Earnings Quality Metrics and What They Measure

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This paper discusses and evaluates the usefulness and appropriateness of commonly used earnings metrics. In a rational expectations equilibrium model, we study the information content of earnings that can be biased by a manager who has market price, earnings, and smoothing incentives. We define earnings quality as the reduction of the market's uncertainty about the firm's terminal value due to the earnings report and compare this measure with value relevance, persistence, predictability, smoothness, and accrual quality. The evaluation is based on their ability to capture the effects of a variation of the manager's incentives and information, and of accounting risk. We find that each metric captures different effects, but some of them, including value relevance and persistence, are closely related to our earnings quality measure. Discretionary accruals are problematic as their behavior depends on specific circumstances. These results provide insights into regulatory changes and guidance for selecting earnings quality metrics in empirical tests of earnings quality.

Keywords: Earnings quality; value relevance; earnings management; accrual quality.

1. Introduction

Earnings quality is one of the most important characteristics of financial reporting systems. High quality is said to improve capital market efficiency, therefore investors and other users should be interested in high-quality financial accounting information. For that reason, standard setters strive to develop accounting standards that improve earnings quality, and many recent changes in auditing, corporate governance, and enforcement have a similar objective.

Earnings quality is used in numerous empirical studies to show trends over time; to evaluate changes in financial accounting standards and in other institutions, such as enforcement and corporate governance; to compare financial reporting systems in different countries; and to study the effect of earnings quality on the cost of capital.

Surprisingly, the concept of earnings quality is quite vague, despite several attempts to make it more precise and to provide a theoretical foundation. For example, Schipper and Vincent [2003] deduce earnings quality from the theory of economic income, and Dechow and Schrand [2004] define earnings quality as a measure of how well earnings reflect the actual performance of a firm. Standard setters such as the Financial Accounting Standards Board (FASB) and the International Accounting Standards Board (IASB) formulate in their draft on the first part of a common conceptual framework the need for high-quality financial reporting [IASB, 2008]. They avoid defining quality, but list a number of qualitative characteristics that should achieve a high quality, including relevance, faithful representation, comparability, verifiability, timeliness, and understandability.

The empirical literature has developed several metrics to proxy for earnings quality (see the surveys in, e.g., Schipper and Vincent [2003]; Dechow and Schrand [2004]; Francis, Olsson and Schipper [2008]; and Dechow, Ge, and Schrand [2010]). The metrics are based on the qualitative characteristics in the conceptual framework, but also on other concepts. Some of them are based on accounting numbers only, while others include market prices, too. Despite their widespread use, there is no agreement on several metrics whether a high value of the metric indicates high or low earnings quality. For example, smoothness is used to proxy for earnings management (indicating low earnings quality) or for additional information incorporated by the manager (indicating high earnings quality). Therefore, some studies use the neutral term “earnings attributes” rather than earnings quality metrics. Nevertheless, even if the different directions of the interpretation are acknowledged,[1] most studies *a priori* assume a particular interpretation in their analyses.

The metrics capture only certain aspects that are considered important for earnings quality, e.g., the time series of earnings or market price reactions on earnings. Therefore, many empirical studies aggregate several metrics into an earnings quality score, often by adding up the ranks of the metrics. Doing so, the researchers implicitly assume equal weights of the metrics, which neglects perhaps different importance of certain effects, a potential overlap of metrics, correlations among the metrics, and the potential information content in the cardinal measurement.

The objective of this paper is to provide a theory of measures of earnings quality to evaluate the usefulness and appropriateness of commonly used earnings quality metrics. We develop a model with a firm whose terminal value (liquidating cash flow) is uncertain and the uncertainty reduces over time. The firm’s manager makes a decision about the bias (accrual, earnings management) in her earnings report, and rational investors in a capital market use the earnings report to make inferences about the value of the firm. We incorporate typical incentives of the manager, increasing the market price, increasing or smoothing reported earnings, and we consider costs of earnings management. We do not model these incentives endogenously because we want to vary them in the analysis of earnings quality metrics. The manager has private information about the future cash flows and uses this information when deciding on the bias. Thus, we consider both earnings management and information content of accruals together. We

[1] See, e.g, the discussion in Barth, Landsman, and Lang [2008].

do not distinguish between quality of financial reporting and earnings quality because our earnings capture all the information about the terminal value in the model.

After characterizing the unique linear rational expectations equilibrium we define earnings quality in our framework as the reduction of the market's assessment of the variance of the terminal value due to the earnings report. This notion arises naturally from our model, and it is consistent with the definition by Francis, Olsson, and Schipper [2008, p. 8], who define quality of information as the precision (and lack of uncertainty) of a measure with respect to a valuation relevant construct, which they assume is to support certain judgments and decisions in the capital markets.

In addition to this benchmark measure, we study value relevance, persistence, predictability, smoothness, and accrual quality, each of them suitably defined in our underlying setting. We examine the behavior of each of these metrics on variations of fundamental determining factors of earnings quality. In particular, we consider four sets of factors: (i) managerial incentives, which affect the earnings report due to earnings management; (ii) operating risk, which is the variation of the future cash flows and the terminal value; (iii) private information of the manager about the operating risk; and (iv) accounting risk which determines the precision of the accounting system.

The main results are as follows: There is no metric of those we study that perfectly traces the behavior of our benchmark measure. Closest to the benchmark's performance are value relevance, followed by persistence. We show conditions when they behave similarly or not and when their relation is monotonic or not. Predictability is generally an appropriate metric except if one is interested in a change of the private information of the manager, where using predictability can lead to false conclusions. Probably the most striking result is that metrics that depend directly on the discretionary accruals are highly problematic metrics for earnings quality and conclusions based on them should be interpreted with caution. Our correlation-based smoothness metric generally does not pick up variations in the incentives and reacts to changes in information and accounting risks in a way that is opposite to our benchmark. Similarly, our metrics for accrual quality, the expected value of accruals or of squared accruals, capture an incentive effect that is not existent for the earnings quality measure and react to variations of other determining factors opposite to the benchmark or ambiguously, depending on specific realizations of other factors. Thus, using these metrics is likely to lead to false conclusions from empirical tests.

An advantage of our model is that we are able to trace any differences that occur in the behavior of the metrics to their economic causes. The model analysis explains why the metrics react similarly or opposite to the behavior of our earnings quality measure, and under which conditions they do so. For example, we show when the information content explanation for smoothing and accrual quality is more consistent with the results than the earnings management explanation. These results caution against using certain metrics as indicating higher or lower earnings quality without controlling for such conditions.

Our results contribute to the empirical literature by providing a framework that guides the formulation of hypotheses and the appropriate selection of earnings quality metrics for specific research questions, such as the impact of tighter accounting standards, stronger enforcement, more precise accounting information, stronger corporate governance, and management compensation schemes, on earnings quality.

There are few theoretical papers that directly address earnings quality measures, although a notion of earnings quality is embedded in many studies. Our linear rational expectations equilibrium is based on Fischer and Verrecchia [2000], who focus on the manager's market price incentive in a one-period model, but do not consider smoothing incentives. Ewert and Wagenhofer [2005] use a similar model to consider value relevance when the manager can manage earnings or affect real cash flows of the firm. Sankar and Subrahmanyam [2001] study a model with a risk averse manager with a time-additive utility function so that smoothing results from the manager's desire to smooth consumption. Similar to our model, they find that allowing for earnings management improves the information in the capital market because the bias allows the manager to incorporate private value relevant information early. However, they do not consider earnings quality metrics. Dye and Sridhar [2004] study relevance and reliability of accounting information and model the accountant as the gatekeeper to trade off these two characteristics in a capital market equilibrium. Marinovic [2010] examines earnings management and capital market reactions when it is uncertain if the manager can bias the earnings report. He shows the existence of an equilibrium with a mixed earnings management strategy and a market price reaction that is bounded from above for increasing earnings reports. He finds that price volatility around earnings announcements and persistence are useful metrics, whereas predictability and smoothness do not capture earnings quality appropriately because they are non-monotonic measures.

Like our paper, these models exclusively focus on decision usefulness of accounting earnings in a capital market equilibrium. They do not consider stewardship uses of accounting information as do, for example, the optimal contracting models by Christensen, Feltham, and Şabac [2005] and Christensen, Frimor, and Şabac [2009]. A recent paper that addresses both stewardship and valuation purposes is by Drymiotes and Hemmer [2009]. They study a multi-period agency model and explicitly consider empirical earnings quality metrics in that setting. Similar in spirit to our results, they find that empirical metrics for earnings quality do not capture many of the real economic effects. Individual results differ, however, due to the different objectives of financial reporting and the model structures used in the two studies.

We do not claim that higher earnings quality in the sense that earnings reports provide more precise information about firm value is necessarily a socially desirable objective. For example, Kanodia, Singh, and Spero [2005] and Göx and Wagenhofer [2010] find situations in which maximizing precision is not in the interest of the firm and investors, even if it is costless to do so. Nagar and Yu [2009] test the prediction of models that "too" precise public information may coordinate speculators' behavior, which can lead to self-fulfilling beliefs and, ultimately, crisis.

In the next section we set up the model and explain main factors that influence earnings quality. Section 3 establishes the rational expectations equilibrium and describes the properties of the earnings reporting and market pricing strategies. In section 4, we define earnings quality within the model structure and show its determinants. Section 5 includes a detailed analysis of the behavior of the earnings quality metrics on variations of the main influencing factors. Section 6 draws together these results and discusses implications. Section 7 concludes.

2. The Model

We consider a firm that has a stochastic terminal value $\tilde{x}$. To focus on the characteristics of accounting information we assume that the firm's operating activities are held constant and do not affect the distribution of $\tilde{x}$. Therefore, we essentially model a pure exchange economy with one risky asset (the firm). The terminal value consists of the following components:

$$\tilde{x} = \mu + \tilde{\varepsilon} + \tilde{\delta} + \tilde{\omega}. \tag{1}$$

μ denotes the prior expected value of the terminal value and $\tilde{\varepsilon}$, $\tilde{\delta}$, and $\tilde{\omega}$ are normally distributed random variables with zero means and variances $\sigma_{\varepsilon}^2, \sigma_{\delta}^2$ and σ_{ω}^2, respectively. The random variables are mutually independent. They depend on the business model of the firm and the environment in which the firm operates. The structure of the terminal value and μ are common knowledge. The random variables realize sequentially over time, starting with $\tilde{\varepsilon}$, then $\tilde{\delta}$, and $\tilde{\omega}$ at the liquidation date. The realizations are not observed before the respective date, but there is some specific information about them as we describe below. We refer to the random components in $\tilde{x}$ as operating risk.

For simplicity, we do not model realizations of cash flows in each period, but assume that all cash flows obtain when the terminal value realizes. Adding observable period cash flows are unlikely to add significant new insights.

Accounting system

The accounting system generates noisy signals of components of the operating risk. In the first period, the manager privately receives a signal

$$\tilde{y}_1 = \mu + \tilde{\varepsilon} + \tilde{n} . \tag{2}$$

This signal includes the expected value of $\tilde{x}$ and one random component, $\tilde{\varepsilon}$, but also includes an accounting noise term $\tilde{n}$, which is normally distributed with zero mean and variance σ_n^2. In the second period, the manager privately receives another accounting signal

$$\tilde{y}_2 = \mu + \tilde{\varepsilon} + \tilde{\delta} + \tilde{u} . \tag{3}$$

It captures the information content of $\tilde{y}_1$ with respect to $\tilde{x}$ plus another random component, $\tilde{\delta}$, and includes a noise term $\tilde{u}$, which has zero mean and variance σ_u^2. All random variables are mutually independent.

Both accounting signals, $\tilde{y}_1$ and $\tilde{y}_2$, are unbiased estimates of the terminal value $\tilde{x}$. $\tilde{y}_1$ provides information about the realized risk, $\tilde{\varepsilon}$, and $\tilde{y}_2$ includes information about both $\tilde{\varepsilon}$ and $\tilde{\delta}$. Ignoring the signal noise, signal $\tilde{y}_2$ is a more precise measure about $\tilde{x}$ because it includes more of the "permanent" operating risk than $\tilde{y}_1$. The noise terms $\tilde{n}$ and $\tilde{u}$ determine the precision of the accounting system, and we refer to their variances σ_n^2 and σ_u^2 as accounting risk, which is the inverse of accounting precision.

In each period, the manager releases a mandatory earnings report based on her private information about $\tilde{y}_1$ and $\tilde{y}_2$. Each report is restricted to a single number ("earnings"), and we assume the manager cannot make additional voluntary disclosures about her private information.[2] However, we allow for earnings management so that the manager can report a biased earnings number. Let *b* denote the bias ("accrual") in the first period, then the earnings report is

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[2] This blocked communication assumption may be a result of the difficulty to verify information other than in mandatory accounting reports. See Sankar and Subrahmanyam [2001] for a similar assumption.

$$\tilde{m}_1 = \tilde{y}_1 + b. \quad (4)$$

The value of b can be positive or negative. Consistent with most accounting standards, we assume a clean surplus condition for the bias, so that the accrual reverses in subsequent periods.[3] In our two-period model, we do not allow the manager to introduce another bias in the second period, hence, the second period earnings report simply is

$$\tilde{m}_2 = \tilde{y}_2 - b.$$

The manager decides on the bias after observing the realized accounting signal y_1 and additional information that is not included in $\tilde{y}_1$. For simplicity and without significant loss of generality, we assume the additional information is a perfect signal of the (future) realization of the operating risk parameter $\tilde{\delta}$. We refer to $\tilde{\delta}$ also as information risk to capture the amount of private information of the manager. $\tilde{\delta}$ is informative about the terminal value $\tilde{x}$ over and above what is included in $\tilde{y}_1$, and it can be embedded in the first-period earnings via the accrual chosen by the manager. In some of the subsequent analyses, we consider a situation in which the manager does not have additional information. We represent this situation by setting the variance of $\tilde{\delta}$ to zero, i.e., $\sigma^2_{\tilde{\delta}} = 0$, which also implies $\tilde{\delta} = 0$. Note that $\tilde{y}_2$ is still informative about $\tilde{x}$ given $\tilde{y}_1$, because it depends on $\tilde{\varepsilon}$.

Investors in the capital market are risk neutral and hold rational expectations. They interpret the earnings reports accordingly and trade so that the market price $P(\cdot)$ of the firm captures the information in the earnings reports, that is, $P(m_1) = \mathrm{E}\left[\tilde{x} \middle| m_1\right]$ and $P(m_1, m_2) = \mathrm{E}\left[\tilde{x} \middle| m_1, m_2\right]$. Figure 1 summarizes the sequence of events and the basic notation of the model.

[3] We assume that the accounting risk (defined in the sequel) is subject to clean surplus only in the terminal period, so that it is outside our analysis. Assuming otherwise adds to complexity without affecting the main results, as accounting risk is not strategic.

[Insert Figure 1 about here]

Management incentives

The choice of the bias b depends on the manager's private information and on her incentives. The literature on earnings management usually assumes that managers are interested in maximizing the (short-term) market price of the firm and/or reported earnings, and they favor smooth earnings over time.[4] To cover a broad set of possible incentives, we assume the following utility function of the manager:

$$U = pP\left(m_1\right) + \tilde{g}m_1 - s\mathrm{E}\left[\left(\tilde{m}_2 - m_1\right)^2 \middle| y_1, \delta\right] - r\frac{b^2}{2}. \qquad (5)$$

The manager cares for up to four different components, market price, reported earnings, smooth earnings, and the cost of biasing the earnings report. The weights attached to each of these components are p, s, $\tilde{g}$ and r, respectively. They are exogenous given because we are interested in the effects of variations of these weights on earnings quality metrics. Therefore, we use four weights rather than three (which would be sufficient to capture the substitution effect between the components) to be able to isolate the effect of individual components in equilibrium. This approach is consistent with empirical studies that identify changes in institutional or economic factors and predict their effects on earnings quality.

The structure of the manager's utility function is common knowledge. All weights except for $\tilde{g}$ are constants; only the manager knows the realization of $\tilde{g}$, whereas investors only know the distribution of $\tilde{g}$. Assuming that the weights are exogenous, we can vary the incentives (or the parameters of their distribution) in the subsequent analysis directly and study their effects on earnings quality.

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[4] For example, Graham, Harvey, and Rajgopal [2005, pp. 24-26, 44-47] report that the overwhelming majority of the managers they surveyed are interested in a high stock price and in smooth earnings.

The first component of the utility function captures the manager's interest in the market price $P_1 = P(m_1)$, which depends on the earnings report m_1. For example, the manager plans to raise external capital after the earnings release and wants to boost the market price. For simplicity, we do not include in the utility function the second-period market price $P_2 = P(m_1, m_2)$. However, since the bias b reverses in the second period, the expected net effect of a bias would depend on the weights attached to P_1 and P_2. In that sense, the weight p can be interpreted as the weight on P_1 relative to P_2. For most of the analysis, we assume $p \geq 0$ in explaining the results, but the analysis is not restricted by p being positive.

The second component is the manager's interest in the reported earnings m_1 directly. This interest may arise from earnings targets the manager wants to reach, from the compensation scheme, from political cost considerations or debt covenants. The weight g is the realization of a random variable, $\tilde{g}$, which is normally distributed with mean $\bar{g}$ and variance σ_g^2. The manager knows g, but the market only knows the distribution. We allow for asymmetric information about the weight to capture an important aspect of reality, in which the manager knows better about some aspects of her incentives.[5]

The third component of the utility function captures the manager's smoothing desire. A smoothing incentive may arise even under risk neutrality to reduce earnings volatility (Trueman and Titman [1988]), from earnings targets, and the like. We use the expected value of the squared differences between (expected) second period earnings and first period earnings (which is known by the manager in period 1), conditional on the set of available information (y_1, δ). The weight $s \geq 0$ denotes the intensity of the smoothing incentive, and the higher s, the more emphasis the manager puts on smooth earnings.

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[5] Arguably, the manager may have private information about each of the weights of the components in her utility function. We include only one such component in our analysis, mainly for tractability reasons. Private information about the weight on market price p is studied in Fischer and Verrecchia [2000].

The fourth component in the utility function denotes the cost that the manager has to bear by reporting earnings that deviate from those provided by the accounting system. In line with many other earnings management models, we assume that this cost increases in the accounting bias b at an increasing rate. A higher weight $r \geq 0$ on the cost captures the difficulty and increasing effort to add bias to the underlying accounting signal. The cost of earnings management can be a result of personal discomfort of earnings management and of institutional factors, such as liability risk or tightness of accounting standards and corporate governance provisions.[6]

To summarize, the model captures in a parsimonious way four fundamental sets of factors that determine earnings quality:

(i) The operating risk of the firm. Operating risk is the "real" volatility of firm value, captured by the three random variables $\tilde{\varepsilon}$, $\tilde{\delta}$, and $\tilde{\omega}$, which realize sequentially over time, so that each realization is a "permanent" (rather than transitory) value-generating factor.

(ii) The private information of the manager about the operating risk. Since we assume that the manager privately obtains δ, we refer to $\tilde{\delta}$ also as information risk because from an ex ante point of view, the variance σ_{δ}^{2} also represents the volatility of the manager's signal. A higher variance σ_{δ}^{2} implies more private information of the manager.[7] The private information depends on the business model, the environment the firm operates in, the

[6] The subsequent results reveal that earnings management has a positive effect in that it allows for communication of the manager's private information to the capital market. In that sense, earnings management can be useful. Nevertheless, corporate governance and enforcement are not responsive to such effects, but penalize earnings management.

[7] In subsequent analyses we vary the information content of the manager's private information, which also varies the total operating risk of the firm. However, our results are not affected by that change in total operating risk in the metrics we study.

expertise of the manager and the availability of a management accounting system that include forward-looking information of different quality.

(iii) The precision of the accounting system. Accounting risk affects the information content of the accounting system as captured by the variance of the noise terms $\tilde{n}$ and $\tilde{u}$ of the accounting signals that provide information about the operating risk. Since we focus on the first period, we refer to the variance of $\tilde{n}$, σ_n^2, as accounting risk. Higher risk is equivalent to a lower precision of the accounting system. It is a result of financial accounting standards, and as such it is a design variable for accounting standard setters and, perhaps, firms as well.

(iii) Management incentives. Incentives are captured by the weights p, $\tilde{g}$, s, and r attached to the market price, reported earnings, the smoothing desire and the cost of earnings management in the manager's utility function, respectively. Incentives drive earnings quality by providing direction on the manager's earnings management choices. They can be affected by the design of corporate governance (in particular, management compensation), by accounting standards, and by institutions in which financial reporting is embedded, including internal control systems, compliance, auditing, and regulatory scrutiny.

In the subsequent analysis, we vary these factors and examine their effects on a variety of earnings quality metrics. We focus on the first period after such a change occurs, rather than on some weighted average of the two periods of our model. This is in line with most of the empirical studies that use critical events as a basis for their analysis.

3. Equilibrium

3.1. Characterization of the linear equilibrium

A rational expectations equilibrium consists of a reporting strategy by the manager (defining the bias of the report) and a capital market reaction that correctly infers the information contained in the earnings report on average. Each of these strategies is an optimal response based on conjectures of the other player's strategy. In equilibrium their conjectures are fulfilled.

