Political Economy in a Changing World*

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Abstract

We provide a general framework for the analysis of the dynamics of institutional change (e.g., democratization, extension of political rights or repression of different groups), and how this dynamics interacts with (anticipated and unanticipated) changes in the distribution of political power and in economic structure. We focus on the Markov voting equilibria, which require that economic and political changes should take place if there exists a subset of players with the power to implement such changes and who will obtain higher expected discounted utility by doing so. Assuming that economic and political institutions as well as individual types can be ordered, and preferences and the distribution of political power satisfy a natural "single crossing" condition, we prove the existence of pure-strategy equilibrium, provide conditions for its uniqueness, and present a number of comparative static results that apply at this level of generality. We then use this framework to study the dynamics of political rights and repression in the presence of radical groups that can stochastically grab power depending on the distribution of political rights in society. We characterize the conditions under which the presence of radicals leads to repression (of less radical groups), show a type of path dependence in politics resulting from radicals coming to power, and identify a novel strategic complementarity in repression.

Comments are very welcome.

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JEL Classification:

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1 Introduction

Most regime transitions take place in the midst of uncertainty and turmoil, which sometimes brings to power — or paves the road for the rise of — the most radical factions such as the militant Jacobins during the Reign of Terror in the French Revolution or the Nazis during the crisis of the Weimar Republic. The possibility of "extreme" outcomes is of interest not only because the resulting regimes have caused much human suffering and powerfully shaped the course of history, but also because, in many episodes, the fear of such radical extremist regimes has been one of the drivers of repression against a whole gamut of opposition groups. The events leading up to the October Revolution of 1917 in Russia illustrate both how an extremist fringe group can rise to power, and the dynamics of repression partly motivated by the desire of ruling elites to prevent the empowerment of extremist — and sometimes also of more moderate — elements.

Russia entered the 20$^{\text{th}}$ century as an absolute monarchy, but started a process of limited political reforms in response to labor strikes and civilian unrest in the aftermath of its defeat in the Russo-Japanese war of 1904-1905. Despite the formation of political parties (for the first time in Russian history) and an election with a wide franchise, the repression against the regime's opponents continued, and the parliament, the Duma, had limited powers and was considered by the tsar as an advisory rather than legislative body (Pipes, 1995). The tsar's intentions appear to have been to introduce minimal power-sharing to appease the opposition, while still retaining control, in part relying on repression against the leftist groups, his veto power, the right to dissolve the Duma, full control of the military and cabinet appointments, and ability to rule by decree when the Duma was not in session. In this instance, therefore, it appears that a fear of further lurch to the left was a major motivation or at the very least excuse for limiting power sharing.

This strategy handicapped the development of a constitutional regime in Russia, perhaps paradoxically further strengthening the two major leftist parties, Social Revolutionaries and Social Democrats (corresponding to communists, consisting of Bolsheviks and the Mensheviks), which together controlled about 2/5 of the 1906 (the "second") Duma and explicitly targeted a revolution. Lenin, the leader of the Bolshevik wing of the Social Democrats, recognized that a revolution was possible only by exploiting turmoil. In the context of the 1906 Duma, he stated:

"Our task is [...] to use the conflicts within this Duma, or connected with it, for choosing the right moment to attack the enemy, the right moment for an insurrection against the autocracy." Later, he argued: "[...] the Duma should be used for the purposes of the revolution, should be used mainly for promulgating the Party's political and socialist views and not for legislative 'reforms,' which, in any case, would mean supporting the counter-revolution and curtailing democracy in every way."

The February Revolution of 1917, in the midst of World War I that had become very unpopular following large casualties, territorial losses and very limited military victories, created the opening for the Bolsheviks, bringing to power the Provisional Government and then later, the moderate Social Revolutionary Alexander Kerensky. Further military defeats of the Russian army in the summer of 1917, the destruction of the military chain of command by emergence of Bolshevick-led soldier committees,[1] and Kerensky's willingness to enter into an alliance with Social Democrats to defeat the attempted coup by the army during the Kornilov affair strengthened the Bolsheviks further. Still, in the elections to the Constituent Assembly in November 1917, the Bolsheviks had only a small fraction of the vote. Nevertheless, they successfully exploited their control of Petrograd Soviets to outmaneuver the more popular Social Revolutionaries, first entering into an alliance with so-called Left Social Revolutionaries, and then coercing them to leave the government so as to form their own one-party dictatorship.

This episode thus illustrates both the possibility of a series of transitions bringing to power some of the most radical groups and the potential implications of the concerns of moderate political transitions further empowering radical groups. Despite a growing literature on political transitions, the issues we have just illustrated in the context of the Bolshevik Revolution cannot be studied with existing models.[2] This is because they necessitate a dynamic model where several groups can form temporary coalitions and a rich set of stochastic shocks create a changing environment, potentially leading to a sequence of political transitions away from current power-

[1] The Bolsheviks were against the war and in fact, favored defeat for the Russian forces. Lenin's programmatic article "The Defeat of One's Own Government in the Imperialist War" published in July 1915, a year into World War I, opens with the statement: "During a reactionary war a revolutionary class cannot but desire the defeat of its government."

[2] These types of political dynamics are not confined to episode in which extreme left groups might come to power. The power struggles between secularists and religious groups in Turkey and more recently in the Middle East and North Africa are also motivated by and illustrate similar concerns on both sides.

holders. Moreover, such a model can also shed further light on key questions in the literature on regime transitions, including those concerning political transitions with several heterogeneous groups, gradual enfranchisement, and the interactions between regime dynamics and coalition formation. We therefore start with developing a framework for the study of dynamic political economy in the presence of stochastic shocks and changing environments. We then apply this framework to the analysis of implications of potential shifts of power to radical groups during tumultuous times.

