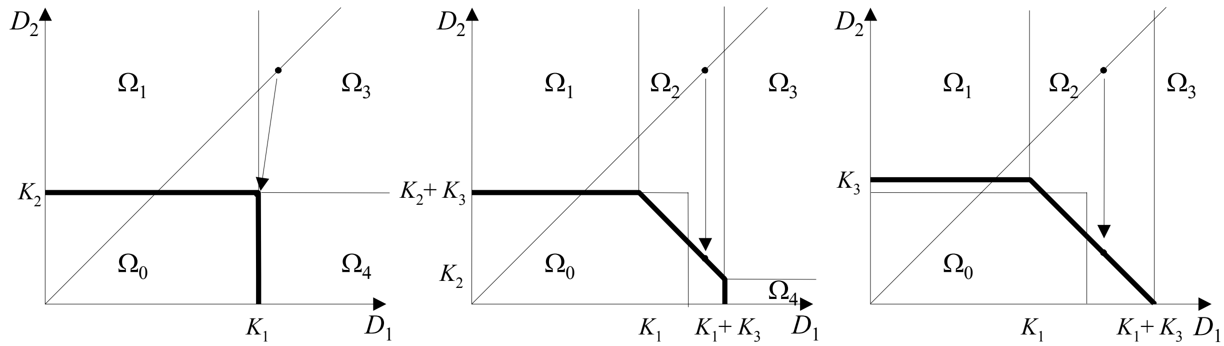

Figure 2: The optimal investment strategies when demands are perfectly positively correlated, $\Delta p > 0$ and $\frac{c_1}{p_1} < \frac{c_2}{p_2}$ for three scenarios: high cost of flexibility (left), medium cost (center), low cost (right).

Proposition 5 *Consider a parametric family of demand density functions $g(\cdot\,|\theta)$, assuming $g(\cdot\,|\theta)$ is differentiable w.r.t. the scalar parameter θ. Let the domains $\Omega_i = \Omega_i(K^*)$ for $i = 1, \cdots, 4$ and $\alpha(K^*)$ be defined as above and define the vector J by $J_i = \int_{\Omega_i} \frac{\partial}{\partial\theta} g(z\,|\theta)\mathrm{d}z$. Then the θ-sensitivity of the optimal investment level K^* under strategy (iii) is*

$$\frac{\partial}{\partial\theta}K^* = \begin{pmatrix} -\alpha_2 p_2 & (\alpha_1+\alpha_4)p_2 & \alpha_4 p_1 + \alpha_1 p_2 & \alpha_4 p_1 \\ \alpha_5 p_2 & (\alpha_1+\alpha_5)p_2 & (\alpha_1+\alpha_5)p_2 - \alpha_3\Delta p & -\alpha_3 p_1 \\ (\alpha_2+\alpha_6)p_2 & (\alpha_6-\alpha_1)p_2 & \alpha_6 p_1 + \alpha_3\Delta p - \alpha_1 p_2 & (\alpha_3+\alpha_6)p_1 \end{pmatrix} J. \tag{19}$$

Comparing the signs in (19) shows that also here a substitution effect is present between dedicated and flexible capacity. A general study of the impact of demand parameter θ requires an analysis of the sum of products of the parameters J and α, each of which itself is a complex function of the demand distribution. For the remainder of this article, we will focus on the correlation between the two product demands.

Proposition 6 *Let product demands be perfectly positively correlated: $P(\{D_1 = D_2\}) = 1$.*

(a) If $\Delta p = 0$, or if $\Delta p > 0$ with $\frac{c_1}{p_1} \geq \frac{c_2}{p_2}$, then $\bar{c}_3 = \max(c_1, c_2)$ and case (i) of Proposition 2 occurs: it is optimal to invest only in dedicated resources regardless *of the cost of flexible capacity c_3 (and of a* positive *price differential Δp).*

(b) If $\Delta p > 0$ with $\frac{c_1}{p_1} < \frac{c_2}{p_2}$, then $\max(c_1, c_2) < \bar{c}_3 = \frac{\Delta p}{p_1}c_1 + c_2 < c_1 + c_2$ and cases (i) and (iii) of Proposition 2 can occur (case (ii) may never occur), always with $K_1^* > K_2^* + K_3^*$. The optimality equations for K^* in case (iii) simplify to

$$P(K_1^* < D_1) = \frac{c_1 + c_2 - c_3}{p_2}, P(K_1^* + K_3^* < D_1) = \frac{c_3 - c_2}{\Delta p} \text{ and } P(K_2^* + K_3^* < D_1) = \frac{c_2}{p_2}.$$

Proposition 6 emphasizes the significance of the "price" differential Δp in the flexibility decision. Contrary to intuition, it shows that it *is* optimal to invest in flexible capacity if there is a positive price (margin) differential $\Delta p > 0$ and $c_1/p_1 < c_2/p_2$ and if $c_3 < \bar{c}_3$. Figure 2 provides a graphical explanation of how flexibility can yield superior performance, even with perfectly positively correlated demands. It portrays the shape of the feasible regions created by *optimal* investment strategies under three different parameter combinations that differ only in the cost of flexible capacity. Only demand pairs on the 45^o line are possible. To compare the three scenarios, consider the specific demand outcome and the corresponding optimal production decision that are connected by the arrow in the pictures. In scenario 1 at the left, the cost of flexible capacity is high and it is optimal to invest only in dedicated capacity, yielding a rectangular feasible region. If the cost of flexible capacity decreases below $\bar{c}_3$, we arrive at scenario 2 in the center. Here, investing in the flexible resource creates the option to produce more (compared to the investment in dedicated resources only shown by the dotted rectangular feasible region) of the more profitable product 1 at the expense of the less profitable product 2 when demand falls in domain Ω_2. The associated profit gain outweighs the increased investment costs so that it is optimal to invest in the flexible resource even though the product demands move in lockstep. As we move from scenario 1 to scenario 2, the maximal product 1 capacity increases, while the maximal product 2 capacity remains constant (K_2 in scenario 1 equals $K_2 + K_3$ in scenario 2). This is in agreement with Proposition 3: as c_3 decreases, the increase in K_3^* partially substitutes the decrease in dedicated capacity. Finally, if the cost of flexibility decreases so much that K_2 becomes zero in scenario 2 (while $P(\Omega_3) > 0$), scenario 3 at the right applies[1].

Notice that the simple necessary and sufficient condition for a firm to invest in flexible resources is independent of the particular probability distribution of $D_1 = D_2$. It thus is independent of the level of variability or "risk" in demand, as long as some variability is present. Obviously, as said earlier, the presence of uncertainty remains key.

Proposition 7 *Let product demands be perfectly negatively correlated:* $P(\{D_1+D_2 = k > 0\}) = 1$, *and let* $\bar{c}_3 = p_2 + \frac{\Delta p}{p_1} c_1$ *and* $c_3^* = p_1 - \frac{\Delta p}{p_2} c_2$.

(a) *If* $\frac{c_1}{p_1} + \frac{c_2}{p_2} > 1$ *and* $c_3 > \bar{c}_3$, *then the optimal strategy invests in dedicated resources only: case (i) of Proposition 2 occurs with* $K_1^* + K_2^* < k$.

(b) *If* $\frac{c_1}{p_1} + \frac{c_2}{p_2} > 1$ *and* $\bar{c}_3 \geq c_3 > c_3^*$ *(this implies a positive price differential* $\Delta p > 0$*), then the optimal strategy invests in all three resources with* $K_1^* + K_2^* + K_3^* < k$ *and*

$$P(D_1 < K_1^*) = \frac{c_3 - c_1}{p_2}, P(D_1 < K_1^* + K_3^*) = \frac{p_1 - c_3}{\Delta p} \text{ and } P(D_1 < k - K_2^*) = \frac{c_2}{p_2}. \tag{20}$$

(c) *If* $\frac{c_1}{p_1} + \frac{c_2}{p_2} > 1$ *and* $c_3^* \geq c_3$, *or if* $\frac{c_1}{p_1} + \frac{c_2}{p_2} \leq 1$, *then the optimal strategy invests in all three resources with* $K_1^* + K_2^* + K_3^* = k$ *and*

$$P(D_1 < K_1^*) = \frac{c_3 - c_1}{p_2} \quad \text{and} \quad P(K_1^* + K_3^* < D_1) = \frac{c_3 - c_2}{p_1}. \tag{21}$$

[1]While scenarios 1 and 2 always occur, scenario 3 may never occur, namely if K_2 remains strictly positive in scenario 2 when the cost of flexibility is at its minimum ($= \max(c_1, c_2)$).