The manager maximizes the earnings report $m_1 = y_1 + b$ by choosing the bias b contingent on her information set (y_1, δ) and her conjecture about the market price reaction on the earnings report, denoted by $\hat{P}(m_1)$. To gain more insight into the choice of b, we rewrite the utility function (5) as follows:

$$\begin{aligned} U(y_1,\delta) &= p\hat{P}(m_1) + gm_1 - s\mathrm{E}\left[(\tilde{m}_2 - m_1)^2 \middle| y_1,\delta\right] - \frac{r}{2}b^2 \\ &= p\hat{P}\left(y_1 + b(y_1,\delta)\right) + g\left(y_1 + b(y_1,\delta)\right) \\ &\quad - s\left[Var\left(\tilde{y}_2 \middle| y_1,\delta\right) + \left(\mathrm{E}\left[\left(\tilde{y}_2 \middle| y_1,\delta\right)\right] - y_1 - 2b(y_1,\delta)\right)^2\right] - \frac{r}{2}b(y_1,\delta)^2. \end{aligned} \tag{6}$$

Assuming a differentiable pricing function, the first-order condition of (6) with respect to $b(y_1, \delta)$ characterizes the optimal accounting bias:

$$b^*(y_1,\delta) = \frac{p}{8s+r}\cdot\frac{d\hat{P}}{dm_1} + \frac{g}{8s+r} + \frac{4s}{8s+r}\left(\mathrm{E}\left[\tilde{y}_2 \middle| y_1,\delta\right] - y_1\right).$$

Note that the bias b is deterministic, contingent on (y_1, δ), so it does not affect the conditional variance of the second-period earnings, $Var\left(\tilde{y}_2 \middle| y_1,\delta\right)$.

We assume a linear rational expectations equilibrium[8] with the manager's conjecture of the market pricing function.

$$\hat{P}(m_1) = \hat{\alpha} + \hat{\beta}m_1,$$

and with investors' conjecture of the manager's earnings report as linear in her private information,

[8] Guttman, Kadan, and Kandel [2006] show that there are other equilibria with non-differentiable reporting strategies in such rational expectations models, but linear equilibria are the only ones that survive stricter equilibrium refinements (Einhorn and Ziv [2010]).

$$\hat{b} = \hat{\xi} + \hat{\tau} g + \hat{\varphi} y_1 + \hat{\gamma}\delta \,.$$

The next result shows the existence of a unique linear equilibrium and characterizes the optimal reporting strategy.

Proposition 1: *There exists a unique linear equilibrium in which the manager reports earnings*

$$m_1^*\left(y_1,\delta\right) = y_1 + b^*\left(y_1,\delta\right) = \left(1-Q\right)\mu + Qy_1 + 4sR\delta + \beta pR + gR \,. \tag{7}$$

The equilibrium market price is $P\left(m_1\right) = \alpha + \beta m_1$ *where*

$$\alpha \equiv \mu\left(1-\beta\right) - \beta^2 pR - \beta\bar{g}R \tag{8}$$

and

$$\beta \equiv \frac{Q\sigma_\varepsilon^2 + Z\sigma_\delta^2}{Q^2\left(\sigma_\varepsilon^2 + \sigma_n^2\right) + Z^2\sigma_\delta^2 + R^2\sigma_g^2} = \frac{Cov\left(\tilde{x},\tilde{m}_1\right)}{Var\left(\tilde{m}_1\right)} > 0\,. \tag{9}$$

The proof is in the appendix. *R*, *Z*, and *Q* are positive constants that be used throughout the paper to save notation *R* and *Z* collect smoothing incentives and the cost of earnings management,

$$R \equiv \frac{1}{8s+r} \text{ and } Z \equiv \frac{4s}{8s+r} = 4sR\,.$$

Since *s* and *r* are greater or equal to zero, $Z \in [0, 1/2]$. *Q* also includes variance terms and is defined as

$$Q \equiv 1 - \frac{4s}{8s+r}\cdot\frac{\sigma_n^2}{\sigma_\varepsilon^2 + \sigma_n^2} = 1 - ZS_n\,,$$

where $S_n \equiv \dfrac{\sigma_n^2}{\sigma_\varepsilon^2 + \sigma_n^2}$. Since $S_n \in (0, 1)$, the range of *Q* is (1/2, 1].

3.2. Properties of the equilibrium strategies

Proposition 1 shows that b^* is a linear function of y_1 and δ, and that the market price is a linear function of m_1, thus confirming the conjectures. We now discuss the properties of the equilibrium strategies. The equilibrium bias is

$$b^*(y_1,\delta) = -S_n Z y_1 + (Z\delta + S_n Z\mu + p\beta R + gR). \quad (10)$$

A first observation is that the equilibrium generally entails a biased earnings report.[9]

The bias b^* is negatively related to y_1 so that b^* smoothes the "shock" of y_1 over the two periods. It does so with a weight $S_n Z$ that lies between 0 and 1/2. To understand why, assume the special case that earnings management is costless ($r = 0$). The manager wants to smooth the reported earnings over the two periods ($s > 0$). Since y_1 is the best estimate for $\tilde{y}_2$ (and for $\tilde{x}$), she would want to include as much of the "permanent" component in y_1, that is the operating risk component ε, into the earnings report. If y_1 measures ε without noise, i.e., $\sigma_n^2 = 0$, then $S_n = 0$ and the equilibrium earnings m_1^* fully includes y_1.[10] Conversely, if y_1 is a very noisy measure of ε, then most of the variation in y_1 is due to the realization of $\tilde{n}$, which is transitory and does not affect $\tilde{y}_2$. If $\sigma_n^2 \to \infty$ then $S_n \to 1$, and since $Z = 1/2$, the weight on y_1 is $S_n Z = 1/2$, which implies an equal spread of the transitory component over the two periods. In more general situations, S_n lies between 0 and 1 and its exact value depends on the relative precision of the permanent ($\tilde{\varepsilon}$) and the transitory ($\tilde{n}$) components. The weight $(1 - S_n)$ is the earnings response coefficient if y_1 was unaffected by incentives and the private information δ of the manager. Finally, if earnings management becomes increasingly costly (higher r), then Z decreases and a lower share of the transitory component in y_1 is shifted to the second-period earnings.

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[9] However, there exists a set of values (y_1, δ) for which $b^* = 0$.

[10] The fixed effect μ that is carried in the signal y_1 is adjusted with the same factor according to (10).

The earnings report m_1^* and the bias b^* increase in the private information δ for a similar reason, although there are two differences. First, the manager learns δ without noise, so no correction (similar to S_n) applies. Second, y_1 does not contain δ, whereas it is included in $\tilde{y}_2$. Therefore, smoothing requires anticipation of a share of δ in the first-period earnings, thus increasing the bias. If earnings management is costless, then half of δ is shifted to period one, and if it is costly, only a lower amount is shifted. The share is precisely Z, and therefore b^* includes $Z\delta$.

The equilibrium bias strongly depends on the manager's incentives. The market price incentive is captured by the incentive parameter p on the first-period market price, P_1. The equilibrium bias is linear increasing in p at with a weight of $p\beta R$, which increases reported earnings in the first period. The weight depends on the smoothing incentive s, the cost of earnings management r, and on β, which is determined by the capital market reactions to a shift in earnings in equilibrium. We discuss β below. The earnings incentive g has a similar, but more direct effect on the equilibrium bias. The bias increases linearly in g with a weight of R. The capital market corrects for the bias by deducting the market price incentive and the *expected* earnings biases in the fixed term α.

A crucial effect on the bias stems from the manager's smoothing incentive, s. For example, assume $s \to 0$, then $Z \to 0$ and the bias only depends on the last terms,

$$b^*\left(y_1,\delta\right)\big|_{s\to 0} = \frac{\beta p}{r} + \frac{g}{r}.$$

For $s \to 0$ β approaches $1 - S_n > 0$, therefore, $b^* \to p(1 - S_n)/r + g/r$ is a positive constant that decreases in the cost of bias. Moreover, the bias becomes completely uninformative about the private information of the manager.

The sole effect of the cost of earnings management r is to somewhat dampen the bias that is induced by the other incentives. It is not itself a source for bias.

Investors are fully aware of the manager's incentives and adjust the earnings report accordingly. If there were no bias, then $m_1 = y_1$ and the market would update its expectation of the terminal value of the firm as follows:

$$\begin{aligned} \mathrm{E}\left[\tilde{x}\middle|y_1\right] &= \mu + \frac{Cov\left(\tilde{y}_1, \tilde{x}\right)}{Var\left(\tilde{y}_1\right)}\left(y_1 - \mathrm{E}\left[\tilde{y}_1\right]\right) \\ &= \mu + \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_n^2}(y_1 - \mu) \\ &= S_n\mu + (1 - S_n)y_1. \end{aligned}$$

The weight of the (unbiased) accounting signal is $(1 - S_n) = \sigma_\varepsilon^2 / (\sigma_\varepsilon^2 + \sigma_n^2)$. Since there is a bias in equilibrium, the sensitivity β depends on its determinants,

$$\beta = \frac{Q\sigma_\varepsilon^2 + Z\sigma_\delta^2}{Q^2\left(\sigma_\varepsilon^2 + \sigma_n^2\right) + Z^2\sigma_\delta^2 + R^2\sigma_g^2}.$$

β is equal to $(1 - S_n)$ only if $s = 0$ (implying $Z = 0$ and $Q = 1$) and if there is no uncertainty with respect to the earnings incentive weight ($\sigma_g^2 = 0$). Further notice that β is strictly positive, but not confined by 1. For example, suppose that the risks σ_ε^2, σ_n^2, and σ_g^2 are small. Then, Q is also small, and β is mainly determined by $Z\sigma_\delta^2$, where $Z \leq 1/2$. If σ_δ^2 is high, , then $\beta > 1$ arises.

Due to smoothing, the manager spreads the transitory component of the first period accounting signal between periods 1 and 2. Investors realize that the relationship between y_1 and m_1 has a smaller slope if smoothing is present, but the market can still infer y_1 after seeing m_1. Hence, the information content is essentially unchanged implying that the sensitivity of the market price with respect to the earnings report becomes larger if smoothing exists.

A bias is the only means by which investors can learn something about the manager's private information on δ. But the existence of a bias is not sufficient. Without a smoothing incentive there is a bias (due to the price and earnings incentives), but it would be uninformative about information that is not incorporated in the accounting signal y_1. For that reason, smoothing can indeed be beneficial from an informational point of view. From the view of the investors, however, reported earnings are no longer a precise indicator of the underlying accounting signal y_1 as they are now mingled with the additional information δ and its interpretation is difficult because of the incentive risk σ_g^2. The combined effect determines the quality of earnings.

4. Earnings quality

In this paper, we adopt a pure decision-usefulness perspective of financial reports: Earnings are of higher quality the more information they contain with respect to the terminal value $\tilde{x}$ of the firm. The following definition of earnings quality arises naturally in this setting:[11]

Definition: Earnings quality *EQ* is the reduction of the market's uncertainty about the terminal value due to the earnings report in period 1, formally:

$$EQ \equiv Var\left(\tilde{x}\right) - Var\left(\tilde{x} \middle| m_1\right). \quad (11)$$

A greater *EQ* implies higher earnings quality.[12] Our definition is similar to that in Francis, Olsson and Schipper [2008] and equivalent to that used in Marinovic [2010] as applied to our setting. Since

$$Var\left(\tilde{x} \middle| m_1\right) = Var\left(\tilde{x}\right) - \frac{Cov\left(\tilde{x}, \tilde{m}_1\right)^2}{Var\left(\tilde{m}_1\right)}$$

EQ can equivalently be written as

$$EQ = \frac{Cov\left(\tilde{x}, \tilde{m}_1\right)^2}{Var\left(\tilde{m}_1\right)}. \quad (12)$$

[11] In order to save notation, we use *b* and m_1 for the equilibrium values and drop the (y_1, δ) dependencies in the analysis if there is no confusion.

[12] An alternative would be to define *EQ* as the proportion of the reduction in the variance of the terminal value over the prior variance. Since the denominator is independent from incentives and the accounting precision, an ordering based on this measure would result in similar results.

Upon the release of the earnings report, investors revise their expectation of the terminal value to $P_1 = \mathrm{E}\left[\tilde{x} \middle| m_1\right]$. *EQ* does not directly capture the level of the price reaction but considers the distribution of the posterior market price relative to that of the previous price.

Although not obvious from the above definition, *EQ* is based on an equilibrium notion, which takes appropriate account of the manager's incentives for earnings management, rational expectations by investors, and the information content of earnings. Thus, *EQ* is determined by several exogenous and endogenous factors that interact in a complex way. The next proposition records the properties of *EQ* with respect to variations in the determining factors.

Proposition 2: *Earnings quality EQ*

(i) strictly increases in the smoothing incentive s;

(ii) strictly decreases in the cost of bias r if $\frac{\sigma_g^2}{\sigma_\delta^2} \leq \Gamma$, *where* $\Gamma > 0$*, and strictly increases otherwise*;

(iii) is unaffected by the market price incentive p and the expected earnings incentive $\bar{g}$ *;*

(iv) strictly increases in the risk σ_δ^2 *of private information* δ*;*

(v) strictly decreases in the accounting risk σ_n^2 *of signal* y_1 *and*

(vi) strictly decreases in the incentive risk σ_g^2*.*

The proof is in the appendix.

The first set of results in the Proposition considers the effects of incentives on earnings quality. Part (i) states that a stronger smoothing incentive enhances earnings quality. While counterintuitive at a first glance, the fundamental reason is that smoothing is necessary for incorporating the manager's private information δ into the first-period earnings. A stronger smoothing incentive induces a greater weight of that information in the earnings. Therefore, (with appropriate interpretation under rational expectations) a stronger smoothing incentive leads to a greater reduction of the residual variance of the terminal value, implying higher earnings quality.

A similar effect arises if the cost of bias increases because it makes a bias more costly and, in equilibrium, there is less bias and, hence, less information about δ. This effect reduces earnings quality. However, there is another effect that stems from the market's uncertainty as to the manager's earnings incentives. Suppose, there is no such uncertainty, then $\sigma_g^2 = 0$, and the reduction in *EQ* surfaces. The uncertainty about the earnings incentive moderates this negative effect, and for high uncertainty, it even outweighs it. The threshold value Γ is defined in (A3) in the proof. If the incentive risk is high relative to the information risk, then the reduction of the uninformative variability of earnings overcompensates the reduction of the informative variability, which eventually results in a net increase in the information content of first period earnings.

Correspondingly, part (vi) shows that *EQ* decreases in the earnings incentive risk, σ_g^2. The higher this risk, the less informative is the earnings report m_1 and the less the market reacts to it, as an earnings "surprise" can be due not only to permanent earnings and accounting errors, but also to certain management incentives.

Part (iii) states that the market price incentive *p* and the *expected* earnings incentive $\bar{g}$ have no effect on earnings quality. The bias resulting from this incentive is determined only by the earnings response coefficient β and the relation of *p* and $\bar{g}$ to the other incentive parameters, but is independent of the information y_1 and δ. Therefore, this part of the bias does not include any useful information to reduce the variance of the terminal value and earnings quality is not affected. Investors are able to back out the bias on average, but since do not know *g*, they cannot back it out fully, so the variance σ_g^2 affects the price reaction, but not the expected earnings incentive $\bar{g}$.

The next set of results in Proposition 2 relates to relevant risk parameters. Part (iv) states that earnings quality strictly increases in the variance σ_δ^2 of the operating risk $\tilde{\delta}$ that the manager learns early. This risk is the component of terminal value that materializes only in the second-period accounting signal y_2, but not in y_1. The bias contains a portion of the realized δ, and the higher the variance of $\tilde{\delta}$, the lower the residual variance after information about the realization is received.

Part (v) states that earnings quality strictly decreases in the noise σ_n^2 in the accounting signal y_1. The higher this noise, the less informative is the signal about the underlying operating risk ε, which is the information in the earnings report that is of interest to investors when updating their expectations. Therefore, a less precise accounting signal reduces earnings quality. Proposition 2 and the subsequent results do not record results on a change in the operating risk $\tilde{\varepsilon}$ because it has the reverse effect of the accounting noise $\tilde{n}$.

A practical difficulty with *EQ* is that it cannot be empirically estimated directly because $\tilde{x}$ is not observable. However, in our equilibrium structure, *EQ* can be expressed in terms of observables[13] because

$$EQ \equiv \frac{Cov\left(\tilde{x},\tilde{m}_1\right)^2}{Var\left(\tilde{m}_1\right)} = \left(\frac{Cov\left(\tilde{x},\tilde{m}_1\right)}{Var\left(\tilde{m}_1\right)}\right)^2 Var\left(\tilde{m}_1\right) = \beta^2 Var\left(\tilde{m}_1\right),$$

where $\beta \equiv \dfrac{Cov\left(\tilde{x},\tilde{m}_1\right)}{Var\left(\tilde{m}_1\right)}$ is the earnings response coefficient (and our value relevance metric that we examine below). Since $Cov\left(\tilde{m}_1,\tilde{P}_1\right) = Cov\left(\tilde{m}_1,\left(\alpha+\beta\tilde{m}_1\right)\right) = \beta Var\left(\tilde{m}_1\right)$ in equilibrium, β can be expressed as

$$\beta = \frac{Cov\left(\tilde{m}_1,\tilde{P}_1\right)}{Var\left(\tilde{m}_1\right)}.$$

Thus, in an empirical study the product of the square of the coefficient β (resulting from regressing price on earnings) and the earnings variance captures the reduction of the market's uncertainty with respect to firm value.

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[13] We thank Jeremy Bertomeu for suggesting this point.

5. Analysis of earnings quality metrics

Our earnings quality measure *EQ* provides a benchmark for evaluating the behavior of commonly used metrics for earnings quality (or earnings attributes). The closer the behavior of a particular metric is in line with that of *EQ* upon a variation of a determining factor of earnings quality, the more appropriate is its use in empirical studies.

In this section we study metrics that are commonly used as proxies for earnings quality in the empirical accounting literature.[14] We begin with value relevance, which is a metric that uses accounting information and market prices, and continue with metrics that rely solely on accounting information, namely, persistence, predictability, smoothness, and accrual quality. We define each of these metrics within the confines of our model and examine their behavior for a variation in the factors that determine earnings quality, incentives (s, p, $\bar{g}$, and r), private information (σ_δ^2), accounting risk (σ_n^2), and incentive risk (σ_g^2), and then we compare this behavior with that of *EQ*.

We do not examine metrics such as conservatism or distributional properties of earnings because our model is not specified so as to capture such metrics. Conditional conservatism introduces a non-linearity that does not fit our linear framework. For example, a metric that uses the asymmetric timeliness for positive and negative market returns introduces a “kink” in the market reaction function. Metrics that examine earnings distributions, such as loss avoidance, are based on deviations from normal distributions that we assume throughout the analysis.

5.1. Value relevance

Value relevance captures the notion that earnings are of high quality if they are capable to explain the firm’s market price and/or market returns. The literature employs several approaches,

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[14] For a survey see, e.g., Dechow, Ge, and Schrand [2010].

most of which can be classified according to Holthausen and Watts [2001] into two categories.[15] One set are relative association studies, which focus on the R^2 from a regression of the market price and/or return on earnings (and potentially other accounting variables). Another set, incremental association studies, concentrate on the ability of a specific accounting item (e.g., earnings, book value) to explain market value or return. This category uses the estimated regression coefficient.

We begin with the incremental association and define value relevance as the slope coefficient from a regression of the market price (or return) on reported earnings,

$$\beta = \frac{Cov(\tilde{x}, \tilde{m}_1)}{Var(\tilde{m}_1)} = \frac{Q\sigma_\varepsilon^2 + Z\sigma_\delta^2}{Q^2(\sigma_\varepsilon^2 + \sigma_n^2) + Z^2\sigma_\delta^2 + R^2\sigma_g^2}.$$

This metric is often referred to as earnings response coefficient. A greater β indicates higher earnings quality.

The value relevance β is closely related to our earnings quality measure EQ because from (12) we have

$$EQ = \beta Cov(\tilde{x}, \tilde{m}_1).$$

The next proposition shows that the behavior of β and EQ on variations of the determining factors is similar, but not fully aligned.

Proposition 3: *Value relevance* β
(i) strictly increases in the smoothing incentive s if $\sigma_\delta^2 \le \sigma_n^2(2 - S_n)$. *Otherwise, if* σ_δ^2 *is sufficiently large, then* β *is inversely u-shaped in s;*
(ii) (1) strictly decreases in the cost of bias r if $\frac{\sigma_g^2}{\sigma_\delta^2} \le \Gamma$ *and either* $\sigma_\delta^2 \le S_n\sigma_\varepsilon^2$ *or both*

[15] In their survey, Holthausen and Watts [2001] mention a third category of value relevance studies, which they refer to as "marginal information content studies."

$S_n\sigma_\varepsilon^2 \le \sigma_\delta^2 \le \sigma_n^2\left(2-S_n\right)$ and $\sigma_g^2 \le -16s^2\left(\sigma_\delta^2-\sigma_n^2\left(2-S_n\right)\right)$ *hold. (2)* β *strictly increases if* $\frac{\sigma_g^2}{\sigma_\delta^2} > \Gamma$ *and* $\sigma_\delta^2 > S_n\sigma_\varepsilon^2$*. (3),If* σ_δ^2 *is sufficiently large, then* β *is inversely u-shaped in r;*

(iii) is unaffected by the market price incentive p and the expected earnings incentive $\overline{g}$ *;*

(iv) strictly increases in the risk σ_δ^2 *of private information* δ*;*

(v) strictly decreases in the accounting risk σ_n^2 *of signal* y_1*; and*

(vi) strictly decreases in the incentive risk σ_g^2*.*

The proof is in the appendix. Parts (iii), (iv) and (v) state that the market price and expected earnings incentives *p* and $\overline{g}$, the operating risk σ_δ^2 and the accounting risk σ_n^2 induce the same ordering of earnings quality as does *EQ*. The difference lies in the first two parts that consider changes in the smoothing incentive *s* and in the cost of bias *r*. Proposition 3 provides a sufficient condition on the risk terms, $\sigma_\delta^2 \le \sigma_n^2\left(2-S_n\right)$, so that β and *EQ* move in the same direction. However, it also states that for large σ_δ^2, value relevance β decreases for an increasingly strong smoothing incentive while *EQ* always increases.

