Dynamics of Rate of Return Regulation

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Abstract: It is commonly accepted in the industrial organization literature that rate of return regulation leads to inefficient outcomes. Since prices are calculated so as to cover the regulated firm's average cost, which includes the periodic fixed costs associated with plant property and equipment, the conventional argument is that prices must exceed the marginal cost of production. This paper examines the dynamic properties of the rate of return regulation process when the regulated firm periodically undertakes new capacity investments. Our analysis identifies prices which can potentially emerge as equilibria of the regulation process. It is shown that the underlying accounting (depreciation) rules determine whether these equilibrium prices will be above, equal to, or below the long run marginal cost. We provide conditions under which the rate of return regulation process is dynamically stable so that prices indeed converge to their equilibrium values.

1 Introduction

Regulatory agencies commonly rely on Rate of Return Regulation (RoR) to determine product prices in a range of industries such as telecommunication services, electricity, gas and water.[1] In its traditional form, RoR regulation sets prices so that the regulated firm's Return on Assets attains a target value in each period. Equivalently, prices are set so that revenues cover not only current operating expenses but also an interest charge on the book value of the firm's operating assets.[2]

A frequently voiced conclusion in the industrial organization literature is that RoR regulation results in inefficient outcomes. In particular, product prices are predicted to be too high relative to the levels that a welfare maximizing planner would choose. One argument pointing to inefficiency is that prices must cover the firm's average cost, including fixed costs which frequently account for a large fraction of total costs. As a consequence, RoR regulation is allegedly not in a position to implement the prices and quantities that correspond to marginal cost pricing; see, for example, Baumol (1971), Nicholson (2005) and Schotter (2008).[3]

Recent studies by Biglaiser and Riordan (2000) and Rogerson (2008b) have examined RoR regulation in dynamic settings in which the firm makes a sequence of overlapping capacity investments. The significance of multiperiod investment settings is it becomes possible to determine a product's long-run marginal cost, which includes the cost of providing production capacity for a single period.[4] To the extent that the regulated firm's fixed costs consist mainly of capacity related depreciation charges, the efficiency of RoR regulation then hinges on how well historical cost reflects long-run marginal cost. Rogerson (2008b) provides a central benchmark result by identifying a particular depreciation schedule with the property that capital costs i.e., the sum of depreciation- and interest charges on the remaining book value of the assets, are indeed equal to the long run marginal cost in each period. This

[1]When governments agencies procure unique items, such as weapon systems, prices are frequently calculated on a cost-plus basis, where the mark-up over cost is again calculated so as to give the contractor a fair return on its invested capital; see, for instance, Laffont and Tirole (1993).

[2]In certain industries, such as telecommunications, traditional RoR regulation has increasingly been replaced with so-called Price Cap regulation. Accordingly, prices are set periodically so as to meet a current RoR constraint. In subsequent years, the regulated firm is then free to charge lower price until such time as a new price cap is calculated according to an updated RoR calculation (Laffont and Tirole, 2000).

[3]Parts of the regulation literature has also investigated inefficiencies related to distortions in the regulated firm's input mix; e.g., Averch and Johnson (1962).

[4]Arrow (1964) shows that despite the inherent joint cost of acquiring an assets with a useful life of T periods, in a setting with overlapping investment decisions it becomes possible to calculate the marginal cost of providing one unit of capacity of capacity for one period of time.

depreciation schedule must not only reflect the anticipated decay of an asset's productive capacity but also anticipated changes in future asset acquisition costs. Falling acquisition costs due to technological progress are an essential feature of some regulated industries such as telecommunications. Rogerson refers to the corresponding intertemporal cost allocation as the *Relative Replacement Cost* rule.

Our analysis in this paper departs from the observation that straight-line depreciation is commonly used by regulatory agencies (Biglaiser and Riordan, 2000). This approach is, of course, consistent with the financial reporting practices that most unregulated firms use in connection with plant, property and equipment. Our principal objective then is to identify the dynamic and asymptotic properties of the RoR regulation process when depreciation is not necessarily calibrated to the underlying fundamentals as required in Rogerson's (2008b) benchmark result.

For a given set of accounting rules, we find that there is at most one equilibrium candidate for the product price that could emerge asymptotically as the long run limit of the RoR regulation process. Put differently, product prices could not possibly converge to any other price over time. The unique equilibrium price candidate is lower than the firm's long-run marginal cost whenever the underlying depreciation schedule for capacity assets is more accelerated than the R.R.C. rule identified in Rogerson (2008b).[5] In particular, straight line depreciation amounts to a more accelerated schedule, i.e., assets are written off more quickly early on, if the underlying asset productivity conforms to the *one-hoss shay* pattern, where productive capacity is undiminished for the entire useful life of the asset.

An equilibrium price below the long-run marginal cost may seem at odds with the requirement that the regulated firm breaks even under RoR regulation. It must be kept in mind, though, that the break-even constraint is applied in term's of the firm's residual income. It is well known that for a firm with no initial assets in place the present values of all operating cash flows is equal to the present value of all residual incomes (Hotelling, 1925, Preinreich, 1938), regardless of the accounting rules.[6] Given an accelerated (front-loaded) depreciation schedule, the resulting product prices will exceed long-run marginal cost early on before prices fall below that value and possibly settle at the long run equilibrium price. Thus, the firm breaks even in each period in accounting terms and does so in economic terms over the entire (infinite) horizon. However, the temporal distortions in product prices either above or below the long-run marginal cost do not cancel each other in terms of welfare losses.

[5]Conversely, the candidate equilibrium price is above long-run marginal cost whenever the underlying depreciation schedule is decelerated relative to the R.R.C.

[6]This identity is sometimes referred to as the *conservation property* of residual income. In a regulation framework, this identity was also observed by Schmalensee (1989).

Our results show that the deviation between long run marginal cost and the unique price that can potentially emerge as an equilibrium price of the RoR process is not only determined by the choice of depreciation schedule but also the growth rate in the product market. Specifically, we find that an expanding external product market tends to close the gap between long run marginal cost and the potential equilibrium price.[7] To quantify these distortions, we find that for a stationary product market (no growth) the long run marginal cost of capacity exceeds the historical cost of capacity by about 20% when capacity assets have an undiminished useful life (one-hoss shay) of 25 years. The induced deviations may be much larger before the resulting prices approach their equilibrium values.

One might expect the RoR process to be stable in the sense that for a "reasonable" class of starting points prices do indeed converge to the unique equilibrium candidate. While we have not observed any examples to the contrary, such global stability turns out to be difficult to establish. For a fairly broad class of environments we establish a local stability result which shows that starting from an initial configuration of past investments for which prices are "close" to the candidate equilibrium price, the resulting trajectory of RoR regulation prices will indeed converge.

Global stability is shown to obtain in our model for a more restrictive class of environments in which assets have an unbounded useful life and both capacity decay and accounting depreciation follow a geometric pattern. For this class of environments, one also obtains monotonic convergence: for a firm that starts out with no assets initially, prices are above long-run marginal cost in early periods, provided the depreciation schedule is accelerated relative to the R.R.C. rule. Prices then decline monotonically and approach the unique equilibrium price below marginal cost.

In terms of prior work, our analysis is most closely related to Biglaiser and Riordan (2000), Friedl (2007) and Rogerson (2008b). At first glimpse some of our findings may contradict those in Friedl (2007) who concludes that RoR regulation will result in prices that are inefficiently high even though the regulator applies straight line depreciation and the one-hoss shay assumption is being imposed. The key difference is that Friedl (2007) allows for only a single capacity investment decision, the acquisition cost of which is then amortized over a finite horizon. Biglaiser and Riordan (2000) confine their analysis to a setting in which capacity decays geometrically and book values also follow a geometric sequence. Contrary to our findings and those on Rogerson (2008b), they conclude that even if the depreciation schedule is set so as to coincide with the R.R.C. rule, product prices will in the limit still exceed the long run marginal cost. This seemingly contradictory conclusion hinges on an

[7]This finding applies only to accelerated depreciation schedules. Growth has the opposite effect for decelerated schedules.

additional feature of their model which relates to the retirement of obsolete assets.[8]

The plan of the paper is as follows. We present the RoR regulation model, including the evolution of product prices and investments in Section 2. The analysis of prices that can potentially emerge as equilibria of the RoR process is contained in Section 3. There we also demonstrate how the choice of accounting rules and growth in the product market create a joint bias in the equilibrium price relative to the underlying marginal cost. Section 4 presents stability results, both local stability and the global convergence result for the class of geometric decay patterns. We conclude in Section 5 and present all proofs in the Appendix.

2 Capacity Investments and Product Prices

We consider a single-product firm that makes its capacity investments and pricing decisions subject to a rate of return regulation constraint. In each period, the output of the firm, q_t, is bounded by the total capacity available in that period, K_t. If feasible, the firm is required to operate at capacity, so that $q_t = K_t$.[9] Production capacity is generated by assets, which have a useful life of T periods. Specifically, an investment expenditure of $v \cdot I_t$ at date t will add capacity to produce an additional $x_\tau \cdot I_t$ units of output at date $t+\tau$, where $1 \leq \tau \leq T$.[10] Thus, new investments come "online" with a lag of one period, yet their productivity may diminish over time. Thus we assume that $1 = x_1 \geq x_2 \geq \cdots \geq x_T > 0$. In the so-called *one-hoss shay* scenario all x_τ are equal to one, as the asset is assumed to have undiminished productive capacity over its entire useful life.

At date t, the capacity available is determined by the current *state*:

$$\boldsymbol{\theta}_t = (I_{t-T}, ..., I_{t-1}),$$

which reflects the relevant history of past investment decisions. In particular, we have:

$$K(\boldsymbol{\theta}_t) = I_{t-T} \cdot x_T + ... + I_{t-1} \cdot x_1.$$

[8]In what Biglaiser and Riordan term naive rate of return regulation (Proposition 3), the regulator provides insufficient incentives for the firm dispose of assets that still have positive book value.

[9]In many industries of interest, such as electricity, there is a natural peak-load pattern in demand so that full capacity utilization effectively amounts to having $q_t = \Delta \cdot K_t$ for some $\Delta < 1$. The presence of peak-load demand does not alter the conclusions obtained in our model in a substantive manner.

[10]To begin with, the unit cost of new capacity investments, v, is held fixed over time. In some industries, like electricity, the cost of new capacity investments has arguably gone up in recent years. Other regulated industries, like telecommunications, are characterized by significant learning curves and therefore the cost of new assets is falling over time. We consider the possibility of time-variant asset acquisition costs in Section 4 below.

Let $P_t(K_t)$ denote the price that consumers are willing to pay in period t if K_t units of output are supplied. The functions $P_t(\cdot)$ are assumed to be decreasing such that

$$\lim_{K_t \to \infty} P_t(K_t) = 0.$$

Furthermore, total revenue $R_t(K_t) = P_t(K_t) \cdot K_t$, is assumed to be concave in K_t.

In terms of costs, the regulated firm is allowed set product prices so as to recover both variable operating costs and capital costs, the latter composed of depreciation charges and interest charges imputed on the book value of the firm's capacity assets. The variable operating cost per unit of output is assumed to be constant and denoted by w. Assets are capitalized in their acquisition period and then fully depreciated over their useful life, according to some schedule $\mathbf{d} = (d_1, \ldots, d_T)$, satisfying $\sum_{\tau=1}^{T} d_\tau = 1$. Given the current state $\boldsymbol{\theta}_t$, the total depreciation charge in period t is then given by:

$$D_t(\boldsymbol{\theta}_t) = v \cdot (d_T \cdot I_{t-T} + \ldots + d_1 \cdot I_{t-1}).$$

Since depreciation is the only accrual in our model, income is given by revenues less variable operating costs and depreciation:

$$Inc_t = P_t(K(\boldsymbol{\theta}_t)) \cdot K(\boldsymbol{\theta}_t) - w \cdot K(\boldsymbol{\theta}_t) - D_t(\boldsymbol{\theta}_t).$$

The book value of one capacity unit acquired at date t is originally recorded at its cost v and then amortized over the next T periods according to the depreciation schedule, $\mathbf{d}$. Specifically the remaining book value at date $t + \tau$, $0 \leq \tau \leq T$, is given by:

$$bv_\tau = (1 - \sum_{i=1}^{\tau} d_i) \cdot v$$

Given the current state, $\boldsymbol{\theta}_t$, the firm's aggregate value of capacity assets at date $t-1$ is given by:

$$BV_{t-1}(\boldsymbol{\theta}_t) = bv_{T-1} \cdot I_{t-T} + \ldots + bv_0 \cdot I_{t-1}.$$

Capital costs, which are the sum of depreciation and an imputed charge on the beginning-of-period book value, will be denoted by C_t:

$$C_t(\boldsymbol{\theta}_t) = D_t(\boldsymbol{\theta}_t) + r \cdot BV_{t-1}(\boldsymbol{\theta}_t),$$

where r denotes the cost of capital.[11] Finally, the sum of variable operating costs and capital costs will be referred to as the *aggregate historical cost* in period t:

[11] The capital charge rate, r, will generally be determined as a weighted average of the costs of equity and debt. Our analysis treats this cost as exogenous and time invariant.

$$H_t(\boldsymbol{\theta}_t) = w \cdot K(\boldsymbol{\theta}_t) + C_t(\boldsymbol{\theta}_t).$$

The constraint imposed by Rate of Return (RoR) regulation is usually represented in terms of the firm's accounting rate of return on assets, that is:

$$\frac{Inc_t}{BV_{t-1}} \leq r. \tag{1}$$

It has long been recognized in both the regulation and the accounting literature that a firm operating consistently under the constraint imposed by (1) will not make any positive economic profits in the sense that the present value of all cash flows is non-negative. This follows from the *conservation property of residual income* which states that, for a firm with no assets at its inception, the present value of cash flows is equal to the present value of all residual incomes, regardless of the applicable depreciation rules.[12] Since the rate of return constraint in (1) is equivalent to:

$$RI_t = Inc_t - r \cdot BV_{t-1} \leq 0,$$

it follows that the firm will break even over its entire life time (in terms of discounted cash flows) only if the inequality constraint in (1) is met as an equality in *every* period. Since residual income is, by definition, equal to the difference between revenues and historical cost, that is $RI_t \equiv P_t(K) \cdot K - H_t$, the firm is instructed to make its investment decision in period t (at date $t-1$) so that if it sells the capacity available at date t, its revenue will cover the full historical cost. Accordingly, we have the following definition of a feasible state:

Definition 1 *A state $\boldsymbol{\theta}_t$ is feasible if the rate-of-return regulation constraint is satisfied with equality at date t, that is:*

$$P_t(K(\boldsymbol{\theta}_t)) \cdot K(\boldsymbol{\theta}_t) = H_t(\boldsymbol{\theta}_t).$$

There may be multiple non-negative investment levels at date t which lead to a feasible state, $\boldsymbol{\theta}_{t+1}$. For the purposes of our analysis, it does not matter how the firm chooses among these alternative investment levels. On the other hand, depending on the history $\boldsymbol{\theta}_t$ and the depreciation rules in place, there may not exist an investment level $I_t > 0$ at date t which makes $\boldsymbol{\theta}_{t+1}$ feasible. In that case, it must be that $P_t(K(\boldsymbol{\theta}_{t+1})) \cdot K(\boldsymbol{\theta}_{t+1}) - H(\boldsymbol{\theta}_{t+1}) < 0$ for

[12] This result dates back to Hotelling (1925) and Preinreich (1938); see also Schmalensee (1989).

all $I_t > 0$.[13] We assume the firm is then required to set $I_t = 0$.[14] Our analysis generally takes the perspective that the investment levels up to date T are exogenously given and that the firm is subject to a rate of regulation constraint from date T onward. In particular, new investment levels must be chosen so as maintain feasibility. If no such investment level exists, the firm is required to set $I_t = 0$. As shown below, it makes sense to set $I_t = 0$ when feasibility is not attainable in the short-run, because the regulator ultimately ensures feasibility in the long-run.