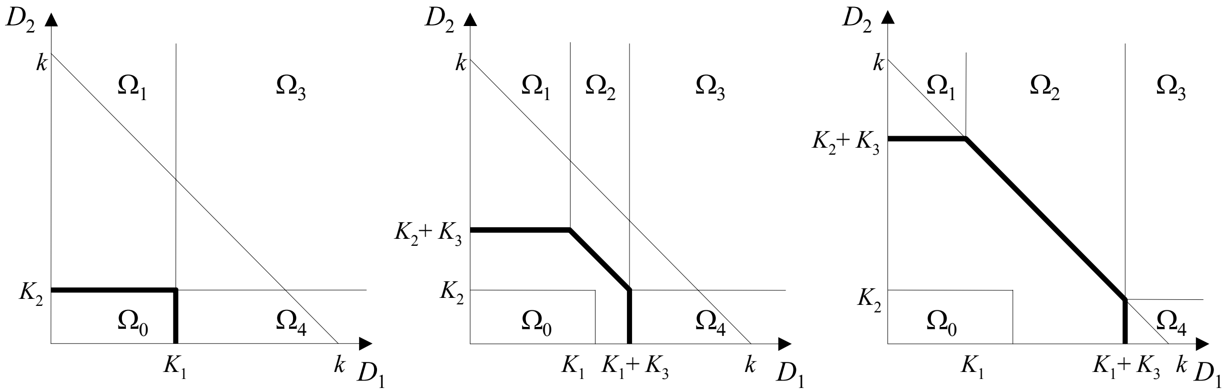

Figure 3: The optimal investment strategies when demands are perfectly negatively correlated and $\frac{c_1}{p_1} + \frac{c_2}{p_2} > 1$ for three scenarios: high cost of flexibility (left), medium cost (center), low cost (right).

Figure 3 portrays the shape of the feasible regions created by the *optimal* investment strategies under the three different possible scenarios of Proposition 7: high cost of flexible capacity (left picture), medium cost (middle), and low cost (right). Now, only demand pairs on the -45^o line are possible. Although more in line with intuition, Proposition 7 shows that if both products are sufficiently profitable (as measured by $\frac{c_1}{p_1} + \frac{c_2}{p_2} \leq 1$), it is optimal to invest in the flexible resource *regardless of its marginal investment cost*, within the obvious limits of our assumptions $(0 < c_1, c_2 < c_3 < c_1 + c_2)$. Also, even with perfect negative correlation, the proposition shows that there are situations where one should not invest in flexible capacity (case (a) requires high flexible cost and low margin products) or where total capacity is less than total a-priori known demand k (case (b) allows medium flexible cost if the price (margin) differential is high). These results, like those of Proposition 4, are strong in that the threshold values $\bar{c}_3$ and c_3^* are independent of the probability distribution of $D_1 = k - D_2$.

In the following section optimal investment plans are derived numerically for a particular family of demand distributions where two parameters, correlation and variability, can be varied continuously and independently. Extending the two boundary cases presented here, it appears that the optimal levels of dedicated capacity increase in a concave manner as correlation increases, while the optimal level of flexible capacity decreases in a convex manner. (The opposite usually, but not always, happens as variability increases.) This is the substitution effect implicit in Proposition 5.

5.1 Modeling Non-Perfect Correlation and Risk in Demand

Finally, we want to consider the situation where D_1 and D_2 are imperfectly correlated. For that purpose a particular family of demand distributions will be considered, and optimal investment plans will be derived numerically. The demand distributions to be considered have two parameters, a correlation parameter $\rho \in (-1, 1)$ and a variability parameter $\alpha > 0$ (not to be confused with $\alpha(K^*)$ of Proposition 3) and the associated probability density functions are denoted by $g(\cdot\,|\alpha, \rho)$.

The density $g(\cdot\,|\alpha,\rho)$ concentrates its probability mass only on the region

$$R(\rho) = \left\{ z \in [0,2]^2 : (z_1-1)^2 - 2\rho(z_1-1)(z_2-1) + (z_2-1)^2 < 1-\rho^2 \right\}. \tag{22}$$

$R(\rho)$ is an ellipse contained in the square $[0,2]\times[0,2]$ with center $(1,1)$. Its axes of symmetry are the diagonal lines passing through $(1,1)$ with slopes -1 and $+1$ and have lengths $\sqrt{1+\rho}$ and $\sqrt{1-\rho}$. The same diagonal lines are axes of symmetry for the density function $g(\cdot\,|\alpha,\rho)$, so its mean is $(1,1)$. The particular form of the distribution is discussed in Appendix B, where it is shown that $g(\cdot\,|\alpha,\rho)$ has covariance matrix

$$\Sigma = \frac{1}{2(\alpha+1)} \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}. \tag{23}$$

The density $g(\cdot\,|\alpha,\rho)$, is symmetric and according to (23) each component (product demands D_1 and D_2) has variance $1/2(\alpha+1)$, so we shall define the "risk" parameter of the demand density g as

$$\text{Risk} = (1+\alpha)^{-1/2}. \tag{24}$$

Note that risk ranges between 0 and 1. The family of densities $g(\cdot\,|\alpha,\rho)$ model demand distributions where one can vary both correlation ρ and risk $(1+\alpha)^{-1/2}$ continuously and independently. Figure 4[2] shows the characteristic shape of the density g when $\alpha > 1$ (left picture) and when $\alpha < 1$ (right picture, which is "cut open" for visual clarity). When $\alpha = 1$, we obtain a uniform distribution over the elliptical domain R described above.

To investigate how the optimal investment depends on the correlation between the demands for the two products, we have solved numerically for the optimal investment, using this family of demand distributions for two cases. Case 1 with contribution margin parameter $p = (1, .9)'$ and marginal investment cost parameter $c = (.3, .28, .5)'$ represents a situation where both products are (almost) equally profitable and flexible capacity is relatively expensive. The optimal investment results for Case 1 are presented in Figure 5 for various levels of the risk parameter. The optimal levels of dedicated capacity increase in a concave manner as correlation increases, while the optimal level of flexible capacity decreases in a convex manner. This is the substitution effect alluded to above, which is implicit in Proposition 5. Dedicated capacity reaches and remains at its maximum when flexible capacity becomes zero. While dedicated capacity seems to be risk-independent for one specific correlation ρ^* (about -0.7 in Case 1), strictly positive flexible capacity levels are risk-dependent for all correlations.

The picture in the upper right corner of Figure 5 shows that the upper threshold value for the cost of flexibility, $\bar{c}_3$, decreases in a concave manner as correlation increases. After the threshold hits the marginal cost of flexibility $c_3 = 0.5$, it is optimal not to invest in flexibility. The pictures in the lower right of Figure 5 show that the optimal value of V, the expected value of operating profits minus resource investment costs, is rather insensitive to correlation. The *value of flexibility*, defined

[2] Figures 4 to 7 are shown on pages 22-25.

as $\frac{V-V_d}{V_d}$, the relative difference between the optimal value V and the optimal value V_d if the firm's investment options are restricted to dedicated resources only, decreases in a convex manner, very similar to the change in the optimal investment level of the flexible resource. Thus, the optimal investment strategy yields a *hedge* against uncertainty in correlation: by optimally investing in the flexible resource, the firm minimizes the sensitivity of its "value" V to changes in correlation.

Case 2 with contribution margin parameter $p = (1, .7)'$ and marginal investment cost parameter $c = (.3, .38, .45)'$ represents a situation where product 1 is more profitable than product 2 and flexible capacity is relatively inexpensive. The optimal investment results for Case 2 are presented in Figure 6. Optimal capacity levels exhibit the same trends as in Case 1. However, the flexible capacity is strictly decreasing but strictly positive everywhere, even with perfectly positively correlated demand in accordance with Proposition 6. Therefore, the substitution effect implies that total dedicated capacity is strictly increasing. Indeed, the optimal dedicated capacity level of resource 2, K_2, is strictly increasing, as opposed to Case 1 where K_2 reaches a maximum and is constant thereafter. Also, only dedicated resource 1 has a "risk-independent" correlation ρ^* of about -0.1 (higher than in Case 1), while the optimal levels of the other dedicated resource always are risk dependent. Because the optimal value function is more sensitive to correlation in Case 2 than in Case 1, the optimal investment provides a less effective hedge when flexible capacity is inexpensive than in Case 1, where flexible capacity is relatively expensive.