The sufficient condition

$$\sigma_\delta^2 \le \sigma_n^2\left(2-S_n\right), \tag{13}$$

where $(2 - S_n) > 1$, is satisfied if the information content of $\tilde{\delta}$ is not much higher than the noise in the accounting system. A large variability of the private information $\tilde{\delta}$ implies that the covariance between the earnings report $\tilde{m}_1$ and the terminal value $\tilde{x}$ increases for a larger smoothing incentive since the manager injects a higher proportion of the signal into the reported earnings, so these earnings covary stronger with the terminal value. On the other hand, the earnings variance initially decreases in *s* because of the lower impact of the accounting risk on the earnings variability[16] and the only moderately increasing effect of δ. However, the greater the smoothing incentive, the more σ_δ^2 determines the earnings variance; and if σ_δ^2 is large enough, then the variability of earnings may increase for sufficiently high smoothing incentives. This effect works against the market response to the earnings report, which can ultimately lead to a reduced value relevance for high values of *s*. The same reasoning applies for the effect of an increase in the cost of bias *r*.

The second deviation from the behavior of *EQ* is in the effect of varying the cost of bias *r*. As discussed in Proposition 2, *EQ* strictly decreases in *r* if

$$\frac{\sigma_g^2}{\sigma_\delta^2} \le \Gamma, \tag{14}$$

and strictly increases otherwise. $\Gamma > 0$ is defined in (A3) in the appendix. The behavior of value relevance β is broadly similar, although additional conditions apply for conforming effects. Proposition 3 states three sufficient conditions for a specific behavior of β on a change in *r*. These conditions are determined by the relative values of the operating risks σ_ε^2 and σ_δ^2 and accounting risk σ_n^2. They are also reminiscent of the condition that also appears in part (i) of the Proposition, $\sigma_\delta^2 \le \sigma_n^2\left(2 - S_n\right)$, for a similar reason. Therefore, the relationship of *EQ* and β cannot be stated unambiguously.

It is instructive to consider the special case when the manager does not have private information about $\tilde{\delta}$ ($\sigma_\delta^2 = 0$) and there is no market uncertainty about the earnings incentive ($\sigma_g^2 = 0$). Value relevance β then becomes

$$\beta = \frac{Q\sigma_\varepsilon^2}{Q^2\left(\sigma_\varepsilon^2 + \sigma_n^2\right)} = \frac{1 - S_n}{Q}.$$

---

[16] Recall that, due to smoothing, the manager attempts to smooth the contemporaneous shock in the firm's earnings across the two periods, thus lowering the portion of variability in first-period reported earnings that results from the noise inherent in the accounting system.

Since Q strictly decreases in s and strictly increases in r, β strictly increases in s and strictly decreases in r. The reason is that the capital market adjusts for the bias that is induced by the manager's incentives. In contrast, EQ does not change with varying s or r in the case of $\sigma_\delta^2 = 0$ and $\sigma_g^2 = 0$. Note that

$$Var(\tilde{x}|m_1) = Var(\tilde{x}) - \frac{Cov(\tilde{x}, \tilde{m}_1)^2}{Var(\tilde{m}_1)} = Var(\tilde{x}) - \frac{Q^2\sigma_\varepsilon^4}{Q^2(\sigma_\varepsilon^2 + \sigma_n^2)} = Var(\tilde{x}|y_1).$$

Therefore, the information content of m_1 (absent δ and $\sigma_g^2 = 0$) does not vary with incentives, and so the earnings quality EQ does not vary either. As Proposition 3 states, this is a limiting case as for small $\sigma_\delta^2 > 0$ the direction of the effects become aligned. However, caution should be taken in interpreting the value relevance as a cardinal metric of earnings quality.

Relative association studies use the R^2 as measure of value relevance. Our linear equilibrium yields an R^2 which always equals 1 because reported earnings are the sole information source for the market and R^2 is the correlation coefficient between the market price and the reported earnings.[17] However, there is a close connection of R^2 with β. To see this, we briefly consider a slight extension of the price formation. Suppose the market price includes a stochastic term $\tilde{v}$, for example due to liquidity or noise trading,

$$P = \mathrm{E}\left[\tilde{x}|\tilde{m}_1\right] + \tilde{v}$$

where $\tilde{v}$ is normally distributed with mean zero and independent from the other random variables.

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[17] This is because R^2 is the correlation coefficient between the market price and the reported earnings, $\frac{Cov\left(\tilde{P}, \tilde{m}_1\right)}{Std\left(\tilde{m}_1\right)Std\left(\tilde{P}\right)} = \frac{\beta Var\left(\tilde{m}_1\right)}{Std\left(\tilde{m}_1\right)\beta Std\left(\tilde{m}_1\right)} = 1$. See Lev [1989] for a linear model based on changes of variables (stock price revisions and unexpected reported earnings). It also applies for the levels version since the *a priori* stock price and expectations are given and irrelevant for the correlation coefficient.

Corollary: Let the market price be $P = \mathrm{E}\left[\tilde{x} \middle| \tilde{m}_1\right] + \tilde{v}$ *. Then the* R^2 *from regressing market returns on earnings is strictly increasing in EQ.*

Proof: R^2 is defined as

$$R^2 = \frac{Cov\left(\tilde{P}, \tilde{m}_1\right)}{Std\left(\tilde{m}_1\right) Std\left(\tilde{P}\right)} = \frac{Cov\left(\alpha + \beta \tilde{m}_1 + \tilde{v}, \tilde{m}_1\right)}{Std\left(\tilde{m}_1\right) Std\left(\alpha + \beta \tilde{m}_1 + \tilde{v}\right)} = \frac{\beta Var\left(\tilde{m}_1\right)}{Std\left(\tilde{m}_1\right) Std\left(\alpha + \beta \tilde{m}_1 + \tilde{v}\right)}$$

$$= \frac{\beta Std\left(\tilde{m}_1\right)}{Std\left(\alpha + \beta \tilde{m}_1 + \tilde{v}\right)} = \frac{\sqrt{\left(\beta^2 Var\left(\tilde{m}_1\right)\right)}}{\sqrt{\left(\beta^2 Var\left(\tilde{m}_1\right) + \sigma_v^2\right)}}$$

$$= \sqrt{\frac{EQ}{EQ + \sigma_v^2}}.$$

(q.e.d.)

This expression shows that R^2 is monotone in *EQ* given the price volatility is caused by noise trading. This result is even stronger than the results we report in Proposition 3 for the earnings response coefficient β, which is closely aligned with *EQ*.

5.2. Persistence

Persistence measures the extent current earnings persist or recur in the future. High persistence is regarded a desirable earnings attribute by investors, and of high earnings quality, since it suggests a stable, sustainable and low-risk earnings process. Accounting standards may support the persistence by separating recurring from non-recurring items; firms often disclose *pro forma* earnings that do not include non-recurring or special items, and analysts often eliminate certain items that they believe are not persistent.

Persistence is commonly estimated by the slope coefficient from a regression of current earnings on lagged earnings (or, sometimes, components of lagged earnings, such as cash flows and accruals). In our setting, we use the slope coefficient from regressing expected second-period earnings $\tilde{m}_2$ on first-period earnings m_1, which is tantamount to an "earnings relevance coefficient" that shows how expected second-period earnings are affected by first-period earnings.[18] Since

$$\mathrm{E}\left[\tilde{m}_2 \middle| m_1\right] = \mathrm{E}\left[\tilde{m}_2\right] + \frac{Cov\left(\tilde{m}_1, \tilde{m}_2\right)}{Var\left(\tilde{m}_1\right)} \cdot \left(m_1 - \mathrm{E}\left[\tilde{m}_1\right]\right),$$

we define persistence *PS* as

$$PS \equiv \frac{Cov\left(\tilde{m}_1, \tilde{m}_2\right)}{Var\left(\tilde{m}_1\right)}.$$

A greater persistence *PS* is supposed to indicate higher earnings quality. Rewriting *PS* yields (see the proof of Proposition 4 in the appendix)

$$PS \equiv \beta + \frac{Q\left(\sigma_\varepsilon^2 + \sigma_n^2\right)}{Var\left(\tilde{m}_1\right)} - 1.$$

This shows that *PS* is a metric that augments the value relevance β by an additional term that results from the impact of smoothing on the time series of earnings. Note that if $s = 0$ and $\sigma_g^2 = 0$, then $Q = 1$ and $Var(\tilde{m}_1) = Var(\tilde{y}_1)$, so that the additional term vanishes and $PS = \beta$. The additional term in the definition of *PS* can be either positive or negative. For example, if σ_δ^2 becomes sufficiently large, the second term in the *PS* equation is less than 1, thus implying $PS < \beta$.

The term $Q(\sigma_\varepsilon^2 + \sigma_n^2)/Var(\tilde{m}_1)$ adds some complexity to the behavior of *PS* with respect to the determining factors. For example, whereas β increases in σ_δ^2, this term decreases; similarly, β decreases in the accounting risk σ_n^2, but the additional term increases.[19] However, the next proposition shows that for many cases, the effect resulting from β dominates the other effect, so

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[18] While this definition may appear as being a two-period metric, it is based strictly on information available in period 1, because it uses the expectation of m_2 conditional on m_1.

[19] This can be verified by differentiating the new term with respect to the accounting noise.

the net effect is often not very different from that of β (as stated in Proposition 3). Proposition 4 provides the results.

Proposition 4: Persistence PS

(i) strictly increases in the smoothing incentive s if $\sigma_\delta^2 \le \sigma_n^2\left(2-S_n\right)$. *Otherwise, if* σ_δ^2 *is sufficiently large, then PS is inversely u-shaped in s;*

(ii) is inversely u-shaped in r if $Cov\left(\tilde{m}_1,\tilde{m}_2\right)\ge 0$ *and* $\sigma_g^2 > \max\left\{0;-16s^2\left(\sigma_\delta^2-\sigma_n^2\left(2-S_n\right)\right)\right\}$. *Otherwise, the effect is ambiguous and depends on the specific parameters;*

(iii) is unaffected by the market price incentive p and the expected earnings incentive $\bar{g}$ *;*

(iv) strictly increases in the risk σ_δ^2 *of private information* δ*; and*

(v) strictly decreases in the accounting risk σ_n^2 *of signal* y_1 *for* $\sigma_g^2 \le \underline{\Lambda}$, *first strictly decreases and then increases in* σ_n^2 *for* $\underline{\Lambda} < \sigma_g^2 \le \overline{\Lambda}$, *and strictly increases in* σ_n^2 *for* $\overline{\Lambda} < \sigma_g^2$*; and*

(vi) strictly decreases in the incentive risk σ_g^2.

The proof is in the appendix. Persistence *PS* behaves similarly to value relevance β for changes in the smoothing parameter, market price and earnings incentives, information risk, and incentive risk.

Differences occur for changes in the cost of bias *r* and of the precision of the accounting system (accounting risk σ_n^2). Note that persistence is the covariance of earnings in the two periods over the variance of first-period earnings. As shown in detail in the proof, $Cov(\tilde{m}_1,\tilde{m}_2)$ first increases and then decreases in *r*. The variance $Var(\tilde{m}_1)$ strictly increases in *r*, but for large σ_g^2 it first decreases and then increases. The total effect of $Cov(\tilde{m}_1,\tilde{m}_2)/Var(\tilde{m}_1)$ is ambiguous. The Proposition gives a sufficient condition for *PS* to be inversely u-shaped in *r* if $Cov\left(\tilde{m}_1,\tilde{m}_2\right)\ge 0$ and $\sigma_g^2 > \max\left\{0;-16s^2\left(\sigma_\delta^2-\sigma_n^2\left(2-S_n\right)\right)\right\}$. But it also implies that persistence *PS* can be negative in equilibrium because the covariance $Cov\left(\tilde{m}_1,\tilde{m}_2\right)<0$ can become negative for low *r*. In any case, the behavior is distinctly different from that of *EQ*.

Another difference occurs for changes in the accounting risk σ_n^2. The effect depends on the uncertainty of investors with respect to the manager's earnings incentive σ_g^2: For low uncertainty, persistence strictly decreases; for intermediate uncertainty, it is u-shaped; and for high uncertainty it strictly increases in σ_n^2. The thresholds are defined in the proof of the Proposition. Notice that *EQ* and β always strictly decrease in accounting risk. Persistence shows the same effect only for low incentive risk. Otherwise, persistence can increase, because for large σ_g^2, the covariance is negative as m_1 and m_2 move in opposite directions if bias is added due to the accounting incentive. Moreover, the resulting noise is completely unrelated to the operating risk of the firm. Therefore, an increase in the accounting risk σ_n^2 raises the importance of the accounting signal relative to the accounting incentive *g*, so that the (negative) covariance increases.

5.3. Predictability

Predictability encompasses the notion that earnings are of high quality the more useful they are to predict itself. Similar to persistence, predictability is generally viewed as a desirable attribute of earnings because it reduces the variability of forecasts of earnings. There are several ways to empirically estimate predictability. One is using the same regression of current earnings on lagged earnings, as for estimating persistence, but taking R^2 as the proxy for predictability. Another way is taking the standard deviation of the residuals from the same regression.

We use the variance of second-period earnings $\tilde{m}_2$ conditional on the earnings report in the first period m_1, that is, $Var\left(\tilde{m}_2 \middle| m_1\right)$. A lower conditional variance implies that investors have more precise information for forecasting second-period earnings. To retain the convention that a higher value of the metric is indicative of higher earnings quality, we define predictability *PD* as the negative value of the conditional variance of second-period earnings $\tilde{m}_2$, that is,

$$PD \equiv -Var\left(\tilde{m}_2 \middle| m_1\right).$$

Rewriting *PD* (see the proof of Proposition 5) shows its close relationship to persistence *PS*:

$$
\begin{aligned}
-PD &= Var(\tilde{m}_2) - \frac{Cov(\tilde{m}_1, \tilde{m}_2)^2}{Var(\tilde{m}_1)} \\
&= 2\sigma_\varepsilon^2 + Var(\tilde{y}_1) + Var(\tilde{y}_2) - (1+PS)^2 Var(\tilde{m}_1).
\end{aligned} \tag{15}
$$

This expression shows that the persistence *PS* is a main driver of predictability, as it captures the bias in both earnings reports, which is driven by incentives and the additional information $\tilde{\delta}$. However, accounting risk σ_n^2 and operating risk σ_δ^2 are also included in the variance terms $Var(\tilde{y}_1)$, $Var(\tilde{y}_2)$, and $Var(\tilde{m}_1)$. Therefore, the behavior of *PD* is not directly apparent. The next proposition establishes the effects of a variation of the determining factors on *PD*.

Proposition 5: Predictability PD

(i) strictly increases in the smoothing incentive s;

(ii) strictly decreases in the cost of bias r if $\sigma_\delta^2 < 1 + (2\sigma_\varepsilon^2 + \sigma_n^2)\left(S_n - \frac{1}{Z}\right)$. *Otherwise, if the incentive risk* σ_g^2 *is sufficiently large, then PD strictly increases in r;*

(iii) is unaffected by the market price incentive p and the expected earnings incentive $\bar{g}$ *;*

(iv) decreases in the risk σ_δ^2 *of private information* δ*;*

(v) strictly decreases in the accounting risk σ_n^2 *of signal* y_1*; and*

(vi) strictly decreases in the incentive risk σ_g^2*.*

The proof is in the appendix. Comparing these results with those obtained for *EQ* in Proposition 2, we find that predictability *PD* corresponds with *EQ* except for part (ii), where the conditions for a decrease or increase differ, and for part (iv), which states a converse relationship for the risk of private information $\tilde{\delta}$. But as shown in the proof, the effects that lead to this correspondence differ somewhat in the details.

To explain the results in the proposition, note that a higher smoothing incentive *s* leads to a greater incorporation of the manager's private information into the bias and, hence, first-period earnings. In the second period, the same information is included in the accounting signal m_2, although somewhat dampened by the reversal of the bias. Therefore, the precision with which the second-period earnings can be predicted increases with higher smoothing incentives. Second, a change in the market price incentive *p* and the earnings incentive $\overline{g}$ have no effect since in this model the market can undo its effect on the bias. A higher magnitude of the accounting risk σ_n^2 increases the conditional variance of second-period earnings $\tilde{m}_2$ since the variability of the bias in the first period also increases with accounting risk, and the same variability increases the variance of second-period earnings due to the clean surplus condition on the bias.

An increase in the cost of bias *r* decreases predictability due to its lower incorporation of the manager's private information δ as long as the private information is not large (low σ_δ^2). Otherwise, if the incentive risk σ_g^2 becomes large, this effect is inverted and predictability increases in higher cost *r* because the high cost dampens the effect of the earnings incentive on the earnings report.

Perhaps surprisingly, the behavior of *PD* for a variation of the risk σ_δ^2 is opposite from that of *EQ*. The reason is that σ_δ^2 also appears as a direct component in *PD* because it is embedded in the second-period accounting signal m_2 and, hence, in the variance $Var(\tilde{y}_2)$. This is due to the fact that $\tilde{\delta}$ stands for information risk, but also for operating risk that is resolved in the second period. Whereas the earlier metrics do not pick up the operating risk effect, predictability does. Hence, a larger operating risk increases the variance of $\tilde{m}_2$, which is not outweighed by the effects of the information transfer that influences the conditional variance of $\tilde{m}_2$. Therefore, using predictability *PD* would lead to a wrong conclusion on earnings quality for a variation of private information held by the manager and embedded in the bias.

5.4. Smoothness

The smoothness of earnings can be considered either as a favorable or an unfavorable earnings attribute. Consistent with persistence and predictability, a smoother earnings series is less volatile and allows for better forecasting. For example, managers can use their private information to smooth out nonrecurring effects. Smoothness is then the converse of volatility and, therefore, smoother earnings are indicative of higher earnings quality. On the other hand, smoothness can be considered as reducing the information content in earnings and mask the firm's "true" performance. Under this view, smoothness is a measure for earnings management, and since earnings management is considered undesirable, smoother earnings are associated with a lower earnings quality. An advantage of the equilibrium model is that we can identify which interpretation is more appropriate in our setting.

Empirical studies use various metrics for smoothness. One metric is the ratio of the standard deviation of earnings and the standard deviation of operating cash flows. Our model does not include a periodic cash flow process and, therefore, we do not employ this metric. Another metric uses the fact that smoothness implies a negative correlation between the change in accounting accruals and the change in operating cash flows. A stronger (negative) correlation indicates greater smoothness. A related metric is the correlation between the change in discretionary accruals and the change in pre-discretionary income.[20] We use the variables themselves and not their changes, as we have no period 0 values and are interested only in relative changes upon varying parameters. We define our metric for smoothness, *SM*, as the negative correlation between discretionary accruals, which is the bias *b*, and pre-discretionary earnings, which is our accounting signal $\tilde{y}_1$, that is,

$$SM \equiv -Corr\left(\tilde{b}, \tilde{y}_1\right) = -\frac{Cov\left(\tilde{b}, \tilde{y}_1\right)}{Std\left(\tilde{b}\right) Std\left(\tilde{y}_1\right)}.$$

A higher *SM* implies greater smoothness. The following results obtain.

Proposition 6: Smoothness SM

(i) strictly increases in the smoothing incentive s;

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[20] See, e.g., Tucker and Zarowin [2006].

(ii) is unaffected by the cost of bias r;

(iii) is unaffected by the market price incentive p and the expected earnings incentive $\bar{g}$ *;*

(iv) strictly decreases in the risk σ_δ^2 *of private information* δ*;*

(v) strictly increases in the accounting risk σ_n^2 *of signal* y_1*; and*

(vi) strictly decreases in the incentive risk σ_g^2*.*

The proof is in the appendix. It shows that *SM* can be expressed as

$$SM = \frac{\sigma_n^2}{\left(\sigma_n^4 + \left(\sigma_\delta^2 + \frac{1}{16s^2}\sigma_g^2\right)\left(\sigma_\varepsilon^2 + \sigma_n^2\right)\right)^{1/2}}.$$

From this equation, it is immediate that *SM* is unaffected by the market price incentive *p*, the expected earnings incentive $\bar{g}$, and the cost of bias *r*. A greater smoothing incentive induces an increase in *SM* because the variance of the equilibrium bias decreases in σ_g^2 as a result of the manager's private knowledge of the importance of the earnings incentive. If there is no incentive risk ($\sigma_g^2 = 0$), then *SM* is independent of any incentive effects, but depends on the operating and accounting risks only.

Therefore, this metric generally does not capture most incentives as a determining factor for earnings quality and earnings management. Intuitively, these results obtain since *SM* is a correlation coefficient that measures the degree of linear dependence between discretionary accruals and pre-discretionary income. As our model studies linear equilibria with linear pricing and reporting strategies, there is always a perfect linear relationship between the interesting variables, which does not depend on the magnitude of *p*, $\bar{g}$, and *r*. It captures smoothing incentives similar to *EQ* only because the variance of the bias decreases in *s*.

Smoothness strictly decreases in the variance σ_δ^2 of private information δ; and strictly increases in the variance σ_n^2 of the accounting signal y_1. Both effects are generally in contrast to other metrics. Consider the dependence on σ_δ^2 first. According to *SM*, a higher variability σ_δ^2 of the private information δ decreases *SM*, which implies a lower smoothness. Even more striking is the fact that increasing accounting risk σ_n^2 implies that *SM* always strictly increases.

These results are surprising not only because of the diametrically opposed conclusions that are drawn on smoothness with respect to changes in factors that are important for earnings quality, but also because it is not obvious whether more smoothing is an indicator of higher or lower earnings quality. We discuss this observation in more detail later.

5.5. Accrual quality

Most of the metrics focus on the information content of earnings about the terminal value, that is, about the operating risk components inherent in the terminal value. Only the smoothness metric *SM* separately considers two components of reported earnings, the accounting signal y_1 and the bias *b*. The bias is the accounting accrual, which is determined by the manager's incentives (earnings management), but also carries additional information. The accrual quality specifically seeks to capture the quality of the accounting system, rather than the quality of the accounting and the operating environment. The idea is that high accrual quality also proxies for high earnings quality, holding the operating environment constant.