The next example provides a first glimpse at the type of abstraction we will use and the strategic interactions that may emerge.

Example 1 Consider a society consisting of n groups, spanning from $-l < 0$ (left-wing) to $r > 0$ (right-wing), with group 0 normalized to contain the median voter. Let us assume that the current powerholder, e.g., the Russian tsar or the elite in general, is the rightmost group r. The stage payoff of each group depends on the current 'political state' which encapsulates the distribution of political and economic rights. Each group maximizes the discounted sum of stage payoffs. Stochastic shocks affect both stage payoffs and the likelihood of shifts in political power in a given political state (e.g., in the Russian context, the possibility of a group inside or outside the Duma grabbing power or sidelining some groups).

Suppose that a shock to the environment starting from the stable dictatorship of the tsar (group r) changes stage payoffs (e.g., protests reduce the payoffs to the tsar from monopolizing power) and makes it desirable to share power with moderate groups, say $j = r-1, ..., r-k$. Now several considerations are potentially important. First, the tsar may not go all the way to including groups $j = r-1, ..., r-k$ or may maintain a veto power if feasible, because he may be worried of a 'slippery slope' — once power shifts to these groups, they may later include additional groups further to the left, which is costly for the tsar. Second, the probability that radical extremist groups may gain power might be higher in states in which additional groups to the left are included in the decision-making process, further discouraging limited power-sharing. Thus, in the first two scenarios, the tsar might be afraid of our stylized description of the Russian path where power gradually (and stochastically) shifts from left liberal groups to the coalition of socialist/communist groups, and then ultimately to the most extreme elements among them, the Bolsheviks. Third, and counteracting the first two, the most moderate central left groups, such as $r-1$ and $r-2$, may be unwilling to enter into alliances with other groups to their left, such as $r - k$, because they are themselves afraid of a yet another switch of power to groups to their left. But if so, the tsar may be more willing to allow power-sharing in this latter case, calculating that further slide down the slope will be limited.

This example provides a simple model that might capture some of the interactions we have outlined above (our general framework will be richer than this in several dimensions). Though stylized, this example communicates the complex strategic interactions involved in dynamic political transitions in the presence of stochastic shocks and changing environments. For example, a strengthening of extremist groups such as Bolsheviks may sometimes discourage power-sharing with moderate groups because of slippery slope considerations. But paradoxically, if slippery slope considerations discourage moderate groups from entering into any sort of alliance with groups to their left, this might induce current powerholders to share power with them.

Against this background, the framework we develop in this paper will show that, under natural assumptions, we can characterize the equilibria of this class of environments fairly tightly and perform comparative statics, shedding light on these and a variety of other dynamic strategic interactions. For instance, in the context of the above example, we will show that a range of shocks that increase the risk of costly power shifts to the most extremist groups will (weakly) discourage power-sharing. We will also show how repression against extremists might be driven by fears of repression from the extremists when they take control (thus highlighting a natural but novel type of strategic complementarity in dynamic political economy).

Formally, we consider a society consisting of $i = 1, 2, ..., n$ groups or individuals and $s = 1, 2, ..., m$ *states*, which represent both different economic arrangements with varying payoffs for different types of individuals, and different political arrangements and institutional choices. Stochastic shocks are modeled as stochastic changes in *environments*, which contain information on preferences of all individuals over states and the distribution of political power within states. This approach is general enough to capture a rich set of permanent and transitory (as well as both anticipated or unanticipated) stochastic shocks depending on the current state and environment. Individuals care about the expected discounted sum of their utility, and they make joint choices among feasible political transitions, based on their political power. Our key assumption is that both preferences and the distribution of political power satisfy a natural *single crossing* property: we assume that individuals and states are "ordered," and higher-indexed individuals relatively prefer higher-indexed states and also tend to have greater political power in such states. (Changes in environments shift these preferences and distribution of political power, but maintain single-crossing.)

Our notion of equilibrium is *Markov voting equilibrium*, capturing two natural requirements: (1) that changes in states should take place if there exists a subset of players with the power to implement them and who will obtain higher continuation utility (along the equilibrium path) by doing so; (2) that strategies and continuation utilities should only depend on payoff-relevant variables and states. Under these assumptions, we establish the existence of pure-strategy equilibria. Furthermore, we show that the stochastic path of states in any Markov voting equilibrium ultimately converges to a steady state—i.e., to a state that does not induce further changes once reached (Theorems 1 and 3). Although Markov voting equilibria are not always unique, we provide sufficient conditions that ensure uniqueness (Theorems 2 and 4). We further demonstrate a close correspondence between these Markov voting equilibria and the pure-strategy Markov perfect equilibria of our environment (Theorem 5).