Rate of return regulation is usually positioned as an attempt to maximize consumer welfare subject to the constraint that the firm does not receive direct monetary transfers from the regulator. Instead, the firm is supposed to break even by being allowed to charge product prices that cover its average cost. Leaving regulatory issues aside for a moment, it is instructive to characterize the efficient capacity levels that a social planner would choose so as to optimize consumer welfare net of production costs. Consumer welfare at date t is given by:

$$B_t(q_t) = \int_0^{q_t} P_t(u)du.$$

Let $\gamma \equiv \frac{1}{1+r}$. Assuming that at date 0 the firm has no assets in place, that is, $q_0 = 0$ and $q_1 = I_0$, the regulator's welfare maximization problem then becomes:[15]

$$\max_{\{q_t; I_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} [B_t(q_t) - w \cdot q_t - v \cdot I_t] \cdot \gamma^t \tag{2}$$

subject to:

$$q_t \leq K_t \equiv I_{t-T} \cdot x_T + ... + I_{t-1} \cdot x_1.$$

A central insight in the papers of Arrow (1964) and Rogerson (2008a, 2008b) is that this dynamic maximization problem is intertemporally separable if market demand keeps expanding over time and therefore the regulator seeks to add capacity in each period. This notion is formalized in the following definition.

[13] If the sign of $\Delta_t(I_t) \equiv P_t(K(\boldsymbol{\theta}_{t+1})) \cdot K(\boldsymbol{\theta}_{t+1}) - H(\boldsymbol{\theta}_{t+1})$ were to vary over the range of $I_t > 0$, then by continuity of the function $\Delta_t(\cdot)$ there would exist an I_t where $\Delta_t(\cdot)$ is zero. To see that it is impossible to have $\Delta_t(I_t) > 0$ for all $I_t > 0$, we note that $P_t(K_t)$ tends to zero for large values of K_t (induced by high values of I_t). Yet, the average cost $\frac{H_t}{K_t}$ tends to a positive number, $w + v \cdot (d_1 + r)$ as I_t gets large.

[14] In order to alleviate the impact of negative residual income, it is conceivable that the regulator would allow the firm to set a higher price at which the quantity demanded is below the available capacity level.

[15] As will become clear, it is of no importance to the following derivation that there are no productive capacity assets in place at date 0. It only matters that the planner does not commence with excess capacity.

Assumption (A1) *The inverse demand curves satisfy:*

$$P_t(K) \leq P_{t+1}(K)$$

for all $K \geq 0$.

Throughout the paper, we assume that demand curves satisfy A1. Intuitively, A1 ensures that the welfare planner never ends up with excess capacity if in each period the investment is chosen myopically optimal so as to balance current welfare and costs. The following insight dates back to Arrow (1964).

Lemma 0: *The optimal capacity levels, K_t^*, that solve the optimization problem in (2) are given by:*

$$P_t(K_t^*) = w + c,$$

where

$$c = \frac{v}{\sum_{i=1}^{T} x_i \cdot \gamma^i}. \tag{3}$$

This result shows that the planner's optimization problem is intertemporally separable. Capacity investments entail an inherent jointness that results from the fact that v dollars spent today generate a T-period stream of future capacity levels. Yet, if new capacity can be added in each period, the cost c becomes effectively the marginal cost of one unit of capacity made *available for one period of time.*[16] It is also useful to note that c is exactly the price that a hypothetical supplier would charge for renting out capacity for one period of time, if the rental business is constrained to make zero economic profit. Accordingly, we will also refer to c as the *competitive rental price of capacity.*

[16]For the special case of $T = 2$, the formula in (3) can be derived as follows . The planner can increase capacity at date t by one unit without affecting capacity levels in subsequent periods through the following "reshuffling" of future capacity acquisitions: buy one more unit of capacity at date $t-1$, buy x_2 fewer units in period t, buy $(x_2)^2$ more units in period $t+1$, and so on. The cost of this variation, evaluated in terms of its present value as of date $t-1$, is given by:

$$v \cdot \left[1 - \gamma \cdot x_2 + \gamma^2 \cdot x_2^2 - \gamma^3 \cdot x_2^3 + \gamma^4 \cdot x_2^4 \ldots\right] = v \cdot \frac{1}{1 + \gamma \cdot x_2},$$

and therefore the present value of the variation at date t is:

$$(1 + r) \cdot v \cdot \frac{1}{1 + \gamma \cdot x_2} \equiv c.$$

Lemma 0 suggests that the efficiency of rate of return regulation hinges on how well the historical cost $H_t(\boldsymbol{\theta}_t)$ approximates the economically relevant cost $(w + c) \cdot K(\boldsymbol{\theta}_t)$. The goodness of this approximation depends in turn on the choice of depreciation rules for the capacity assets. Rogerson (2008b) provides an important insight in this regard by focusing on the capital costs, i.e., the sum of depreciation and imputed interest charges. Let

$$z_\tau \equiv v \cdot d_\tau + r \cdot bv_{\tau-1}, \tag{4}$$

so that $z_\tau \cdot I_{t-\tau}$ is the capital cost charged in period t for investment $I_{t-\tau}$ undertaken at date $t - \tau$. Since an investment in one unit of capacity undertaken at date $t - \tau$ still contributes x_τ units of capacity at date t, the economically relevant capital cost at date t is:

$$z^*_\tau = x_\tau \cdot c. \tag{5}$$

Rogerson (2008b) refers to the cost allocation rule in (5) as the *Relative Replacement Cost* (R.R.C.) rule, reflecting that the current replacement cost of an asset is v and that the cost charge in period t is given by the proportion of current capacity relative to the total discounted capacity of the asset:[17]

$$z^*_\tau = v \cdot \frac{x_\tau}{\sum_{i=1}^{T} x_i \cdot \gamma^i}.$$

Earlier literature has shown that there is a one-to-one relation between depreciation schedules $\mathbf{d} = (d_1, ..., d_T)$ and the capital cost charges $(z_1, ..., z_T)$. Formally, the linear mapping defined by (4) is one-to-one: for any intertemporal cost charges $(z_1, ..., z_T)$, with the property that $\sum_{\tau=1}^{T} z_\tau \cdot \gamma^\tau = v$, there exists a unique depreciation schedule $\mathbf{d}$ such that (4) is satisfied. We denote by $\mathbf{d}^*$ the unique depreciation schedule corresponding to the R.R.C. cost allocation rule.[18] It is readily verified that in the one-hoss shay scenario ($x_\tau = 1$), the corresponding depreciation schedule is the *annuity depreciation* method, where $d_\tau = (1 + r) \cdot d_{\tau-1}$. The R.R.C. rule does coincide with *straight-line depreciation* if the productive

[17] The concept of replacement cost is more complex in Rogerson (2008b), since in his model the acquisition cost of new assets declines exponentially over time. We consider this possibility in Section 5 below.

[18] The R.R.C. depreciation schedule conforms to Hotelling's (1925) concept economic depreciation if one posits the existence of a competitive rental market for assets. The R.R.C. rule is also closely related to so-called *Relative Benefit Depreciation* rule which has played a central role in earlier studies on managerial incentives for investment decisions; see, for example, Reichelstein (1997), Rogerson (1997), Baldenius and Ziv (2003), Bareket and Mohnen (2007) and Pfeiffer and Schneider (2007). As the name suggests, the charges under the relative benefit depreciation rule apportion the initial investment expenditure in proportion to the subsequent expected future cash inflows, while the R.R.C. rule, in contrast, refers "only" to the decay in production capacity over time.

capacity of assets capacity declines linearly over time such that $x_\tau = 1 - \frac{r}{1+r\cdot T}\cdot(\tau - 1)$. Finally, if $T = \infty$ and productive capacity declines geometrically such that $x_\tau = (1-\alpha)\cdot x_{\tau-1}$ for $0 < \alpha < 1$, the corresponding R.R.C. depreciation schedule is also geometric with $d_\tau = (1-\alpha)\cdot d_{\tau-1}$.

Given the R.R.C. depreciation rule, we then conclude that the firm's historical cost reflects precisely the economically relevant cost that a social planner would impute at date t:

$$\begin{aligned}
H_t(\boldsymbol{\theta}_t) =& w\cdot K(\boldsymbol{\theta}_t) + D_t(\boldsymbol{\theta}_t) + r\cdot BV_{t-1}(\boldsymbol{\theta}_t) \\
=& w\cdot K(\boldsymbol{\theta}_t) + z_T^*\cdot I_{t-T} + \ldots + z_1^*\cdot I_{t-1} \\
=& w\cdot K(\boldsymbol{\theta}_t) + c\cdot x_T\cdot I_{t-T} + \ldots + c\cdot x_1\cdot I_{t-1} \\
=& (w+c)\cdot K(\boldsymbol{\theta}_t).
\end{aligned}$$

To state Rogerson's (2008b) benchmark result formally in the context of our price regulation model, we refer to a *trajectory* $\boldsymbol{\theta}$ as an infinite sequence of investments levels:

$$\boldsymbol{\theta} = (I_0, I_1, ..., I_t, ...)$$

For any $t \geq T$ the trajectory defines a sequence of states $\boldsymbol{\theta}_t = (I_{t-T}, ..., I_{t-1})$. A trajectory will be said to be *feasible* if $\boldsymbol{\theta}_t$ is feasible for all $t \geq T$. In stating the following result, it will be convenient to consider a setting where the firm started with no capacity investments at date 0. Thus $q_1 = I_0$, $q_2 = I_1 + x_2\cdot I_0$ and so forth. For any date $t < T$, we can thus extend the notion of a feasible state by letting $\boldsymbol{\theta}_t = (0, ..., 0, I_0, ..., I_{t-1})$.

Proposition 0: *Suppose the firm started with no capacity at date* 0*. If assets have been depreciated consistently according to the R.R.C. rule, then rate-of-return regulation results in a trajectory* $\boldsymbol{\theta}$ *that is feasible and attains marginal cost pricing, i.e.,* $P_t(K(\boldsymbol{\theta}_t)) = c + w$ *for all* $t \geq 1$.

This result dispels the economic logic articulated in many microeconomics textbooks regarding the efficiency of rate of return regulation, e.g., Baumol (1971), Nicholson (2005) and Schotter (2008). The conventional argument is that because direct transfers to the regulated firm are politically undesirable, regulatory agencies cannot implement marginal cost pricing. In order for the firm to break even, prices must therefore cover average costs which include the (historical) fixed costs associated with the regulated firm's investments in property, plant and equipment. Proposition 0 shows that with suitably chosen depreciation rules the historical cost of sunk asset expenditures aligns precisely with the long-run marginal cost of capacity.

Of course, Proposition 0 does not yet speak to the efficiency of rate of return regulation if for the purposes of calculating the firm's cost basis regulatory agencies commonly calculate depreciation charges according to the straight line rule. As observed above, straight line depreciation is consistent with R.R.C. rule in the exceptional case where the x_τ decline linearly at a specific rate. The natural question for our analysis, though, concerns the dynamic of the rate of regulation process and the type of distortions, both in terms of direction and magnitude, one should expect to see over time.

3 Price Equilibria

In this section, we characterize prices that can emerge as equilibrium prices of the rate of return regulation process for a given set of accounting rules which may not conform to the R.R.C. depreciation rule. In particular, we seek to understand which prices can emerge as limit points of a trajectory determined jointly by an initial state and the rules of the regulation process.

While our analysis allows for the possibility of growth in the product market, we do restrict attention to so-called *proportionate growth* requiring that:

$$P_{t+1}\big((1+\mu_t)\cdot K_t\big) = P_t(K_t), \text{ for } 1 \leq t \leq T. \tag{6}$$

The parameter μ_t then represents the growth rate of the market in period t. We assume $\mu_t < r$. We note that the proportionate growth assumption will be met by constant elasticity demand curves if the elasticity factor remains constant, yet the size of the market expands. The proportionate growth assumption can also be met by linear demand curves of the form $P_t = a_t - b_t \cdot q_t$ if the intercept a_t remains constant, yet the slope parameter b_t contracts over time at the rate $(1+\mu_t)^{-1}$. As a consequence, demand will become less elastic over time. We note in passing that if an unregulated monopolist has constant marginal (long-run) cost and faces a demand curve exhibiting proportionate growth, the profit maximizing capacity levels will grow precisely at the rate μ_t in period t. Similarly, in the context of Lemma 0 above, a social planner would ideally choose capacity levels that expand at the rate μ_t.

Definition 2 *A trajectory $\boldsymbol{\theta}$ and a price p_∞ constitute a price-equilibrium if $\boldsymbol{\theta}$ is feasible and for all $t \geq T$*

$$p_\infty = P_t(K(\boldsymbol{\theta}_t)).$$

A price p_∞ will be said to be an *equilibrium price* if there exists a trajectory $\boldsymbol{\theta}$ such that $(\boldsymbol{\theta}, p_\infty)$ constitute a price equilibrium. Such a trajectory $\boldsymbol{\theta}$ is then said to *support* p_∞. The following result identifies an equilibrium price for the special case in which demand grows at a constant rate.

Proposition 1 *If market demand grows at some constant rate μ, then*

$$p^* = w + c \cdot \frac{\sum_{\tau=1}^{T} (1+\mu)^{T-\tau} \cdot z_\tau(\mathbf{d})}{\sum_{\tau=1}^{T} (1+\mu)^{T-\tau} z_\tau^*} \tag{7}$$

is an equilibrium price of the rate of return regulation process, provided that $P_T(0) > p^*$. This price is supported by a constant growth trajectory of the form: $I_{t+1} = I_t \cdot (1+\mu)$.

From hereon, it will be assumed that the price p^*, identified in (7), is in the range of admissible prices, that is $P_T(0) > p^*$ and therefore $P_t(0) > p^*$ for all $t \geq T$. We note that the characterization of the equilibrium price p^* in Proposition 1 recovers the benchmark result in Proposition 0 insofar as p^* is equal to the long-run marginal cost $w + c$ whenever depreciation conforms to the R.R.C. rule. We refer to the fraction:

$$\Gamma \equiv \frac{\sum_{\tau=1}^{T} (1+\mu)^{T-\tau} \cdot z_\tau(\mathbf{d})}{\sum_{\tau=1}^{T} (1+\mu)^{T-\tau} \cdot z_\tau^*} \tag{8}$$

as the *accounting bias*. Clearly, this bias factor is a joint function of the depreciation schedule and the growth rate in the product market, μ_t. Before we delineate the properties of Γ, we seek to characterize the prices that can emerge as limit points of the RoR process for some initial state $\boldsymbol{\theta}_T$.