Finally, Figure 7 shows how the optimal investment strategy depends on the variability or risk in the demand distribution. Because the results for Cases 1 and 2 above are very similar, only the results for Case 2 are shown. Where the optimal capacity levels for the dedicated resources are increasing in correlation, there is no unique trend in the change in dedicated capacity as risk changes: it seems that the optimal level of dedicated capacity 1 is decreasing in risk for correlations smaller than the correlation threshold ρ^* identified above, while it increases for correlations larger than ρ^*. On the other hand, the optimal level of flexible capacity seems to be strictly increasing in the level of risk. [One must be careful about extrapolating these conclusions to other sets of demand distributions. For example, Fine and Freund [11, pp. 460–461] find that for a particular family of uncorrelated discrete demand distributions, the optimal level of flexible capacity may not be monotone in the level of risk.] The upper threshold $\bar{c}_3$ of the flexible marginal capacity cost is rather insensitive to risk, while the optimal value V decreases almost linearly with risk. The *value of flexibility* increases (in a convex manner) with the level of risk. Comparing the sensitivity of the optimal value V to risk with its sensitivity to correlation, one may conclude that in these two examples the optimal investment strategy can provide a more effective hedge against correlation than against risk.

These results can be related to the general framework of de Groote [6]. He considers three elements:

- A set of *technologies* whose flexibility is compared. A technology in this paper is identified by its capacity vector K.

- A set of *environments* in which these technologies might be operated. If we are interested in risk, we can identify environments by the parameter α.

- A *performance criterion* for the evaluation of different technologies in different environments. We have chosen the firm value $V(K, \alpha)$ as the criterion.

If we continue to follow intuition and use the capacity level of the flexible resource, K_3, as a measure of flexibility of the system, then K_3 induces a total ordering on the set of technologies parameterized by the vector K. Because this ordering differs from the ordering that may be induced by the vector-valued function V, de Groote's results cannot be applied here directly. However, our examples for the family of densities g imply a result similar to de Groote's Property 2 on operations strategy: an increase in the diversity of the environment makes it more desirable to select a more flexible technology—indeed K_3^* increases with risk[3].

Acknowledgments: I am grateful to Mike Harrison, Sunil Chopra and the anonymous referees for their helpful suggestions on preliminary versions.

References

[1] S. Athey and A. Schmutzler. Product and process flexibility in an innovative environment. *Working paper, Stanford Business School*, pages 1–24, Apr. 1995.

[2] S. Benjaafar. Models for performance evaluation of flexibility in manufacturing systems. *Intl. J. of Production Research*, 32(6):1383–1402, 1994.

[3] M. C. Burstein. Finding the economical mix of rigid and flexible automation for manufacturing. In K. E. Stecke and R. Suri, editors, *Proc. Second ORSA/TIMS Conf. Flexible Manufacturing Systems*. Elsevier, New York, 1986.

[4] B. Carlsson. Flexibility and the theory of the firm. *Int. J. of Industrial Organization*, 7:179–203, 1989.

[5] J. P. Caulkins and C. H. Fine. Seasonal inventories and the use of product-flexibile manufacturing technology. 26:351–375, 1990.

[6] X. de Groote. The flexibility of production processes: A general framework. *Management Science*, 40(7):933–945, July 1994.

[7] A. K. Dixit and R. S. Pindyck. *Investment Under Uncertainty*. Princeton University Press, Princeton, NJ, 1994.

[8] B. C. Eaton and N. Schmitt. Flexible manufacturing and market structure. *The American Economic Review*, pages 875–888, Sept. 1994.

[9] J. C. Eberly and J. A. Van Mieghem. Multi-factor dynamic investment under uncertainty. *J. of Economic Theory*, 75(2):345–387, Aug. 1997.

[10] A. V. Fiacco. *Introduction to Sensitivity and Stability Analysis in Nonlinear Programming*, volume 165 of *Mathematics in Science and Engineering*. Academic Press, New York, 1983.

[11] C. H. Fine and R. M. Freund. Optimal investment in product-flexible manufacturing capacity. *Management Science*, 36(4):449–466, April 1990.

[3] de Groote's theory could be applied to produce a measure of system flexibility that depends on all components of the vector K consistent with his framework. But since Fine and Freund [11, pp. 460–461] reported a particular family of uncorrelated discrete demand distributions for which de Groote's Property 2 does not hold, there may not exist a measure of system flexibility for which all of de Groote's results would apply to our model.

[12] Y. P. Gupta and S. Goyal. Flexibility of manufacturing systems: Concepts and measurements. *European J. of Operational Research*, 43:119–135, 1989.

[13] J. M. Harrison and J. A. Van Mieghem. Multi-resource investment strategies: Operational hedging under demand uncertainty. *European J. of Operational Research*, pages 1–20, 1998. forthcoming.

[14] H. He and R. S. Pindyck. Investments in flexible producton capacity. *J. of Economic Dynamics and Control*, 16:575–599, Jul–Oct 1992.

[15] D. Heyman and M. Sobel. *Stochastic Models in Operations Research*, volume 2. McGraw-Hill, 1984.

[16] R. A. Jones and J. M. Ostroy. Flexibility and uncertainty. *Review of Economic Studies*, 51:13–32, 1984.

[17] N. Kulatilaka. Valuing the flexibility in flexible manufacturing systems. *IEEE Trans. Engineering Management*, 35(4):250–257, 1988.

[18] K. Laengle, P. M. Griffin, and S. O. Griffin. A quantification of the economic value of flexible capacity. *Intl. J. of Production Research*, 32(6):1421–1430, 1994.

[19] S. Li and D. Tirupati. Dynamic capacity expansion problem with multiple products: Technology selection and timing of capacity additions. *Operations Research*, 42(5):958–976, 1994.

[20] P. Milgrom and J. Roberts. The economics of modern manufacturing: Technology, strategy, and organization. *The American Economic Review*, 80:511–528, June 1990.

[21] P. Milgrom and C. Shannon. Monotone comparative statics. *Econometrica*, 62(1):157–180, Jan. 1994.

[22] L.-H. Röller and M. M. Tombak. Competition and investment in flexible technologies. *Management Science*, 39(1):107–114, Jan. 1993.

[23] A. K. Sethi and S. P. Sethi. Flexibility in manufacturing: A survey. *Int. J. of Flexible Manufacturing Systems*, 2:289–328, 1990.

[24] G. Stigler. Production and distribution in the short run. *J. of Political Economy*, 47:305–327, June 1939.

[25] X. Vives. Information, flexibility, and competition. *J. of the Japanese and International Economies*, 7:219–237, 1993.

A Proofs

Proposition 1 Proof. The firm seeks to solve $\max_{K \in \mathbb{R}^3_+} V(K) = \max_{K \in \mathbb{R}^3_+} \mathrm{E}\pi(K,D) - c'K$. Because D is finite with probability 1 and is a continuous random vector, Proposition 2 in [13] shows that $\mathrm{E}\pi(\cdot, D)$ (and thus $V(\cdot)$) is differentiable and that the shadow prices $\lambda(K,D)$ exist and satisfy $\nabla \mathrm{E}\pi(\cdot,D) = \mathrm{E}\lambda(\cdot,D)$. Thus, a vector K^* that solves this concave optimization problem also solves the sufficient Kuhn-Tucker conditions (9)–(10) where $\nu \in \mathbb{R}^3_+$ is the Lagrange multiplier of the non-negativity constraint on K. ■

Lemma: Optimal solution K^* is unique because $V(K)$ is strictly concave. **Proof.** Denote the line integrals of the probability density g of P over the boundaries of the domains $\Omega_i(K)$:

$$I_1 = \int_0^{K_1} g(x, K_2 + K_3)\,\mathrm{d}x, \qquad I_4 = \int_0^{K_2} g(K_1 + K_3, y)\,\mathrm{d}y, \tag{25}$$

$$I_2 = \int_{K_1}^{K_1+K_3} g(x, K_1 + K_2 + K_3 - x)\,\mathrm{d}x, \qquad I_5 = \int_{K_2+K_3}^{\infty} g(K_1, y)\,\mathrm{d}y, \tag{26}$$

$$I_3 = \int_{K_1+K_3}^{\infty} g(x, K_2)\,\mathrm{d}x, \qquad I_6 = \int_{K_2}^{\infty} g(K_1 + K_3, y)\,\mathrm{d}y. \tag{27}$$