Despite the generality of the framework described here and the potential countervailing forces highlighted by our example above, we also establish a number of strong comparative static results. First, consider a change in environment (either anticipated or unanticipated) which leaves unchanged preferences or the allocation of political power in any of the states $s = 1, ...s'$, but potentially changes them in states $s = s' + 1, ..., m$. The result is that if the steady state of equilibrium dynamics described above, x, was at a state that did not experience change (i.e., $x \leq s'$), then the new steady state emerging after the change in environment can be no smaller than this steady state (Theorem 6). Intuitively, a transition to any of the smaller states $s \leq x$ could have been chosen, but was not, before the change. Now, given that preferences and political power did not change for these states, they have not become more attractive. In contrast, some of the higher-ranked states may have become more attractive, which may induce a transition to a higher state. In fact, perhaps somewhat surprisingly, transition to a state $s \geq s' + 1$ can take place even if all states $s = s' + 1, ..., m$ become less attractive for all agents in society. An interesting and novel implication of this result is that in some environments, there may exist critical states, such as a "stable democracy," and if these critical states are reached before the arrival of certain major shocks or changes (which might have otherwise led to their collapse), there will be no turning back (see Corollary 1).

Second, our framework implies a related result on dynamic equilibrium trajectories (Theorem

7). Consider a similar shock to that discussed in the previous paragraph, leaving preferences and the distribution of political power the same in states $s = 1, ..., s'$. Our results is that if this shock arrives before the steady state $x \leq s'$ is reached and if the discount factor sufficiently small (smaller than some threshold β_0), then the direction of changes in states will remain the same as before (i.e., if there were transitions towards higher states before, this will continue, and vice versa). Intuitively, this happens because, with sufficiently small discount factor, all agents care about the payoffs in the next period most, and by assumption, these payoffs have not changed (though payoffs of states to the right of s' may have changed very significantly). This result again has a range of important and novel implications, which can be illustrated with an application to the dynamics of democracy. Suppose that those currently holding power were considering a limited extension of the franchise, and a shock reduces their power in full democracy. Then this shock will not deter enfranchisement provided that agents are not very forward-looking, but may do so otherwise (i.e., when they have high discount factors).

Third, suppose that a change in environment makes extreme states "sticky," for example, high-indexed individuals, who prefer the highest-indexed states, increase their political power (but preferences remain unchanged). Another result shows that if the shock happened when the society was away from these extreme states (e.g., in this example, it was in a sufficiently low-indexed state), then the equilibrium trajectory is not affected (Theorem 8). This once again has interesting implications in the context of the dynamics of democracy. For example, suppose that this change makes the poor sufficiently powerful in democracy that any move away from democracy becomes impossible if the poor oppose it. Then our result implies that this change will only impact the equilibrium if we were currently in democracy and considering a move away from it. For example, if the equilibrium, before the change, involved a transition from limited democracy to a more democratic state, then this change in environment does not affect the equilibrium path (Corollary 2).

The second part of the paper applies our framework to the issues discussed at the beginning of the Introduction — the emergence and implications of radical politics. After establishing that our framework and comparative statics can be directly applied to the class of problems described in Example 1, we derive a number of additional results for this application. These include the following. First, we characterize some of the factors making radicals themselves choose to repress the rest of society when they come to power, and how society will react to the possibility that radicals may come to power (in terms of its repression of the political rights of different groups). Second, applying our general comparative statics, we show that, starting from a steady state, changes in probabilities of transition (affecting the likelihood of a takeover by radicals) or changes in preferences in states where radicals are in power or have significant power, will weakly increase repression against them. This result is notable for the same reasons as our Theorem 6: it is true regardless of the direction of change). Third, applying our Theorem 7, we characterize the conditions under which an increase in the likelihood of radicals grabbing power may change the dynamics of political liberalization. Fourth, we show a particular type of path dependence: if the steady state before the arrival of a shock involved repression against the left, then the arrival of the left radical, even if reversed, will reduce future repression against the left. Fifth, we identify a new strategic complementarity: if the radicals' cost of using repression against other groups declines, then this will tend to increase repression against the radicals. The intuition is simple: the expectation that radicals, once in power, will use repression and solidify their power encourages repression against them (and vice versa).[3]

Our paper is related to a large political economy literature. First, our previous work, in particular Acemoglu, Egorov, and Sonin (2012), takes one step in this direction by introducing a model for the analysis of the dynamics and stability of different political rules and constitutions. However, that approach not only heavily relies on deterministic and stationary environments (thus ruling out changes in political power) but also focuses on environments in which the discount factor is sufficiently close to 1 so that all agents just care about the payoff from a stable state (that will emerge and persists) if such a state exists. Here, in contrast, it is crucial that political change and choices are motivated by potentially short-term gains.[4]

[3] This result is also interesting as it provides a perspective on why repression differs markedly across societies. For example, Russia before the Bolshevik Revolution repressed the leftists, and after the Bolshevik Revolution systematically repressed the rightists and centrists, while the extent of repression of either extreme has been more limited in the United Kingdom. Such differences are often ascribed to differences in "political culture". Our result instead suggests that (small) differences in economic interests or political costs of repression can lead to significantly different repression outcomes.

[4] In Acemoglu, Egorov and Sonin (2010), we study political selection and government formation in a population with heterogeneous abilities and allow stochastic changes in the competencies of politicians. Nevertheless, this is done under two assumptions, which significantly simplify the analysis and make it inapplicable to the general sets of issues we are interested in here: stochastic shocks are assumed to be very infrequent and the discount factor is again taken to be large (close to 1).