Definition 3: *A price p_∞ is an asymptotic equilibrium price if there exists a supporting trajectory $\boldsymbol{\theta}$ such that:*

$$\lim_{t \to \infty} P_t\left(K\left(\boldsymbol{\theta}_t\right)\right) = p_\infty.$$

As stated, the concept of an asymptotic equilibrium price does not require that the supporting trajectory is feasible in the sense that the firm does earn the targeted rate of return on its investment. Depending on the initialization of the dynamic system and the accounting rules in place, the trajectory emerging under the RoR regulation process may not be feasible simply because no non-negative investment level I_t exists at which the firm obtains the targeted rate of return. It turns out, however, that under relatively mild regularity conditions feasibility is only a temporary concern if prices do approach some limit value p_∞. The productive capacity of assets is said to decline *strictly* over time if $1 = x_1 > x_2 > ... > x_T > 0$.

Lemma 1 *Suppose the productive capacity of assets decays strictly over time and market demand grows at a constant rate $\mu \geq 0$. Then any trajectory $\boldsymbol{\theta}$ that supports some p_∞ as an asymptotic equilibrium price is feasible at all dates t beyond some $\hat{T}$.*

The proof of the lemma uses the fact that if prices converge, then so must the growth-adjusted capacity levels. When each asset's productive capacity is strictly lower over time, the presence of an infeasible state implies an absence of investment and consequently a strict drop in capacity levels. Beyond some cutoff date, however, this is inconsistent with the convergence of the scaled capacity levels. Stated differently, there can exist only a finite number of infeasible states in a trajectory that supports an asymptotic equilibrium price.

In light of the above lemma, we can effectively put aside concerns regarding infeasibility by focusing attention on time periods sufficiently far into the future. This enables us to derive the following general characterization result regarding the set of possible asymptotic price equilibria.

Proposition 2 *Suppose the production capacity of assets decays strictly over time and market demand grows at a constant rate $\mu \geq 0$. The price p^*, given in (7), is then the unique asymptotic equilibrium price.*

It should be noted that Proposition 2 does not imply that prices will converge to p^* under the rate of return regulation process; it merely shows that p^* is the only candidate for an equilibrium price in the limit. While this result is extremely powerful, it should be noted that, by restricting attention to strictly declining x vectors, it does not cover the one-hoss shay scenario commonly considered in the regulation literature. However, if the periodic growth rate μ is strictly positive, then the proof of Proposition 2 can be easily adapted to extend to the one-hoss shay setting. Moreover, even if $\mu = 0$, it is possible to derive a variant of Proposition 2 that does cover the one-hoss shay case. What is required then is the inclusion of a non-degeneracy condition on the depreciation rule in place. This is stated formally in Proposition 2' below.

Proposition 2': *Suppose the depreciation schedule has the property that $z_1 > z_2 > ... > z_T > 0$ and market demand grows at a constant rate $\mu \geq 0$. Then, the price p^*, given in (7), is the unique asymptotic equilibrium price.*

To illustrate the condition on the $\mathbf{z}$ vector, we note that with one-hoss shay productivity and straight-line depreciation, the capital costs z_τ are indeed decreasing over time (in fact, linearly so) since the periodic depreciation charges are constant and the imputed interest charges decrease with the drop in book values.

From Definitions 2 and 3, it is evident that any equilibrium price must also be an asymptotic equilibrium price. Combining Propositions 1, 2 and 2', we then have the following result, which we state without proof.

Corollary to Proposition 2: *Under the conditions of either Proposition 2 or 2', p^*, as given in (7), is the unique equilibrium price.*

We now characterize the accounting bias factor Γ identified in (8). A natural first question is whether Γ acts as a mark-up or discount on the long-run marginal cost of capacity, c. To address this question, we follow the approach in Rajan and Reichelstein (2008) and introduce a partial ordering on the class of depreciation schedules. For one unit of capacity investments, let $bv_\tau(\mathbf{d})$ denote the book values that emerge for the depreciation schedule $\mathbf{d}$, that is, $bv_\tau(\mathbf{d}) = v \cdot (1 - \sum_{i=1}^{\tau} d_i)$.

Definition 4 *A depreciation schedule* $\mathbf{d}$ *is more accelerated than* $\mathbf{d}'$ *if*

$$bv_\tau(\mathbf{d}) \leq bv_\tau(\mathbf{d}')$$

for any $\tau \leq T - 1$.

An accelerated depreciation schedule writes assets off too fast in comparison to the unbiased benchmark of R.R.C. depreciation. As a consequence, the capital costs z_τ of a new investments are too high early in the asset life cycle. The following result shows how this bias impacts the asymptotic equilibrium price under RoR regulation.

Proposition 3 *The unique equilibrium price p^*, given in (7), satisfies:*

$$p^* < w + c$$

provided the depreciation schedule is accelerated relative to the R.R.C. rule.

The proof of this result shows that the accounting bias Γ is less than 1 whenever the depreciation schedule is accelerated relative to the R.R.C. rule, while $\Gamma > 1$ for a decelerated depreciation rule. At first glance, the result in Proposition 3 appears implausible. How is it possible for a regulated firm to break even over an infinite horizon and, at the same time, for consumers to buy the product at a price that is below the social marginal cost $w + c$? The answer is that these two conditions can co-exist only once the system evolves from an appropriate initial state. Yet, there are also costs and benefits associated with obtaining such an initial state. Put differently, if the firm starts with no capacity at date 0, then prices must early on exceed $w + c$ before they ultimately approach $p^* < w + c$. This point will be illustrated further in the simulations reported in Section 4 below.

Our finding in Proposition 3 is seemingly at odds with the results obtained by Friedl (2007). His claim is that if assets are depreciated under the straight line rule and the productivity pattern corresponds to the one-hoss shay scenario, then the resulting capacity level will be too low and the resulting product price too high relative to the socially efficient levels. Essential to this result, however, is that the regulated firm is assumed to make only a single capacity investment at date 0. Since historical costs fall over time, due to straight line depreciation, product prices fall as well under the RoR constraint. As a consequence, demand exceeds the available supply in subsequent periods, yet the regulated firm is assumed not to respond with new capacity investments.

We now turn to a further examination of the accounting bias factor Γ. Clearly, it is a function of both the depreciation schedule and the rate of growth in the product market, μ. To assess the impact of μ on Γ, we consider a stronger notion of accelerated depreciation. The notion put forth in Definition 4 is essentially a "second-order dominance" condition referring to the cumulative amount amortized at a particular point in time. The following criterion provides a more stringent "first-order dominance" criterion.

Definition 5 *A depreciation schedule* $\mathbf{d}$ *is uniformly more accelerated than the R.R.C. rule if* $\frac{z_\tau(\mathbf{d})}{x_\tau}$ *is monotonically decreasing in* τ*.*

By definition, the depreciation schedule corresponding to the R.R.C. rule, $\mathbf{d}^*$, has the property that

$$\frac{z_\tau^*}{x_\tau} = c.$$

In contrast, a depreciation schedule that is uniformly accelerated relative to the R.R.C. rule starts out with a ratio $\frac{z_1^*}{x_1} > c$ and then intersects the value c at some point in time $\hat{\tau} < T$.

Corollary to Proposition 3 *If depreciation is uniformly more accelerated than the R.R.C. rule, the equilibrium* p^* *is increasing in* μ*.*

To conclude this section, we provide a numerical illustration of the bias factor Γ in a scenario where depreciation is held fixed at the straight-line rule. The productivity factors x_τ are assumed to decline linearly over time, so that $x_\tau = 1 - \beta(\tau - 1)$. Therefore $\beta = 0$ corresponds to the one-hoss shay scenario while, as observed above, when $\beta = \frac{r}{1+r\cdot T}$ straight line depreciation amounts to the R.R.C. rule. As a consequence, one then obtains $\Gamma = 1$. Figure 1 shows the level sets of the function $\Gamma(\mu, \beta | r, T)$ for values of $r = .12$ and $T = 25$.

Consistent with the above results, we note that $\Gamma(\cdot)$ assumes values below (above) 1 whenever $\beta < (>) \frac{r}{1+r\cdot T} = .03$. Furthermore, it can be verified that the depreciation schedule is in fact uniformly accelerated for $\beta \leq \frac{r}{1+r\cdot T}$ and therefore $\Gamma(\cdot)$ is indeed increasing in μ.

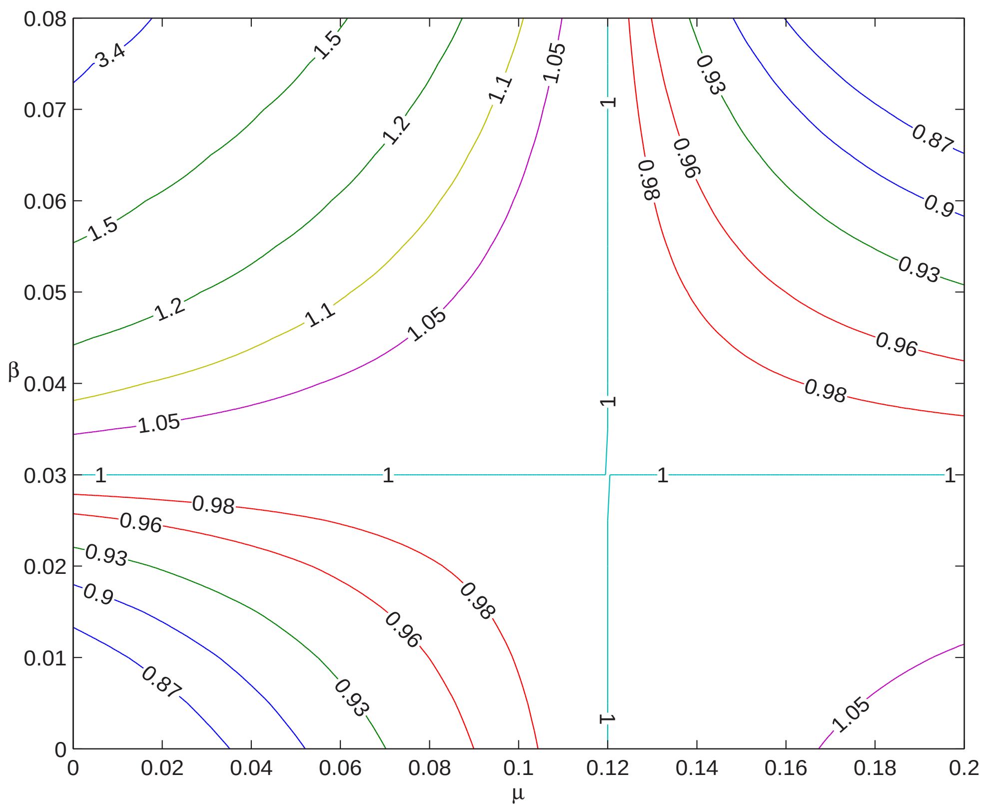

Figure 1: Level sets (isoquants) of the bias function $\Gamma(\mu, \beta | r = .12, T = 25)$ under straight line depreciation. Here, μ represents growth in the product market while β captures the periodic decline in productive capacity.

Figure 1 shows that there is no accounting bias if the growth rate in the product market happens to equal the cost of capital, r. This observation follows directly from the observation that

$$\sum_{\tau=1}^{T} (1+r)^{T-\tau} \cdot z_\tau(\mathbf{d}) = v \cdot \gamma^{-T},$$

for all depreciation schedules $\mathbf{d}$.[19]

With accelerated depreciation (the lower half of the picture), the largest accounting bias occurs in the South-West corner of Figure 1, corresponding to no growth and one-hoss shay.

[19]Directly related to this observation, Rajan and Reichelstein (2008) show that historical cost and marginal cost coincide, regardless of the accounting rules, whenever new investments grow at the firm's cost of capital.

The accounting bias then becomes $\Gamma(0, 0|r = .12, T = 25) = .803$. While this factor may not seem to entail that significant a distortion, we recall that it applies only to the asymptotic equilibrium price. To gauge the overall inefficiency associated with accounting biases under RoR regulation, one must consider the losses in the present value of all consumer surpluses resulting from the use of some biased depreciation rule as opposed to the unbiased R.R.C. rule.

4 Stability of the RoR Regulation Process

The above results naturally raise the following questions: (i) Starting from any feasible initial state, will the RoR regulation process generate a trajectory such that the resulting product prices converge to the asymptotic equilibrium price p^* identified above? (ii) More modestly, can such convergence be obtained if the firm starts out without any capacity assets at date 0? (iii) If global convergence cannot be established, is the RoR process at least locally stable, i.e., it converges if the initial state is close to the state that is known to support p^*?

While we are not aware of examples where global stability cannot be obtained, it also appears difficult to establish any of the above three claims at the level of generality maintained in the previous sections. In the following analysis we will therefore consider specific combinations of productivity patterns and depreciation schedules. We start by assuming that productive capacity follows the one-hoss shay pattern and depreciation is straight-line. It is also assumed that demand is stable over time. Under these assumptions, we know from the Corollary to Proposition 2 that p^* is the unique equilibrium price. Let $\boldsymbol{\theta}^* = (I^*, I^*, I^*, ...)$ denote the constant investment trajectory supporting p^*. Finally, let $\xi(\cdot, \cdot)$ denote the Euclidean distance in $\mathbb{R}^T$.

Definition 6 *The price equilibrium $(p^*, \boldsymbol{\theta}^*)$ is said to be locally stable if there exist an $\epsilon > 0$, such that for all $\boldsymbol{\theta}_T$ for which $\xi(\boldsymbol{\theta}_T^*, \boldsymbol{\theta}_T) < \epsilon$, any trajectory $\boldsymbol{\theta}$ generated by $\boldsymbol{\theta}_T$ supports p^*, that is:*

$$\lim_{t\to\infty} P_t((K(\boldsymbol{\theta}_t)) = p^*.$$

In other words, local stability of the price equilibrium requires that if any other trajectory comes close to the constant growth trajectory that supports p^*, that trajectory must also support p^* as an asymptotic equilibrium price. We have the following result.

Proposition 4 *Assume one-hoss shay productivity, straight-line depreciation, and constant demand. Then, the price equilibrium* $(\boldsymbol{\theta}^*, p^*)$ *is locally stable.*

For the remainder of this section, we consider a setting in which assets are infinitely lived, and both productive capacity and accounting book values decay geometrically. In addition, we allow for constant growth in the product market over time. The key assumptions in this parametric specification are as follows:

- The price in the product market is a linear function of firm output (or capacity). The intercept, a, is stable over time, while the slope declines over time at the rate $(1+\mu)^{-1}$, where μ is the constant rate at which the overall market grows. Note that this is consistent with the proportionate growth assumption made earlier. Stated formally, the inverse demand curve at time t is given by:

$$P_t(q) = a - b_t \cdot q$$

  where $b_t = \frac{b}{(1+\mu)^t}$, $\mu \geq 0$.