The negative Hessian of V, $-H = -\nabla^2 V = -\nabla \mathrm{E}\lambda' =$

$$-H = \begin{pmatrix} p_1 I_4 + p_2(I_2 + I_5) + \Delta p I_6 & p_2 I_2 & p_1 I_4 + p_2 I_2 + \Delta p I_6 \\ p_2 I_2 & p_2(I_1 + I_2 + I_3) & p_2(I_1 + I_2) \\ p_1 I_4 + p_2 I_2 + \Delta p I_6 & p_2(I_1 + I_2) & p_1 I_4 + p_2(I_1 + I_2) + \Delta p I_6 \end{pmatrix}, \tag{28}$$

is positive, symmetric, and strictly diagonally dominant and thus positive definite. Thus V is strictly concave and the optimal capacity level K^* is unique. ■

Proposition 2 Proof. The three strategies correspond to an interior point solution and two boundary solutions to the general optimality equations (9)–(10). Thus, it only remains to show that no other of the remaining five boundary sets are optimal.

Boundary $K_1^* = 0, K_2^*, K_3^* > 0$ has $P(\Omega_1) = 0$ and $\mathrm{E}\lambda_1 = c_3$. Because $c_3 > c_1$ by assumption, there is no positive ν_1 that solves optimality equations (9)–(10).

Boundary $K_1^* > 0, K_2^* = K_3^* = 0$ has $P(\Omega_2) = P(\Omega_4) = 0$ and $\mathrm{E}\lambda_2 = p_2$. Because $p_2 > c_2$ by assumption, there is no positive ν_2 that solves optimality equations (9)–(10).

Boundary $K_2^* > 0, K_1^* = K_3^* = 0$ has $P(\Omega_1) = P(\Omega_2) = 0$ and $\mathrm{E}\lambda_1 = p_1$. Because $p_1 > c_1$ by assumption, there is no positive ν_1 that solves optimality equations (9)–(10).

Boundary $K_3^* > 0, K_1^* = K_2^* = 0$ has $P(\Omega_1) = P(\Omega_4) = 0$ and $\mathrm{E}\lambda_1 = c_3$. Because $c_3 > c_1$ by assumption, there is no positive ν_1 that solves optimality equations (9)–(10).

Finally, boundary $K_1^* = K_2^* = K_3^* = 0$ has $P(\Omega_1) = P(\Omega_2) = P(\Omega_4) = 0, P(\Omega_3) = 1$ and $\mathrm{E}\lambda_1 = p_1$. Because $p_1 > c_1$ by assumption, there is no positive ν_1 that solves optimality equations (9)–(10). ■

Proposition 3 Proof. Because $V(K;c) = \mathrm{E}\pi(K,D) - c'K$ is convex in c on the convex set $\mathbb{R}^3_+$ for each $K \in \mathbb{R}^3_+$, convexity of $V^*(c)$ follows from convexity preservation under maximization [15, p. 525, Prop. B-3]. Because $\nabla_c V(K;c) = -K$ is independent of c and the optimal level K^* is unique, Fiacco [10, Theorem 2.3.1., p. 25] (a form of the "envelope theorem") directly yields that V^* is differentiable with gradient $-K^*$. Using the integrals (25)–(25) evaluated at $K = K^*$, implicit differentiation of the optimality equations for all three cases in Proposition 2 yields:

For case (iii):

$$H \nabla_c K^{*\prime} = \nabla_c\,(c, c, c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \nabla_c K^{*\prime} = H^{-1}, \tag{29}$$

where

$$H^{-1} = \begin{pmatrix} -\alpha_1 - \alpha_2 - \alpha_4 & -\alpha_1 & \alpha_1 + \alpha_2 \\ -\alpha_1 & -\alpha_1 - \alpha_3 - \alpha_5 & \alpha_1 + \alpha_3 \\ \alpha_1 + \alpha_2 & \alpha_1 + \alpha_3 & -\alpha_1 - \alpha_2 - \alpha_3 - \alpha_6 \end{pmatrix}, \tag{30}$$

where the non-negative vector $\alpha(K^*) \in \mathbb{R}^6_+$ is defined as:

$$\begin{aligned}
\alpha_1 &= -p_2|H|^{-1}[p_1 I_1 I_4 + \Delta p I_1 I_6], \\
\alpha_2 &= -p_2|H|^{-1}[p_1 I_2 I_4 + p_1 I_3 I_4 + p_2 I_2 I_3 + \Delta p I_3 I_6 + \Delta p I_2 I_6], \\
\alpha_3 &= -p_2|H|^{-1}[p_2 I_1 I_2 + p_2 I_1 I_5 + p_2 I_2 I_5], \\
\alpha_4 &= -p_2|H|^{-1}[p_2 I_1 I_3], \\
\alpha_5 &= -p_2|H|^{-1}[p_1 I_4 I_5 + \Delta p I_5 I_6], \\
\alpha_6 &= -p_2|H|^{-1}[p_2 I_3 I_5].
\end{aligned}$$

(Note that $|H| < 0$ because H is negative definite.)

For case (ii) with $K_2 = 0$ we have that $I_4 = 0$ and $\frac{\partial}{\partial c_2} K^* = 0$ and

$$G \begin{pmatrix} \frac{\partial}{\partial c_1} \\ \frac{\partial}{\partial c_3} \end{pmatrix} (K_1^*, K_3^*) = \begin{pmatrix} \frac{\partial}{\partial c_1} \\ \frac{\partial}{\partial c_3} \end{pmatrix} \begin{pmatrix} c_1 & c_1 \\ c_3 & c_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{\partial}{\partial c_1} \\ \frac{\partial}{\partial c_3} \end{pmatrix} (K_1^*, K_3^*) = G^{-1}, \tag{31}$$

where

$$\begin{aligned}
G &= -\begin{pmatrix} p_2(I_2 + I_5) + \Delta p I_6 & p_2 I_2 + \Delta p I_6 \\ p_2 I_2 + \Delta p I_6 & p_2(I_1 + I_2) + \Delta p I_6 \end{pmatrix} && (32) \\
G^{-1} &= -|G|^{-1} \begin{pmatrix} p_2 I_1 + p_2 I_2 + \Delta p I_6 & -p_2 I_2 - \Delta p I_6 \\ -p_2 I_2 - \Delta p I_6 & p_2 I_2 + p_2 I_5 + \Delta p I_6 \end{pmatrix} && (33) \\
&= \begin{pmatrix} -\alpha_2 - \alpha_4 & \alpha_2 \\ \alpha_2 & -\alpha_2 - \alpha_6 \end{pmatrix} && (34)
\end{aligned}$$

where $\alpha_2, \alpha_4, \alpha_6 \geq 0$ because G is negative definite (and thus $|G| > 0$) and $\alpha_1 = \alpha_3 = \alpha_5 = 0$.