Second, a diverse range of papers in dynamic political economy and in dynamics of clubs emerge as special cases of our paper. Among these, Roberts (1999) deserves special mention as an important precursor of our analysis.[5] Roberts studies a dynamic model of club formation in which current members of the club vote about whether to admit new members and whether to contract the club. Roberts also makes single-crossing type assumptions and focuses on non-stochastic environments and majoritarian voting (see also Barberà, Maschler, and Shalev, 2001, for a related setup). Both our framework and characterization results are more general, not only because they incorporate stochastic elements, but also because we provide conditions for uniqueness, convergence to steady states, and general comparative static results. In addition, Gomes and Jehiel's (2005) paper, which studies dynamics in a related environment with side transfers, is also noteworthy. This paper, however, does not include stochastic elements or similar general characterization results either. Strulovici (2010), who studies a voting model with stochastic arrival of new information, is also related but his focus is on information leading to inefficient dynamics, while changes in political institutions or voting rules are not part of the model.

Third, our motivation is also related to the literature on political transitions. Acemoglu and Robinson (2000a, 2001) consider environments in which institutional change is partly motivated by a desire to reallocate political power in the future to match the current distribution of power. Acemoglu and Robinson's analysis is simplified by focusing on a society consisting of two social groups (and in Acemoglu and Robinson, 2006, with three social groups). In Acemoglu and Robinson (2001), Fearon (2005), Powell (2005), and Acemoglu, Ticchi and Vindigni (2010), anticipation of future changes in political power leads to inefficient policies, civil war or collapse of democracy. There is a growing literature that focuses on situations where decisions of the current policy makers affect the future allocation of political power (see also Besley and Coate, 1998). In Acemoglu and Robinson (2000a), the current elite decides whether to extend the franchise to change the future distribution of political power as a commitment to future policies (and thus potentially staving off costly social unrest or political revolution).[6]

[5] Other important contributions here include Barberà and Jackson (2004), Burkart and Wallner (2000), Jehiel and Scotchmer (2001), Alesina, Angeloni, and Etro (2005), Bordignon and Brusco (2003), Lizzeri and Persico (2004), and Lagunoff (2006).

[6] See also Bourguignon and Verdier (2000), where the choice of educational policy today affects political participation in the future.

Fourth, there is a small literature on strategic use of repression, which includes Acemoglu and Robinson (2000b), Gregory, Schroeder, and Sonin (2011) and Wolitzky (2011). In Wolitzky (2011), different political positions (rather than different types of individuals) are repressed in order to shift the political equilibrium in the context of a two-period model of political economy. In Acemoglu and Robinson (2000b), repression arises because political concessions can be interpreted as a sign of weakness. None of the papers discussed in the previous three paragraphs study the issues we focus on or make progress towards a general framework of the sort presented here.

The rest of the paper is organized as follows. In Section 2, we formulate a general framework of political economy with institutional changes and shocks: the environment, assumptions and definitions we will use throughout the paper, and the concept of Markov Voting Equilibirum. Section 3 contains the analysis of Markov Voting Equilibria. We start with the stationary case (without shocks), then extend the analysis to the general case where shocks are possible, and then compare the concepts of Markov Voting Equilibrium to Markov Perfect Equilibrium in a properly defined dynamic game. We establish several comparative statics results that hold even at this level of generality; this allows us to study the society's reactions to shocks in applied models. Section 4 applies the general model to issues of social mobility and dynamic (dis)enfranchisement. Section 5 discusses possible extensions and limitations of the general framework. Section 6 concludes.

2 General Framework

Time is discrete and infinite, indexed by $t \geq 1$. The society consists of n agents, $N = \{1, \ldots, n\}$. The set of agents is ordered, and the order reflects the initial distribution of some variable of interest: agents with higher numbers may be the elite (and pro-authoritarian rule), while those with lower numbers may be workers or peasants favoring democracy; other possible scales include rich-vs.-poor or secular-vs.-religious. In each period, the society may find itself in one of the h environments $E^1, \ldots, E^h$; we denote the set of environments by $\mathcal{E}$. The environment that the society finds itself in encapsulates agent's economic payoffs and political rules, which are described below in detail. Most importantly, the transitions between environments are stochastic and follows a Markov chain: the probability that the society which lived period t in environment

E will find itself in environment E' equals

$$\pi\left(E, E'\right). \tag{1}$$

Naturally,

$$\sum\nolimits_{E' \in \mathcal{E}} \pi\left(E, E'\right) = 1.$$

Importantly, changes between environments are stochastic and beyond the control of agents. This implies, for example, that a stochastic shock by itself cannot abolish a constitution in favor of another one, but it can change the economic payoffs or reallocate political power in a way that might induce such an outcome that powerful agents in the society decide to undertake these acts.

We make the following assumption on the probabilities of transitions between environments:

Assumption 1 *If* $1 \leq x < y \leq h$, *then*

$$\pi\left(E^y, E^x\right) = 0. \tag{2}$$

In other words, Assumption 1 stipulates only a finite number of shocks.[7] Moreover, it assumes that environments are numbered so that only transitions to higher-numbered environments are possible; this is without loss of generality, but enables us to use the convention that once the last environment, here E^h, has been reached, there are no further stochastic shocks.[8]

Fix an environment E. In this environment, there is a finite set of states $S = \{1, \ldots, m\}$, which we assume to be the same for all environments.[9] (the number of states is m). By states we mean political or social arrangements, distribution of political power or of means of production, over which the society, in principle, has control, at least if it gets support of sufficiently many powerful agents. Importantly, the set of states is *ordered*: this may be interpreted as a sequence of political arrangements which gives less and less power to the poor and more and more power

[7] Notice that Assumption 1 does not preclude the possibility that the environment returns to the state where it was before, but requires that it happens a finite number of times. Indeed, to model the possibility of q transitions between E^1 and E^2, we can define $E^3 = E^1$, $E^4 = E^2$, etc.