- Each asset lives for infinitely many periods, but the productive capacity of any investment declines geometrically over time at rate α. As before, we normalize x_1 to equal 1. Productivity in future periods is then defined recursively as:

$$x_{\tau+1} = (1-\alpha)\, x_\tau$$

  with $0 < \alpha < 1$.

- In addition to the geometric decline in asset productivity, we posit that the accounting rules mandate declining-balance depreciation at periodic rate δ, $0 < \delta < 1$. In other words, depreciation expense in each period is a constant proportion δ of start of period book value. This implies that the book value of one unit of capacity acquired at date 0 equals the acquisition cost, v, while the remaining book value at any future date $\tau \geq 0$, is given by:

$$bv_\tau = v \cdot (1-\delta)^\tau$$

  Given this specification, the relationship between the depreciation rate and beginning book value can be expressed as

$$d_{\tau+1} = \delta \cdot \frac{bv_\tau}{v}$$

An immediate implication of these assumptions is that book values, depreciation charges, and capital costs all evolve analogously over time:

$$bv_{\tau+1} = (1-\delta)\, bv_\tau;$$

$$d_{\tau+1} = (1-\delta)\, d_\tau;$$

and, therefore,

$$z_{\tau+1} = (1-\delta)\, z_\tau.$$

Since investments now remain productive indefinitely, rather than having a finite life of T years, the definition of a state has to be adjusted. In particular, the relevant history includes not just investments made in the prior T periods, but the entire vector of new investments from the inception of the firm. Accordingly, we redefine $\boldsymbol{\theta}_t$ as follows:

$$\boldsymbol{\theta}_t = (I_0, ..., I_{t-1})$$

Note that, because of the geometric structure of the model, the future capacity levels and capital costs of assets in place at date t are completely defined by the current capacity, K_t, and the current capital cost, C_t, where, as before

$$C_t(\boldsymbol{\theta}_t) = \sum_{\tau=1}^{t} z_\tau \cdot I_{t-\tau}$$

In other words, the composition of the current set of assets in place is no longer of relevance. As in previous sections, we will call a state feasible if the regulation constraint is satisfied as an equality in that state:

$$P_t\left(K(\boldsymbol{\theta}_t)\right) \cdot K(\boldsymbol{\theta}_t) = C(\boldsymbol{\theta}_t) + w \cdot K(\boldsymbol{\theta}_t).$$

From prior literature, we know that the R.R.C. rule in this setting corresponds to $\delta = \alpha$; i.e., the depreciation rate matches the rate of decay in productivity. Further, $\delta < \alpha$ amounts to liberal accounting and $\delta > \alpha$ represents conservative accounting. As before, we focus on the latter scenario.

In the more general settings studied in sections 2 and 3, a key technical concern was the possibility that there might exist infeasible states along the trajectory of investments, i.e., periods in which no positive investment level would satisfy the rate of return constraint. We show next that this issue does not arise in the geometric scenario. In particular, given a feasible state, there exists one, and only one, positive investment level that meets the following period's rate of return constraint as an equality.

Lemma 2 *In the geometric decay model, suppose depreciation is more accelerated than the R.R.C. rule. If market demand grows at a constant rate, $\mu \geq 0$, and $\boldsymbol{\theta}_t$ is feasible, then there exists a unique $I_t > 0$ such that $\boldsymbol{\theta}_{t+1}$ is feasible.*

Observe also that if the firm starts with no capacity assets at date 0, there will be a unique I_0 such that $\boldsymbol{\theta}_1$ is feasible. By induction, it follows then that when a firm enters the market with no assets in place, there is a unique chain of future investment levels that lead to period-by-period feasibility under rate of return regulation.

We now compute the marginal cost of capacity, c, and the capacity level corresponding to p^* in the geometric case. Applying equation (3) and recognizing that $T = \infty$, the marginal cost of capacity is equal to:

$$c = \frac{v}{\sum_{\tau=1}^{\infty} (1-\alpha)^{\tau-1} \gamma^{\tau}} = v\,(\alpha + r)\,.$$

The unique equilibrium price (see (7)) is:

$$p^* = w + c \cdot \frac{r+\delta}{r+\alpha} \cdot \frac{\sum_{\tau=1}^{\infty} (1+\mu)^{-\tau} (1-\delta)^{\tau-1}}{\sum_{\tau=1}^{\infty} (1+\mu)^{-\tau} (1-\alpha)^{\tau-1}} = w + c \cdot \Gamma,$$

where

$$\Gamma = \frac{(r+\delta)\cdot(\alpha+\mu)}{(r+\alpha)\cdot(\delta+\mu)}.$$

Let K^* denote the capacity level corresponding to p^* at date 0:

$$a - bK^* = p^*$$

Therefore,

$$K^* = \frac{1}{b}\,(a - w - c \cdot \Gamma)$$

Proposition 5 *In the geometric decay model, suppose depreciation is more accelerated than the R.P.C. rule. If the firm is initially in a feasible state and market demand grows at a constant rate μ, then*

$$\frac{K(\boldsymbol{\theta}_t)}{(1+\mu)^t} \to K^*$$

and

$$P(K(\boldsymbol{\theta}_t)) \to p^*.$$

We know from Lemma 2 that, starting from any feasible history of prior investments, there is a unique subsequent sequence of positive levels of investment that satisfy the rate of return constraint in each period. Proposition 5 shows that this investment trajectory must eventually result in prices converging to the unique equilibrium price, p^*. Moreover, the growth-adjusted levels of investment also converge to a unique value, K^*. Proposition 5 thus establishes a global convergence result, as opposed to the more limited notion of local stability addressed in Proposition 4.

The proof of Proposition 5 demonstrates not just the above-documented convergence properties, but also the nature of the equilibrium trajectory of investment levels and prices. Specifically, the proof of the result shows that the convergence to the equilibrium occurs in a monotonic fashion. Consider a feasible vector $\boldsymbol{\theta}_t$ and the concomitant market price at date t. If this price lies below the equilibrium price, p^*, then the unique trajectory of future feasible levels of investment will result in a sequence of prices that are strictly monotone increasing, eventually converging to p^*. On the other hand, if the price is greater than p^*, then every future price will be strictly lower and the sequence will again reach p^* in the limit.

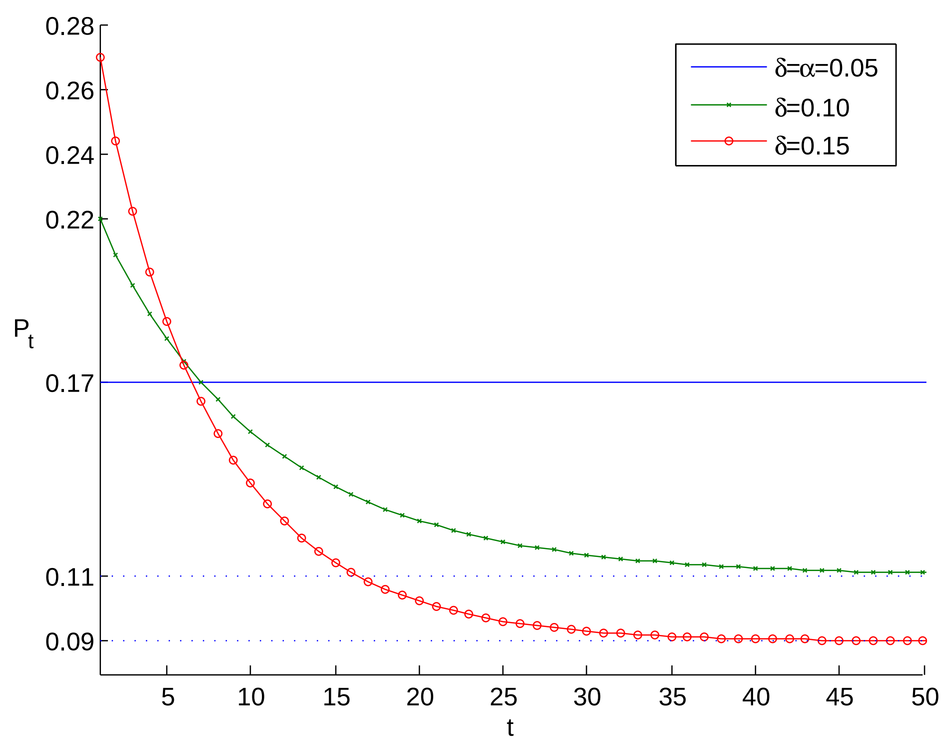

Figure 2: Trajectory of prices under rate of return regulation for various depreciation policies when the firm has no assets in place.

An especially interesting case of the latter scenario occurs when a firm enters the market with no capacity assets. In this case, the initial market price is always above the marginal cost. In each subsequent period, the price is strictly lower. It eventually equals and then drops below the marginal cost, and finally converges to the equilibrium price, p^*. Figure 2 illustrates this result for the following set of parameters: $w = 0, v = 1, \mu = 0, r = 12\%, \alpha = 5\%$, and $P_t(q) = 1 - q$. The horizontal line represents the marginal cost of 0.17. If the depreciation policy followed the R.R.C. rule, i.e., if δ were set equal to the periodic decay in productive assets ($\delta = \alpha = 0.05$), this is indeed the price that would be realized in every period. The green curve represents an accelerated depreciation method ($\delta = 10\%$). Note that it intersects the marginal cost line exactly once, from above, and then converges in a strictly decreasing manner to the equilibrium price, 0.11. The red curve depicts the impact of using an even more accelerated depreciation schedule ($\delta = 15\%$). In this case, the beginning price is greater than before, but the price subsequently drops much faster, cuts the marginal cost line earlier, and eventually converges to the new, lower equilibrium price of 0.09.

Figure 3 illustrates the impact of starting with a market price that is lower than the equilibrium price level. The parameters are the same as those assumed for Figure 2, and the depreciation rate is set at ($\delta = 10\%$). The key difference is that the firm is assumed to have substantial assets in place when it enters the market. In particular, suppose that the firm's initial capacity is 0.9, thereby yielding a market price of 0.1. In this setting, note that prices increase monotonically and converge to the equilibrium price ($p^* = 0.11$). This implies, of course, that the price stays below the marginal cost for the entire duration.

There are several questions of interest that we have not addressed as yet. The first is the speed of convergence, i.e., the rate at which the trajectory of prices approaches the equilibrium level. Related to this is the issue of how long prices stay above marginal cost when a firm starts out with no capacity in place. Finally, what is the percentage efficiency of alternative depreciation methods in terms of the discounted consumer welfare under the induced trajectory of market prices? This is an issue of great interest to academic researchers since it gets to the heart of identifying the real effects of accounting policies. It is also obviously a matter of importance to policymakers and one that we hope our model will enable us to shed some light on, at least in the geometric setting.

The reader may wonder how one reconciles the result in Proposition 1 with those characterized above for the geometric case. In the general setting, with assets that operate for T periods, Proposition 1 established the existence of a constant growth investment schedule that supports the equilibrium price. The proof of this result constructs an appropriate initialization in terms of investment choices for the first T periods that can then be continued in a constant growth trajectory that yields p^* as the price for all periods from T onwards. In the geometric case, a similar sequence can be shown to exist. In fact, given the uniqueness of

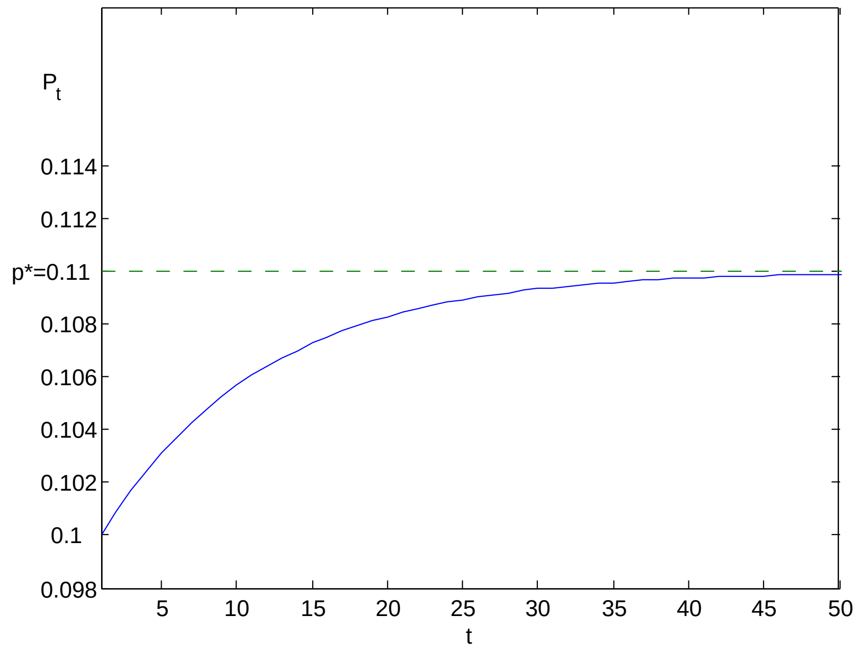

Figure 3: Trajectory of prices under rate of return regulation for an accelerated depreciation policy when the firm enters the market with significant capacity assets.

feasible continuation investment levels (Lemma 2) and the structure of the geometric setting, it is not necessary to specify the first T investment levels. It suffices to identify an initial level of capacity assets and associated historical costs, as well as an appropriate first-period choice of investment. These in turn ensure that future levels of feasible investment always increase at rate μ and, further, that the market price in each subsequent period equals the equilibrium price, p^*.

A related question is whether, leaving aside this particular constant growth trajectory, convergence from any starting point in the general finite-life model also takes place in a monotonic fashion. The answer is an emphatic no. We have conducted a variety of simulations in the general T-period setting. With linearly declining productivity (including the one-hoss shay scenario) and straight line depreciation, the trajectory of prices is highly non-monotonic. The market price that arises from rate of return regulation tends to oscillate around the equilibrium price. Convergence to the equilibrium price then occurs as the amplitude of the wave attenuates when the number of time periods gets sufficiently large.

5 Concluding Remarks

Under construction!

6 Appendix

Proof of Proposition 1:

We first state and prove a useful intermediate result.

Observation: Assume that market demand grows at rate μ and that, for some trajectory $\boldsymbol{\theta}$, investments grow at the same rate:

$$I_{t+1} = (1+\mu) \cdot I_t$$

for all $t \geq 0$. Then, if $\boldsymbol{\theta}_T$ is feasible, trajectory $\boldsymbol{\theta}$ and price

$$p^* = P_T(K(\boldsymbol{\theta}_T))$$

constitute a price-equilibrium.