Finally, for case (i) with $K_3 = 0$, we have $I_2 = 0$ and $I_5 = I_6$ and $\frac{\partial}{\partial c_3} K^* = 0$ and

$$F \begin{pmatrix} \frac{\partial}{\partial c_1} \\ \frac{\partial}{\partial c_2} \end{pmatrix} (K_1^*, K_2^*) = \begin{pmatrix} \frac{\partial}{\partial c_1} \\ \frac{\partial}{\partial c_2} \end{pmatrix} \begin{pmatrix} c_1 & c_1 \\ c_2 & c_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{\partial}{\partial c_1} \\ \frac{\partial}{\partial c_2} \end{pmatrix} (K_1^*, K_2^*) = F^{-1}, \tag{35}$$

where

$$\begin{aligned}
F &= -\begin{pmatrix} p_1 I_4 + p_1 I_6 & 0 \\ 0 & p_2(I_1 + I_3) \end{pmatrix} && (36) \\
F^{-1} &= -|F|^{-1} \begin{pmatrix} p_2(I_1 + I_3) & 0 \\ 0 & p_1 I_4 + p_1 I_6 \end{pmatrix} = \begin{pmatrix} -\alpha_4 & 0 \\ 0 & -\alpha_6 \end{pmatrix} && (37)
\end{aligned}$$

where $\alpha_4, \alpha_6 \geq 0$ because F is negative definite (and thus $|F| > 0$) and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_5 = 0$. Note that ∇ is a subgradient on the two sets of threshold costs $\{c \in \mathbb{R}^3_+ : c_3 = \bar{c}_3 \text{ or } c_3 = \underline{c}_3\}$ of Lebesgue measure zero. Clearly α is a non-negative function of K^* for all three cases. ■

Proposition 4 Proof. Because $V(K;p) = (p_1, p_2, p_1, p_2)' \mathrm{E}x(K, D) - c'K$ is linear (and thus convex; note that $x(K, D)$ are independent of $p_1 \geq p_2$) in p on the convex set $\{p : p_1 \geq p_2 \geq 0\}$ for each $K \in \mathbb{R}^3_+$, convexity of $V^*(p)$ follows from convexity preservation under maximization [15, p. 525, Prop. B-3]. Because $\nabla_p V(K; c) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \mathrm{E}x(K, D)$ is independent of c and the optimal level K^* is unique, Fiacco [10, Theorem 2.3.1., p. 25] (a form of the "envelope theorem") directly yields that V^* is differentiable with gradient $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \mathrm{E}x^*(K, D)$. Using the integrals (25)–(25) evaluated at $K = K^*$, implicit differentiation of the optimality equations for all three cases in Proposition 2 yields: abbreviating $P(\Omega_i(K^*))$ by P_i and the corresponding vector by $\vec{P}$, for case (iii):

$$\frac{\partial}{\partial p_1} K^* = -H^{-1} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \vec{P} = \begin{pmatrix} 0 & 0 & \alpha_4 & \alpha_4 \\ 0 & 0 & -\alpha_3 & -\alpha_3 \\ 0 & 0 & \alpha_3+\alpha_6 & \alpha_3+\alpha_6 \end{pmatrix} \vec{P} \tag{38}$$

$$\frac{\partial}{\partial p_2} K^* = -H^{-1} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \vec{P} = \begin{pmatrix} -\alpha_2 & \alpha_1+\alpha_4 & \alpha_1 & 0 \\ \alpha_5 & \alpha_1+\alpha_5 & \alpha_1+\alpha_3+\alpha_5 & 0 \\ \alpha_2+\alpha_6 & -\alpha_1+\alpha_6 & -\alpha_1-\alpha_3 & 0 \end{pmatrix} \vec{P} \tag{39}$$

and thus:

$$\nabla_p K^{*'} = \begin{pmatrix} \beta_2 & -\beta_1 & \beta_1+\beta_3 \\ -\beta_5+\beta_6+\beta_7 & \beta_4 & \beta_5-\beta_6+\beta_8-\beta_9 \end{pmatrix} \tag{40}$$

and

$$\frac{\partial}{\partial p_2}(K_1^*+K_3^*) = \begin{pmatrix} \alpha_6 & \alpha_4+\alpha_6 & -\alpha_3 & 0 \end{pmatrix} \vec{P} = \beta_7+\beta_8-\beta_9, \tag{41}$$

$$\frac{\partial}{\partial p_2}(K_1^*+K_2^*+K_3^*) = \begin{pmatrix} \alpha_5+\alpha_6 & \alpha_1+\alpha_4+\alpha_5+\alpha_6 & \alpha_1+\alpha_5 & 0 \end{pmatrix} \vec{P} \tag{42}$$

$$= \beta_4+\beta_7+\beta_8-\beta_9 > 0 \tag{43}$$

so that $\beta > 0$ with $\beta_4+\beta_7+\beta_8 > \beta_9$.

For case (ii) with $K_2 = 0$ we have that $P_4 = 0$ and $I_4 = 0$:

$$\frac{\partial}{\partial p_1}\begin{pmatrix} K_1^* \\ K_3^* \end{pmatrix} = -G^{-1} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \vec{P}_{1:3} = \begin{pmatrix} 0 & 0 & \alpha_4 \\ 0 & 0 & \alpha_6 \end{pmatrix} \vec{P}_{1:3}, \tag{44}$$

$$\frac{\partial}{\partial p_2}\begin{pmatrix} K_1^* \\ K_3^* \end{pmatrix} = -G^{-1} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \vec{P}_{1:3} = \begin{pmatrix} -\alpha_2 & \alpha_4 & 0 \\ \alpha_2+\alpha_6 & \alpha_6 & 0 \end{pmatrix} \vec{P}_{1:3}, \tag{45}$$

thus $\beta_1 = \beta_4 = \beta_6 = \beta_9 = 0$.

Finally, for case (i) with $K_3 = 0$, we have $P_2 = 0$, $I_2 = 0$ and $I_5 = I_6$:

$$\frac{\partial}{\partial p_1}\begin{pmatrix} K_1^* \\ K_2^* \end{pmatrix} = -F^{-1} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \vec{P}_{1,3:4} = \begin{pmatrix} 0 & \alpha_4 & \alpha_4 \\ 0 & 0 & 0 \end{pmatrix} \vec{P}_{1,3:4} = \begin{pmatrix} \frac{c_1}{p_1^2(I_4+I_6)} \\ 0 \end{pmatrix}, \tag{46}$$

$$\frac{\partial}{\partial p_2}\begin{pmatrix} K_1^* \\ K_2^* \end{pmatrix} = -F^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \vec{P}_{1,3:4} = \begin{pmatrix} 0 & 0 & 0 \\ \alpha_6 & \alpha_6 & 0 \end{pmatrix} \vec{P}_{1,3:4} = \begin{pmatrix} 0 \\ \frac{c_2}{p_2^2(I_1+I_3)} \end{pmatrix}, \tag{47}$$

so that $\beta_2 = \frac{c_1}{p_1^2(I_4+I_6)}, \beta_1 = \beta_3 = 0, \beta_4 = \frac{c_2}{p_2^2(I_1+I_3)}$ and $\beta_{5:10} = 0$. ■

Proposition 5 Proof. Implicit differentiation of the optimality equation (9) with $\nu = 0$ yields $H\frac{\partial}{\partial\theta}K^* + \sum_{i=1}^4 \lambda_i J_{\Omega_i(K^*)} = 0$ or

$$\frac{\partial}{\partial\theta} K^* = -H^{-1} \begin{pmatrix} 0 & p_2 & p_1 & p_1 \\ p_2 & p_2 & p_2 & 0 \\ p_2 & p_2 & p_1 & p_1 \end{pmatrix} \vec{J} \tag{48}$$

$$= \begin{pmatrix} -\alpha_2 p_2 & (\alpha_1+\alpha_4)p_2 & \alpha_4 p_1 + \alpha_1 p_2 & \alpha_4 p_1 \\ \alpha_5 p_2 & (\alpha_1+\alpha_5)p_2 & (\alpha_1+\alpha_5)p_2 - \alpha_3 \Delta p & -\alpha_3 p_1 \\ (\alpha_2+\alpha_6)p_2 & (\alpha_6-\alpha_1)p_2 & \alpha_6 p_1 + \alpha_3 \Delta p - \alpha_1 p_2 & (\alpha_3+\alpha_6)p_1 \end{pmatrix} \vec{J} \tag{49}$$

■

Proposition 6 Proof. First note that we obviously could have taken any linear function with positive slope: $D_2 = \xi_1 D_1 + \xi_2$ with $\xi_1 > 0$ to define perfectly positively correlated random demands.

Solving (11)–(12) if $\frac{c_1}{p_1} \geq \frac{c_2}{p_2}$ yields $\bar{c}_3 = c_1$, such that only strategy (i) exists with $K_1^* \leq K_2^*$. If $\frac{c_1}{p_1} < \frac{c_2}{p_2}$, the $\bar{c}_3 = (1 - \frac{p_2}{p_1})c_1 + c_2 > c_1$ such that both strategy (i) and (ii) can exist (depending on c_3) with $K_1^* > K_2^*$, in which case the optimality equations for strategy (iii) simplify to those given in Proposition 6. ■

Proposition 7 Proof. (Same note as in previous proof: we could have taken $D_2 = \xi_3 D_1 + \xi_4$ with $\xi_3 < 0$ to define perfectly negatively correlated demands.)