To estimate the accrual quality, many empirical studies employ a metric suggested by Dechow and Dichev [2002], which is based on the standard deviation of the residuals from a regression of working capital accruals on lagging, contemporaneous and leading operating cash flows. This metric focuses on measurement errors in specific accruals. Our model is not sufficiently specified to study this metric.

Another metric separates earnings into "normal" and discretionary (or "abnormal") accruals, based on some model that predicts "normal" accruals from other variables, e.g., based on the Jones [1991] model. Accrual quality is estimated based on the statistical properties of the discretionary accruals. In our setting, we directly use the accounting signal y_1 as "normal" accruals and the bias *b* as discretionary accruals. In an empirical study the econometrician must estimate these “normal” accruals from other observables, which introduces noise in the estimate and decreases the “quality” of the accrual metric relative to our model.[21]

This separation between “normal” and discretionary accruals is in line with the common interpretation that discretionary accruals are the result of earnings management because our bias is a consequence of the manager’s incentives; were there no incentives (*p*, *s*, *g* and *r* equal to zero), the bias would be zero. Therefore, most literature considers high discretionary accruals as indicating low accrual quality and, ultimately, low earnings quality. However, we note that the bias is the carrier for the private information of the manager, so it cannot be considered as necessarily indicating low earnings quality. This latter effect is more prominent as much of the earnings management is backed out by the rational investors.

We define two different, but related, metrics for discretionary accruals. The first metric is the negative expected value of the bias,

$$DA_1 \equiv -\mathrm{E}\left[\tilde{b}\right].$$

We use the negative value to interpret DA_1 in conformance with the notion that a larger value of the metric indicates higher accrual quality and higher earnings quality.

A potential problem with this metric is that positive and negative realizations of the bias cancel out in expectation. In the extreme, DA_1 can remain constant although the bias is significantly affected by a change in a determining factor, which affects the variance, but not the expected value. Therefore, some studies use the expected value of the absolute amount of bias. Since taking absolutes introduces a deviation from our linear setting, we define the second metric for discretionary accruals as the negative expected value of the squared bias,

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[21] One may argue that if an econometrician is able to estimate discretionary accruals from observable data, then investors can do so, too. Such an assumption can be included in our model by assuming that the market receives a noisy signal of *b* in addition to the firm’s earnings. We conjecture that our results would qualitatively carry over to such an extended model.

$$DA_2 \equiv -\mathrm{E}\left[\tilde{b}^2\right].$$

Again, a higher value of DA_2 is supposed to indicate higher accrual quality and higher earnings quality.

To see the behavior of the first accrual quality metric, DA_1, it is instructive to write DA_1 in the following form:

$$DA_1 = -\mathrm{E}\left[(1-Q)\mu + (Q-1)\tilde{y}_1 + Z\tilde{\delta} + \beta pR + R\overline{g}\right] = -\frac{\beta p + \overline{g}}{8s + r}. \quad (16)$$

Proposition 7: Assuming p > 0 and $\overline{g} > 0$*, accrual quality* DA_1

(i) strictly increases in the smoothing incentive s if β *decreases (which occurs for* $\sigma_\delta^2 > \sigma_n^2(2 - S_n)$ *and large s). Otherwise, the effect depends on the parameter values;*

(ii) strictly increases in the cost of bias r if β *decreases (see Proposition 3 (ii)). Otherwise, the effect depends on the parameter values;*

(iii) strictly decreases in the market price incentive p and the expected earnings incentive $\overline{g}$ *;*

(iv) strictly decreases in the risk σ_δ^2 *of private information* δ*;*

(v) strictly increases in the accounting risk σ_n^2 *of signal* y_1*;and*

(vi)strictly increases in the incentive risk σ_g^2*.*

If p < 0 and $\overline{g} < 0$*, the direction of the effects reverse.*

To explain these results, note first that DA_1 is affected by both the market price incentive *p* and the expected earnings incentive $\overline{g}$, whereas *p* and $\overline{g}$ are irrelevant for earnings quality *EQ* as the market can completely adjust for it. For example, if the manager is particularly interested in a high market price in the first period ($p > 0$), DA_1 strictly decreases in *p* because higher *p* increases the bias, which reduces the accrual quality metric. A similar reasoning holds for a change in $\overline{g}$. The direction of the effect of the other factors depends on whether the sum ($\beta p + \overline{g}$) is positive or negative. For the following discussion, we assume $p > 0$ and ($\beta p + \overline{g}$) > 0; otherwise, the results reverse.

To see the other effects, note that the behavior of DA_1 is essentially determined by the behavior of β, moderated by a base term $\overline{g}R$. A general observation is that a higher value relevance β is generally seen as indicative for high earnings quality, although we establish in Proposition 3 that there are situations where higher β is associated with lower EQ. Since DA_1 varies negatively with β, the effects of changes in the determining factors are reversed.

The effects of varying the private information σ_δ^2 and accounting risk σ_n^2 of signal y_1 follow immediately from (16) because the other terms are not affected by these factors. Therefore, accrual quality DA_1 strictly decreases in σ_δ^2 and strictly increases in σ_n^2, just the opposite of most of the other metrics. The reason is that an increase in σ_δ^2 increases the informative bias (thus leading to an increase in value relevance), whereas an increase in σ_n^2 reduces it, as there is less information content about the operating risk ε in the accounting signal (implying a decrease in value relevance).

The behavior of DA_1 is less obvious for variations of the smoothing incentive s and the cost parameter r because they affect not only β but also the denominator $(8s + r)$. The term $1/(8s + r) = R$ decreases in s and r. The proposition records cases in which β also decreases in s or r, because then DA_1 is unambiguously increasing and accrual quality strictly increases. Proposition 7 (i) and (ii) essentially repeat the conditions for a decrease in β (see Proposition 3). These conditions also imply that DA_1 and β are affected in opposite directions (as long as $p > 0$ and $(p + \overline{g}) > 0$). In other cases the effect of varying s and r is indeterminate. It is easy to find parameter values for which the effect on DA_1 is positive or negative. We do not record more conditions as they would involve a combination of conditions on most of the determining factors, and no general patterns emerge.

One special case is of interest, though. It is the case if the manager does not have private information about $\tilde{\delta}$ and there is no earnings incentive, which is equivalent to setting $\sigma_\delta^2 = 0$, $\sigma_g^2 = 0$, $\overline{g} = 0$, and $\sigma_g^2 = 0$. Then DA_1 is determined by

$$DA_1 = -\frac{p\sigma_\varepsilon^2}{(8s+r)\sigma_\varepsilon^2 + (4s+r)\sigma_n^2}.$$

In this case, DA_1 strictly increases in both *s* and *r*. The increase for *s* obtains since a higher smoothing incentive essentially curbs the manager's desire to boost first-period earnings to induce a higher market price. That would increase the difference between second and first-period earnings, which is negatively valued by the manager. Thus, discretionary accruals decrease on average, and DA_1 increases as it is defined as the negative expected value of these accruals. The increase in *r* follows from the fact that higher cost for earnings management directly lead to a reduction of earnings management. Recall that in the case of $\sigma_\delta^2 = 0$ and $\sigma_g^2 = 0$, earnings quality *EQ* is unaffected by the smoothing and cost parameter, and this again demonstrates that the use of DA_1 as a metric for earnings quality may be misleading.

The accrual quality metric DA_2can be expressed as follows:

$$DA_2 = -Var\left(\tilde{b}\right) - \Bigg(\underbrace{\mathrm{E}\left[\tilde{b}\right]}_{-DA_1}\Bigg)^2 = -Z^2\left(\sigma_n^2 S_n + \sigma_\delta^2 + \frac{1}{16s^2}\sigma_g^2\right) - \frac{\left(p\beta + \bar{g}\right)^2}{\left(8s+r\right)^2}. \tag{17}$$

Proposition 8: Assuming p > 0 and $\bar{g} > 0$, *accrual quality* DA_2

(i) is ambiguous in the smoothing incentive s;

(ii) strictly increases in the cost of bias r if β *decreases (see Proposition 3 (ii)). Otherwise, the effect depends on the parameter values;*

(iii) strictly decreases in the absolute values of the market price incentive p and the expected accounting incentive $\bar{g}$ *;*

(iv) strictly decreases in the risk σ_δ^2 *of private information* δ*;*

(v) is ambiguous in the accounting risk σ_n^2 *of signal* y_1*; and*

(vi) is ambiguous in the incentive risk σ_g^2.

The results follow from the fact that DA_2 is determined by two terms, $Var(\tilde{b})$ and $\mathrm{E}[\tilde{b}]$, as shown in equation (17). Equation (17) also shows a close relationship with the behavior of DA_1.

Part (iii) is related to the behavior of $\mathrm{E}[\tilde{b}]$ but differs in that p and $\bar{g}$ are included in squared form. Therefore, the more pronounced the market price or the expected earnings incentive in either direction, the lower is the metric DA_2. This effect indicates a higher magnitude of discretionary accruals, which is intuitive since a larger absolute market price incentive induces more bias (in absolute terms) to either boost the firm's reported earnings (if $p > 0$ and $\bar{g} > 0$) or to reduce them (if $p < 0$ and $\bar{g} < 0$). However, this effect does not mirror the behavior of our earnings quality measure EQ as investors are able to anticipate and back out the effect of the market price incentive p and the expected earnings incentive $\bar{g}$ (see Proposition 3).

Part (iv) follows from the fact that a change in σ_δ^2 moves the two terms in equation (17) in the same direction. Both results are similar to the behavior of DA_1, but again opposite to most other metrics. To see part (v) note that the first term in the equation of DA_2, $Var(\tilde{b})$, increases in the accounting risk σ_n^2, and the second term, $\mathrm{E}[\tilde{b}]$, decreases. Therefore, the effect cannot be signed unambiguously, but depends on the specific parameter values.

An increase in the cost of bias r reduces Z and, thus, $Var(\tilde{b})$, and it reduces $\mathrm{E}[\tilde{b}]$ under the conditions stated in Proposition 7 (ii), so that the terms are affected in the same direction. This means that DA_2 increases in r. Similarly, an increase in the smoothing incentive s increases $Var(\tilde{b})$, but the effect on $\mathrm{E}[\tilde{b}]$ is ambiguous. This ambiguity and the interaction with the effect on $Var(\tilde{b})$ prohibit a more general statement on the behavior of DA_2.[22] And finally, an increase in the uncertainty about the manager's earnings incentive σ_g^2 results in an ambiguous effect on DA_2 because $Var(\tilde{b})$ increases in σ_g^2, but β and, thus, the second term in (17) decrease in σ_g^2.

6. Discussion

In this section, we summarize the results of the propositions and discuss some implications for the choice of earnings quality metrics. Table 1 shows the main results of the behavior of the

[22] Even for the special case of no private information of the manager, the effect of varying s is ambiguous.

earnings quality metrics we study upon a variation of the factors that determine earnings quality. Note that we derive results only on the direction of the effects and not on their amounts so that quantitative comparisons cannot be made.

[Insert Table 1 about here]

Table 1 provides a number of insights into the appropriateness of earnings quality metrics in empirical studies. A first and important insight is that there is no metric that perfectly traces our measure *EQ* that we consider the most appropriate earnings quality metric in our model.[23] While some metrics are closely aligned with *EQ*, other measures are non-monotonic in a change of earnings quality factors.

Second, closest to the benchmark is value relevance, followed by persistence, which react similarly because they are closely linked to *EQ* and to each other. This similarity obtains in the model, which predicts the direct link between market price reactions and future expected earnings. In practice, these two metrics are likely to perform somewhat differently, depending on the efficiency of the market and the existence of effects such as noise trading, among others. This result lends credibility to the use of value relevance as a metric for earnings quality.[24] The only difference in prediction between value relevance and *EQ* occurs for the smoothing incentive and the cost of bias: An increase in the smoothing incentive has an unambiguous effect on *EQ* whereas the direction of the effect on value relevance can depend on the different risk

[23] Although we find that value relevance R^2 is strongly monotonic in our earnings quality measure, we show this result for a slight variation of the original setting.

[24] Holthausen and Watts [2001] are very critical as to the inferences one can draw from value relevance studies. However, our results do not directly speak to their arguments, because our model focuses on the information content of earnings, while Holthausen and Watts largely consider contractual roles of accounting (e.g., stewardship).

parameters. For variations in the cost of bias, the effects of both *EQ* and value relevance depends on the relation between private information and accounting risk, albeit the conditions are not the same. Therefore, empirical studies that examine the effect of a variation of incentives and cost of earnings management on earnings quality should make sure that the conditions under which these metrics behave similarly are satisfied in the sample.

Third, the behavior of predictability deviates from that of earnings quality *EQ* is similar, except for the effect of information risk on the metrics and the cost of bias (in some cases). The information risk stands for the private information endowment of the manager when choosing the bias. Since the behavior of predictability is strictly opposite to that of *EQ* for a variation of information risk, when studying the effect of different levels of private information on earnings quality, predictability would lead to false conclusions. On the other hand, if changes in the incentives are the focus of a study, predictability is generally an appropriate metric.

Fourth, metrics that depend directly on the discretionary accruals (our bias *b*) are highly problematic metrics for earnings quality. The smoothness metric *SM*, which is the negative correlation between discretionary accruals and pre-discretionary income, does not react on changes in the incentives except for smoothing, which are the driving force behind discretionary accruals and much of the earnings management. The reason is that incentives cancel out in equilibrium. If it reacts to a variation of information risk and accounting risk, it is in a direction opposite of that of *EQ*.

Fifth, our two metrics for accrual quality also do not capture the actual effects of a variation of factors. Again, the reason is that they use discretionary accruals as the basis for the metric. They are the only two metrics we study that are affected by the market price and earnings incentives because an increase in these incentives increases the level of bias, albeit there is no such effect on *EQ* because the information content of earnings is not affected. The behavior of both metrics is mostly ambiguous and does not allow gaining more general insights for a variation of the smoothing incentive. Even worse, the majority of the effects that are systematic are opposite to those of *EQ* and most of the other earnings quality metrics, if one assumes a positive incentive increases the short-term market price of the firm. This result suggests that the common interpretation of both measures in a way that large discretionary accruals reduce the accrual quality and earnings quality is not appropriate. It is the discretionary accrual that incorporates the manager's private information in the reported earnings, thus, making them decision-useful.[25] Empirical results on earnings management that rely on discretionary accruals metrics should be interpreted with caution.

The results summarized in Table 1 this paper contributes to the empirical literature in that they provide a theory for selecting suitable earnings quality metrics for specific analyses. Some metrics are more appropriate than others for studying certain economic effects and interpreting the results.

The results also provide conditions under which the effect of varying determining factors of earnings quality reverses. As Table 1 shows, these conditions require an assessment of underlying private information of the manager and the precision of the accounting system, but also the incentive structure (long-term versus short-term market price incentives). Controlling for these conditions and their effects may help sharpen empirical tests.

The results also provide insights into interdependencies of earnings quality metrics. For example, Francis et al. [2004] provide a table of correlations among earnings attributes. Their definition of metrics differs somewhat from ours; for example, they use the R^2 for value relevance. However, persistence and predictability are defined similarly, and their data show a significantly positive correlation between these two metrics.[26] This is consistent with our results except for the case of varying the information risk (and less so the cost of bias), but a direct

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[25] This view is also supported by empirical studies, such as Tucker and Zarowin [2006].

[26] See Francis et al. [2004], p. 982.

comparison is difficult as our results are based on a variation of single determining factors, whereas the correlation is calculated for their entire data set.

Many studies on earnings quality employ more than one metric acknowledging possible deficiencies in each metric. Often, these studies aggregate these metrics by using the sum of their ranks. Suppose a study employs discretionary accruals (DA_1) and value relevance (β), then our results predict that they generally react in opposite directions, so the aggregate metric based on the sum of ranks would be unaffected by the regulatory change. Such effects can also explain why studies can find insignificant results.

We do note that some of our results on the effect of determining factors on earnings quality EQ and other metrics appear counterintuitive at first sight. However, considering the market reactions in equilibrium, they become understandable. A particular example is the null effect of the market price incentive p and on the expected earnings incentive $\bar{g}$. The reason is that the capital market is aware of these incentives and can back out the bias on average. Therefore, the information content of earnings does not change, although the earnings level changes.

The results are also helpful to predict the effects of a regulatory change, for example, the introduction of tighter accounting standards. This change can be captured (*ceteris paribus*) by an increase in the cost of earnings management, which translates into an increase in the cost of bias r. Our results suggest that the effect of tighter accounting standards is neither straightforward nor generally robust. Consider again Proposition 2, which states that earnings quality EQ decreases for tighter standards because it prohibits that the accruals carry private information. However, if the uncertainty of the manager's earnings incentives is high this relationship reverses, so that tighter accounting standards increase earnings quality. Interestingly, a change in the corporate governance to increase the transparency of management compensation reduces the uncertainty of the manager's incentives, which again implies that earnings quality is negatively affected by tighter standards. Results like this indicate that earnings quality is determined by many interacting factors, and a regulatory change in one factor may lead to unintended consequences, if the change is not coordinated with other changes.

7. Conclusions

The aim of this paper is to provide a theory about the usefulness and appropriateness of earnings metrics commonly used in empirical studies. The analysis is in the context of a rational expectations equilibrium model in which we analyze the information content, or decision usefulness, of earnings reported by a manager who has market price, reported earnings, and smoothing incentives. Our benchmark measure of earnings quality captures the reduction of the market's assessment of the variance of the terminal value due to the earnings report. The earnings quality metrics examined are value relevance, persistence, predictability, smoothness, and accrual quality. While we define these metrics suitably within our model, we believe these definitions are faithful to the statistical constructs used in empirical studies.

We describe the behavior of these earnings quality metrics upon a variation of private information, accounting risk and management incentives. The results show that these parameters jointly determine earnings quality and that the various metrics capture variations in the underlying factors very differently. We find that value relevance is closely aligned with earnings quality, followed by persistence, whereas other metrics react differently, often even non-monotonic in earnings quality. The most striking result is that metrics that depend on discretionary accruals are difficult to interpret as their behavior depends on the specific circumstances, despite there is a clear effect of the variation of the factors that determine earnings quality in equilibrium.

The model provides a rich setting for studying the interaction of reporting strategies and the information content of earnings in a capital market setting. However, we stress that our model uses several simplifying assumptions that are necessary to keep the analysis tractable, so that our results may not be robust for changes in the set of assumptions. One assumption is a simple (pure exchange) economic setting in which uncertainty is exogenous and there are no productive activities of the manager. Of particular importance for our results are smoothing incentives, which are the cause for the conveyance of the manager's private information into earnings. We also assume that only one of the incentives of the manager is not known by the market. Even so, the market is able to back out on average the bias that results from the manager's incentives. We consider a two-period model in which the initial uncertainty about the liquidating firm value resolves over time; and we concentrate on the effects of a change in the earnings quality factors in the period of that change. This event window is often the basis for empirical studies, so our results should provide predictions usable for empirical tests. Finally, we focus on decision usefulness of reported earnings in capital markets and abstract from contracting uses of accounting information.

The model could be extended along various dimensions. The manager may be able to affect the cash flow distribution, have additional communication channels available, or can voluntarily disclosure her private information outside the reported earnings. Another extension is to consider a full-fledged multi-period or a steady-state model, where new information can arise or being released at different times. Such extensions may affect our results, particularly for time-series based metrics.

This paper is a first step to provide a theory to understand what commonly used earnings metrics really measure, to offer explanations for the relationship between these metrics, to guide the selection and form predictions on the behavior of earnings quality metrics in empirical studies.

Appendix

Proof of Proposition 1

The optimal bias conditional on (y_1, δ) and the conjecture about the market price reaction $\hat{P}$ is

$$b^*\left(y_1,\delta\right)=\frac{p}{8s+r}\cdot\frac{d\hat{P}}{dm_1}+\frac{g}{8s+r}+\frac{4s}{8s+r}\left(\mathrm{E}\left[\tilde{y}_2|y_1,\delta\right]-y_1\right).$$

Due to normality,

$$\mathrm{E}\left[\tilde{y}_2|y_1,\delta\right]=\mu+\frac{Cov\left(\tilde{y}_1,\tilde{y}_2\right)}{Var\left(\tilde{y}_1\right)}\left(y_1-\mathrm{E}\left[\tilde{y}_1\right]\right)+\frac{Cov\left(\tilde{\delta},\tilde{y}_2\right)}{Var\left(\tilde{\delta}\right)}\left(\delta-\mathrm{E}\left[\tilde{\delta}\right]\right)$$

$$=\mu+\frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2+\sigma_n^2}\left(y_1-\mu\right)+\frac{\sigma_\delta^2}{\sigma_\delta^2}\left(\delta-0\right).$$

Therefore,

$$b^*\left(y_1,\delta\right)=\frac{p\dfrac{d\hat{P}}{dm_1}}{8s+r}+\frac{g}{8s+r}+\frac{4s}{8s+r}\left(\left(\mu-y_1\right)\frac{\sigma_n^2}{\sigma_\varepsilon^2+\sigma_n^2}+\delta\right).$$

Let $R\equiv\frac{1}{8s+r}$, $Z\equiv 4sR$, $S_n\equiv\frac{\sigma_n^2}{\sigma_\varepsilon^2+\sigma_n^2}$, and $Q\equiv 1-ZS_n$. Given linear conjectures, $\frac{d\hat{P}}{dm_1}=\hat{\beta}$, and the optimal bias is

$$b^*\left(y_1,\delta\right)=\left(1-Q\right)\mu-\left(1-Q\right)y_1+Z\delta+p\hat{\beta}R+gR.$$

It is easy to see that b^* is linear in g, y_1 and δ.