[8] This does not mean that the society must reach E^h: for example, it is permissible to have three environments with $\pi\left(E^1, E^2\right) = \pi\left(E^1, E^3\right) > 0$, and all other transition probabilities between being equal to zero.

[9] Acemoglu, Egorov, and Sonin (2011) allows for emergence of new states, but no other shocks are considered. We keep the set of states the same mainly for convenience and to save on notation. In fact, emergence of a new state is equivalent to a shock that makes transitions to/from this state feasible.

to the elite as $s \in S$ increases. To each state we assign stage payoff $u_i(s) = u_{E,i}(s)$, which individual i gets in a period which ends in state s if the current environment is E.

Definition 1 ***(Increasing Differences)*** *Vector* $\{w_i(s)\}_{i \in A}^{s \in B}$*, where* $A, B \subset \mathbb{R}$*, satisfies* Increasing Differences *condition (ID) if for any agents* $i, j \in A$ *such that* $i > j$ *and any states* $x, y \in B$ *such that* $x > y$*,*

$$w_i(x) - w_i(y) \geq w_j(x) - w_j(y). \tag{3}$$

We assume that the stage payoffs, with properly ordered individuals and states, satisfy the *ID* property.

Assumption 2 *In every environment* $E \in \mathcal{E}$*, the vector of utility functions,* $\{u_{E,i}(s)\}_{i \in N}^{s \in S}$*, satisfies the ID property.*

Here, payoffs $\{u_{E,i}(s)\}$ are assigned to combinations of environments, states and individuals rather than endogenously determined; this is made to simplify notation and the game. Implicitly, we think that in every state there is some economic interaction that results in (expected) payoffs $\{u_i(s)\}$, and this will be modeled in Section 4. Any such interaction is permissible in our model, as long as Assumption 2 is satisfied.

Apart from stage payoffs, states are characterized by political power. We capture this by the set of winning coalitions, $W_s = W_{E,s}$. As standard, we make the following assumption:

Assumption 3 ***(Winning Coalitions)*** *For environment* $E \in \mathcal{E}$ *and state* $s \in S$*, the set of winning coalitions* $W_s = W_{E,s}$ *satisfies:*

1. *(monotonicity) if* $X \subset Y \subset N$ *and* $X \in W_s$*, then* $Y \subset W_s$*;*

2. *(properness) if* $X \in W_s$*, then* $N \setminus X \notin W_s$*;*

3. *(decisiveness)* $W_s \neq \varnothing$*.*

The first part of Assumption 3 states that if some coalition has the capacity to implement (social or political) change, then a larger coalition also does. The second part ensures that if some coalition has the capacity to implement change, then the coalition of the remaining players (its complement) does not have one. Finally, the third part, in the light of monotonicity propery, is equivalent to $N \in W_s$, and it thus states that if all players want to implement a change, they can do so.

Assumption 3 puts minimal and natural restrictions on the set of winning coalitions W_s in each given state $s \in S$. We next impose a between-state (albeit still within-environment) restriction on the sets of winning coalitions. We do so to capture the idea that states ranked higher are also likely to be governed by people ranked higher. More formally, we adopt the following definition of quasi-median voter from Acemoglu, Egorov, and Sonin (2012).

Definition 2 ***(Quasi-Median Voter)*** *Player ranked i is a* quasi-median voter *(QMV) in state s (in environment E) if for any winning coalition $X \in W_s$, $\min X \leq i \leq \max X$.*

If we let $M_s = M_{E,s}$ denote the set of QMVs in state s in environment E then, by Assumption 3, $M_s \neq \varnothing$ for any $s \in S$ and $E \in \mathcal{E}$; moreover, the set M_s is connected: whenever $i < j < k$ and $i, k \in M_s$, then $j \in M_s$.[10] In many cases, the set of quasi-median voters is a singleton, $|M_s| = 1$. This will hold in the following two important cases: whenever one individual is the dictator, i.e., $X \in W_s$ if and only if $i \in X$ (and then $M_s = \{i\}$), and when decisions are made by majority voting among the entire set of players or some subset, as long as the number of players is odd. If this holds, we would be able to prove stronger uniqueness results. An example when M_s is not a singleton is unanimity rule, provided that there are at least two players.

The monotonicity assumption we impose is the following.

Assumption 4 ***(Monotone Quasi-Median Voter Property, MQMV)*** *The sequences $\{\min M_s\}_{s \in S}$ and $\{\max M_s\}_{s \in S}$ are non-decreasing in S.*[11]

Assumption 4 is mild and very intuitive; it ensures that states are ordered consistently with agents' power. It suggests that if a certain number of higher-ranked agents is sufficient to implement a change in some state, then they are enough to implement a change in an even higher state. This would hold in a variety of applications, including the one in this paper (Section 4) and Roberts (1999). Trivially, if M_s is a singleton in every state, it is equivalent to M_s being nondecreasing (where M_s is treated as the single element).

[10] There are other, equivalent ways to define QMV. For example, $i \in M_s$ if and only if $\{j \in N : j < i\} \notin W_s$ and $\{j \in N : j > i\} \notin W_s$, or $i \in M_s$ if and only if i belongs to any "connected" winning coalition.

[11] Equivalently, the set-valued function M_s is monotone nondecreasing on S (with respect to the strong set order); see Milgrom and Shannon (1994).