If *Case (i) of Proposition 2* holds: Solving (11)–(12) with $P(\Omega_3) = 0$, yields $\bar{c}_3 = c_1 + c_2$ so that case (i) is never optimal ($c_3 < c_1 + c_2$ by Assumption (A1)). Thus, $P(\Omega_3) > 0$ and solving (11)–(12) with $K_1^* + K_2^* \leq k$ yields $\bar{c}_3 = p_2 + (1 - \frac{p_2}{p_1})c_1$. And $\bar{c}_3 \leq c_1 + c_2$ if and only if $\frac{c_1}{p_1} + \frac{c_2}{p_2} \leq 1$, in which case Case (i) is never optimal. Thus, we need $c_3 > \bar{c}_3 = p_2 + (1 - \frac{p_2}{p_1})c_1$ and $\frac{c_1}{p_1} + \frac{c_2}{p_2} > 1$ for Case (i) to be optimal.

If *Case (iii) of Proposition 2* holds: If $K_1^* + K_2^* + K_3^* > k$, $P(\Omega_2) = P(\Omega_3) = 0$ and the optimality equations would imply that $c_3 = c_1 + c_2$, in contradiction to assumption (A1). Thus, $K_1^* + K_2^* + K_3^* \leq k$. If $K_1^* + K_2^* + K_3^* < k$, solving the optimality equations (with the additional constraint that $P(\Omega_1) + P(\Omega_2) + P(\Omega_3) + P(\Omega_4) = 1$) yields the simplified optimality equations (20). Clearly, $K_1^* + K_2^* + K_3^* < k$ if $P(\Omega_3) = \frac{c_2}{p_2} - \frac{p_1 - c_3}{p_1 - p_2} > 0$ or, equivalently, $c_3 > c_3^* = p_1 - (p_1 - p_2)\frac{c_2}{p_2}$. Finally, if $c_3 \leq c_3^*$, $K_1^* + K_2^* + K_3^* = k$ (notice that $c_3^* \leq \bar{c}_3$ if and only if $\frac{c_1}{p_1} + \frac{c_2}{p_2} > 1$, so that Case (iii) with $K_1^* + K_2^* + K_3^* = k$ is optimal if $\frac{c_1}{p_1} + \frac{c_2}{p_2} \leq 1$) and solving the optimization problem directly (or setting a subgradient of $\nabla \mathrm{E}\pi(K_1^* + K_2^* + K_3^* = k, D) = (x + p_1 P(\Omega_4) - c_1, x + p_2 P(\Omega_1) - c_2, x + p_2 P(\Omega_1) + p_1 P(\Omega_4) - c_3)'$, where $0 \leq x \leq p_2 P(\Omega_2) \leq p_2$), equal to zero) yields the simplified optimality equations (21). ■

B A Two-Parameter Family of Distributions

In this section, we introduce a family of probability distributions that are parameterized by two parameters: a variability or risk parameter $\alpha > 0$ and the correlation coefficient $-1 < \rho < 1$. The interesting fact is that these two parameters are orthogonal in the sense that correlation ρ can be varied while variability (represented by α) remains unchanged and vice-versa. Consider the family of probability distributions with density $h(\cdot\,|\alpha,\rho) : \mathbb{R}^2 \to \mathbb{R}_+$ where, for $z \in \mathbb{R}^2$,

$$h(z\,|\alpha,\rho) = \begin{cases} \frac{\alpha}{\pi}(1-\rho^2)^{\frac{1}{2}-\alpha}\left(1-\rho^2-z_1^2-z_2^2+2\rho z_1 z_2\right)^{\alpha-1} & \text{if } z_1^2-2\rho z_1 z_2+z_2^2 < 1-\rho^2 \\ 0 & \text{elsewhere.} \end{cases} \tag{50}$$

We will show that $h(\cdot\,|\alpha,\rho)$ has mean vector $(0,0)'$ and correlation matrix

$$\Sigma = \frac{1}{2(\alpha+1)}\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}. \tag{51}$$

Before proving this, let us first give some intuition by discussing how this density was constructed. We used three facts. First, a uniform density over an ellipse $z_1^2 - 2\rho z_1 z_2 + z_2^2 = \xi$ has a correlation ρ. Second, the symmetric beta distribution with parameter(s) α has a coefficient of variation (risk) equal to $(1+2\alpha)^{-1/2}$. Third, the distribution obtained by a weighted superposition of a uniform distribution on the family of ellipses with parameter ξ with weight $f(\xi)d\xi$ has the same correlation ρ. Finally, the family of distributions $h(\cdot\,|\alpha,\rho)$ was obtained by superposition of the ellipses with weights proportional to the symmetric beta density.

Translating the family $h(\cdot\,|\alpha,\rho)$, so as to center on the point $(1,1)$, yields the family $g(\cdot\,|\alpha,\rho)$ of demand distributions used above. Implicitly we have made two assumptions: First, demand is finite with probability one and thus, without loss of generality, we can scale the demand axes such that the maximum demand minus minimum demand equals 2. Secondly, to limit the number of parameters in the model, we have assumed that the (scaled) demand distribution is symmetric in D_1 and D_2.

We will now calculate some functionals of the density h. First, note that for $-1 \le u_1 \le u_2 \le 1$,

$$\int_{u_1}^{u_2} du\,(1-u^2)^{\alpha-1} = 2^{2\alpha-1}\left[B_{\frac{1-u_1}{2}}(\alpha,\alpha) - B_{\frac{1-u_2}{2}}(\alpha,\alpha)\right], \tag{52}$$

where $B_x(\alpha,\alpha)$ denotes the incomplete beta function. The marginal density in the first coordinate is independent of ρ:

$$\begin{aligned} h_1(x_1|\alpha,\rho) &= \int_{-1}^{+1} dx_2\, h(x|\alpha,\rho) &(53)\\ &= \frac{\alpha}{\pi}(1-x_1^2)^{\alpha-1/2}\int_{-1}^{+1} dz\,(1-z^2)^{\alpha-1} &(54)\\ &= \frac{\alpha}{\pi}(1-x_1^2)^{\alpha-1/2}2^{2\alpha-1}B(\alpha,\alpha), &(55)\end{aligned}$$

where the substitution $z = \frac{\rho x_1 - x_2}{\sqrt{(1-\rho^2)(1-x_1^2)}}$ was used. In the same way, we have that

$$\begin{aligned} &\int_{-1}^{+1} dx_2\, x_2 h(x\,|\alpha,\rho) &(56)\\ =\ & \frac{\alpha}{\pi}(1-x_1^2)^{\alpha-1/2}\int_{-1}^{+1} dz\,\left(\rho x_1 - z\sqrt{(1-\rho^2)(1-x_1^2)}\right)(1-z^2)^{\alpha-1} &(57)\\ =\ & \rho x_1 h_1(x_1\,|\alpha,\rho). &(58)\end{aligned}$$

The marginal distribution is

$$\begin{aligned} H_1(x_1\,|\alpha,\rho) &= \int_{-1}^{x_1} du\, h_1(u\,|\alpha,\rho) &(59)\\ &= \frac{\alpha}{\pi}2^{2\alpha-1}B(\alpha,\alpha)2^{2\alpha}B_{\frac{1+x_1}{2}}(\alpha+1/2,\alpha+1/2) &(60)\\ &= \frac{B_{\frac{1+x_1}{2}}(\alpha+1/2,\alpha+1/2)}{B(\alpha+1/2,\alpha+1/2)}, &(61)\end{aligned}$$

where the fact $\alpha 2^{4\alpha-1}B(\alpha,\alpha)B(\alpha+1/2,\alpha+1/2)=\pi$ was used. Since h is symmetric, its first moments are zero. The second moments are

$$\begin{aligned}
\mathrm{E}\left[x_1^2|\alpha,\rho\right] &= \int_{-1}^{+1} du\, u^2 h_1(u\,|\alpha,\rho) && (62)\\
&= 2^{2\alpha}\frac{\alpha}{\pi}B(\alpha,\alpha)\int_0^1 dt\, t^{1/2}(1-t)^{\alpha-1/2} && (63)\\
&= 2^{2\alpha}\frac{\alpha}{\pi}B(\alpha,\alpha)B(3/2,\alpha+1/2) && (64)\\
&= \frac{1}{2(\alpha+1)}, && (65)
\end{aligned}$$

where the substitution $t=u^2$ was used. Using (56) we have that

$$\begin{aligned}
\mathrm{E}\left[x_1x_2|\alpha,\rho\right] &= \int_{-1}^{+1} dx_1 x_1 \int_{-1}^{+1} dx_2\, x_2 h(x\,|\alpha,\rho) && (66)\\
&= \int_{-1}^{+1} dx_1 x_1 \rho x_1 h_1(x_1\,|\alpha,\rho) && (67)\\
&= \rho\mathrm{E}\left[x_1^2|\alpha,\rho\right]. && (68)
\end{aligned}$$