The equilibrium earnings report is

$$m_1^*\left(y_1,\delta\right)=y_1+b^*\left(y_1,\delta\right)=\left(1-Q\right)\mu+Qy_1+Z\delta+\frac{p\hat{\beta}}{8s+r}+gR.$$

In a rational expectations equilibrium, the market price equals the expected terminal value conditional on the earnings report, The conditional expectation is

$$P(m_1) = \mathrm{E}\left[\tilde{x} \middle| m_1\right] = \mu + \frac{Cov(\tilde{m}_1, \tilde{x})}{Var(\tilde{m}_1)}\left(m_1 - \mathrm{E}[\tilde{m}_1]\right)$$

$$= \mu + \underbrace{\frac{Q\sigma_\varepsilon^2 + Z\sigma_\delta^2}{Q^2\left(\sigma_\varepsilon^2 + \sigma_n^2\right) + Z^2\sigma_\delta^2 + R^2\sigma_g^2}}_{\beta} \cdot \left(m_1 - \mu - p\hat{\beta}R - \overline{g}R\right).$$

Thus, the market price is linear in m_1, $P(m_1) = \alpha + \beta m_1$, where

$$\beta \equiv \frac{Q\sigma_\varepsilon^2 + Z\sigma_\delta^2}{Q^2\left(\sigma_\varepsilon^2 + \sigma_n^2\right) + Z^2\sigma_\delta^2 + R^2\sigma_g^2}$$

$$\alpha \equiv \mu(1-\beta) - \beta\hat{\beta}pR - \beta\overline{g}R.$$

In equilibrium $\alpha = \hat{\alpha}$ and $\beta = \hat{\beta}$. Inserting these conditions implies

$$\alpha \equiv \mu(1-\beta) - \beta^2 pR - \beta\overline{g}R$$

and β as defined above.

(q.e.d.)

The next two lemmas are useful for proving the subsequent results.

Lemma 1: $Cov(\tilde{m}_1, \tilde{x})$

(i) strictly decreases in s if $\sigma_\delta^2 < \sigma_\varepsilon^2 S_n$ *and strictly increases otherwise;*

(ii) strictly increases in r if $\sigma_\delta^2 < \sigma_\varepsilon^2 S_n$ *and strictly decreases otherwise;*

(iii) strictly increases in σ_δ^2*;*

(iv) strictly decreases in σ_n^2*; and*

(v) is unaffected by σ_g^2*.*

Proof of Lemma 1

$$Cov(\tilde{m}_1, \tilde{x}) = Q\sigma_\varepsilon^2 + Z\sigma_\delta^2 = \sigma_\varepsilon^2 + Z\left(\sigma_\delta^2 - \frac{\sigma_n^2\sigma_\varepsilon^2}{\left(\sigma_\varepsilon^2 + \sigma_n^2\right)}\right) = \sigma_\varepsilon^2 + Z\left(\sigma_\delta^2 - \sigma_\varepsilon^2 S_n\right).$$

s and *r* affect the covariance only via *Z*. Note that *Z* increases in *s* and decreases in *r*. Therefore, the effect on the covariance depends on the sign of the term in parentheses. If $\sigma_\delta^2 > \sigma_\varepsilon^2 S_n$ then $Cov(\tilde{m}_1, \tilde{x})$ increases in *s* and decreases in *r*; and vice versa. This proves (i) and (ii). Parts (iii), (iv) and (v) are immediate from the above expression.

(q.e.d.)

Lemma 2: $Var(\tilde{m}_1)$

(i) strictly decreases in s if $\sigma_\delta^2 \le \sigma_n^2(2-S_n)$. *If* $\sigma_\delta^2 > \sigma_n^2(2-S_n)$, *then* $Var(\tilde{m}_1)$ *achieves a unique minimum at some* $\hat{s} > 0$ *and increases for* $s > \hat{s}$;

(ii) strictly increases in r if $\sigma_g^2 \le \max\left\{0; -16s^2\left(\sigma_\delta^2 - \sigma_n^2(2-S_n)\right)\right\}$.

If $\sigma_g^2 > \max\left\{0; -16s^2\left(\sigma_\delta^2 - \sigma_n^2(2-S_n)\right)\right\}$, *then* $Var(\tilde{m}_1)$ *achieves a unique minimum at some* $\hat{r} > 0$ *and increases for* $r > \hat{r}$;

(iii) strictly increases in σ_δ^2;

(iv) strictly increases in σ_n^2 *and*

(v) strictly increases in σ_g^2.

Proof of Lemma 2

The variance can be written as follows:

$$Var(\tilde{m}_1) = Q^2\left(\sigma_\varepsilon^2 + \sigma_n^2\right) + Z^2\sigma_\delta^2 + R^2\sigma_g^2 = \sigma_\varepsilon^2 + \sigma_n^2(1-2Z) + Z^2\left(\sigma_n^2 S_n + \sigma_\delta^2\right) + R^2\sigma_g^2$$

Since 1 – 2Z ≥ 0, parts (iii), (iv) and (v) follow immediately. To prove parts (i) and (ii), take the partial derivative with respect to variable j ($j = s$ or r):

$$\frac{\partial Var(\tilde{m}_1)}{\partial j} = 2Z_j \underbrace{\left[\sigma_n^2 \underbrace{(ZS_n - 1)}_{<0} + Z\sigma_\delta^2\right]}_{\equiv T} + 2RR_j\sigma_g^2 .$$

Part (i): $j = s$. T strictly increases in Z and achieves its maximum at the upper bound Z =1/2. Setting Z = 1/2 and solving for σ_δ^2 yields a threshold $\sigma_n^2(2-S_n)$. If $\sigma_\delta^2 \le \sigma_n^2(2-S_n)$, then T is always non-positive, which implies that the variance always strictly decreases with respect to s due to $Z_s > 0$ and $R_s < 0$.

Next consider $\sigma_\delta^2 > \sigma_n^2(2-S_n)$. Substituting $Z_s = RZ\, r/s$ and $R_s = -RZ\, 2/s$ in the first derivative of the variance yields

$$\frac{\partial Var(\tilde{m}_1)}{\partial s} = 2RZ\frac{1}{s}\left(rT - 2R\sigma_g^2\right)$$

Since $\sigma_\delta^2 > \sigma_n^2\left(2 - S_n\right)$, there always exist (*s*, *r*) such that *T* is positive. Fix some $r > 0$, then *s* must be large enough to result in $T > 0$, and if $T > 0$ for some $\underline{s}$, then $T > 0$ for any $s > \underline{s}$. Furthermore, *R* strictly decreases in *s* and converges to zero if *s* approaches infinity. Thus, if the combined expression in parentheses is positive for some *s'*, it is positive for all *s* > *s'*. Therefore, the variance achieves a unique minimum at some positive $\hat{s}$ and then increases for increasing *s* further. This completes the proof of part (i).

Part (ii): $j = r$. If $\sigma_\delta^2 \le \sigma_n^2\left(2 - S_n\right)$ then $T \le 0$, and due to $Z_r < 0$ and $R_r < 0$, the sign of

$$\frac{\partial Var\left(\tilde{m}_1\right)}{\partial r} = \underbrace{2Z_r T}_{>0} + \underbrace{2RR_r\sigma_g^2}_{<0}$$

depends on the magnitude of σ_g^2.The variance increases (decreases) in *r* if σ_g^2 is relatively small (large). Suppose that the derivative is positive at $r = 0$, then it remains positive for all $r > 0$. If $Var\left(\tilde{m}_1\right)$ has a stationary point, then it must be a minimum since the second derivative is positive:

$$\begin{aligned}
\frac{\partial^2 Var\left(\tilde{m}_1\right)}{\partial r^2} &= 2Z_{rr}T + 2Z_r T_r + 2R_r^2\sigma_g^2 + 2RR_{rr}\sigma_g^2 \\
&= -2R\left(2Z_r T + 2\sigma_g^2 RR_r\right) + 2Z_r T_r + 2R_r^2\sigma_g^2 \\
&= -2R\frac{\partial Var\left(\tilde{m}_1\right)}{\partial r} + 2Z_r T_r + 2R_r^2\sigma_g^2,
\end{aligned}$$

where

$$Z_{rr} = \frac{8s}{\left(8s + r\right)^3} = \frac{2}{\left(8s + r\right)}\frac{4s}{\left(8s + r\right)^2} = -2RZ_r \text{ and}$$

$$RR_{rr} = \frac{2}{\left(8s + r\right)^4} = \frac{2}{\left(8s + r\right)}\frac{1}{\left(8s + r\right)^3} = -2R^2 R_r.$$

At a stationary point, $\partial Var\left(\tilde{m}_1\right)/\partial r = 0$, and since $Z_r < 0$ and $T_r < 0$, the second derivative is positive. Therefore, any stationary point of $Var\left(\tilde{m}_1\right)$ with respect to *r* is a minimum. The threshold that σ_g^2 must not exceed to guarantee $\partial Var\left(\tilde{m}_1 \middle| r = 0\right)/\partial r > 0$ can be calculated by inserting $r = 0$ (and $Z = 1/2$) into the expression for $\partial Var\left(\tilde{m}_1\right)/\partial r$ and solving for σ_g^2 which eventually leads to

$$\sigma_g^2 < -32s^2 \underbrace{T(r=0)}_{<0} = -32s^2\left[\sigma_n^2\left(\frac{1}{2}S_n - 1\right) + \frac{1}{2}\sigma_\delta^2\right] = -16s^2\left[\sigma_\delta^2 - \sigma_n^2(2-S_n)\right].$$

Now assume that σ_g^2 exceeds this threshold but we still have $\sigma_\delta^2 \le \sigma_n^2(2-S_n)$. Then the variance strictly decreases for small *r*, attains a minimum at some positive value $\hat{r}$ and then increases in *r*. To see this observe that a positive sign of the derivative requires

$$\frac{\partial Var(\tilde{m}_1)}{\partial r} > 0 \Leftrightarrow Z_r T > -RR_r\sigma_g^2 \Leftrightarrow \frac{Z_r T}{-RR_r} > \sigma_g^2.$$

Using $Z_r = -ZR$, $R_r = -R^2$ and $R^2 = Z^2/(16s^2)$, the inequality becomes

$$-16s^2\left(\sigma_\delta^2 - \sigma_n^2\left(\frac{1}{Z} - S_n\right)\right) > \sigma_g^2.$$

The left hand side of this inequality strictly increases in *r* and approaches infinity as $r \to \infty$. Thus, for any σ_g^2, there exists a value $\bar{r}(\sigma_g^2)$ such that $Var(\tilde{m}_1)$ strictly increases for $r > \bar{r}(\sigma_g^2)$. The same argument holds for the case $\sigma_\delta^2 > \sigma_n^2(2-S_n)$, as the threshold $-16s^2\left[\sigma_\delta^2 - \sigma_n^2(2-S_n)\right] < 0$, thus, the derivative is negative for any $\sigma_g^2 > 0$ and small *r* and changes sign for large *r*. This completes the proof of part (ii). (q.e.d.)

Proof of Proposition 2

The partial derivative of *EQ* with respect to any parameter *j* in the proposition (*j* = *s*, *r*, *p*, $\bar{g}$, σ_n^2, σ_δ^2 and σ_g^2) can be written as follows:

$$\begin{aligned}\frac{\partial EQ}{\partial j} &= \frac{\partial \beta}{\partial j}Cov(\tilde{m}_1,\tilde{x}) + \beta\frac{\partial Cov(\tilde{m}_1,\tilde{x})}{\partial j} \\ &= \left(\frac{\frac{\partial Cov(\tilde{m}_1,\tilde{x})}{\partial j}Var(\tilde{m}_1) - Cov(\tilde{m}_1,\tilde{x})\frac{\partial Var(\tilde{m}_1)}{\partial j}}{Var(\tilde{m}_1)^2}\right)Cov(\tilde{m}_1,\tilde{x}) + \beta\frac{\partial Cov(\tilde{m}_1,\tilde{x})}{\partial j} \\ &= \frac{\beta}{Var(\tilde{m}_1)}\underbrace{\left(2\frac{\partial Cov(\tilde{m}_1,\tilde{x})}{\partial j}Var(\tilde{m}_1) - Cov(\tilde{m}_1,\tilde{x})\frac{\partial Var(\tilde{m}_1)}{\partial j}\right)}_{\equiv B}. \end{aligned} \quad \text{(A1)}$$

It is sufficient to evaluate the sign of the term in parentheses, *B*, to to sign the derivative of *EQ*.

Part (i) $j = s$: The two additive terms of B are

$$2\frac{\partial Cov(\tilde{m}_1,\tilde{x})}{\partial s}Var(\tilde{m}_1) = 2Z_s\left(\sigma_\delta^2 - \sigma_\varepsilon^2 S_n\right)\left(Q^2\left(\sigma_\varepsilon^2+\sigma_n^2\right)+Z^2\sigma_\delta^2+R^2\sigma_g^2\right)$$

$$Cov(\tilde{m}_1,\tilde{x})\frac{\partial Var(\tilde{m}_1)}{\partial s} = \left(Q\sigma_\varepsilon^2 + Z\sigma_\delta^2\right)\left(2Z_s\left(Z\sigma_\delta^2 - Q\sigma_n^2\right)+2RR_s\sigma_g^2\right).$$

Taking the difference and rearranging yields (B_1 and B_2 collects all terms without and with the incentive risk σ_g^2, respectively):

$$B = B_1 + B_2$$

$$\begin{aligned} B_1 &= 2Z_s\left(Q^2\sigma_\delta^2\sigma_\varepsilon^2 - QZ\sigma_\delta^2\sigma_\varepsilon^2 + Q^2\sigma_\delta^2\sigma_n^2 - Z^2\sigma_\delta^2\sigma_\varepsilon^2 S_n + ZQ\sigma_n^2\sigma_\delta^2\right) \\ B_2 &= 2Z_s\left(\sigma_\delta^2 - \sigma_\varepsilon^2 S_n\right)R^2\sigma_g^2 - \left(Q\sigma_\varepsilon^2 + Z\sigma_\delta^2\right)2RR_s\sigma_g^2 \\ &= 2R^2\sigma_g^2\frac{1}{s}\left[ZRr\left(\sigma_\delta^2 - \sigma_\varepsilon^2 S_n\right) + 2Z\left(Q\sigma_\varepsilon^2 + Z\sigma_\delta^2\right)\right] \\ &= 2R^2\sigma_g^2\frac{1}{s}\left[\sigma_\delta^2\left(ZRr + 2Z^2\right) + \sigma_\varepsilon^2 Z\left(2Q - S_n Rr\right)\right] \end{aligned} \tag{A2}$$

B_1 is positive since $Q > Z$ and $Z_s > 0$, and $\sigma_\varepsilon^2 S_n < \sigma_n^2\left(2 - S_n\right)$, which holds because $\sigma_\varepsilon^2 S_n = \frac{\sigma_\varepsilon^2\sigma_n^2}{\sigma_\varepsilon^2+\sigma_n^2} < \sigma_n^2 < \sigma_n^2\left(2-S_n\right)$ and $S_n < 1$.

B_2 ist positive because

$$2Q - S_n Rr = Q + Q - S_n Rr = Q + 1 - S_n\left(Z + Rr\right) = Q + 1 - \left(\frac{\sigma_n^2}{\sigma_\varepsilon^2+\sigma_n^2}\right)\left(\frac{4s+r}{8s+r}\right) > 0$$

which proves part (i).

Part (ii): $j = r$: Note that $Z_r = -Z_s\frac{s}{r} < 0$. The sign of B_1 is obtained by substituting this expression for Z_s in B_1 in (A2), which shows that $B_1 < 0$.

$$\begin{aligned} B_2 &= 2Z_r\left(\sigma_\delta^2 - \sigma_\varepsilon^2 S_n\right)R^2\sigma_g^2 - \left(Q\sigma_\varepsilon^2 + Z\sigma_\delta^2\right)2RR_r\sigma_g^2 \\ &= 2R^3\sigma_g^2\left[-Z\left(\sigma_\delta^2 - \sigma_\varepsilon^2 S_n\right) + \left(Q\sigma_\varepsilon^2 + Z\sigma_\delta^2\right)\right] \\ &= 2R^3\sigma_g^2\sigma_\varepsilon^2\left(ZS_n + Q\right) = 2R^3\sigma_g^2\sigma_\varepsilon^2\left(ZS_n + 1 - ZS_n\right) \\ &= 2R^3\sigma_g^2\sigma_\varepsilon^2 > 0. \end{aligned}$$

The sign of the effect of an increase in r on EQ depends on the specific parameters of the problem. Notice that the incentive risk σ_g^2 only appears in B_2. Thus, if σ_g^2 is relatively small (large), the sign of $\partial EQ/\partial r$ will be negative (positive). In particular, we have

$$\frac{\partial EQ}{\partial r}\begin{cases}\leq 0 \text{ if } \sigma_g^2 \leq \dfrac{-B_1}{2R^3\sigma_\varepsilon^2} \\ > 0 \text{ if } \sigma_g^2 > \dfrac{-B_1}{2R^3\sigma_\varepsilon^2}\end{cases}.$$

Inserting B_1 from (A2) (with Z_r instead of Z_s) , using $Z_r = -ZR$ and $Z = 4sR$ and rearranging yields

$$\frac{\partial EQ}{\partial r}\begin{cases}\leq 0 \text{ if } \dfrac{\sigma_g^2}{\sigma_\delta^2} \leq \Gamma \\ > 0 \text{ if } \dfrac{\sigma_g^2}{\sigma_\delta^2} > \Gamma\end{cases}$$

where

$$\Gamma \equiv \frac{Z}{\sigma_\varepsilon^2}\left(Q^2\sigma_\varepsilon^2 - QZ\sigma_\varepsilon^2 + Q^2\sigma_n^2 - Z^2\sigma_\varepsilon^2 S_n + ZQ\sigma_n^2\right) > 0. \qquad \text{(A3)}$$

This proves part (ii).

Part (iii): $j = p$ *or* $\bar{g}$. Neither $Cov(\tilde{x}, \tilde{m}_1)$ nor $Var(\tilde{x})$ depend on p and $\bar{g}$, hence EQ is unaffected by a change in p and $\bar{g}$.

Part (iv): $j = \sigma_\delta^2$. The relevant terms are

$$2\frac{\partial Cov\left(\tilde{m}_1, \tilde{x}\right)}{\partial \sigma_\delta^2} Var\left(\tilde{m}_1\right) = 2Z\left(Q^2\left(\sigma_\varepsilon^2 + \sigma_n^2\right) + Z^2\sigma_\delta^2 + R^2\sigma_g^2\right),$$

$$Cov\left(\tilde{m}_1, \tilde{x}\right)\frac{\partial Var\left(\tilde{m}_1\right)}{\partial \sigma_\delta^2} = \left(Q\sigma_\varepsilon^2 + Z\sigma_\delta^2\right)Z^2.$$

Taking the difference yields

$$2ZQ^2\sigma_\varepsilon^2 - Z^2Q\sigma_\varepsilon^2 + 2ZQ^2\sigma_n^2 + 2Z^3\sigma_\delta^2 - Z^3\sigma_\delta^2 + 2ZR^2\sigma_g^2 > 0$$

since $Q > Z$. This proves part (iv).

Part (v): $j = \sigma_n^2$. Since $Cov(\tilde{m}_1, \tilde{x})$ strictly decreases in σ_n^2 (see Lemma 1 (iv)) and $Var(\tilde{m}_1)$ strictly increases in σ_n^2 (see Lemma 2 (iv)), the sign of the difference is always negative.

Part (vi): $j = \sigma_g^2$. This result is immediate as the incentive risk has no impact on $Cov(\tilde{m}_1, \tilde{x})$ but increases the variance $Var(\tilde{m}_1)$.

(q.e.d.)

Proof of Proposition 3

The partial derivative of β with respect to any factor j can be written as:

$$\begin{aligned}\frac{\partial \beta}{\partial j} &= \left(\frac{\frac{\partial Cov(\tilde{m}_1, \tilde{x})}{\partial j} Var(\tilde{m}_1) - Cov(\tilde{m}_1, \tilde{x}) \frac{\partial Var(\tilde{m}_1)}{\partial j}}{Var(\tilde{m}_1)^2} \right) \\ &= \frac{1}{Var(\tilde{m}_1)^2} \underbrace{\left(\frac{\partial Cov(\tilde{m}_1, \tilde{x})}{\partial j} Var(\tilde{m}_1) - Cov(\tilde{m}_1, \tilde{x}) \frac{\partial Var(\tilde{m}_1)}{\partial j} \right)}_{\equiv C}\end{aligned} \quad \text{(A4)}$$

We first show that the sign of the term C is positive for $j = s$ if $\sigma_\delta^2 \le \sigma_n^2 (2 - S_n)$. Next, we show that for $\sigma_\delta^2 >> \sigma_n^2 (2 - S_n)$, β achieves a maximum at some positive value for $j = s$, r respectively.

Case 1: $\sigma_\delta^2 \le \sigma_n^2 (2 - S_n)$. Let $j = s$. Then

$$C = B - \frac{\partial Cov(\tilde{m}_1, \tilde{x})}{\partial s} Var(\tilde{m}_1) \quad \text{(A5)}$$

where B is defined in (A1). Recall that according to (A2) $B > 0$, if $j = s$. From Lemma 1,

$$\frac{\partial Cov(\tilde{m}_1, \tilde{x})}{\partial s} \le 0$$

for $\sigma_\delta^2 \le \sigma_\varepsilon^2 S_n < \sigma_n^2 (2 - S_n)$, which implies C is positive. For $\sigma_\varepsilon^2 S_n < \sigma_\delta^2 \le \sigma_n^2 (2 - S_n)$, $Cov(\tilde{m}_1, \tilde{x})$ strictly increases and $Var(\tilde{m}_1)$ strictly decreases in s, hence $\beta = Cov(\tilde{m}_1, \tilde{x}) / Var(\tilde{m}_1)$ strictly increases in s. Therefore, β strictly increases in s for $\sigma_\delta^2 \le \sigma_n^2 (2 - S_n)$.