For some applications, one might want to restrict transitions between states that the society may implement; for example, it might be realistic to assume that only transitions to adjacent states are possible. For other applications, the society should be allowed to make any transitions it wants. To capture both possibilities, we introduce a (environment-specific) mapping $F = F_E : S \to 2^S$, which maps every $x \in S$ into the set of states the transition to which is feasible. In other words, $y \in F(x)$ means that the society may transit from x to y. We do not assume that $y \in F(x)$ implies $x \in F(y)$, so irreversible transitions may be modeled. The only requirements we impose are the following.

Assumption 5 *For each environment $E \in \mathcal{E}$, the binary relation $F = F_E$ satisfies:*

1. *For any $x \in S$, $x \in F(x)$;*
2. *For any states $x, y, z \in S$ such that $x < y < z$ or $x > y > z$: If $z \in F(x)$, then $y \in F(x)$ and $z \in F(y)$.*

The first part is almost tautological; it allows the society to remain in the same state. The second part is another mild and natural requirement, implying that if a long transition between two environments is feasible, then any transitions (in the same direction) between intermediate environments also are. Cases that satisfy Assumption 5 and are important for applications include: (a) any transitions possible: $F(x) = S$ for any x; (b) one-step transitions: $y \in F(x)$ if and only if $|x - y| \leq 1$; (c) one-direction transitions: $y \in F(x)$ if and only if $x \leq y$.[12]

The last part of environment characterization is the discount factor, β, which we assume to be the same for all players and across all environments and allow it to take any value such that $\beta \in [0, 1)$. To summarize, the full description of each environment $E \in \mathcal{E}$ is

$$E = \left(N, S, \beta, \{u_i(s)\}_{i \in N}^{s \in S}, \{W_s\}_{s \in S}, \{F(s)\}_{s \in S}\right). \tag{4}$$

In the game, each period t starts with environment $E_{t-1} \in \mathcal{E}$ and with state s_{t-1} inherited from the previous period; then Nature determines E_t according to the Markov chain rule (1)

[12] In an earlier version, we allowed for costs of transitions between states, and imposed a certain increasing differences condition similar to part 2 of Assumption 5. A transition between states x and y may be thought as infeasible if it is sufficiently costly. In the current version we do not model costs of transitions to simplify notation and analysis, and it is not required for the Application (Section 4). All results, statements, and proofs involving costs of transition are available from the authors upon request.

and after that the society decides on s_t. In the beginning, the environment $E_0 \in \mathcal{E}$ and the state $s_0 \in S$ are exogenously given. The society may face a shock (change of the environment) and then decides which state to move to, thereby determining state s_t. At the end of period t, an individual ranked i gets instantaneous payoff

$$v_i^t = u_{E_t,i}(s_t). \tag{5}$$

Denoting the expectation at time t by $\mathbb{E}_t$, the expected discounted payoff of individual i by the end of period t can be written as

$$V_i^t = \mathbb{E}_t \sum\nolimits_{k=0}^{\infty} \beta^k u_{E_{t+k},i}(s_{t+k}). \tag{6}$$

The following sums up the within-period timing in period t.

1. The environment E_{t-1} and state s_{t-1} are inherited from period $t-1$.
2. Shock which determines E_t may occur: $E_t = E \in \mathcal{E}$ with probability $\pi(E_{t-1}, E)$.
3. The society (collectively) decides on state s^t, subject to $s_t \in F_E(s_{t-1})$.
4. Each individual gets instantaneous payoff given by (5).

We deliberately do not describe in detail how the society makes collective decisions (in step 3) as this is not required for the Markov Voting Equilibrium concept; we introduce the detailed steps when we study the noncooperative foundations of MVE.

The equilibria will be characterized by a family of transition mappings $\phi = \{\phi_E : S \to S\}_{E \in \mathcal{E}}$. We let ϕ_E^k be k^{th} iteration of ϕ_E, and we denote throughout $\phi_E^0(s) = s$ for any $s \in S$. With each family of transition mappings we can associate continuation payoffs $V_{E,i}^{\phi}(s)$ for individual i if the state is s, which are recursively given by

$$V_{E,i}^{\phi}(s) = u_{E,i}(s) + \beta_E \sum_{E' \in \mathcal{E}} \pi(E, E') V_{E',i}^{\phi}(\phi_{E'}(s)). \tag{7}$$

(as $0 \le \beta < 1$, the values $V_{E,i}^{\phi}(s)$ are uniquely defined by (7)).

Definition 3 ***(Markov Voting Equilibrium, MVE)*** *A set of transition mappings $\phi = \{\phi_E : S \to S\}_{E \in \mathcal{E}}$ is a* Markov Voting Equilibrium *if the three properties hold:*

1. *(feasibility) for any environment $E \in \mathcal{E}$ and for any state $x \in S$, $\phi_E(x) \in F_E(x)$;*

2. *(core) for any environment $E \in \mathcal{E}$ and for any states $x, y \in S$ such that $y \in F_E(x)$,*

$$\left\{i \in N : V^{\phi}_{E,i}(y) > V^{\phi}_{E,i}(\phi_E(x))\right\} \notin W_{E,x}; \tag{8}$$

3. *(persistence) for any environment $E \in \mathcal{E}$ and for any state $x \in S$,*

$$\left\{i \in N : V^{\phi}_{E,i}(\phi_E(x)) \geq V^{\phi}_{E,i}(x)\right\} \in W_{E,x}. \tag{9}$$

Property 1 requires that MVE involves only feasible transitions (in the current environment). Property 2 is satisfied if no (feasible) alternative $y \neq \phi(x)$ is supported by a winning coalition in x over $\phi_E(x)$ prescribed by the transition mapping ϕ_E. This is analogous to a "core" property: no alternative should be both preferred to the proposed transition by some coalition of players and at the same time enforceable by this coalition; notice, however, that the payoffs involved are continuation utilities and are therefore endogenous to (collective) decisions of the future periods, as well as stochastic shocks . Property 3 requires that it takes a winning coalition to move from any state to some alternative. This requirement singles out the status quo if there is no alternative which some winning coalition would prefer. To put it another way, it takes a winning coalition to move away from a status quo. Both properties will be required for Markov Perfect Equilibria of noncooperative game that we study below.