Proof:

We need to check that $\boldsymbol{\theta}_t$ is feasible and implements price p^* for all t. First note, that as investments grow geometrically over time, $\boldsymbol{\theta}_{t+1} = (1+\mu) \cdot \boldsymbol{\theta}_t$ for all t. Since capacity, $K(\boldsymbol{\theta}_t)$ and historic cost, $H(\boldsymbol{\theta}_t)$, are linear in past investment levels, we have

$$K(\boldsymbol{\theta}_t) = K((1+\mu)^{t-T}\boldsymbol{\theta}_T) = (1+\mu)^{t-T} \cdot K(\boldsymbol{\theta}_T)$$

$$H(\boldsymbol{\theta}_t) = H((1+\mu)^{t-T}\boldsymbol{\theta}_T) = (1+\mu)^{t-T} \cdot H(\boldsymbol{\theta}_T)$$

The assumption that market demand grows at rate μ implies:

$$P_t(K(\boldsymbol{\theta}_t)) = P_T((1+\mu)^{-(t-T)}K(\boldsymbol{\theta}_t)) = P_T(K(\boldsymbol{\theta}_T)) = p^*$$

Therefore, trajectory $\boldsymbol{\theta}$ always results in price p^*. Finally, applying the feasibility assumption of $\boldsymbol{\theta}_T$, one obtains

$$P_t(K(\boldsymbol{\theta}_t))K(\boldsymbol{\theta}_t) = p^* \cdot (1+\mu)^{t-T} \cdot K(\boldsymbol{\theta}_T) = (1+\mu)^{t-T} \cdot H(\boldsymbol{\theta}_T) = H(\boldsymbol{\theta}_t)$$

and state $\boldsymbol{\theta}_t$ is feasible for all t. ■

Now, let K_T be the amount of capacity such that

$$P_T(K_T) = p^*$$

and let

$$I = \frac{K_T}{\sum_{\tau=1}^{T} (1+\mu)^{T-\tau} x_\tau}$$

Consider trajectory, $\boldsymbol{\theta}$, for which investments start at $I_0 = I$ and then grow at rate μ:

$$I_t = (1+\mu)^t I$$

For this trajectory, we have:

$$K(\boldsymbol{\theta}_T) = I \cdot \sum_{\tau=1}^{T} (1+\mu)^{T-\tau} x_\tau = K_T$$

Therefore,

$$P_T(K(\boldsymbol{\theta}_T)) = P_T(K_T) = p^*$$

From the observation proved above, to check that p^* and $\boldsymbol{\theta}$ constitute a price-equilibrium, it suffices to check that $\boldsymbol{\theta}_T$ is feasible. Indeed,

$$H(\boldsymbol{\theta}_T) = w \cdot K(\boldsymbol{\theta}_T) + \sum_{\tau=1}^{T} (1+\mu)^{T-\tau} z_\tau^o(\mathbf{d}) \cdot I$$

On the other hand,

$$P_T(K(\boldsymbol{\theta}_T)) \cdot K(\boldsymbol{\theta}_T) = p^* \cdot K(\boldsymbol{\theta}_T) = w \cdot K(\boldsymbol{\theta}_T) + \frac{\sum_{\tau=1}^{T} (1+\mu)^{T-\tau} z_\tau^o(\mathbf{d})}{\sum_{\tau=1}^{T} (1+\mu)^{T-\tau} z_\tau^*} \cdot c \cdot K(\boldsymbol{\theta}_T)$$

It remains to show that

$$I = \frac{c \cdot K(\boldsymbol{\theta}_T)}{\sum_{\tau=1}^{T} (1+\mu)^{T-\tau} z_\tau^*}$$

Recalling that, by equation (5), $z_\tau^* = c \cdot x_\tau$, we get the required result:

$$\sum_{\tau=1}^{T} (1+\mu)^{T-\tau} z_\tau^* I = c \cdot \sum_{\tau=1}^{T} (1+\mu)^{T-\tau} x_\tau I = K(\boldsymbol{\theta}_T)$$

$\blacksquare$

Proof of Lemma 1:

Consider trajectory $\boldsymbol{\theta}$ for which

$$\lim_{t\to\infty} P_t\left(K\left(\boldsymbol{\theta}_t\right)\right) = p_\infty$$

for some p_∞. Let $\boldsymbol{\theta}_{t+1} = (I_{t-T+1}, ..., I_t)$ be an infeasible state in this trajectory. Then, it has to be that $I_t = 0$ and

$$K_{t+1} = x_2 I_{t-1} + ... + x_T I_{t-T+1}$$

By definition,

$$K_t = x_1 I_{t-1} + ... + x_T I_{t-T} \tag{9}$$

Therefore, we have:

$$K_{t+1} = K_t - (x_1 - x_2) I_{t-1} - ... - (x_T - x_{T+1}) I_{t-T} \tag{10}$$

where we defined x_{T+1} to be zero. All the terms that are subtracted from K_t in the RHS of (10) are non-negative since x_τ's are decreasing. Let

$$\tau_0 = \arg\max_{\tau=1..T} x_\tau I_{t-\tau}$$

Equation (9) implies that

$$x_{\tau_0} I_{t-\tau_0} \geq \frac{K_t}{T}$$

Therefore,

$$(x_{\tau_0} - x_{\tau_0+1}) I_{t-\tau_0} \geq \frac{(x_{\tau_0} - x_{\tau_0+1})}{T x_{\tau_0}} K_t$$

Plugging this inequality into (10), we obtain:

$$K_{t+1} \leq K_t - \frac{(x_{\tau_0} - x_{\tau_0+1})}{T x_{\tau_0}} K_t$$

Let

$$\psi = \frac{1}{2} \min_{\tau=1..T} \frac{(x_\tau - x_{\tau+1})}{T x_\tau}$$

By strict monotonicity of x's, ψ is greater than zero. We have shown that

$$K_{t+1} < (1-\psi) K_t$$

for any infeasible state $\boldsymbol{\theta}_{t+1}$. Let

$$K_\infty = P_0^{-1}(p_\infty)$$

Then, the assumption that prices converge to p_∞ implies that

$$\lim_{t\to\infty} \frac{K_t}{(1+\mu)^t} = K_\infty$$

For any $\epsilon > 0$, there must exist t_0 such that for any $t > t_0$

$$(1+\mu)^t (K_\infty - \epsilon) < K_t < (1+\mu)^t (K_\infty + \epsilon)$$

Let us pick a sufficiently small ϵ and show that infeasible states cannot happen after the corresponding t_0. Indeed, any infeasible state will satisfy

$$K_{t+1} < (1-\psi) K_t < (1-\psi)(1+\mu)^t (K_\infty + \epsilon)$$

On the other hand, it has to be that

$$K_{t+1} > (1+\mu)^{t+1}(K_\infty - \epsilon)$$

These two inequalities can hold simultaneously only if

$$(1-\psi)(K_\infty + \epsilon) > (1+\mu)(K_\infty - \epsilon)$$

which is equivalent to

$$(2+\mu-\psi)\epsilon > (\psi+\mu) K_\infty$$

Since $\psi > 0$, this inequality will not hold for sufficiently small values of ϵ. ■

Proof of Proposition 2:

We state and prove two results that together comprise the proof of Proposition 2.

Step 1: Under the conditions of Proposition 2, if for some trajectory $\boldsymbol{\theta}$ prices converge to p_∞, then there must exist a feasible trajectory $\boldsymbol{\theta}'$ such that

$$P_t(K(\boldsymbol{\theta}'_t)) = p_\infty$$

for all $t \geq T$.

Proof of Step 1:

Recall that $\boldsymbol{\theta}_t = (I_{t-T}, ..., I_{t-1})$ and consider the following sequence of T-dimensional vectors. Define $\boldsymbol{s}(t)$ to be the zero vector for $t < T$ and

$$\boldsymbol{s}(t) = \frac{\boldsymbol{\theta}_t}{(1+\mu)^t}$$

for $t \geq T$. Letting $\boldsymbol{s}_\tau(t)$ denote the τ-th component of $\boldsymbol{s}(t)$, we have:

$$\boldsymbol{s}_\tau(t) = \frac{I_{t-T-1+\tau}}{(1+\mu)^t}$$

Assume that prices converge for trajectory $\boldsymbol{\theta}$, i.e.,

$$\lim_{t\to\infty} P_t(K(\boldsymbol{\theta}_t)) = p_\infty$$

Note that

$$K(\boldsymbol{s}(t)) = (1+\mu)^{-t} K(\boldsymbol{\theta}_t)$$

The fact that the market grows at rate μ implies:

$$P_0\left(K\left(\boldsymbol{s}\left(t\right)\right)\right)=P_t\left(\left(1+\mu\right)^t K\left(\boldsymbol{s}\left(t\right)\right)\right)=P_t\left(K\left(\boldsymbol{\theta}_t\right)\right)$$

Therefore,

$$P_0\left(K\left(\boldsymbol{s}\left(t\right)\right)\right)\to p_\infty \tag{11}$$

Since $P_0\left(\cdot\right)$ is monotonic and continuous, the equation above implies

$$K\left(\boldsymbol{s}\left(t\right)\right)\to K_\infty \tag{12}$$

where $K_\infty=P_0^{-1}\left(p_\infty\right)$. Hence,

$$P_0\left(K\left(\boldsymbol{s}\left(t\right)\right)\right)K\left(\boldsymbol{s}\left(t\right)\right)\to p_\infty K_\infty \tag{13}$$

In Lemma 1, we have shown that if prices converge for some trajectory, then this trajectory can pass through only a finite number of infeasible states. Therefore, for values of t greater than some t_0, all states $\boldsymbol{\theta}_t$ are feasible:

$$P_t\left(K\left(\boldsymbol{\theta}_t\right)\right)K\left(\boldsymbol{\theta}_t\right)=H\left(\boldsymbol{\theta}_t\right)$$

For $t>t_0$, we have:

$$P_0\left(K\left(\boldsymbol{s}\left(t\right)\right)\right)K\left(\boldsymbol{s}\left(t\right)\right)=\left(1+\mu\right)^{-t}P_t\left(K\left(\boldsymbol{\theta}_t\right)\right)K\left(\boldsymbol{\theta}_t\right)=\left(1+\mu\right)^{-t}H\left(\boldsymbol{\theta}_t\right)=H\left(\boldsymbol{s}\left(t\right)\right)$$

Applying (13), we obtain:

$$H\left(\boldsymbol{s}\left(t\right)\right)\to p_\infty K_\infty \tag{14}$$

Let us now show that the vectors $\boldsymbol{s}\left(t\right)$ are bounded. Recall that

$$K\left(\boldsymbol{s}\left(t\right)\right)=x_1\boldsymbol{s}_T\left(t\right)+...+x_T\boldsymbol{s}_1\left(t\right)$$

All x's are strictly greater than zero, so condition (12) implies that $\boldsymbol{s}_\tau\left(t\right)$ are bounded for every τ.

Let us now construct trajectory $\boldsymbol{\theta}'=\left(I_0',I_1',...,I_t',...\right)$ such that $\boldsymbol{\theta}_t'$ is feasible for every $t\geq T$ and

$$P_t\left(K\left(\boldsymbol{\theta}_t'\right)\right)=p_\infty$$

for every $t\geq T$. Since all vectors $\boldsymbol{s}\left(t\right)$ are drawn from a bounded set, there must exist a converging subsequence $\boldsymbol{s}\left(t_i\right)$. Let

$$\boldsymbol{\theta}_T'=\left(1+\mu\right)^T\lim_{i\to\infty}\boldsymbol{s}\left(t_i\right)$$

Applying the linearity of $K(\cdot)$ and equation (12), we can compute capacity and revenues in state $\boldsymbol{\theta}'_T$ as follows:

$$K(\boldsymbol{\theta}'_T) = (1+\mu)^T K\left(\lim_{i\to\infty} \boldsymbol{s}(t_i)\right) = (1+\mu)^T \lim_{i\to\infty} K(\boldsymbol{s}(t_i)) = (1+\mu)^T K_\infty$$

$$P_T(K(\boldsymbol{\theta}'_T))\, K(\boldsymbol{\theta}'_T) = P_T\left((1+\mu)^T K_\infty\right)(1+\mu)^T K_\infty = (1+\mu)^T P_0(K_\infty) K_\infty$$

The cost function is also linear in investments, so we have:

$$H(\boldsymbol{\theta}'_T) = (1+\mu)^T H\left(\lim_{i\to\infty} \boldsymbol{s}(t_i)\right) = (1+\mu)^T \lim_{i\to\infty} H(\boldsymbol{s}(t_i)) = (1+\mu)^T p_\infty K_\infty$$

From these equations, one can see that state $\boldsymbol{\theta}'_T$ is feasible and $P_T(K(\boldsymbol{\theta}'_T)) = p_\infty$. It remains to be shown that this trajectory can be continued indefinitely into the future so that state feasibility and prices are preserved. We will prove the following statement. Assume that state $\boldsymbol{\theta}'_t$ is feasible for some t, $P_t(K(\boldsymbol{\theta}'_t)) = p_\infty$, and there exists a subsequence of $\boldsymbol{s}(t)$, $\boldsymbol{s}(t_i)$, such that

$$\boldsymbol{\theta}'_t = (1+\mu)^t \lim_{i\to\infty} \boldsymbol{s}(t_i) \tag{15}$$

Then, we will show that there exists an investment I'_t such that state $\boldsymbol{\theta}'_{t+1}$ is feasible,

$$P_{t+1}\left(K\left(\boldsymbol{\theta}'_{t+1}\right)\right) = p_\infty$$

and there exists a subsequence of $\boldsymbol{s}(t)$, $\boldsymbol{s}(t'_i)$, such that

$$\boldsymbol{\theta}'_{t+1} = (1+\mu)^{t+1} \lim_{i\to\infty} \boldsymbol{s}(t'_i) \tag{16}$$

The initial state, $\boldsymbol{\theta}'_T$ was constructed to satisfy all of the above-mentioned conditions, so proving the claim above will conclude the proof of this lemma.