Case 2: $\sigma_\delta^2 > \sigma_n^2 (2 - S_n)$.

First we show that for each combination $s > 0$ and $r > 0$, there exists a value of σ_δ^2 such that $\beta\left(Z\left(s,r\right),\sigma_\delta^2\right) > \beta\left(Z = 1/2,\sigma_\delta^2\right)$, i.e., for that σ_δ^2, β under $Z(s, r)$ exceeds its corresponding value if Z is set to its maximum of 1/2. As $Z \to 1/2$ for $s \to \infty$ and/or $r \to 0$, β must have at least one stationary point with respect to s and r, respectively. We then show that if β has a stationary point, then it is a maximum.

β is defined as

$$\beta = \frac{Q\sigma_\varepsilon^2 + Z\sigma_\delta^2}{Q^2\left(\sigma_\varepsilon^2 + \sigma_n^2\right) + Z^2\sigma_\delta^2 + R^2\sigma_g^2} .$$

β strictly increases in σ_δ^2, and $\lim_{\sigma_\delta^2 \to \infty} \beta = \frac{Z}{Z^2} = \frac{1}{Z} > 2$ for $Z < 1/2$. Thus, for each combination ($s > 0$, $r > 0$), σ_δ^2 can be chosen large enough such that $\beta\left(Z\left(s,r\right),\sigma_\delta^2\right) > \beta\left(Z = 1/2,\sigma_\delta^2\right)$. To see that a stationary point of β is a maximum, consider the second partial derivatives of β:

$$\frac{\partial^2 \beta}{\partial j^2} = \frac{1}{Var\left(\tilde{m}_1\right)^2} \underbrace{\left(\frac{\partial^2 Cov\left(\tilde{m}_1, \tilde{x}\right)}{\partial j^2} Var\left(\tilde{m}_1\right) - Cov\left(\tilde{m}_1, \tilde{x}\right) \frac{\partial^2 Var\left(\tilde{m}_1\right)}{\partial j^2} \right)}_{\equiv D} - \frac{2C}{Var\left(\tilde{m}_1\right)^3} \frac{\partial Var\left(\tilde{m}_1\right)}{\partial j} .$$

At a stationary point, $C = 0$, and it suffices to evaluate the sign of D. First let $j = s$ and collect the relevant terms as follows:

$$\frac{\partial^2 Cov\left(\tilde{m}_1, \tilde{x}\right)}{\partial s^2} = Z_{ss}\left(\sigma_\delta^2 - \sigma_\varepsilon^2 S_n\right) = -16RZ_s\left(\sigma_\delta^2 - \sigma_\varepsilon^2 S_n\right) = -16R\frac{\partial Cov\left(\tilde{m}_1, \tilde{x}\right)}{\partial s}$$

$$\begin{aligned}
\frac{\partial^2 Var\left(\tilde{m}_1\right)}{\partial s^2} &= 2Z_{ss}\left(Z\sigma_\delta^2 - Q\sigma_n^2\right) + 2Z_s\left(Z_s\sigma_\delta^2 - Q_s\sigma_n^2\right) + 2\sigma_g^2\left(R_s^2 + RR_{ss}\right) \\
&= -16R\left[2Z_s\left(Z\sigma_\delta^2 - Q\sigma_n^2\right) + 2\sigma_g^2 RR_s\right] + 2Z_s\left(Z_s\sigma_\delta^2 - Q_s\sigma_n^2\right) + 2\sigma_g^2 R_s^2 \\
&= -16R\frac{\partial Var\left(\tilde{m}_1\right)}{\partial s} + \underbrace{2Z_s\left(Z_s\sigma_\delta^2 - Q_s\sigma_n^2\right)}_{>0} + 2\sigma_g^2 R_s^2
\end{aligned}$$

where $RR_{ss} = \frac{1}{\left(8s+r\right)} \frac{16 \cdot 8}{\left(8s+r\right)^3} = \frac{16}{\left(8s+r\right)} \frac{1}{\left(8s+r\right)} \frac{8}{\left(8s+r\right)^2} = -16R^2 R_s$.

The sign of $Z_s\left(Z_s\sigma_\delta^2 - Q_s\sigma_n^2\right)$ is positive due to $Z_s > 0$ and $Q_s = -Z_s S_n$. Rewrite D as follows:

$$D = -16R\left(\frac{\partial Cov(\tilde{m}_1,\tilde{x})}{\partial s}Var(\tilde{m}_1) - Cov(\tilde{m}_1,\tilde{x})\frac{\partial Var(\tilde{m}_1)}{\partial s}\right)$$
$$-Cov(\tilde{m}_1,\tilde{x})\left(2Z_s\left(Z_s\sigma_\delta^2 - Q_s\sigma_n^2\right) + 2\sigma_g^2R_s^2\right)$$
$$= -16RC - Cov(\tilde{m}_1,\tilde{x})\left(2Z_s\left(Z_s\sigma_\delta^2 - Q_s\sigma_n^2\right) + 2\sigma_g^2R_s^2\right)$$
$$= -Cov(\tilde{m}_1,\tilde{x})\left(2Z_s\left(Z_s\sigma_\delta^2 - Q_s\sigma_n^2\right) + 2\sigma_g^2R_s^2\right) < 0.$$

Thus, if β has a stationary point in *s*, it is a maximum. It follows that if σ_δ^2 is large enough, then β first strictly increases in *s*, reaches a maximum at some value $\hat{s}$ and then decreases in *s*. This completes the proof of part (i).

Part (ii): $j = r$. If $\frac{\sigma_g^2}{\sigma_\delta^2} \le \Gamma$, *EQ* decreases according to Proposition 2 (ii) and we have $B \le 0$. If $\sigma_\delta^2 \le S_n\sigma_\varepsilon^2$ also holds, then $\partial Cov(\tilde{m}_1,\tilde{x})/\partial r \ge 0$ (see Lemma 1 (ii)) implying that $C < 0$ if at least one of the above inequalities is strict. If $S_n\sigma_\varepsilon^2 < \sigma_\delta^2 \le \sigma_n^2(2 - S_n)$ and $\sigma_g^2 \le \max\left\{0; -16s^2\left(\sigma_\delta^2 - \sigma_n^2(2 - S_n)\right)\right\} = -16s^2\left(\sigma_\delta^2 - \sigma_n^2(2 - S_n)\right)$ are satisfied, then $\partial Cov(\tilde{m}_1,\tilde{x})/\partial r < 0$ and $\partial Var(\tilde{m}_1,\tilde{x})/\partial r > 0$ (Lemma 2-(ii)), which implies $C < 0$. In this case, *EQ* strictly decreases in *r*, hence $\frac{\sigma_g^2}{\sigma_\delta^2} \le \Gamma$ must additionally be satisfied. If $\frac{\sigma_g^2}{\sigma_\delta^2} > \Gamma$ and $S_n\sigma_\varepsilon^2 < \sigma_\delta^2$, then $B > 0$ (Proposition 2 (ii)) and $\partial Cov(\tilde{m}_1,\tilde{x})/\partial r < 0$ (Lemma 1 (ii)), thus, $C > 0$ and β strictly increases in *r*.

To prove the last statement of part (ii), note that

$$\frac{\partial^2 Cov(\tilde{m}_1,\tilde{x})}{\partial r^2} = Z_{rr}\left(\sigma_\delta^2 - \sigma_\varepsilon^2 S_n\right) = -2R\frac{\partial Cov(\tilde{m}_1,\tilde{x})}{\partial r},$$

$$\frac{\partial^2 Var(\tilde{m}_1)}{\partial r^2} = 2Z_{rr}\left(Z\sigma_\delta^2 - Q\sigma_n^2\right) + 2Z_r\left(Z_r\sigma_\delta^2 - Q_r\sigma_n^2\right) + 2\sigma_g^2\left(R_r^2 + RR_{rr}\right)$$
$$= -2R\frac{\partial Var(\tilde{m}_1)}{\partial r} + \underbrace{2Z_r\left(Z_r\sigma_\delta^2 - Q_r\sigma_n^2\right)}_{>0} + 2\sigma_g^2R_r^2.$$

Hence,

$$D = -2R\left(\frac{\partial Cov(\tilde{m}_1,\tilde{x})}{\partial r}Var(\tilde{m}_1) - Cov(\tilde{m}_1,\tilde{x})\frac{\partial Var(\tilde{m}_1)}{\partial r}\right)$$
$$-Cov(\tilde{m}_1,\tilde{x})\left(2Z_r\left(Z_r\sigma_\delta^2 - Q_r\sigma_n^2\right) - 2\sigma_g^2 R_r^2\right)$$
$$= -2RC - Cov(\tilde{m}_1,\tilde{x})\left(2Z_r\left(Z_r\sigma_\delta^2 - Q_r\sigma_n^2\right) - 2\sigma_g^2 R_r^2\right)$$
$$= -Cov(\tilde{m}_1,\tilde{x})\left(2Z_r\left(Z_r\sigma_\delta^2 - Q_r\sigma_n^2\right) - 2\sigma_g^2 R_r^2\right) < 0.$$

Thus, if β has a stationary point in *r*, then it is a maximum. It follows that if σ_δ^2 is large enough, then β first strictly increases in *r*, reaches a maximum at some value $\hat{r}$ and then decreases in *r*.

Part (iii) follows from the fact that all terms in β are independent from *p* and $\overline{g}$. Part (iv) obtains from evaluating the sign of the partial derivative of β with respect to σ_δ^2:

$$\frac{\partial\beta}{\partial\sigma_\delta^2} = \frac{Z\left(Q^2\left(\sigma_\varepsilon^2+\sigma_n^2\right)+Z^2\sigma_\delta^2+R^2\sigma_g^2\right)-\left(Q\sigma_\varepsilon^2+Z\sigma_\delta^2\right)Z^2}{Var(\tilde{m}_1)^2}$$
$$= \frac{ZQ^2\left(\sigma_\varepsilon^2+\sigma_n^2\right)+R^2\sigma_g^2-Z^2Q\sigma_\varepsilon^2}{Var(\tilde{m}_1)^2} > 0.$$

where the inequality follows due to *Q* > *Z*. Finally, parts (v) and (vi) follow from Lemmas 1 and 2.

(q.e.d.)

Proof of Proposition 4

First, we show that $PS = \beta + \frac{Q\left(\sigma_\varepsilon^2+\sigma_n^2\right)}{Var(\tilde{m}_1)} - 1.$

$$PS = \frac{Cov\left(\left(Q\tilde{y}_1+Z\tilde{\delta}+R\tilde{g}\right),\left(\tilde{y}_2-(Q-1)\tilde{y}_1-Z\tilde{\delta}-R\tilde{g}\right)\right)}{Q^2\left(\sigma_\varepsilon^2+\sigma_n^2\right)+Z^2\sigma_\delta^2+R^2\sigma_g^2}$$
$$= \frac{Cov\left(\left(Q\tilde{y}_1+Z\tilde{\delta}+R\tilde{g}\right),\left(\tilde{\varepsilon}+\tilde{u}+(1-Q)\tilde{y}_1+(1-Z)\tilde{\delta}-R\tilde{g}\right)\right)}{Q^2\left(\sigma_\varepsilon^2+\sigma_n^2\right)+Z^2\sigma_\delta^2+R^2\sigma_g^2}$$
$$= \frac{Q\sigma_\varepsilon^2+Q(1-Q)\left(\sigma_\varepsilon^2+\sigma_n^2\right)+Z(1-Z)\sigma_\delta^2-R^2\sigma_g^2}{Q^2\left(\sigma_\varepsilon^2+\sigma_n^2\right)+Z^2\sigma_\delta^2+R^2\sigma_g^2}$$
$$= \beta + \frac{Q\left(\sigma_\varepsilon^2+\sigma_n^2\right)}{Var(\tilde{m}_1)} - 1.$$

To prove parts (i) and (ii), consider the first derivative of the last representation of *PS* with respect to variable $j = s, r$.

$$PS = \frac{Cov(\tilde{m}_1, \tilde{m}_2)}{Var(\tilde{m}_1)} = \frac{\left(\sigma_\varepsilon^2 + Z\sigma_n^2\right)Q + Z\left(1-Z\right)\sigma_\delta^2 - R^2\sigma_g^2}{Q^2\left(\sigma_\varepsilon^2 + \sigma_n^2\right) + Z^2\sigma_\delta^2 + R^2\sigma_g^2}$$

Differentiating the numerator yields:

$$\begin{aligned}
\frac{\partial Cov\left(\tilde{m}_1, \tilde{m}_2\right)}{\partial j} &= Z_j\left(-\frac{\left(\sigma_\varepsilon^2 + Z\sigma_n^2\right)}{\left(\sigma_\varepsilon^2 + \sigma_n^2\right)}\sigma_n^2 + \sigma_n^2\left(1 - Z\frac{\sigma_n^2}{\sigma_\varepsilon^2 + \sigma_n^2}\right) + \left(1-2Z\right)\sigma_\delta^2\right) - 2RR_j\sigma_g^2 \\
&= Z_j\left(\frac{\sigma_n^2\left(\sigma_\varepsilon^2 + \sigma_n^2\left(1-Z\right)\right) - \sigma_n^2\left(\sigma_\varepsilon^2 + Z\sigma_n^2\right)}{\left(\sigma_\varepsilon^2 + \sigma_n^2\right)} + \left(1-2Z\right)\sigma_\delta^2\right) - 2RR_j\sigma_g^2 \\
&= Z_j\left(\frac{\sigma_n^4\left(1-2Z\right)}{\left(\sigma_\varepsilon^2 + \sigma_n^2\right)} + \left(1-2Z\right)\sigma_\delta^2\right) - 2RR_j\sigma_g^2 \\
&= Z_j\left(1-2Z\right)\left(S_n\sigma_n^2 + \sigma_\delta^2\right) - 2RR_j\sigma_g^2.
\end{aligned}$$

Let $j = s$. Since $Z_s > 0$, $R_s < 0$, and $Z \in [0, 1/2]$, the covariance always increases in *s*. Case 1: Assume $Cov\left(\tilde{m}_1, \tilde{m}_2\right) \geq 0$. Together with Lemma 2 this proves that *PS* is strictly increasing in *s* if $\sigma_\delta^2 \leq \sigma_n^2\left(2 - S_n\right)$.

If $\sigma_\delta^2 > \sigma_n^2\left(2 - S_n\right)$, for each combination of $s > 0$ and $r > 0$, $PS(Z(s, r)) > PS(Z = 1/2)$ if σ_δ^2 is sufficiently large. The proof is similar to that in Proposition 3: Take the limit of *PS* as $\sigma_\delta^2 \to \infty$:

$$\lim_{\sigma_\delta^2 \to \infty} PS\left(s, r\right) = \frac{Z\left(1-Z\right)}{Z^2} = \frac{1-Z}{Z} > \frac{1-0.5}{0.5} = 1 = \lim_{\sigma_\delta^2 \to \infty} PS\left(Z = 1/2\right)$$

since $Z \leq 1/2$. Uniqueness of the stationary point follows from the sign of the second derivative of *PS* with respect to *s* at the stationary point where

$$\frac{\partial PS}{\partial s} = \frac{\dfrac{\partial Cov\left(\tilde{m}_1, \tilde{m}_2\right)}{\partial s}Var\left(\tilde{m}_1\right) - Cov\left(\tilde{m}_1, \tilde{m}_2\right)\dfrac{\partial Var\left(\tilde{m}_1\right)}{\partial s}}{Var\left(\tilde{m}_1\right)^2} = 0.$$

Evaluated at the stationary point, the second derivative is

$$\frac{\partial^2 PS}{\partial s^2} = \frac{\frac{\partial^2 Cov(\tilde{m}_1,\tilde{m}_2)}{\partial s^2} Var(\tilde{m}_1) - Cov(\tilde{m}_1,\tilde{m}_2)\frac{\partial^2 Var(\tilde{m}_1)}{\partial s^2}}{Var(\tilde{m}_1)^2}.$$

$$\begin{aligned}
\frac{\partial^2 Cov(\tilde{m}_1,\tilde{m}_2)}{\partial s^2} &= Z_{ss}\left((1-2Z)\left(S_n\sigma_n^2+\sigma_\delta^2\right)\right) - 2Z_s^2\left(S_n\sigma_n^2+\sigma_\delta^2\right) - 2R_s^2\sigma_g^2 - 2RR_{ss}\sigma_g^2 \\
&= -16RZ_s\left((1-2Z)\left(S_n\sigma_n^2+\sigma_\delta^2\right) - 2RR_s\sigma_g^2\right) - 2Z_s^2\left(S_n\sigma_n^2+\sigma_\delta^2\right) - 2R_s^2\sigma_g^2 \\
&= -16R\frac{\partial Cov(\tilde{m}_1,\tilde{m}_2)}{\partial s} - 2Z_s^2\left(S_n\sigma_n^2+\sigma_\delta^2\right) - 2R_s^2\sigma_g^2
\end{aligned}$$

$$\frac{\partial^2 Var(\tilde{m}_1)}{\partial s^2} = -16R\frac{\partial Var(\tilde{m}_1)}{\partial s} + \underbrace{2Z_s\left(Z_s\sigma_\delta^2 - Q_s\sigma_n^2\right)}_{>0} + 2\sigma_g^2R_s^2.$$

Taking together, the sign of the numerator of $\frac{\partial^2 PS}{\partial s^2}$ evaluated at a stationary point is

$$\begin{aligned}
&\frac{\partial^2 Cov(\tilde{m}_1,\tilde{m}_2)}{\partial s^2} Var(\tilde{m}_1) - Cov(\tilde{m}_1,\tilde{m}_2)\frac{\partial^2 Var(\tilde{m}_1)}{\partial s^2} \\
&= -16R\left(\frac{\partial Cov(\tilde{m}_1,\tilde{m}_2)}{\partial s} Var(\tilde{m}_1)\right) - Var(\tilde{m}_1)\left(2Z_s^2\left(S_n\sigma_n^2+\sigma_\delta^2\right) + 2R_s^2\sigma_g^2\right) \\
&\quad - \left(-16R\frac{\partial Var(\tilde{m}_1)}{\partial s} Cov(\tilde{m}_1,\tilde{m}_2) + Cov(\tilde{m}_1,\tilde{m}_2)\left(\underbrace{2Z_s\left(Z_s\sigma_\delta^2 - Q_s\sigma_n^2\right)}_{>0} + 2\sigma_g^2R_s^2\right)\right) \\
&= -16R\underbrace{\left(\frac{\partial Cov(\tilde{m}_1,\tilde{m}_2)}{\partial s} Var(\tilde{m}_1) - Cov(\tilde{m}_1,\tilde{m}_2)\frac{\partial Var(\tilde{m}_1)}{\partial s}\right)}_{=0\,(\text{stationary point})} \\
&\quad - Var(\tilde{m}_1)\left(2Z_s^2\left(S_n\sigma_n^2+\sigma_\delta^2\right) + 2R_s^2\sigma_g^2\right) - Cov(\tilde{m}_1,\tilde{m}_2)\left(\underbrace{2Z_s\left(Z_s\sigma_\delta^2 - Q_s\sigma_n^2\right)}_{>0} + 2\sigma_g^2R_s^2\right) \\
&= -\left(Var(\tilde{m}_1) + Cov(\tilde{m}_1,\tilde{m}_2)\right)\left(2Z_s^2\left(S_n\sigma_n^2+\sigma_\delta^2\right) + 2R_s^2\sigma_g^2\right) \\
&= -\left(Q\left(2\sigma_\varepsilon^2+\sigma_n^2\right) + Z\sigma_\delta^2\right)\left(2Z_s^2\left(S_n\sigma_n^2+\sigma_\delta^2\right) + 2R_s^2\sigma_g^2\right) < 0.
\end{aligned}$$

Thus, *PS* achieves a unique maximum in *s* if $\sigma_\delta^2 >> \sigma_n^2\left(2-S_n\right)$. Notice that the proof of this result is independent of the sign of $Cov(\tilde{m}_1,\tilde{m}_2)$.

Case 2: $Cov(\tilde{m}_1,\tilde{m}_2) < 0$ and $\sigma_\delta^2 \le \sigma_n^2\left(2-S_n\right)$. $Var(\tilde{m}_1)$ strictly decreases in *s* (Lemma 2 (i)). $Cov(\tilde{m}_1,\tilde{m}_2)$ strictly increases in *s* and approaches a positive value if $s \to \infty$, thus $Cov(\tilde{m}_1,\tilde{m}_2) < 0$ can only hold for a lower interval of *s*. Suppose that $\frac{\partial PS}{\partial s} < 0$ for small *s*, then the numerator of $\frac{\partial PS}{\partial s}$ changes sign at some level *s'* > 0 and then remains positive. This implies that *PS* would achieve a minimum, contradicting the above result that a stationary point of *PS* is a maximum. Hence, we must have $\frac{\partial PS}{\partial s} > 0$ for small *s*, and since $\frac{\partial PS}{\partial s} > 0$ holds for large *s* (due to $Cov(\tilde{m}_1, \tilde{m}_2) > 0$), $\frac{\partial PS}{\partial s} > 0$ for $\sigma_\delta^2 \le \sigma_n^2(2 - S_n)$ and all $s \ge 0$ even if we have $Cov(\tilde{m}_1, \tilde{m}_2) < 0$ for small *s*. If $\sigma_\delta^2 > \sigma_n^2(2 - S_n)$ and $Cov(\tilde{m}_1, \tilde{m}_2) < 0$, a similar argument implies that *PS* achieves a unique maximum in s if $\sigma_\delta^2 >> \sigma_n^2(2 - S_n)$. This completes the proof of part (i).