Finally, it is interesting to see what the limiting distribution is for the extreme points of perfect correlation ($\rho\to-1,+1$) and maximum risk ($\alpha\to 0$). The marginal density $h_1(\cdot\,|\alpha,\rho)$ is independent of ρ and therefore is also the limiting distribution for $\rho\to-1,+1$, which means that with perfect correlation, the demand D_1 is distributed as a beta random variable with symmetric parameters $\alpha+1/2$. This, remains valid for the limit point $\alpha\to 0$. For non-perfect correlation, the probability mass concentrates on the perimeter of the ellipse when $\alpha\to 0$. The limiting distribution is thus essentially one-dimensional and in polar coordinates can be found as follows. Let $r^o(\theta)$ be the radius of the point $(x,y)=(r^o,\theta)$ on the ellipse. We have that $r^{o2}=(1-\rho^2)/(1-\rho\sin 2\theta)$. Denote the limiting distribution for $\alpha\to 0$ and $\rho\not=\pm 1$ by $h^0(\theta\,|\rho)$. We have that for any fixed $\Delta r>0$ with $\Delta r<r^o$:

$$\begin{aligned}
h^0(\theta\,|\rho) &= \lim_{\alpha\to 0}\int_{r^o-\Delta r}^{r^o} h(r,\theta\,|\alpha,\rho)\mathrm{d}r, && (69)\\
&= \lim_{\alpha\to 0}\frac{\alpha}{\pi\sqrt{1-\rho^2}}\int_{r^o-\Delta r}^{r^o}\left(1-(r/r^o)^{\alpha-1}\right) r\mathrm{d}r, && (70)\\
&= \lim_{\alpha\to 0}\frac{r^{o2}}{2\pi\sqrt{1-\rho^2}}\left(2(\Delta r/r^o)+o(\Delta r/r^o)\right)^{\alpha}, && (71)\\
&= \frac{\sqrt{1-\rho^2}}{2\pi(1-\rho\sin 2\theta)}. && (72)
\end{aligned}$$

The cumulative $H^0(\theta\,|\rho)=\int_{-\pi}^{\theta}h^0(\theta\,|\rho)\mathrm{d}\theta$ also can be represented in closed form:

$$H^0(\theta\,|\rho)=\frac{1}{2\pi}\left[\arctan\frac{\tan\theta-\rho}{\sqrt{1-\rho^2}}-\arctan\frac{-\rho}{\sqrt{1-\rho^2}}\right]. \qquad (73)$$

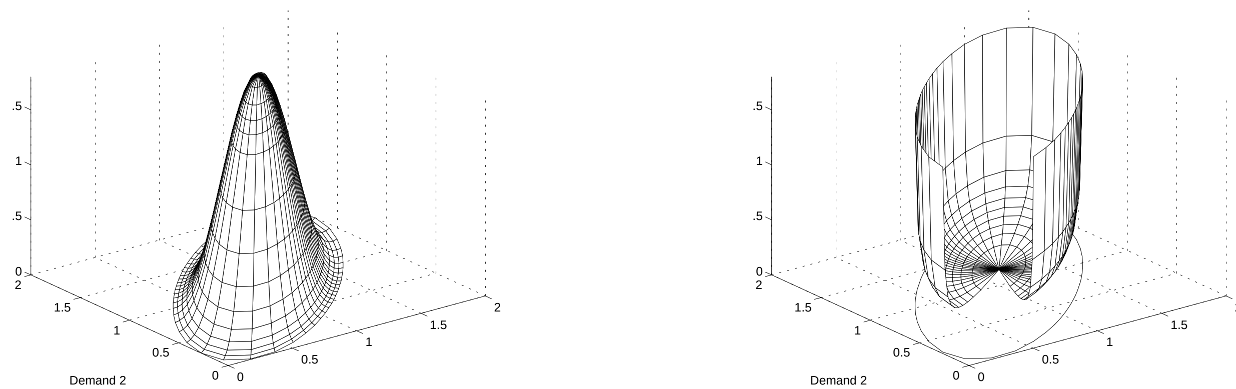

Figure 4: Two members of the family of demand densities $g(\cdot\,|\alpha, \rho)$ both with correlation $\rho = 0.75$ but different risk: low risk (left) and high risk (right).

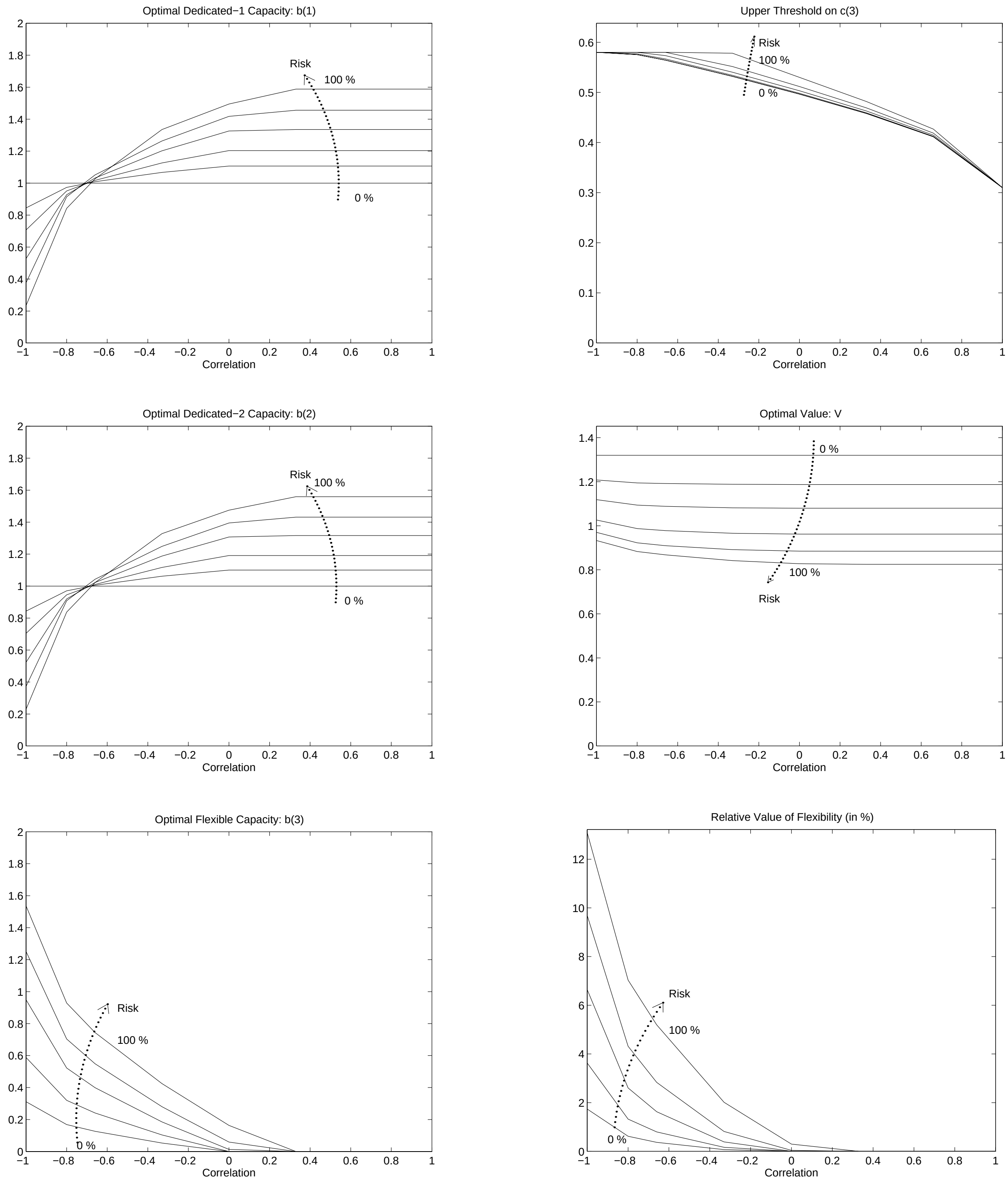

Figure 5: Optimal Investment as a function of correlation–Case 1: flexible capacity is expensive and parameterized by risk for $p = (1, .9)'$ and $c = (.3, .28, .5)'$.