Let us consider sequence $\{\boldsymbol{s}(t_i+1)\}_{i=1}^\infty$. This sequence is comprised of elements of $\{\boldsymbol{s}(t)\}$ immediately following the elements of $\{\boldsymbol{s}(t_i)\}$ from (15). Sequence $\{\boldsymbol{s}(t_i+1)\}_{i=1}^\infty$ must have a converging subsequence (since all vectors $\boldsymbol{s}(\cdot)$ are drawn from a compact set). Let $\{\boldsymbol{s}(t'_i)\}_{i=1}^\infty$ denote this converging subsequence and let

$$\boldsymbol{\varphi} = (1+\mu)^{t+1} \lim_{i\to\infty} \boldsymbol{s}(t'_i)$$

To show that $\boldsymbol{\varphi}$ is a legitimate candidate for $\boldsymbol{\theta}'_{t+1}$, we need to check that the first $T-1$ elements of $\boldsymbol{\varphi}$ correspond to the last $T-1$ elements of $\boldsymbol{\theta}'_t$. Letting $\boldsymbol{\varphi}_\tau$ denote the τ-th coordinate of $\boldsymbol{\varphi}$, we have:

$$\boldsymbol{\varphi}_\tau = (1+\mu)^{t+1} \lim_{i\to\infty} \boldsymbol{s}_\tau(t'_i)$$

By construction, for $\tau < T$

$$\boldsymbol{s}_\tau(t'_i) = (1+\mu)^{-1} \boldsymbol{s}_{\tau+1}(t'_i - 1)$$

Therefore, for $\tau < T$,

$$\boldsymbol{\varphi}_\tau = (1+\mu)^{t+1} \lim_{i\to\infty} \boldsymbol{s}_\tau (t_i') = (1+\mu)^t \lim_{i\to\infty} \boldsymbol{s}_{\tau+1} (t_i' - 1) = (1+\mu)^t \lim_{i\to\infty} \boldsymbol{s}_{\tau+1} (t_i) = I'_{t-T+\tau}$$

Setting $I'_t = \boldsymbol{\varphi}_T$, we will have

$$\boldsymbol{\theta}'_{t+1} = \boldsymbol{\varphi} = (1+\mu)^{t+1} \lim_{i\to\infty} \boldsymbol{s} (t_i')$$

It remains to check that $\boldsymbol{\theta}'_{t+1}$ is feasible and that $P_{t+1}\left(K\left(\boldsymbol{\theta}'_{t+1}\right)\right) = p_\infty$. To that end, we apply the assumption of constant market growth, linearity of $K(\cdot)$ and $H(\cdot)$, and equations (12) and (14) to complete the proof:

$$K\left(\boldsymbol{\theta}'_{t+1}\right) = (1+\mu)^{t+1} K_\infty$$

$$P_{t+1}\left(K\left(\boldsymbol{\theta}'_{t+1}\right)\right) K\left(\boldsymbol{\theta}'_{t+1}\right) = (1+\mu)^{t+1} P_{t+1}\left((1+\mu)^{t+1} K_\infty\right) K_\infty = (1+\mu)^{t+1} p_\infty K_\infty$$

$$H\left(\boldsymbol{\theta}'_{t+1}\right) = (1+\mu)^{t+1} p_\infty K_\infty$$

■

Step 2: Under the conditions of Proposition 2, p^* is the unique asymptotic equilibrium price.

Proof of Step 2:

Let us consider some equilibrium price p_∞ and show that $p_\infty = p^*$. First, we will show that there exists some η such that

$$I_t - \eta K_t \to s^*$$

when $t \to \infty$, for some constant s^*. To sustain constant prices, the firm needs to grow capacity at rate μ:

$$K_{t+1} = (1+\mu) K_t$$

The latest investment can be expressed as:

$$I_t = \frac{\mu}{x_1} K_t + \frac{x_1 - x_2}{x_1} I_{t-1} + ... + \frac{x_T - x_{T+1}}{x_1} I_{t-T} \tag{17}$$

where $x_{T+1} = 0$. Let us find η satisfying:

$$\frac{\mu}{x_1} = \eta - \frac{x_1 - x_2}{x_1 (1+\mu)} \eta - ... - \frac{x_T - x_{T+1}}{x_1 (1+\mu)^T} \eta \tag{18}$$

If $\mu = 0$, then any η satisfies the equation above and we will proceed with $\eta = 0$. If $\mu > 0$, then the coefficient on η in the right-hand-side is strictly greater than zero and the equation defines a unique (positive) η. Since

$$K_{t-\tau} = \frac{K_t}{(1+\mu)^\tau}$$

equations (17) and (18) imply:

$$(I_t - \eta K_t) = \frac{x_1 - x_2}{x_1}(I_{t-1} - \eta K_{t-1}) + ... + \frac{x_T - x_{T+1}}{x_1}(I_{t-T} - \eta K_{t-T})$$

Let $s_t = I_t - \eta K_t$, and let us show the sequence $\{s_t\}$ converges. We know that

$$s_t = \frac{x_1 - x_2}{x_1} s_{t-1} + ... + \frac{x_T - x_{T+1}}{x_1} s_{t-T} \tag{19}$$

All coefficients in the right-hand-side are strictly positive and add up to unity. Let

$$\overline{s}_t = \max_{\tau=0,...,T-1} s_{t-\tau}$$

and

$$\underline{s}_t = \min_{\tau=0,...,T-1} s_{t-\tau}$$

Equation (19) implies that $\underline{s}_{t-1} \leq s_t \leq \overline{s}_{t-1}$. Therefore, it has to be that $\underline{s}_{t-1} \leq \underline{s}_t \leq \overline{s}_t \leq \overline{s}_{t-1}$. These inequalities imply that both sequences $\{\underline{s}_t\}$ and $\{\overline{s}_t\}$ are monotonic and must converge to some limits, $\underline{s}^*$ and $\overline{s}^*$, respectively. Let us assume that $\underline{s}^* \neq \overline{s}^*$. Then, by equation (19),

$$s_{t+1} \geq \underline{s}_t + \left(\min_{\tau=1..T} \frac{x_\tau - x_{\tau+1}}{x_1}\right)(\overline{s}_t - \underline{s}_t)$$

For sufficiently large values of t, this becomes

$$s_{t+1} \geq \underline{s}^* + \left(\min_{\tau=1..T} \frac{x_\tau - x_{\tau+1}}{x_1}\right)(\overline{s}^* - \underline{s}^*) - \epsilon$$

Therefore, for large values of t,

$$\underline{s}_{t+1} \geq \underline{s}^* + \left(\min_{\tau=1..T} \frac{x_\tau - x_{\tau+1}}{x_1}\right)(\overline{s}^* - \underline{s}^*) - \epsilon$$

On the other hand, $\underline{s}_{t+1}$ converges to $\underline{s}^*$, so for large t's, $\underline{s}_{t+1}$ has to be not greater than $\underline{s}^* + \epsilon$. Choosing ϵ so that

$$\underline{s}^* + \epsilon < \underline{s}^* + \left(\min_{\tau=1..T} \frac{x_\tau - x_{\tau+1}}{x_1}\right)(\overline{s}^* - \underline{s}^*) - \epsilon$$

ensures a contradiction. We have shown that $\underline{s}^* = \overline{s}^* \equiv s^*$ and, since $\underline{s}_t \leq s_t \leq \overline{s}_t$, s_t converges to s^*.

If $\mu = 0$, then $s_t = I_t$ and, by the previous step, we know that investment levels converge. Then,

$$p_\infty - w = \frac{C(\boldsymbol{\theta}_t)}{K(\boldsymbol{\theta}_t)} = \lim_{t\to\infty} \frac{C(\boldsymbol{\theta}_t)}{K(\boldsymbol{\theta}_t)} = \frac{s^* \sum_{\tau=1}^T z_\tau}{s^* \sum_{\tau=1}^T x_\tau} = p^* - w$$

Now assume that $\mu > 0$ and let $K_\infty = P_0^{-1}(p_\infty)$. Then, $K_t = (1+\mu)^t K_\infty$, and, therefore,

$$K_\infty = \frac{K_t}{(1+\mu)^t} = \lim_{t\to\infty} \frac{K_t}{(1+\mu)^t}$$

The result obtained in the previous step implies:

$$\lim_{t\to\infty} \frac{I_t - \eta K_t}{(1+\mu)^t} = 0$$

Hence,

$$\lim_{t\to\infty} \frac{I_t}{(1+\mu)^t} = \eta K_\infty \tag{20}$$

Now let us compute η:

$$K_\infty = \lim_{t\to\infty} \frac{K_t}{(1+\mu)^t} = \lim_{t\to\infty} \frac{x_1 I_{t-1} + ... + x_T I_{t-T}}{(1+\mu)^t} = \left(\frac{x_1}{(1+\mu)} + ... + \frac{x_T}{(1+\mu)^T} \right) \eta K_\infty$$

and

$$\eta = \frac{1}{\sum_{\tau=1}^T x_\tau (1+\mu)^{-\tau}}$$

Since prices are constant, capacity grows at rate μ, and the rate-of-return constraint is satisfied in every state, capital costs must also grow at rate μ. Let $C_\infty = \frac{C_t}{(1+\mu)^t}$. Then,

$$C_\infty = \frac{C_t}{(1+\mu)^t} = \lim_{t\to\infty} \frac{C_t}{(1+\mu)^t} = \lim_{t\to\infty} \frac{z_1 I_{t-1} + ... + z_T I_{t-T}}{(1+\mu)^t}$$

Let us now apply (20):

$$C_\infty = \lim_{t\to\infty} \frac{z_1 I_{t-1} + ... + z_T I_{t-T}}{(1+\mu)^t} = \sum_{\tau=1}^T z_\tau (1+\mu)^{-\tau} \eta K_\infty$$

To conclude, observe that

$$p_\infty - w = \frac{C_\infty}{K_\infty} = \frac{\sum_{\tau=1}^T z_\tau (1+\mu)^{-\tau}}{\sum_{\tau=1}^T x_\tau (1+\mu)^{-\tau}} = p^* - w$$

This completes the proof of Proposition 2. ■

Proof of Proposition 2':

As with the proof of Proposition 2, we provide a sequence of intermediate findings that together demonstrate the proof of the result.

Step A: Under the conditions of Proposition 2', if for some trajectory $\boldsymbol{\theta}$ prices converge, then this trajectory can pass through only a finite number of infeasible states.

Proof of Step A:

Consider trajectory $\boldsymbol{\theta}$ for which

$$\lim_{t\to\infty} P_t\left(K\left(\boldsymbol{\theta}_t\right)\right) = p_\infty$$

Let $\boldsymbol{\theta}_{t+1} = (I_{t-T+1}, ..., I_t)$ be an infeasible state in this trajectory. Then, it has to be that $I_t = 0$ and

$$C_{t+1} = z_2 I_{t-1} + ... + z_T I_{t-T+1}$$

By definition,

$$C_t = z_1 I_{t-1} + ... + z_T I_{t-T} \tag{21}$$

This implies that:

$$C_{t+1} = C_t - \left(z_1 - z_2\right) I_{t-1} - ... - \left(z_T - z_{T+1}\right) I_{t-T} \tag{22}$$

where we have defined z_{T+1} to be zero. All the terms that are subtracted from C_t in the RHS of (21) are non-negative since z_τ's are decreasing. Let

$$\tau_0 = \arg\max_{\tau=1..T} z_\tau I_{t-\tau}$$

Equation (21) implies that

$$z_{\tau_0} I_{t-\tau_0} \geq \frac{C_t}{T}$$

Therefore,

$$\left(z_{\tau_0} - z_{\tau_0+1}\right) I_{t-\tau_0} \geq \frac{\left(z_{\tau_0} - z_{\tau_0+1}\right)}{T z_{\tau_0}} C_t$$

Plugging this inequality into (21), we obtain:

$$C_{t+1} \leq C_t - \frac{\left(z_{\tau_0} - z_{\tau_0+1}\right)}{T z_{\tau_0}} C_t$$

Let

$$\psi = \frac{1}{2} \min_{\tau=1..T} \frac{\left(z_{\tau_0} - z_{\tau_0+1}\right)}{T z_{\tau_0}}$$

By strict monotonicity of z_τ's, ψ is greater than zero. We have shown that

$$C_{t+1} < \left(1 - \psi\right) C_t$$

for any infeasible state $\boldsymbol{\theta}_{t+1}$.

Let

$$K_\infty = P_0^{-1}\left(p_\infty\right)$$

Then, the assumption that prices converge to p_∞ implies that

$$\lim_{t\to\infty} \frac{K_t}{(1+\mu)^t} = K_\infty$$

and that

$$\lim_{t\to\infty} \frac{(P_t(K_t) - w)K_t}{(1+\mu)^t} = (p_\infty - w) K_\infty$$

Let us fix a sufficiently small $\epsilon > 0$. Then there must exist t_0 such that for any $t > t_0$

$$(1+\mu)^t ((p_\infty - w) K_\infty + \epsilon) > (P_t(K_t) - w)K_t > (1+\mu)^t ((p_\infty - w) K_\infty - \epsilon)$$

If there are infinitely many infeasible states, then there will exist $t > t_0$ such that $\boldsymbol{\theta}_t$ is feasible and $\boldsymbol{\theta}_{t+1}$ is not (recall that feasible states happen at least every T periods because $P_0(0) > w + z_1$). For this value of t, the following set of conditions holds:

$$(P_t(K_t) - w) K_t = C_t \tag{23}$$

$$(P_t(K_t) - w) K_t < (1+\mu)^t ((p_\infty - w) K_\infty + \epsilon) \tag{24}$$

$$(P_{t+1}(K_{t+1}) - w) K_{t+1} > (1+\mu)^{t+1} ((p_\infty - w) K_\infty - \epsilon) \tag{25}$$

$$C_{t+1} < (1-\psi) C_t \tag{26}$$

Applying conditions (23),(24), and (26), we obtain:

$$C_{t+1} < (1-\psi)(P_t(K_t) - w) K_t < (1-\psi)(1+\mu)^t ((p_\infty - w) K_\infty + \epsilon) \tag{27}$$

Therefore, if ϵ is small enough that

$$(1-\psi)(1+\mu)^t ((p_\infty - w) K_\infty + \epsilon) < (1+\mu)^{t+1} ((p_\infty - w) K_\infty - \epsilon)$$

then (25) and (27) imply

$$C_{t+1} < (P_{t+1}(K_{t+1}) - w) K_{t+1}$$

Let us now reach a contradiction by showing that there must have existed a positive I_t such that $\boldsymbol{\theta}_{t+1}$ would have been feasible. At $I_t = 0$, we have shown that

$$\frac{C_{t+1}}{K_{t+1}} < P_{t+1}(K_{t+1}) - w$$

When $I_{t+1} \to \infty$,

$$\frac{C_{t+1}}{K_{t+1}} \to \frac{z_1}{x_1} > 0$$

and

$$P_{t+1}(K_{t+1}) - w \to -w$$

Therefore, these two functions have to intersect at some positive level of investment I_t. ■

Step B: Under the conditions of Proposition 2', if for some trajectory $\boldsymbol{\theta}$ prices converge to p_∞, then there must exist a feasible trajectory $\boldsymbol{\theta}'$ such that

$$P_t\left(K\left(\boldsymbol{\theta}_t'\right)\right) = p_\infty$$

for all $t \geq T$.

Proof of Step B:

Similar to that of Step 1 in the proof of Proposition 2, and hence omitted. ■

Step C: Under the conditions of Proposition 2', p^* is the unique asymptotic equilibrium price.

Proof of Step C:

Analogous to the proof of Step 2 in the proof of Proposition 2, with x_τ replaced by z_τ and K_t replaced by C_t.

This concludes the proof of Proposition 2'. ■

Proof of Proposition 3:

Let $ACC_t(\boldsymbol{\theta}_t)$ denote the average capital cost of capacity in period t, defined as:

$$ACC_t(\boldsymbol{\theta}_t) = \frac{z_T \cdot I_{t-T} + \ldots + z_1 \cdot I_{t-1}}{K(\boldsymbol{\theta}_t)}$$

Then, by the rate-of-return regulation constraint, it is easy to see that

$$P(K(\boldsymbol{\theta}_t)) = w + ACC_t(\boldsymbol{\theta}_t)$$

in every period when $\boldsymbol{\theta}_t$ is feasible. Let $\boldsymbol{\theta}$ be the equilibrium trajectory from Proposition 1. Then,

$$p^* = w + ACC_T(\boldsymbol{\theta}_T)$$

and the past T investment growth rates are equal to $1 + \mu$ for $\boldsymbol{\theta}_T$.