Part (ii): *j* = *r*. The partial derivative of the covariance with respect to *r* is

$$\frac{\partial Cov(\tilde{m}_1, \tilde{m}_2)}{\partial r} = Z_r\left((1-2Z)\left(S_n\sigma_n^2 + \sigma_\delta^2\right)\right) - 2RR_r\sigma_g^2.$$

Using $Z_r = -ZR$ and $R_r = -R^2$ gives

$$\frac{\partial Cov(\tilde{m}_1, \tilde{m}_2)}{\partial r} = -ZR\left((1-2Z)\left(S_n\sigma_n^2 + \sigma_\delta^2\right)\right) + 2R^3\sigma_g^2.$$

Substituting $Z = 4sR$ and $1 - 2Z = rR$ yields

$$\frac{\partial Cov(\tilde{m}_1, \tilde{m}_2)}{\partial r} = 2R^3\left(\sigma_g^2 - 2sr\left(S_n\sigma_n^2 + \sigma_\delta^2\right)\right).$$

Thus,

$$\frac{\partial Cov(\tilde{m}_1, \tilde{m}_2)}{\partial r}\begin{cases} > 0 \text{ if } \sigma_g^2 > 2sr\left(S_n\sigma_n^2 + \sigma_\delta^2\right) \\ \le 0 \text{ if } \sigma_g^2 \le 2sr\left(S_n\sigma_n^2 + \sigma_\delta^2\right) \end{cases}.$$

Hence, for any positive value of σ_g^2, increasing *r* from *r* = 0, $Cov(\tilde{m}_1, \tilde{m}_2)$ first strictly increases in *r*, reaches a unique maximum and then strictly decreases.

Now assume $Cov(\tilde{m}_1, \tilde{m}_2) \ge 0$ and $\sigma_\delta^2 > \sigma_n^2(2 - S_n)$. By Lemma 2 (ii) $Var(\tilde{m}_1)$ first strictly decreases in *r*, reaches a unique minimum and then strictly increases. Hence, for all $\sigma_g^2 > 0$, *PS* first strictly increases in *r* and decreases for large *r*. This implies that there is a maximum, which can be shown to be the unique stationary point by a procedure similar to the one used above for *s*. If $\sigma_\delta^2 \le \sigma_n^2(2 - S_n)$, the sign of the derivative of *PS* depends on the level of the incentive risk σ_g^2. If $\sigma_g^2 > -16s^2\left(\sigma_\delta^2 - \sigma_n^2(2 - S_n)\right)$, then the same result obtains since the behavior of $Var(\tilde{m}_1)$ is qualitatively similar according to Lemma 2 (ii). If $\sigma_g^2 \le -16s^2\left(\sigma_\delta^2 - \sigma_n^2\left(2-S_n\right)\right)$, then the variance always strictly increases in *r*. Here, the sign of the change in *PS* depends on the specific parameters. If it turns out that *PS* first increases in *r*, the behavior is similar to the two cases above. If *PS* decreases in *r* for *r* = 0, it always decreases in *r*. Otherwise *PS* would have to reach a minimum at some positive $\hat{r}$, contradicting the fact that any stationary point of *PS* is a maximum.

Now consider the combination $Cov(\tilde{m}_1,\tilde{m}_2)<0$ and $\sigma_\delta^2 > \sigma_n^2\left(2-S_n\right)$. Similar to part (i), the sign of $\frac{\partial PS}{\partial r}$ seems ambiguous for small *r*, and $Cov(\tilde{m}_1,\tilde{m}_2)<0$ can only hold for a lower interval of *r* as $Cov(\tilde{m}_1,\tilde{m}_2)\to\sigma_\varepsilon^2$ if $r\to\infty$ and $Cov(\tilde{m}_1,\tilde{m}_2)$ achieves a unique maximum for some $\hat{r}<\infty$. It follows that at the point r' for which $Cov(\tilde{m}_1,\tilde{m}_2)=0$ holds, we must have $r'<\hat{r}$ and $Cov(\tilde{m}_1,\tilde{m}_2)$ still strictly increases in *r*. Hence, we have

$$\frac{\partial PS(r')}{\partial r} = \frac{\frac{\partial Cov(\tilde{m}_1,\tilde{m}_2 | r')}{\partial r} Var(\tilde{m}_1)}{Var(\tilde{m}_1)^2} > 0 .$$

Thus, it cannot be the case that $\frac{\partial PS}{\partial r}<0$ for some $r<r'$ as this would imply that *PS* achieves a minimum in the interval $(0;r')$ contradicting the result that a stationary point of *PS* is a maximum. Therefore, *PS* strictly increases in *r* for small *r* and then decreases.

Finally, consider $Cov(\tilde{m}_1,\tilde{m}_2)<0$ and $\sigma_\delta^2 \le \sigma_n^2\left(2-S_n\right)$. If $\sigma_g^2 > -16s^2\left(\sigma_\delta^2 - \sigma_n^2\left(2-S_n\right)\right)$, *PS* first strictly increases and then decreases in *r*. If $\sigma_g^2 \le -16s^2\left(\sigma_\delta^2 - \sigma_n^2\left(2-S_n\right)\right)$, then $Var(\tilde{m}_1)$ always strictly increases in *r*. If $Cov(\tilde{m}_1,\tilde{m}_2)<0$ then

$$\frac{\partial PS}{\partial r} = \frac{\frac{\partial Cov(\tilde{m}_1,\tilde{m}_2)}{\partial r} Var(\tilde{m}_1) - Cov(\tilde{m}_1,\tilde{m}_2)\frac{\partial Var(\tilde{m}_1)}{\partial r}}{Var(\tilde{m}_1)^2} > 0 \;.$$

for $0 \le r \le r'$ and σ_g^2 sufficiently small.

We conclude that for $Cov(\tilde{m}_1,\tilde{m}_2)<0$, *PS* always first strictly increases and then decreases in *r*, completing the proof of part (ii).

Part (iii) follows because all terms appearing in *PS* are independent from *p* and $\bar{g}$.

Part (iv): The derivative of *PS* with respect to σ_δ^2 is

$$\frac{\partial PS}{\partial \sigma_\delta^2} = \frac{Z(1-Z)\left(Q^2\left(\sigma_\varepsilon^2+\sigma_n^2\right)+Z^2\sigma_\delta^2+R^2\sigma_g^2\right)-\left(\left(\sigma_\varepsilon^2+Z\sigma_n^2\right)Q+Z(1-Z)\sigma_\delta^2-R^2\sigma_g^2\right)Z^2}{Var\left(\tilde{m}_1\right)^2}$$

$$= \frac{Z(1-Z)Q^2\left(\sigma_\varepsilon^2+\sigma_n^2\right)-\left(\sigma_\varepsilon^2+Z\sigma_n^2\right)QZ^2+ZR^2\sigma_g^2}{Var\left(\tilde{m}_1\right)^2}.$$

Expanding the individual terms results in:

$$Z(1-Z)Q^2\left(\sigma_\varepsilon^2+\sigma_n^2\right) = Z(1-Z)\left(1-Z\frac{\sigma_n^2}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)}\right)^2\left(\sigma_\varepsilon^2+\sigma_n^2\right)$$

$$= Z(1-Z)\left(\sigma_\varepsilon^2+\sigma_n^2(1-2Z)+Z^2\frac{\sigma_n^4}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)}\right)$$

$$= Z(1-Z)\sigma_\varepsilon^2+Z(1-Z)\sigma_n^2(1-2Z)+\frac{\sigma_n^4}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)}\left(Z^3-Z^4\right).$$

$$\left(\sigma_\varepsilon^2+Z\sigma_n^2\right)QZ^2 = \left(\sigma_\varepsilon^2+Z\sigma_n^2\right)\left(1-Z\frac{\sigma_n^2}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)}\right)Z^2$$

$$= Z^2\sigma_\varepsilon^2+Z^3\sigma_n^2-Z^3\sigma_n^2\frac{\left(\sigma_\varepsilon^2+Z\sigma_n^2\right)}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)}$$

$$= Z^2\sigma_\varepsilon^2+\frac{Z^3\sigma_n^2}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)}\left(\sigma_\varepsilon^2+\sigma_n^2-\sigma_\varepsilon^2-Z\sigma_n^2\right)$$

$$= Z^2\sigma_\varepsilon^2+\frac{\sigma_n^4}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)}\left(Z^3-Z^4\right).$$

Taking the difference yields:

$$Z(1-Z)Q^2\left(\sigma_\varepsilon^2+\sigma_n^2\right)-\left(\sigma_\varepsilon^2+Z\sigma_n^2\right)QZ^2 = (1-2Z)\left(Z\sigma_\varepsilon^2+Z(1-Z)\sigma_n^2\right)\geq 0,$$

where the equality holds for $s = 0$ (then $Z = 0$) and/or $r = 0$ (then $Z = 1/2$). Hence, since $ZR^2\sigma_g^2 > 0$ for $s > 0$ and $r > 0$,

$$\frac{\partial PS}{\partial \sigma_\delta^2} > 0.$$

Part (v): Write *PS* as

$$PS = \beta + \frac{Q\left(\sigma_\varepsilon^2 + \sigma_n^2\right)}{Var\left(\tilde{m}_1\right)} - 1 = \frac{Q\left(2\sigma_\varepsilon^2 + \sigma_n^2\right) + Z\sigma_\delta^2}{Q^2\left(\sigma_\varepsilon^2 + \sigma_n^2\right) + Z^2\sigma_\delta^2 + R^2\sigma_g^2} - 1.$$

The partial derivative of *PS* with respect to σ_n^2 is

$$\frac{\partial PS}{\partial \sigma_n^2} = \frac{\left(Q'\left(2\sigma_\varepsilon^2 + \sigma_n^2\right) + Q\right)Var\left(\tilde{m}_1\right) - \left(Q\left(2\sigma_\varepsilon^2 + \sigma_n^2\right) + Z\sigma_\delta^2\right)\left(2QQ'\left(\sigma_\varepsilon^2 + \sigma_n^2\right) + Q^2\right)}{Var\left(\tilde{m}_1\right)^2}$$

where $Q' \equiv \partial Q / \partial \sigma_n^2$. This expression is equivalent to

$$\frac{\partial PS}{\partial \sigma_n^2} = \frac{E + F + \left(Q'\left(2\sigma_\varepsilon^2 + \sigma_n^2\right) + Q\right)R^2\sigma_g^2}{Var\left(\tilde{m}_1\right)^2},$$

where
$$E \equiv -Q'Q^2\left(2\sigma_\varepsilon^2 + \sigma_n^2\right)\left(\sigma_\varepsilon^2 + \sigma_n^2\right) - Q^3\sigma_\varepsilon^2$$

$$F \equiv Z\sigma_\delta^2\left[Z\left(Q'\left(2\sigma_\varepsilon^2 + \sigma_n^2\right) + Q\right) - \left(2QQ'\left(\sigma_\varepsilon^2 + \sigma_n^2\right) + Q^2\right)\right].$$

Consider the sign of *E*, using $Q' = -\frac{Z\sigma_\varepsilon^2}{\left(\sigma_\varepsilon^2 + \sigma_n^2\right)^2}$:

$$\begin{aligned} E &= \frac{ZQ^2\sigma_\varepsilon^2\left(2\sigma_\varepsilon^2 + \sigma_n^2\right)}{\left(\sigma_\varepsilon^2 + \sigma_n^2\right)} - Q^3\sigma_\varepsilon^2 = Q^2\sigma_\varepsilon^2 Z\left(1 + \frac{\sigma_\varepsilon^2}{\left(\sigma_\varepsilon^2 + \sigma_n^2\right)}\right) - Q^3\sigma_\varepsilon^2 \\ &= Q^2\sigma_\varepsilon^2\left(Z\left(2 - S_n\right) - Q\right) < 0 \end{aligned}$$

where the inequality holds since

$$Z\left(2 - S_n\right) - Q = Z\left(2 - S_n\right) - \left(1 - ZS_n\right) = 2Z - 1 < 0$$

as $Z < 1/2$ for $r > 0$.

$$F \equiv Z\sigma_\delta^2\left[\frac{\partial\left(ZQ\left(2\sigma_\varepsilon^2 + \sigma_n^2\right)\right)}{\partial \sigma_n^2} - \frac{\partial\left(Q^2\left(\sigma_\varepsilon^2 + \sigma_n^2\right)\right)}{\partial \sigma_n^2}\right].$$

$$\begin{aligned} ZQ\left(2\sigma_\varepsilon^2 + \sigma_n^2\right) &= 2Z\sigma_\varepsilon^2 + Z\sigma_n^2\left(1 - Z\left(1 + \frac{\sigma_\varepsilon^2}{\left(\sigma_\varepsilon^2 + \sigma_n^2\right)}\right)\right) \\ &= 2Z\sigma_\varepsilon^2 + Z\sigma_n^2\left(1 - Z\left(2 - S_n\right)\right) \\ &= 2Z\sigma_\varepsilon^2 + Z\sigma_n^2\left(1 - 2Z\right) + Z^2\sigma_n^2 S_n. \end{aligned}$$

$$\frac{\partial\left(ZQ\left(2\sigma_\varepsilon^2+\sigma_n^2\right)\right)}{\partial\sigma_n^2}=Z\left(1-2Z\right)+Z^2\frac{\partial\left(\frac{\sigma_n^4}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)}\right)}{\partial\sigma_n^2}$$

Similarly, substituting for Q in $Q^2\left(\sigma_\varepsilon^2+\sigma_n^2\right)$ yields

$$Q^2\left(\sigma_\varepsilon^2+\sigma_n^2\right)=\sigma_\varepsilon^2+\sigma_n^2\left(1-2Z\right)+Z^2\frac{\sigma_n^4}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)},$$

and the partial derivative with respect to σ_n^2 is:

$$\frac{\partial\left(Q^2\left(\sigma_\varepsilon^2+\sigma_n^2\right)\right)}{\partial\sigma_n^2}=\left(1-2Z\right)+Z^2\frac{\partial\left(\frac{\sigma_n^4}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)}\right)}{\partial\sigma_n^2}.$$

The term F becomes:

$$F\equiv Z\sigma_\delta^2\left[\frac{\partial\left(ZQ\left(2\sigma_\varepsilon^2+\sigma_n^2\right)\right)}{\partial\sigma_n^2}-\frac{\partial\left(Q^2\left(\sigma_\varepsilon^2+\sigma_n^2\right)\right)}{\partial\sigma_n^2}\right]=Z\sigma_\delta^2\left[\left(1-2Z\right)\left(Z-1\right)\right]<0$$

as $Z\in(0, 1/2)$ for $s>0$ and $r>0$. Since both E and F are negative,

$$\frac{\partial PS}{\partial\sigma_n^2}=\frac{E+F}{Var\left(\tilde{m}_1\right)^2}<0\text{ if }\sigma_g^2=0.$$

Further,

$$\begin{aligned}Q'\left(2\sigma_\varepsilon^2+\sigma_n^2\right)+Q&=-Z\frac{\sigma_\varepsilon^2}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)^2}\left(\sigma_\varepsilon^2+\sigma_\varepsilon^2+\sigma_n^2\right)+1-Z\frac{\sigma_n^2}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)}\\&=-Z\frac{\sigma_\varepsilon^4}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)^2}-Z\frac{\sigma_\varepsilon^2}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)}+1-Z\frac{\sigma_n^2}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)}\\&=1-Z\left(1+\left(1-S_n\right)^2\right)\geq 1-2Z\geq 0,\end{aligned}$$

due to $0\leq Z\leq ½$ and $0\leq S_n<1$, and it follows that

$$\left(Q'(2\sigma_\varepsilon^2+\sigma_n^2)+Q\right)R^2\sigma_g^2>0$$

if $\sigma_n^2>0$ and $Z<1/2$. This term strictly increases in σ_g^2 so that for any combination of the other parameters, σ_g^2 can be chosen sufficiently large that PS increases in σ_n^2. The threshold is

$$E+F+\left(Q'\left(2\sigma_\varepsilon^2+\sigma_n^2\right)+Q\right)R^2\sigma_g^2 \geq 0$$

or
$$\sigma_g^2 \geq \frac{(1-2Z)\left(Q^2\sigma_\varepsilon^2+Z(1-Z)\sigma_\delta^2\right)}{\left(1-Z\left(1+(1-S_n)^2\right)\right)R^2}.$$

Due to $\partial S_n / \partial \sigma_n^2 > 0$ and $Q = 1 - ZS_n$, the threshold strictly decreases in σ_n^2. Hence, if σ_g^2 exceeds the threshold for $\sigma_n^2 = 0$, it exceeds it for all values of σ_n^2. Inserting $\sigma_n^2 = 0$ in the threshold yields

$$\frac{(1-2Z)\left(Q^2_{\left(\sigma_n^2=0\right)}\sigma_\varepsilon^2+Z(1-Z)\sigma_\delta^2\right)}{\left(1-Z\left(1+\left(1-S_{n\left(\sigma_n^2=0\right)}\right)^2\right)\right)R^2}=\frac{(1-2Z)\left(\sigma_\varepsilon^2+Z(1-Z)\sigma_\delta^2\right)}{(1-2Z)R^2}$$

$$=\frac{\left(\sigma_\varepsilon^2+Z(1-Z)\sigma_\delta^2\right)}{R^2}\equiv\overline{\Lambda}.$$

If σ_g^2 is smaller than the threshold for $\sigma_n^2 \to \infty$, then *PS* always decreases in σ_n^2. Taking the limit of the threshold for $\sigma_n^2 \to \infty$ gives

$$\frac{(1-2Z)\left(Q^2_{\left(\sigma_n^2\to\infty\right)}\sigma_\varepsilon^2+Z(1-Z)\sigma_\delta^2\right)}{\left(1-Z\left(1+\left(1-S_{n\left(\sigma_n^2\to\infty\right)}\right)^2\right)\right)R^2}=\frac{(1-2Z)\left((1-Z)^2\sigma_\varepsilon^2+Z(1-Z)\sigma_\delta^2\right)}{(1-Z)R^2}$$

$$=\frac{(1-2Z)\left((1-Z)\sigma_\varepsilon^2+Z\sigma_\delta^2\right)}{R^2}=\underline{\Lambda}.$$

This completes the proof of part (v).

Part (vi): *PS* can be written as

$$PS=\beta+\frac{Q\left(\sigma_\varepsilon^2+\sigma_n^2\right)}{Var\left(\tilde{m}_1\right)}-1=\frac{Q\left(2\sigma_\varepsilon^2+\sigma_n^2\right)+Z\sigma_\delta^2}{Var\left(\tilde{m}_1\right)}-1.$$

Only $Var\left(\tilde{m}_1\right)$ is affected by σ_g^2 and strictly increases in σ_g^2. Hence, *PS* strictly decreases in σ_g^2.

(q.e.d.)