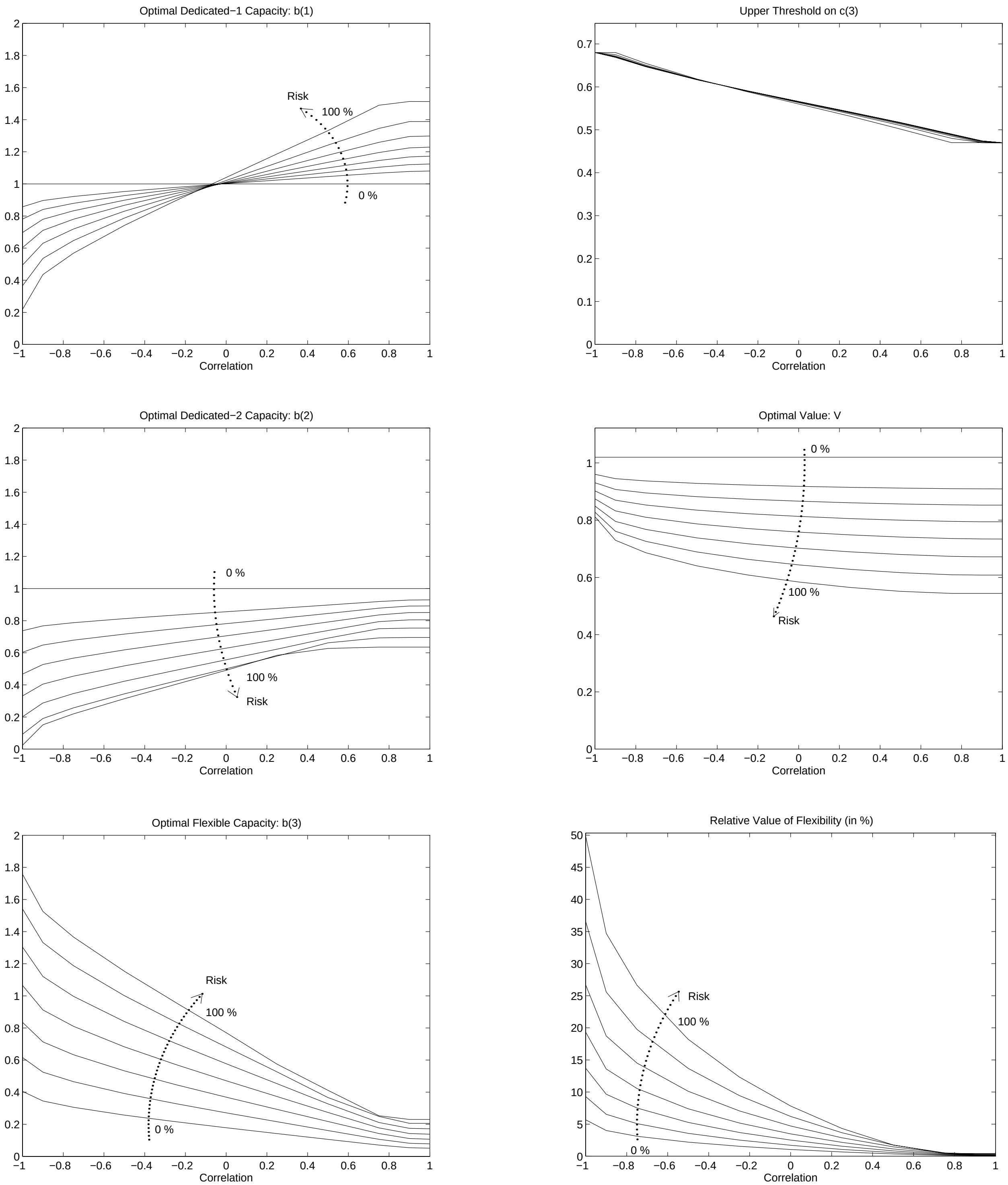

Figure 6: Optimal Investment as a function of correlation–Case 2: flexible capacity is inexpensive. and parameterized by risk for $p = (1, .7)'$ and $c = (.3, .38, .45)'$.

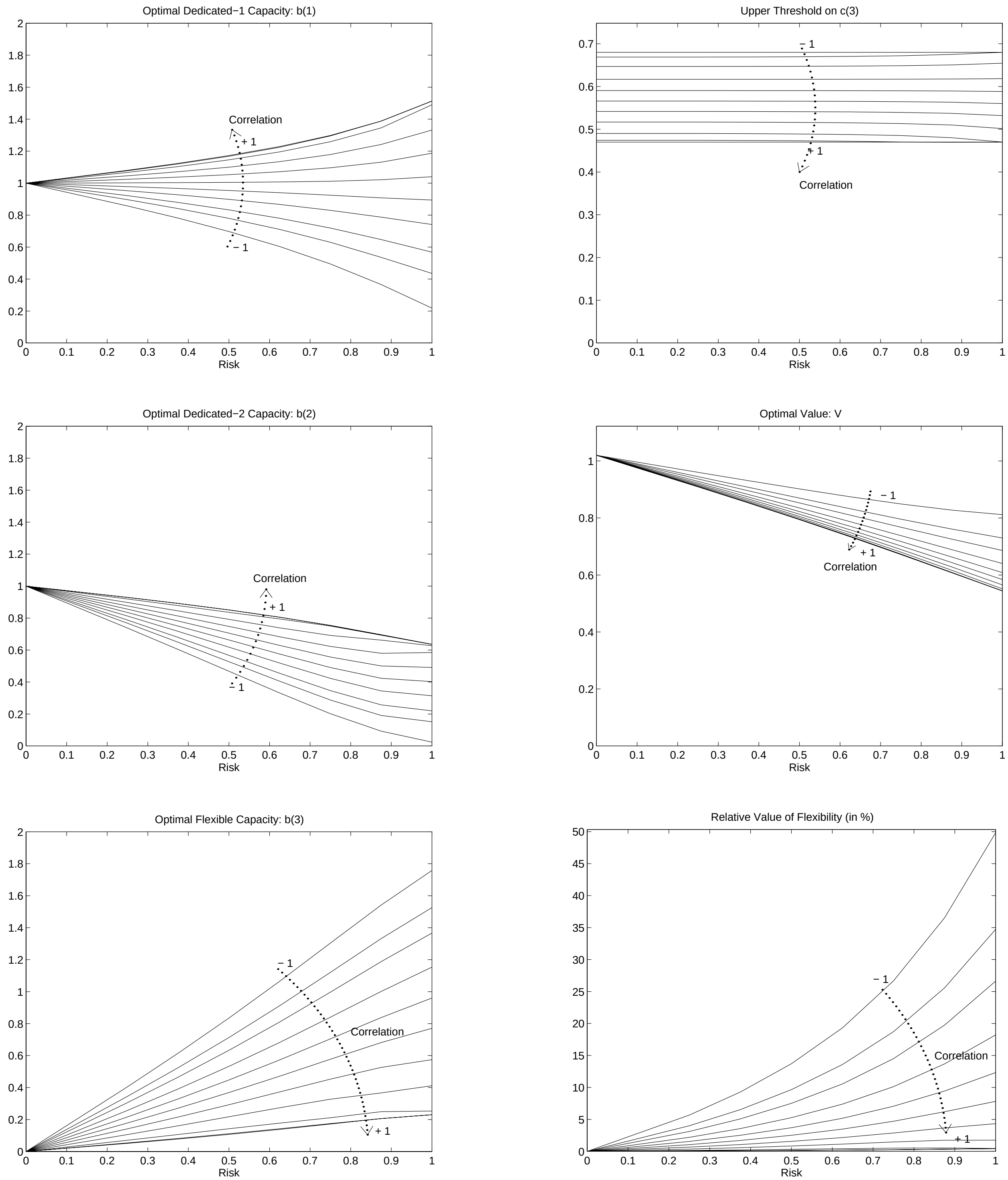

Figure 7: Optimal Investment as a function of risk and parameterized by correlation for $p = (1, .7)'$ and $c = (.3, .38, .45)'$.