The first claim of Proposition 5 then follows immediately from Proposition 2 of Rajan and Reichelstein (2008); the second claim is also an immediate consequence of their Proposition 3. ■

Proof of Proposition 4:

Consider the equilibrium trajectory, $\boldsymbol{\theta}^*$, and some other trajectory, $\boldsymbol{\theta}$, such $\boldsymbol{\theta}_T$ is close to $\boldsymbol{\theta}^*_T$. Let $\varepsilon_t = I_t - I^*_t$ and $\boldsymbol{\varepsilon}_t = \boldsymbol{\theta}_t - \boldsymbol{\theta}^*_t = (\varepsilon_{t-T}, ..., \varepsilon_{t-1})$ denote the difference between the two investment paths. We will consider the mapping, Φ, which defines transitions from vector $\frac{\boldsymbol{\varepsilon}_t}{(1+\mu)^t}$ to vector $\frac{\boldsymbol{\varepsilon}_{t+1}}{(1+\mu)^{t+1}}$. The proof consists of three major steps. First, it will be shown that, when $\frac{\boldsymbol{\varepsilon}_t}{(1+\mu)^t}$ is sufficiently small, the mapping Φ is uniquely defined and that it is constant over time. In the second step, this mapping will be linearized around the zero-vector. The proof will be concluded by showing that the zero-vector is an attractor for the linearized mapping.

Step 1: Φ is well-defined.

First, let us show that mapping Φ is constant over time. Let $\boldsymbol{\varepsilon}^i_t$ denote the i-th component of $\boldsymbol{\varepsilon}_t$, and let $s_t = \frac{\boldsymbol{\varepsilon}_t}{(1+\mu)^t}$ and $s^i_t = \frac{\boldsymbol{\varepsilon}^i_t}{(1+\mu)^t}$. By definition,

$$\boldsymbol{\varepsilon}^i_t = \varepsilon_{t-1-T+i} = I_{t-1-T+i} - I^*_{t-1-T+i}$$

Therefore, for the first $T-1$ components of $s_{t+1} = \Phi(s_t)$, we have:

$$s^i_{t+1} = \frac{\boldsymbol{\varepsilon}^i_{t+1}}{(1+\mu)^{t+1}} = \frac{1}{1+\mu} \cdot \frac{\boldsymbol{\varepsilon}^{i+1}_t}{(1+\mu)^t} = \frac{s^{i+1}_t}{1+\mu} \tag{28}$$

This part of the mapping clearly does not depend on t. The last component of $\boldsymbol{\varepsilon}_{t+1}$ is:

$$\boldsymbol{\varepsilon}^T_{t+1} = I_t - I^*_t$$

Since, on the equilibrium path, the investments grow at rate μ, the vector

$$v \equiv \frac{\boldsymbol{\theta}^*_t}{(1+\mu)^t}$$

is constant over time. Investment I_t is chosen so as to satisfy the RoR constraint:

$$R_{t+1}(K(\boldsymbol{\theta}_{t+1})) = C(\boldsymbol{\theta}_{t+1}) \tag{29}$$

We also know that

$$R_{t+1}(K(\boldsymbol{\theta}^*_{t+1})) = C(\boldsymbol{\theta}^*_{t+1}) \tag{30}$$

Subtracting (30) from (29), one obtains:

$$R_{t+1}(K(\boldsymbol{\theta}_{t+1})) - R_{t+1}(K(\boldsymbol{\theta}^*_{t+1})) = C(\boldsymbol{\varepsilon}_{t+1}) \tag{31}$$

Dividing both sides by $(1+\mu)^{t+1}$, we get:

$$\frac{1}{(1+\mu)^{t+1}} \cdot R_{t+1}(K(\boldsymbol{\theta}_{t+1})) - \frac{1}{(1+\mu)^{t+1}} \cdot R_{t+1}(K(\boldsymbol{\theta}^*_{t+1})) = C(s_{t+1})$$

As demand grows at rate μ, this is equivalent to:

$$R_0(K(\frac{\boldsymbol{\theta}_{t+1}}{(1+\mu)^{t+1}})) - R_0(K(\frac{\boldsymbol{\theta}^*_{t+1}}{(1+\mu)^{t+1}})) = C(s_{t+1})$$

Or,

$$R_0(K(s_{t+1}+v)) - R_0(K(v)) = C(s_{t+1}) \tag{32}$$

The last component of s_{t+1}, s^T_{t+1}, is defined by the equation above. This equation does not depend on t, therefore Φ is constant over time.

To check that Φ is uniquely defined when s_t is close to zero, we need to show that equation (32) defines s^T_{t+1} uniquely in that case. Let us expand (32) in the following way:

$$R_0(s^T_{t+1} + \frac{s^T_t + ... + s^2_t}{1+\mu} + K(v)) - R_0(K(v)) - (w+z^o_1)s^T_{t+1} - \frac{(w+z^o_2)s^T_t + ... + (w+z^o_T)s^2_t}{1+\mu} = 0 \tag{33}$$

First, let us apply the implicit function theorem to show that, in the neighborhood of the zero-vector, there exists a continuous function $s^T_{t+1}(s_t)$ satisfying (32), such that $s^T_{t+1}(\mathbf{0}) = 0$. According to this theorem, we need to check that the partial derivative of the left-hand-side of (33) with respect to s^T_{t+1} is different from zero. This derivative is equal to:

$$R'_0(s^T_{t+1} + \frac{s^T_t + ... + s^2_t}{1+\mu} + K(v)) - w - z^o_1$$

At $s_t = \mathbf{0}$ and $s^T_{t+1} = 0$, this expression reduces to:

$$R'_0(K(v)) - w - z^o_1$$

Note that

$$R'_0(K(v)) = p^* + P'_0(K(v)) < p^* < w + z^o_1$$

Therefore, the partial derivative of the LHS of (33) with respect to s^T_{t+1} is negative, and there exists a solution to (32), $s^T_{t+1}(s_t)$, which is close to 0 when s_t is close to $\mathbf{0}$. Let us now show that this solution is unique. Assume that there exists some other $\hat{s}^T_{t+1}$ also satisfying (33). Let $\Delta = \hat{s}^T_{t+1} - s^T_{t+1}$. Then, subtracting two versions of (32) from one another, we have:

$$R_0(K(s_{t+1}+v)+\Delta) - R_0(K(s_{t+1}+v)) = \Delta \cdot (w + z^o_1)$$

Since $R_0(\cdot)$ is concave, $R_0(0) = 0$, and both $K(s_{t+1}+v)+\Delta$ and $K(s_{t+1}+v)$ are greater than 0, it follows that

$$\frac{R_0(K(s_{t+1}+v)+\Delta) - R_0(K(s_{t+1}+v))}{\Delta} < \frac{R_0(K(s_{t+1}+v))}{K(s_{t+1}+v)} = P_0(K(s_{t+1}+v))$$

Now recall that we defined s^T_{t+1} to be the root close to zero, so

$$P_0(K(s_{t+1}+v)) \approx p^* < w + z^o_1$$

Therefore,

$$R_0(K(s_{t+1}+v)+\Delta) - R_0(K(s_{t+1}+v)) < \Delta \cdot (w + z^o_1)$$

and there exists a neighborhood of $\mathbf{0}$ where Φ is uniquely defined.

Step 2: Linearization of Φ.

As was shown in equation (28), for $1 \leq i \leq T-1$

$$s^i_{t+1} = \frac{s^{i+1}_t}{1+\mu}$$

The last component s_{t+1}, s^T_{t+1} is defined by equation (33). To linearize Φ around $\mathbf{0}$, we will apply the first-order expansion to s^T_{t+1}. As $s^T_{t+1}(\mathbf{0}) = 0$,

$$s^T_{t+1} \approx \frac{\partial s^T_{t+1}}{\partial s^1_t} \cdot s^1_t + ... + \frac{\partial s^T_{t+1}}{\partial s^T_t} \cdot s^T_t$$

The partial derivatives can be computed from equation (33) using the implicit function differentiation theorem. At $s_t = \mathbf{0}$ they are equal to:

$$\frac{\partial s^T_{t+1}}{\partial s^i_t} = \frac{w + z^o_{T-i+2} - R'_0(K(v))}{(1+\mu)(R'_0(K(v)) - w - z^o_1)}$$

for $2 \leq i \leq T$ and $\frac{\partial s^T_{t+1}}{\partial s^1_t} = 0$.

For brevity, let us write R' for $R'_0(K(v))$. For future reference, we note that $R'_0(K(v)) < P_0(K(v)) = p^*$. The linearized mapping takes the following form:

$$\begin{pmatrix} s^1_{t+1} \\ s^2_{t+1} \\ ... \\ ... \\ s^T_{t+1} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{1+\mu} & 0 & ... & ... & 0 \\ 0 & 0 & \frac{1}{1+\mu} & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & ... & 0 & \frac{1}{1+\mu} \\ 0 & \frac{w+z^o_T-R'}{(1+\mu)(R'-w-z^o_1)} & \frac{w+z^o_{T-1}-R'}{(1+\mu)(R'-w-z^o_1)} & ... & ... & \frac{w+z^o_2-R'}{(1+\mu)(R'-w-z^o_1)} \end{pmatrix} \cdot \begin{pmatrix} s^1_t \\ s^2_t \\ ... \\ ... \\ s^T_t \end{pmatrix} \quad (34)$$

Step 3: Proof of stability.

To show that the zero-vector is an attractor for the linear system (34), we will apply the Jury stability criterion. The first step is to compute the characteristic polynomial of this system. Let $a_0, ..., a_T$ be the coefficients of this polynomial, $F(u)$, on degrees $T, ..., 0$, respectively, with a_0 normalized to unity. Given the form of the transition matrix in (34), one can see that:

$$a_0 = 1$$

$$a_1 = -\frac{1}{(1+\mu)} \cdot \frac{w + z_2^o - R'}{R' - w - z_1^o}$$
$$\ldots$$
$$a_i = -\frac{1}{(1+\mu)^i} \cdot \frac{w + z_{i+1}^o - R'}{R' - w - z_1^o}$$
$$\ldots$$
$$a_{T-1} = -\frac{1}{(1+\mu)^{T-1}} \cdot \frac{w + z_T^o - R'}{R' - w - z_1^o}$$
$$a_T = 0$$

Jury's criterion states that zero-vector is an attractor for a linear system if all the roots of the characteristic polynomial, corresponding to that system, lie in the zero circle. The Jury test is a procedure for checking that this condition holds for a specific polynomial. To use this test, we first construct the following array:

$$\begin{array}{ccccc}
a_T^{(0)} & a_{T-1}^{(0)} & \ldots & a_1^{(0)} & a_0^{(0)} \\
a_0^{(0)} & a_1^{(0)} & \ldots & a_{T-1}^{(0)} & a_T^{(0)} \\
a_{T-1}^{(1)} & a_{T-2}^{(1)} & \ldots & a_0^{(1)} & \\
a_0^{(1)} & a_1^{(1)} & \ldots & a_{T-1}^{(1)} & \\
\ldots & & & & \\
a_2^{(T-2)} & a_1^{(T-2)} & a_0^{(T-2)} & & \\
a_0^{(T-2)} & a_1^{(T-2)} & a_2^{(T-2)} & &
\end{array}$$

In this array $a_t^{(0)} = a_t$ for all t, and, then, each next odd row is computed as follows:

$$a_t^{(i)} = \begin{vmatrix} a_{t+1}^{(i-1)} & a_0^{(i-1)} \\ a_{T-i-t}^{(i-1)} & a_{T-i+1}^{(i-1)} \end{vmatrix} \tag{35}$$

for $0 \leq t \leq T - i$. Four conditions have to be met, for the polynomial $F(u)$ to have all the roots inside the unit circle. First, $|a_T^{(0)}|$ must not exceed $a_0^{(0)}$. This condition is obviously satisfied as $a_T^{(0)} = 0$ and $a_0^{(0)} = 1$. Second, we need to check that $F(1)$ is greater than zero. Third, $F(-1)$ must be greater than zero for even T and less than zero for odd T. Finally, for each $i \geq 1$, $|a_{T-i}^{(i)}|$ must be greater than $|a_0^{(i)}|$. In what follows, we check these conditions one by one.

First let us compute $F(1)$. Using expressions for coefficients obtained earlier, we get:

$$F(1) = 1 - \frac{\sum_{i=1}^{T-1} (1+\mu)^{-i} (w + z_{i+1}^o - R')}{(R' - w - z_1)}$$

Since, as was noted earlier, $R' - w - z_1 < 0$, we need to show that

$$(R' - w - z_1) - \sum_{i=1}^{T-1} (1+\mu)^{-i} (w + z_{i+1}^o - R') < 0$$

or, that

$$\sum_{i=0}^{T-1}(1+\mu)^{-i}(w+z^o_{i+1}-R')>0$$

Note that $R'<p^*$, which, under the assumptions made, is equal to:

$$p^*=w+\frac{\sum_{i=1}^{T}(1+\mu)^{(T-i)}z^o_i}{\sum_{i=1}^{T}(1+\mu)^{T-i}}$$

Therefore,

$$\sum_{i=1}^{T}(1+\mu)^{T-i}R'<\sum_{i=1}^{T}(1+\mu)^{T-i}(w+z^o_i)$$

Dividing by $(1+\mu)^{T-1}$ and rearranging, we obtain the required result.

Let us now check that $F(-1)$ is greater than zero for even t and less than zero for odd T. First, note that $F(-1)$ has the same sign as:

$$(w+z^o_1-R')F(-1)=(w+z^o_1-R')(-1)^T+...+(w+z^o_T-R')(1+\mu)^{-(T-1)}(-1)^1 \quad (36)$$

Since the firm uses straight-line depreciation to account for its assets, we have:

$$z^0_i=v\left(\frac{1}{T}+\frac{T+1-i}{T}r\right)$$

Therefore, z^o_i form an arithmetic progression.

From now on, we use the assumption that $\mu=0$. By equation (36), for even values of T, we have:

$$(w+z^o_1-R')F(-1)=v\cdot\frac{T}{2}\cdot\frac{1}{T}r=vr/2>0$$

For odd values of T, the expression is:

$$(w+z^o_1-R')F(-1)=-(w+(\frac{1}{T}+r)v-R')+\frac{T-1}{2}\cdot\frac{1}{T}rv=R'-w-\frac{1}{T}v-\frac{T+1}{2}rv$$

But we know that

$$R'<p^*=w+\frac{1}{T}v+\frac{T(T+1)}{2T}rv=w+\frac{1}{T}v+\frac{T+1}{2}r$$

Therefore, in this case,

$$(w+z^o_1-R')F(-1)<0$$

It remains to be shown that for each $i\geq 1$,

$$|a^{(i)}_{T-i}|>|a^{(i)}_0|$$

Let b denote the following quantity:

$$b = \frac{\frac{1}{T} rv}{z_1^o + w - R'}$$

Then, the coefficients of the characteristic polynomial can be rewritten as:

$$a_t^{(0)} = 1 - bt$$

for $0 \leq t \leq T-1$ and $a_T^{(0)} = 0$.