Throughout the paper, we focus on monotone MVE, i.e., MVE with monotone transition mappings for each $E \in \mathcal{E}$. In many cases this is without loss of generality, and Theorem 9 states mild sufficient conditions for when all MVEs are (generically) monotone. However, Example 5 shows that a nonmonotone MVE may exist if these conditions fail.

3 Analysis

In this section, we analyze the structure of equilibria and a general framework introduced in Section 2. We first (Subsection 3.1) prove existence of monotone MVE in a stationary (deterministic) environment. We then (Subsection 3.2) extend these results to situations in which there are stochastic shocks and non-stationary elements. In Subsection 3.3, we study the relation between MVE and Markov Perfect Equilibria (MPE) of a dynamic game representing the framework of Section 2. We then derive a number of comparative static results for the general model in Subsection 3.4.

3.1 Stationary environment

We first study the case of only one environment ($|\mathcal{E}| = 1$); this will form the induction base later. For this part, we drop the index for the environment and assume that the only environment persists.

For any mapping $\phi : S \to S$, the continuation utility of player i after a transition to s has taken place is given by

$$V_i^{\phi}(s) = u_i(s) + \sum\nolimits_{k=1}^{\infty} \beta^k u_i\left(\phi^k(s)\right). \tag{10}$$

We start our analysis with some preliminary lemmas which we think are of independent interest. The next lemma emphasizes the critical role of quasi-median voters (QMV) in our theory.

Lemma 1 *Suppose that vector $\{w_i(s)\}$ satisfies Increasing Differences property for $S' \subset S$. Take $x, y \in S'$, $s \in S$ and $i \in N$ and let*

$$P = \{i \in N : w_i(y) > w_i(x)\}.$$

Then $P \in W_s$ if and only if $M_s \subset P$. A similar statement is true for relations $\geq$, $<$, $\leq$.

Lemma 1 is an immediate consequence of the ID property. If $w_i(y) > w_i(x)$ for members of W_s, then this holds for all $i \leq \max M_s$ if $y < x$ and for all $i \geq \min M_s$ if $y > x$. In either case, this holds for members of some winning coalition. The "only if" part also follows from ID property: it implies that $w_i(y) > w_i(x)$ must hold for a connected coalition, and therefore it holds for all members of M_s by the Definition 2 of quasi-median voter.

For each $s \in S$, let us introduce the binary relation $>_s$ on the set of n-dimensional vectors:

$$w^1 >_s w^2 \Leftrightarrow \left\{i \in N : w_i^1 > w_2^1\right\} \in W_s,$$

and let us introduce notation $\geq_s$ in a similar way. Lemma 1 now implies that if a vector $\{w_i(x)\}$ satisfies ID, then for any $s \in S$, the relations $>_s$ and $\geq_s$ are transitive on $\{w_\cdot(x)\}_{x \in S}$.

The next result shows that if ϕ is monotone, then continuation utilities $\left\{V_i^{\phi}(s)\right\}_{i \in N}^{s \in S}$ and $\left\{V_i^{\phi}(s \mid x)\right\}_{i \in N}^{s \in S}$ satisfy increasing differences, provided that Assumption 2 and Assumption 5 are satisfied.

Lemma 2 *For a mapping $\phi : S \to S$, vector $\left\{V_i^{\phi}(s)\right\}_{i \in N}^{s \in S}$, given by* (10)*, satisfies ID if at least one of the two properties hold:*

1. ϕ is monotone;

2. for all $x \in S$, $|\phi(x) - x| \leq 1$.

This result, though simple, is critical for what follows. It establishes that if instantaneous utilities satisfy the ID property, so do continuation utilities, provided that either of the conditions on the transition mapping ϕ holds. In other words, once we ensure that ϕ satisfies one of these properties, continuation utilities are guaranteed to satisfy ID.

For mapping ϕ to constitute a MVE, it must satisfy the three properties of Definition 3, and of these, the 'core' property is apparently hardest to satisfy. The next lemma simplifies the analysis considerably by proving that if for a monotone mapping ϕ the 'core' property is violated (i.e., there is a deviation that makes all members of some winning coalition in the current state better off), then one can find a *monotone deviation*, i.e., a deviation such that the resulting mapping after the deviation is also monotone. We call it Monotone Deviation Principle by analogy with One-Stage Deviation Principle in extensive form games, which also states that if some deviation makes a player better off, then there is a one-stage deviation which does so.

Lemma 3 ***(Monotone Deviation Principle)*** *Suppose that mapping $\phi : S \to S$ is monotone and satisfies property (1-feasibility) of Definition 3, but property (2-core) is violated, i.e., for some $x, y \in S$ (such that $y \in F(x)$),*

$$V^{\phi}(y) >_x V^{\phi}(\phi(x)) . \tag{11}$$

Then one can pick $x, y \in S$ such that $y \in F(x)$, (11) *holds, and the mapping $\phi' : S \to S$ given by*

$$\phi'(s) = \begin{cases} \phi(s) & \text{if } s \neq x \\ y & \text{if } s = x \end{cases} \tag{12}$$

is monotone.