Proof of Proposition 5

Derivation of (15)

$$Var\left(\tilde{m}_2\right)=Var\left(\tilde{y}_2-\tilde{b}\right)=Var\left(\tilde{y}_2\right)+Var\left(\tilde{b}\right)-2Cov\left(\tilde{y}_2,\tilde{b}\right),$$

where

$$Var\left(\tilde{y}_2\right)=\sigma_\varepsilon^2+\sigma_\delta^2+\sigma_u^2$$

$$\begin{aligned}Var\left(\tilde{b}\right)&=\left(Q-1\right)^2\left(\sigma_\varepsilon^2+\sigma_n^2\right)+Z^2\sigma_\delta^2+R^2\sigma_g^2\\&=\left(Q^2-2Q+1\right)\left(\sigma_\varepsilon^2+\sigma_n^2\right)+Z^2\sigma_\delta^2+R^2\sigma_g^2\\&=Var\left(\tilde{m}_1\right)+Var\left(\tilde{y}_1\right)-2Q\left(\sigma_\varepsilon^2+\sigma_n^2\right)\end{aligned}$$

$$2Cov\left(\tilde{y}_2,\tilde{b}\right)=2\left(Q-1\right)\sigma_\varepsilon^2+2Z\sigma_\delta^2=2Cov\left(m_1,\tilde{x}\right)-2\sigma_\varepsilon^2$$

Inserting these terms yields

$$Var\left(\tilde{m}_2\right)=2\sigma_\varepsilon^2+Var\left(\tilde{y}_1\right)+Var\left(\tilde{y}_2\right)+Var\left(\tilde{m}_1\right)-2Cov\left(\tilde{m}_1,\tilde{x}\right)-2Q\left(\sigma_\varepsilon^2+\sigma_n^2\right).$$

As shown in the proof of Proposition 4

$$Cov\left(\tilde{m}_1,\tilde{m}_2\right)=Q\sigma_\varepsilon^2+Q\left(1-Q\right)\left(\sigma_\varepsilon^2+\sigma_n^2\right)+Z\left(1-Z\right)\sigma_\delta^2-R^2\sigma_g^2$$

or

$$Cov\left(\tilde{m}_1,\tilde{m}_2\right)=\left(Cov\left(\tilde{m}_1,\tilde{x}\right)+Q\left(\sigma_\varepsilon^2+\sigma_n^2\right)\right)-Var\left(\tilde{m}_1\right).$$

$$\frac{Cov\left(\tilde{m}_1,\tilde{m}_2\right)^2}{Var\left(\tilde{m}_1\right)}=\frac{\left(Cov\left(\tilde{m}_1,\tilde{x}\right)+Q\left(\sigma_\varepsilon^2+\sigma_n^2\right)\right)^2}{Var\left(\tilde{m}_1\right)}-2\left(Cov\left(\tilde{m}_1,\tilde{x}\right)+Q\left(\sigma_\varepsilon^2+\sigma_n^2\right)\right)+Var\left(\tilde{m}_1\right).$$

Inserting this expressions into the definition of *PD* yields

$$-PD=2\sigma_\varepsilon^2+Var\left(\tilde{y}_1\right)+Var\left(\tilde{y}_2\right)-\frac{\left(Cov\left(\tilde{m}_1,\tilde{x}\right)+Q\left(\sigma_\varepsilon^2+\sigma_n^2\right)\right)^2}{Var\left(\tilde{m}_1\right)}.$$

Recall that *PS* can be written as

$$PS=\frac{Cov\left(\tilde{m}_1,x\right)+Q\left(\sigma_\varepsilon^2+\sigma_n^2\right)}{Var\left(\tilde{m}_1\right)}-1.$$

Therefore,

$$\left(1+PS\right)^2Var\left(\tilde{m}_1\right)^2=\left(Cov\left(\tilde{m}_1,x\right)+Q\left(\sigma_\varepsilon^2+\sigma_n^2\right)\right)^2.$$

Dividing by $Var\left(\tilde{m}_1\right)$ and inserting in the term for *PD* results in

$$-PD = 2\sigma_\varepsilon^2 + Var\left(\tilde{y}_1\right) + Var\left(\tilde{y}_2\right) - \left(1+PS\right)^2 Var\left(\tilde{m}_1\right).$$

Proof of the statements in Proposition 5

To prove parts (i) and (ii) consider the partial derivative of *PD* with respect to $j = s, r$. Note that only *PS* and $Var\left(\tilde{m}_1\right)$ depend on *s* and *r*. Define *G* and *H* as follows:

$$G \equiv \left(1+PS\right)^2 Var\left(\tilde{m}_1\right) = \frac{H^2}{Var\left(\tilde{m}_1\right)},$$

where

$$H \equiv \left(Cov\left(\tilde{m}_1, x\right) + Q\left(\sigma_\varepsilon^2 + \sigma_n^2\right)\right) = Q\left(2\sigma_\varepsilon^2 + \sigma_n^2\right) + Z\sigma_\delta^2$$
$$= 2\sigma_\varepsilon^2 + \sigma_n^2 - Z\sigma_n^2 \frac{\left(2\sigma_\varepsilon^2 + \sigma_n^2\right)}{\left(\sigma_\varepsilon^2 + \sigma_n^2\right)} + Z\sigma_\delta^2.$$

The partial derivative with respect to a parameter *j* is

$$G_j = \frac{2HH_j Var\left(\tilde{m}_1\right) - H^2 Var_j\left(\tilde{m}_1\right)}{Var\left(\tilde{m}_1\right)^2} = \frac{H}{Var\left(\tilde{m}_1\right)^2}\left(2H_j Var\left(\tilde{m}_1\right) - HVar_j\left(\tilde{m}_1\right)\right). \qquad \text{(A6)}$$

Part (i): $j = s$. The term in parentheses on the right hand side of (A6) is

$$2Z_s\left(\sigma_\delta^2 - \sigma_n^2\frac{\left(2\sigma_\varepsilon^2 + \sigma_n^2\right)}{\left(\sigma_\varepsilon^2 + \sigma_n^2\right)}\right)\left(Q^2\left(\sigma_\varepsilon^2 + \sigma_n^2\right) + Z^2\sigma_\delta^2 + R^2\sigma_g^2\right)$$
$$-\left(Q\left(2\sigma_\varepsilon^2 + \sigma_n^2\right) + Z\sigma_\delta^2\right)2\left(Z_s\left(Z\sigma_\delta^2 - Q\sigma_n^2\right) + RR_s\sigma_g^2\right).$$

After some rearrangement this expression is equal to

$$2Z_s\sigma_\delta^2\left(Q^2\left(\sigma_\varepsilon^2 + \sigma_n^2\right) - Z^2\sigma_n^2 - Z^2S_n\sigma_\varepsilon^2 - 2Z\sigma_\varepsilon^2 + 2S_nZ^2\sigma_\varepsilon^2\right)$$
$$+2\sigma_g^2\left(Z_sR^2 - \left(Q\left(2\sigma_\varepsilon^2 + \sigma_n^2\right) + Z\sigma_\delta^2\right)RR_s\right).$$

Using $Q = 1 - S_nZ$ in the first term leads to $2Z_s\sigma_\delta^2(\sigma_\varepsilon^2 + \sigma_n^2)\left(1-2Z\right) > 0$. The sign of the second term is positive due to $Z_s > 0$ and $R_s < 0$. Thus,

$$\frac{\partial G}{\partial s} = \frac{\partial\left(\left(1+PS\right)^2 Var\left(\tilde{m}_1\right)\right)}{\partial s} > 0.$$

This proves part (i).

Part (ii): $j = r$. The term in parantheses on the right hand side of (A6) is

$$2Z_r\left(\sigma_\delta^2-\sigma_n^2\frac{\left(2\sigma_\varepsilon^2+\sigma_n^2\right)}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)}\right)\left(Q^2\left(\sigma_\varepsilon^2+\sigma_n^2\right)+Z^2\sigma_\delta^2+R^2\sigma_g^2\right)$$
$$-\left(Q\left(2\sigma_\varepsilon^2+\sigma_n^2\right)+Z\sigma_\delta^2\right)2\left(Z_r\left(Z\sigma_\delta^2-Q\sigma_n^2\right)+RR_r\sigma_g^2\right),$$

which, after some rearrangements, is equal to

$$2Z_r\sigma_\delta^2\left(Q^2\left(\sigma_\varepsilon^2+\sigma_n^2\right)-Z^2\sigma_n^2-Z^2S_n\sigma_\varepsilon^2-2Z\sigma_\varepsilon^2+2S_nZ^2\sigma_\varepsilon^2\right)$$
$$+2\sigma_g^2\left(Z_rR^2-\left(Q\left(2\sigma_\varepsilon^2+\sigma_n^2\right)+Z\sigma_\delta^2\right)RR_r\right).$$

Using $Q=1-S_nZ$, the first summand is $2Z_r\sigma_\delta^2(\sigma_\varepsilon^2+\sigma_n^2)\left(1-2Z\right)<0$. Using $R_r=-R^2$ and $Z_r=-ZR$, the second summand is equal to

$$2\sigma_g^2\left(Z_rR^2-\left(Q\left(2\sigma_\varepsilon^2+\sigma_n^2\right)+Z\sigma_\delta^2\right)RR_r\right)=2\sigma_g^2R^3\left(Q\left(2\sigma_\varepsilon^2+\sigma_n^2\right)+Z\sigma_\delta^2-Z\right).$$

This term is non-negative if

$$\sigma_\delta^2\geq 1-\frac{Q}{Z}\left(2\sigma_\varepsilon^2+\sigma_n^2\right)=1+\left(2\sigma_\varepsilon^2+\sigma_n^2\right)\left(S_n-\frac{1}{Z}\right)$$

because $S_n < 0$ and $Z \leq 1/2$ imply $S_n - 1/Z < 0$. Thus, if σ_δ^2 exceeds this threshold, then *PD* strictly increases in *r* if σ_g^2 is large. If σ_δ^2 falls short of the threshold, then *PD* always strictly decreases in *r*. This completes the proof of part (ii).

Part (iii) follows because no term in *PD* depends on *p* and $\bar{g}$.

Part (iv): Using primes to denote the first partial derivatives with respect to σ_δ^2, observe that $H'=Z$ and $Var\left(\tilde{m}_1\right)'=Z^2$. Then, the partial derivative of –*PD* with respect to σ_δ^2 is

$$
\begin{aligned}
\frac{\partial(-PD)}{\partial\sigma_\delta^2} &= Var\left(\tilde{y}_2\right)' - G' \\
&= 1 - \frac{H}{Var\left(\tilde{m}_1\right)^2}\left(2ZVar\left(\tilde{m}_1\right) - HZ^2\right) \\
&= 1 + \frac{H^2}{Var\left(\tilde{m}_1\right)^2}Z^2 - 2\frac{H}{Var\left(\tilde{m}_1\right)}Z \\
&= 1 + \left(1+PS\right)^2 Z^2 - 2\left(1+PS\right)Z \\
&= \left(1-\left(1+PS\right)Z\right)^2 \geq 0,
\end{aligned}
$$

where the last inequality is strict except for special cases in which $1/Z = 1 + PS$. Therefore, PD decreases in σ_δ^2, which proves part (iv).

Part (v): Using primes to denote the first partial derivatives with respect to σ_n^2, obtain

$$
H' = 1 - Z\left(1 + \frac{\sigma_\varepsilon^4}{\left(\sigma_\varepsilon^2 + \sigma_n^2\right)^2}\right) \text{ and}
$$

$$
Var\left(\tilde{m}_1\right)' = 1 - 2Z + Z^2\sigma_n^2\frac{\left(2\sigma_\varepsilon^2 + \sigma_n^2\right)}{\left(\sigma_\varepsilon^2 + \sigma_n^2\right)^2}.
$$

Then

$$
\begin{aligned}
\frac{\partial(-PD)}{\partial\sigma_n^2} &= Var\left(\tilde{y}_1\right)' - G' \\
&= 1 - \frac{H}{Var\left(\tilde{m}_1\right)^2}\left(2H'Var\left(\tilde{m}_1\right) - HVar\left(\tilde{m}_1\right)'\right) \\
&= 1 + \frac{H^2}{Var\left(\tilde{m}_1\right)^2}\left(1 - 2Z + Z^2\sigma_n^2\frac{\left(2\sigma_\varepsilon^2 + \sigma_n^2\right)}{\left(\sigma_\varepsilon^2 + \sigma_n^2\right)^2}\right) - 2\frac{H}{Var\left(\tilde{m}_1\right)}\left(1 - Z\left(1 + \frac{\sigma_\varepsilon^4}{\left(\sigma_\varepsilon^2 + \sigma_n^2\right)^2}\right)\right) \\
&= 1 + \left(1+PS\right)^2\left(1 - 2Z + Z^2\sigma_n^2\frac{\left(2\sigma_\varepsilon^2 + \sigma_n^2\right)}{\left(\sigma_\varepsilon^2 + \sigma_n^2\right)^2}\right) - 2\left(1+PS\right)\left(1 - Z\left(1 + \frac{\sigma_\varepsilon^4}{\left(\sigma_\varepsilon^2 + \sigma_n^2\right)^2}\right)\right).
\end{aligned}
$$

This expression is positive for $Z = 0$ because

$$
1 + \left(1 + PS\left(Z=0\right)\right)^2 - 2\left(1 + PS\left(Z=0\right)\right) = \left(1 - \left(1 + PS\left(Z=0\right)\right)\right)^2 = PS\left(Z=0\right)^2 > 0.
$$

The proof that it is positive for all feasible Z is by contradiction. Assume to the contrary that there exists a Z for which the derivative is negative, then due to continuity it must change signs somewhere in the interval [0, 1/2] and equal zero at some positive $Z \le 1/2$, implying that there exists a $Z \le 1/2$ for which the equation

$$1+\left(1+PS\right)^2 Var'\left(\tilde{m}_1\right)-2\left(1+PS\right)H'=0$$

holds. This equation is quadratic in (1 + *PS*) for any given values of $Var'\left(\tilde{m}_1\right)$ and H'. Therefore, holding $Var'\left(\tilde{m}_1\right)$ and H' fixed, the solution contains two roots ρ_1 and ρ_2, provided that a solution exists in the set of real numbers $\Re$. Choose one of the roots, ρ, we demonstrate that the resulting quadratic equation

$$\rho^2 Var'\left(\tilde{m}_1\right)-\rho\left(2H'\right)+1=0$$

has no solution in $\Re$ for $0 < Z \le 1/2$. A necessary condition for the existence of real roots is

$$\left(2H'\right)^2 \ge 4Var'\left(\tilde{m}_1\right) \Leftrightarrow \left(H'\right)^2 - Var'\left(\tilde{m}_1\right) \ge 0.$$

$$\left(H'\right)^2 = 1-2Z\left(1+\frac{\sigma_\varepsilon^4}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)^2}\right)+Z^2\left(1+\frac{\sigma_\varepsilon^4}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)^2}\right)^2$$

$$\begin{aligned}
&\left(H'\right)^2 - Var'\left(\tilde{m}_1\right) = \\
&= 1-2Z\left(1+\frac{\sigma_\varepsilon^4}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)^2}\right)+Z^2\left(1+\frac{\sigma_\varepsilon^4}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)^2}\right)^2 - 1 + 2Z - Z^2\left(\frac{2\sigma_\varepsilon^2\sigma_n^2+\sigma_n^4}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)^2}\right) \\
&= -2Z\frac{\sigma_\varepsilon^4}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)^2}+Z^2\left(\left(1+\frac{\sigma_\varepsilon^4}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)^2}\right)^2-\left(\frac{2\sigma_\varepsilon^2\sigma_n^2+\sigma_n^4}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)^2}\right)\right) \\
&= -2Z\frac{\sigma_\varepsilon^4}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)^2}+Z^2\left(\frac{\left(\left(\sigma_\varepsilon^2+\sigma_n^2\right)^2+\sigma_\varepsilon^4\right)^2-\left(\sigma_\varepsilon^2+\sigma_n^2\right)^2\left(2\sigma_\varepsilon^2\sigma_n^2+\sigma_n^4\right)}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)^2\left(\sigma_\varepsilon^2+\sigma_n^2\right)^2}\right) \\
&= -2Z\frac{\sigma_\varepsilon^4}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)^2}+Z^2\left(\frac{\left(\sigma_\varepsilon^2+\sigma_n^2\right)^2 3\sigma_\varepsilon^4+\sigma_\varepsilon^8}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)^2\left(\sigma_\varepsilon^2+\sigma_n^2\right)^2}\right) \\
&= -2Z\frac{\left(\sigma_\varepsilon^4+2\sigma_\varepsilon^2\sigma_n^2+\sigma_n^4\right)\sigma_\varepsilon^4}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)^2\left(\sigma_\varepsilon^2+\sigma_n^2\right)^2}+Z^2\left(\frac{4\sigma_\varepsilon^8+6\sigma_\varepsilon^6\sigma_n^2+3\sigma_\varepsilon^4\sigma_n^4}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)^2\left(\sigma_\varepsilon^2+\sigma_n^2\right)^2}\right) \\
&= \frac{2\sigma_\varepsilon^8 Z\left(2Z-1\right)+4\sigma_\varepsilon^6\sigma_n^2 Z\left(\tfrac{3}{2}Z-1\right)+2\sigma_\varepsilon^4\sigma_n^4 Z\left(\tfrac{3}{2}Z-1\right)}{\left(\sigma_\varepsilon^2+\sigma_n^2\right)^2\left(\sigma_\varepsilon^2+\sigma_n^2\right)^2}<0
\end{aligned}$$

The last inequality follows since 0 < Z ≤ 1/2. Thus, no real roots exist for the above quadratic equation. We conclude that the derivative of *PD* with respect to σ_n^2 is negative, which completes the proof of part (v).

Part (vi): To see the impact of a change of σ_g^2 on *PD*, observe that $\partial G / \partial \sigma_g^2 < 0$ since the variance $Var(\tilde{m}_1)$ increases in σ_g^2 and *H* remains unaffected. Hence *PD* decreases in the incentive risk.

(q.e.d.)

Proof of Proposition 6

Using $R = \dfrac{Z}{4s}$,

$$
\begin{aligned}
SM &\equiv -\frac{Cov\left(\tilde{b}, \tilde{y}_1\right)}{Std\left(\tilde{b}\right) Std\left(\tilde{y}_1\right)} \\
&= -\frac{Cov\left(\left((Q-1)\tilde{y}_1 + Z\tilde{\delta} + R\tilde{g}\right), \tilde{y}_1\right)}{Std\left((Q-1)\tilde{y}_1 + Z\tilde{\delta} + R\tilde{g}\right) Std\left(\tilde{y}_1\right)} \\
&= -\frac{(Q-1)\left(\sigma_\varepsilon^2 + \sigma_n^2\right)}{\left((Q-1)^2\left(\sigma_\varepsilon^2 + \sigma_n^2\right) + Z^2\sigma_\delta^2 + R^2\sigma_g^2\right)^{1/2}\left(\sigma_\varepsilon^2 + \sigma_n^2\right)^{1/2}} \\
&= \frac{Z\sigma_n^2}{\left(Z^2\dfrac{\sigma_n^4}{\left(\sigma_\varepsilon^2 + \sigma_n^2\right)} + Z^2\sigma_\delta^2 + \dfrac{Z^2}{16s^2}\sigma_g^2\right)^{1/2}\left(\sigma_\varepsilon^2 + \sigma_n^2\right)^{1/2}} \\
&= \frac{Z\sigma_n^2}{Z\left(\dfrac{\sigma_n^4}{\left(\sigma_\varepsilon^2 + \sigma_n^2\right)} + \sigma_\delta^2 + \dfrac{1}{16s^2}\sigma_g^2\right)^{1/2}\left(\sigma_\varepsilon^2 + \sigma_n^2\right)^{1/2}} \\
&= \frac{\sigma_n^2}{\left(\sigma_n^4 + \left(\sigma_\delta^2 + \dfrac{1}{16s^2}\sigma_g^2\right)\left(\sigma_\varepsilon^2 + \sigma_n^2\right)\right)^{1/2}} > 0.
\end{aligned}
$$

This term is independent of *r* and *p*, and parts (ii) and (iii) follow immediately. As the denominator decreases in *s*, *SM* increases in *s* which proves part (i). Parts (iv), (v) and (vi) follow by observing that $\dfrac{\partial SM}{\partial \sigma_\delta^2} < 0$, $\dfrac{\partial SM}{\partial \sigma_n^2} > 0$ and $\dfrac{\partial SM}{\partial \sigma_g^2} < 0$.

(q.e.d.)

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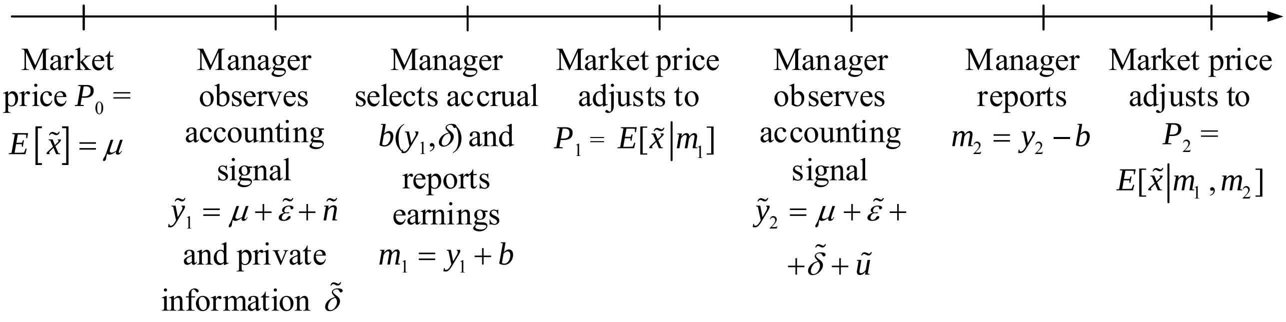
Fig. 1: Sequence of events



Tab. 1: Behavior of earnings quality metrics for an increase in incentives, information risk and accounting risk

Earnings quality metric	Smoothing incentive *s*	Cost of bias *r*	Price and earnings incentives *p* and $\bar{g}$	Information risk σ_δ^2	Accounting risk σ_n^2	Incentive risk σ_g^2
Benchmark: Earnings quality *EQ*	+	– / + [1]	no	+	–	–
Value relevance β	+ [2]	– / + [3]	no	+	–	–
Persistence *PS*	+ [2]	+ / – [4]	no	+	– / + [5]	–
Predictability *PD*	+	– or + [6]	no	–	–	–
Smoothness *SM*	+	no	no	–	+	–
Discretionary accruals DA_1 [7]	±	+	–	–	+	+
Discretionary accruals DA_2 [7]	±	+	–	–	±	±

[1] Decrease if $\sigma_g^2 / \sigma_\delta^2$ is low; and vice versa.

[2] If $\sigma_\delta^2 \leq \sigma_n^2 (2 - S_n)$; otherwise increase and then decrease.

[3] Decrease if $\sigma_g^2 / \sigma_\delta^2$ is low and other conditions are satisfied; otherwise sign depends on other variables.

[4] Inverse u-shaped if certain conditions apply; ambiguous otherwise.

[5] Increase, u-shaped, or decrease for low, intermediate, and high σ_g^2.

[6] Decrease if σ_δ^2 is low; increase if σ_g^2 is very high.

[7] If $p > 0$ and $\bar{g} > 0$; otherwise, effects may reverse.

This table shows how each earnings quality metric is affected for an increase in incentives, information risk and accounting risk. "+" indicates an increase, "–" a decrease and "±" an ambiguous effect that depends on several other parameters. The earnings report consists of the sum of an (unbiased) accounting signal and on a bias (accrual) that is determined by the manager based on the incentives she has from her utility function. Incentives include smoothing (higher *s* implies a larger smoothing incentive), an incentive to influence the market price and reported earnings (higher p implies a higher effect of the contemporaneous market price on the utility, higher $\bar{g}$ increases the expected importance of reported earnings), and the cost of the bias (larger r implies more costly accruals or earnings management). A higher information risk σ_{δ}^{2} implies more private information of the manager. A higher accounting risk σ_{n}^{2} indicates a less precise accounting system producing the first-period accounting signal. A higher incentive risk σ_{g}^{2} increases the investors' uncertainty with respect to the manager's earnings incentive.

Definitions:
Earnings quality is the reduction of the variance of the terminal value due to the earnings report.
Value relevance is the slope coefficient from regressing market price on earnings.
Persistence is the slope coefficient from regressing expected second-period earnings on first-period earnings.
Predictability is the variance of second-period earnings conditional on first-period earnings.
Smoothness *SM* is the negative correlation between discretionary accruals and pre-discretionary earnings.
Discretionary accruals DA_1 is the negative expected value of the bias (discretionary accruals).
Discretionary accruals DA_2 is the negative expected value of the squared bias (discretionary accruals).