In the rest of the proof, the following upper-bound on b will be employed. Observe that

$$R' < p^* = w + \frac{z_1^o + ... + z_T^o}{T} = w + \frac{z_1^o + z_T^o}{2}$$

After simple algebra, the inequality above translates into:

$$b(T-1) = \frac{(T-1)rv}{T(z_1^o + w - R')} < 2$$

Let us now apply formula (37) to compute the third row in the Jury array:

$$a_t^{(1)} = \begin{vmatrix} a_{t+1}^{(0)} & 1 \\ a_{T-1-t}^{(0)} & 0 \end{vmatrix} = -a_{T-i-t}^{(0)} = (T-1-t)b - 1$$

Therefore, $|a_{T-1}^{(1)}| = 0$. On the other hand,

$$|a_0^{(1)}| = |(T-1)b - 1| < 1$$

by the upper-bound on b derived earlier. We have shown that $|a_0^{(1)}| < |a_{T-1}^{(1)}|$.

The fifth row of the array can be computed as follows:

$$a_t^{(2)} = \begin{vmatrix} a_{t+1}^{(1)} & a_0^{(1)} \\ a_{T-2-t}^{(1)} & a_{T-1}^{(1)} \end{vmatrix} = \begin{vmatrix} (T-2-t)b - 1 & (T-1)b - 1 \\ (t+1)b - 1 & -1 \end{vmatrix}$$

The determinant above reduces to:

$$a_t^{(2)} = (2 - (T-1)b)T \cdot (t+1) = \delta \cdot (t+1)$$

where $\delta = (2 - (T-1)b)T > 0$. Clearly, $|a_0^{(2)}| < |a_{T-2}^{(2)}|$. Let us now show that all subsequent odd rows will have the same structure, namely that, if $a_t^{(i-1)} = \delta(t+1)$ for some positive δ, then

$$a_t^{(i)} = \delta'(t+1)$$

for some positive δ'. Indeed,

$$a_t^{(i)} = \begin{vmatrix} (t+2)\delta & \delta \\ (T-i-t+1)\delta & (T-i+2)\delta \end{vmatrix} = \delta^2(T+3-i)(t+1) \tag{37}$$

All remaining inequalities follow immediately. ■

Proof of Lemma 2:

First, we will establish an auxiliary lemma that will convert the case of constant market growth to the case of stationary markets ($\mu = 0$). Then, we will prove Lemma 2 under the assumption of no growth.

Auxiliary Lemma: Given a model with constant market growth, consider the following model with stationary markets:

$$P'_t(K'_t) = a - bK'_t$$

$$x'_\tau = \frac{x_\tau}{(1+\mu)^\tau}$$

$$z'_\tau = \frac{z_\tau}{(1+\mu)^\tau}$$

Then, state $\boldsymbol{\theta}_t = (I_0, ..., I_{t-1})$ is feasible in the constant growth model if and only if state $\boldsymbol{\theta}'_t = (\frac{I_0}{(1+\mu)^0}, ..., \frac{I_{t-1}}{(1+\mu)^{t-1}})$ is feasible in the stationary model.

Proof:

First observe that

$$K'_t(\boldsymbol{\theta}'_t) = \sum_{\tau=1}^{t} I_{t-\tau}(1+\mu)^{-(t-\tau)}x_\tau(1+\mu)^{-\tau} = (1+\mu)^{-t}K_t$$

Hence,

$$P'_t(K'_t) = P_t(K_t)$$

and

$$P'_t(K'_t)K'_t = (1+\mu)^{-t}P_t(K_t)K_t$$

It remains to show that

$$H'_t(\boldsymbol{\theta}'_t) = (1+\mu)^{-t}H_t(\boldsymbol{\theta}_t)$$

Indeed,

$$H'_t(\boldsymbol{\theta}'_t) = \sum_{\tau=1}^{t} I_{t-\tau}(1+\mu)^{-(t-\tau)}z_\tau(1+\mu)^{-\tau} = (1+\mu)^{-t}H_t$$

■

Now it remains to establish Lemma 2 in the case of no growth. Since state $\boldsymbol{\theta}_t$ is feasible, we have:

$$P_t(K_t)K_t = C_t + wK_t \tag{38}$$

State $\boldsymbol{\theta}_{t+1}$ is feasible if and only if

$$P_{t+1}(K_{t+1})K_{t+1} = C_{t+1} + wK_{t+1} \tag{39}$$

Capital costs in period $t+1$ can be decomposed as the sum of capital costs of investments prior to I_t and the capital cost of the latest investment:

$$C_{t+1} = C_t\left(1-\delta\right) + z_1 I_t$$

Applying (38), the first term in the RHS of the equation above can be rewritten as

$$C_t\left(1-\delta\right) = \left(1-\delta\right)\left(P_t\left(K_t\right)K_t - wK_t\right)$$

By the definition of declining balance accounting

$$z_1 = \delta v + rv$$

Since $x_1 = 1$ and capacity declines at rate $(1-\alpha)$,

$$I_t = K_{t+1} - \left(1-\alpha\right)K_t$$

Putting these observations together, we obtain that state $\boldsymbol{\theta}_{t+1}$ is feasible if and only if

$$P_{t+1}\left(K_{t+1}\right)K_{t+1} = \left(1-\delta\right)\left(P_t\left(K_t\right)K_t - wK_t\right) + \left(\delta+r\right)v\left(K_{t+1} - \left(1-\alpha\right)K_t\right) + wK_{t+1}$$

Applying the assumption about inverse demand curves, this equality can be rewritten as:

$$\left(a - bK_{t+1}\right)K_{t+1} = \left(1-\delta\right)\left(\left(a-bK_t\right)K_t - wK_t\right) + \left(\delta+r\right)v\left(K_{t+1} - \left(1-\alpha\right)K_t\right) + wK_{t+1}$$

This is a quadratic equation in K_{t+1} and it has two solutions:

$$K_{t+1}^{(1)} = \frac{1}{2b}\left(a - w - v(r+\delta)\right)$$
$$+\frac{1}{2b}\sqrt{(a-w-v(r+\delta))^2 + 4b(-a+rv+w-rv\alpha+(a+v-w-v\alpha)\delta)K_t + 4b^2(1-\delta)K_t^2}$$

and

$$K_{t+1}^{(2)} = \frac{1}{2b}\left(a - w - v(r+\delta)\right) -$$
$$-\frac{1}{2b}\sqrt{(a-w-v(r+\delta))^2 + 4b(-a+rv+w-rv\alpha+(a+v-w-v\alpha)\delta)K_t + 4b^2(1-\delta)K_t^2}$$

Let us now recall that $I_t = K_{t+1} - \left(1-\alpha\right)K_t$ and compute the corresponding investments:

$$I_t^{(1)} = \frac{1}{2b}(a - w - v(r+\delta) - 2\left(1-\alpha\right)bK_t) +$$
$$+\frac{1}{2b}\sqrt{(a-w-v(r+\delta))^2 + 4b(-a+rv+w-rv\alpha+(a+v-w-v\alpha)\delta)K_t + 4b^2(1-\delta)K_t^2}$$

$$I_t^{(2)} = \frac{1}{2b}\left(a - w - v(r+\delta) - 2\left(1-\alpha\right)bK_t\right) -$$
$$-\frac{1}{2b}\sqrt{(a-w-v(r+\delta))^2 + 4b(-a+rv+w-rv\alpha+(a+v-w-v\alpha)\delta)K_t + 4b^2(1-\delta)K_t^2}$$

First we will show that $I_t^{(2)}$ is negative and, therefore, if a positive feasible continuation exists, it is unique. Second, we will show that a positive feasible I_t exists (this will imply that $I_t^{(1)}$ is positive). For the former statement, it suffices to check that

$$\begin{aligned}&\left(a - w - v(r+\delta) - 2\left(1-\alpha\right)bK_t\right)^2 < \\ &(a-w-v(r+\delta))^2 + 4b(-a+rv+w-rv\alpha+(a+v-w-v\alpha)\delta)K_t + 4b^2(1-\delta)K_t^2\end{aligned}$$

Expanding and subtracting right-hand-side from the left-hand-side, the inequality above can be rewritten as:

$$4bK_t\left(bK_t(\alpha^2 - 2\alpha + \delta) + (a-w)(\alpha-\delta)\right) < 0 \tag{40}$$

Let us assume that $\alpha^2 - 2\alpha + \delta > 0$. Then, the left-hand-side of (40) is less than zero if

$$K_t < \frac{(a-w)(\delta-\alpha)}{b(\alpha^2 - 2\alpha + \delta)}$$

Note that

$$\delta - \alpha > \alpha^2 - 2\alpha + \delta$$

Therefore,

$$\frac{(a-w)(\delta-\alpha)}{b(\alpha^2 - 2\alpha + \delta)} > \frac{a-w}{b}$$

On the other hand, we know that

$$a - bK_t > w$$

which implies

$$K_t < \frac{a-w}{b}$$

and inequality (40) has to hold. Now let us assume that $\alpha^2 - 2\alpha + \delta < 0$. Then, inequality (40) holds when

$$K_t > \frac{(a-w)(\delta-\alpha)}{b(\alpha^2 - 2\alpha + \delta)}$$

In that case, note that by the conservatism assumption $\delta > \alpha$, and, since $P_t\left(K_t\right) > w$, $a > w$. Therefore,

$$\frac{(a-w)(\delta-\alpha)}{b(\alpha^2 - 2\alpha + \delta)} < 0$$

and K_t has to be greater than this number.

To show that a positive feasible I_t exists, we first observe that, if the company does not invest anything in period t, its total costs will be lower than revenues:

$$\begin{aligned}
P_{t+1}\left(K_{t+1}\right)K_{t+1} &= (1-\alpha)\,P_{t+1}\left((1-\alpha)\,K_t\right)K_t > (1-\alpha)\,P_t\left(K_t\right)K_t = \\
&= (1-\alpha)\left(C_t + wK_t\right) > (1-\delta)\,C_t + (1-\alpha)\,wK_t = \\
&= C_{t+1} + wK_{t+1}
\end{aligned}$$

On the other hand, if the company invests such that $K_{t+1} = \frac{a}{b}$, its revenues will be equal to zero, while its costs will be positive. Therefore, there must exist a positive I_t which equates revenues to total costs. ■

Proof of Proposition 5:

By the same auxiliary argument that was employed in the proof of Lemma 2, the case of constant market growth can be converted to the case of no growth by adjusting x_τ and z_τ appropriately. We therefore restrict attention to the case when $\mu = 0$.

First we will show that $K_{t+1} > K_t$ if $K_t < K^*$ and $K_{t+1} < K_t$ otherwise. We know that K_{t+1} will be given by the expression for $K_{t+1}^{(1)}$ from the proof of the previous lemma:

$$\begin{aligned}
K_{t+1} &= \frac{1}{2b}\left(a - w - v(r+\delta)\right) \\
&+ \frac{1}{2b}\sqrt{(a-w-v(r+\delta))^2 + 4b(-a+rv+w-rv\alpha+(a+v-w-v\alpha)\delta)K_t + 4b^2(1-\delta)K_t^2}
\end{aligned}$$

It is clear from this formula that, starting from K_2, it will be the case that

$$K_t \geq \frac{1}{2b}\left(a - w - v(r+\delta)\right)$$

Therefore, K_{t+1} is greater than K_t whenever

$$\begin{aligned}
&\left(a - w - v(r+\delta) - 2bK_t\right)^2 < \\
&(a-w-v(r+\delta))^2 + 4b(-a+rv+w-rv\alpha+(a+v-w-v\alpha)\delta)K_t + 4b^2(1-\delta)K_t^2
\end{aligned}$$

This is equivalent to

$$4bK_t\left(-\left(a-w\right)\delta + v\alpha\left(r+\delta\right) + b\delta K_t\right) < 0$$

The latter inequality holds when

$$0 < K_t < \frac{\left(a-w\right)\delta - v\alpha\left(r+\delta\right)}{b\delta} = K^*$$

and the converse holds when $K_t > K^*$. We have shown that $K_{t+1} > K_t$ if $K_t < K^*$ and $K_{t+1} < K_t$ otherwise. Let us now show that $K_{t+1} > K^*$ if $K_t > K^*$ and $K_{t+1} < K^*$ otherwise. First note that, since $\delta > \alpha$

$$\frac{1}{2b}\left(a - w - v(r+\delta)\right) < \frac{\left(a-w\right)\delta - v\alpha\left(r+\delta\right)}{b\delta} = K^*$$

Therefore, $K_{t+1} > K^*$ if and only if

$$\left(a - w - v(r+\delta) - 2\frac{(a-w)\,\delta - v\alpha\,(r+\delta)}{\delta}\right)^2 <$$
$$(a - w - v(r+\delta))^2 + 4b(-a + rv + w - rv\alpha + (a + v - w - v\alpha)\delta)K_t + 4b^2(1-\delta)K_t^2$$

This condition is equivalent to:

$$\frac{4\left((w-a)\delta + v\alpha(r+\delta) + bK_t\delta\right)\left(v(\alpha-\delta)(r+\delta) - b(1-\delta)\delta K_t\right)}{\delta^2} < 0 \qquad (41)$$

The quadratic function in the numerator of the left-hand-side has two roots:

$$\left\{\frac{(a-w)\,\delta - v\alpha(r+\delta)}{b\delta}, \frac{v\,(\alpha-\delta)\,(r+\delta)}{b\,(1-\delta)}\right\}$$

The first root is equal to K^* and the second root is negative due to the conservatism assumption, $\delta > \alpha$. The coefficient on K_t^2 in (41) is negative, so the function will be greater than zero for $0 < K_t < K^*$ and less than zero for $K_t > K^*$. We have shown that if $K_t < K^*$, then $K_t < K_{t+1} < K^*$, and if $K_t > K^*$, then $K_t > K_{t+1} > K^*$. These inequalities imply that sequence $\{K_t\}$ converges to some limit K_∞. Let us show that $K_\infty = K^*$. Since all states $\boldsymbol{\theta}_t$ are feasible, $P\,(K_t)\,K_t = wK_t + C_t$, it has to be that

$$wK_t + C_t \to P\,(K_\infty)\,K_\infty$$

Since $I_t = K_{t+1} - (1-\alpha)\,K_t$,

$$I_t \to \alpha K_\infty$$

On the other hand, $(\delta + r)\,vI_t = C_{t+1} - (1-\delta)\,C_t$, therefore

$$(\delta + r)\,vI_t \to \delta\,(P\,(K_\infty)\,K_\infty - wK_\infty)$$

Hence,

$$\alpha K_\infty = \frac{\delta\,(P\,(K_\infty)\,K_\infty - wK_\infty)}{(\delta + r)\,v}$$

and

$$P\,(K_\infty) = w + \frac{\alpha\,(\delta + r)}{\delta}v = p^*$$

■

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