The idea of proof is to take the "shortest" deviation, i.e., a pair (x, y) with minimal $|y - \phi(x)|$ such that (11) holds. Without loss of generality, $y > \phi(x)$. Since ϕ is monotone and ϕ', given by (12), is not, there must be some $z < x$ such that $y < \phi(z) \leq \phi(x)$; take the minimal of such z. A deviation at z from $\phi(z)$ to y is monotone, and by assertion must hurt at least one QMV at z, and thus by Assumption 2 it would hurt individual $i = \max M_x$ as $z < x$. If so, this individual i should benefit not only from deviation from $\phi(x)$ to y at state x, but also from $\phi(x)$ to $\phi(z)$. Now, $\phi(z) \leq \phi(x)$ implies that a winning coalition at x benefits from such deviation. But $|\phi(z) - \phi(x)| < |y - \phi(x)|$, which contradicts that we took the shortest deviation.

With the help of Monotone Deviation Principle, we can prove the following result, which will be critical for the proof of existence of MVE. Suppose that we split the set of states into two subsets, $[1, a]$ and $[a+1, m]$, and find (by induction) the MVEs on these respective domains. The question is whether these two mappings, when combined, form an MVE on the entire domain. Clearly, feasibility and persistence (properties (1) and (3) of Definition 3) would hold, and the Monotone Deviation Principle tells us that either a winning coalition in a prefers to move to some state in $[a+1, m]$, or a winning coalition in $a+1$ preferes to move to some state in $[1, a]$. The contribution of the next lemma is that the latter two possibilities are mutually exclusive, and this is the key to the proof of existence (Theorem 1).

Lemma 4 ***(No Double Deviation)*** *Let $a \in [1, m-1]$, and let $\phi_1 : [1, a] \to [1, a]$ and $\phi_2 : [a+1, m] \to [a+1, m]$ be two monotone mappings which are MVE on their respective domains. Let $\phi : S \to S$ be defined by*

$$\phi(s) = \begin{cases} \phi_1(s) & \text{if } s \leq a \\ \phi_2(s) & \text{if } s > a \end{cases} \tag{13}$$

Then exactly one of the following is true:

1. *ϕ is a MVE on S;*

2. *there is $z \in [a+1, \phi(a+1)]$ such that $z \in F(a)$ and $V^{\phi}(z) >_a V^{\phi}(\phi(a))$;*

3. *there is $z \in [\phi(a), a]$ such that $z \in F(a+1)$ and $V^{\phi}(z) >_{a+1} V^{\phi}(\phi(a+1))$.*

We are now ready to prove the existence result.

Theorem 1 ***(Existence)*** *There exists a monotone MVE. Moreover, if ϕ is a monotone MVE, then evolution $s_0, s_1 = \phi(s_1), s_2 = \phi(s_2), \ldots$ is monotone, and there exists a limit state $s_\tau = s_{\tau+1} = \ldots = s_\infty$.*

We use induction on the number of states. If $m = 1$, then $\phi : S \to S$ given by $\phi(1) = 1$ is an MVE for trivial reasons. For $m > 1$, we assume, to obtain a contradiction, that there is no MVE. Take any of $m-1$ possible splits of S into nonempty $C_a = \{1, \ldots, a\}$ and $D_a = \{a+1, \ldots, m\}$, where $a \in \{1, \ldots, m-1\}$, and then take MVE ϕ_1^a on C_a and MVE ϕ_2^a on D_a (assume for simplicity that they are unique; the Appendix describes the way we select ϕ_1^a and ϕ_2^a in the general case). Lemma 4 implies that either there is a deviation from a to $[a+1, \phi_2^a(a+1)]$ or a deviation from $a+1$ to $[\phi_1^a(a), a]$, but not both (the first option is impossible because we assumed that there is no MVE). Let us say that $g(a) = r$ (for "right") in the former case, and that $g(a) = l$ in the latter, then g is a well-defined single-valued function. We then have the following possibilities.

If $g(1) = r$, we can "extend" the MVE ϕ_2^1 onto the entire domain by assigning $\phi(1) \in [2, m]$ appropriately; similarly, if $g(m-1) = l$, we can extend ϕ_1^{m-1} by choosing $\phi(m) \in [1, m-1]$ appropriately (details provided in the Appendix). It remains to consider the case where $g(1) = l$ and $g(m-1) = r$. Then there must exist $a \in \{2, \ldots, m-1\}$ such that $g(a-1) = l$ and $g(a) = r$. We take equilibria ϕ_1^{a-1} on $[1, a-1]$ and ϕ_2^a on $[a+1, m]$, and consider let us define $\phi : S \to S$ by

$$\phi(s) = \begin{cases} \phi_1^{a-1}(s) & \text{if } s < a \\ b & \text{if } s = a \\ \phi_2^a(s) & \text{if } s > a \end{cases}, \tag{14}$$

where $b \in \left[\phi_1^{a-1}(a-1), a-1\right] \cup [a+1, \phi_2^a(a+1)]$ is picked so that $V_i^\phi(b)$ is maximized for some $i \in M_a$ (and $b \in F(a)$). Suppose, without loss of generality, that $b < a$, then $\phi|_{[1,a]}$ is a MVE on $[1, a]$. By Lemma 3, to show that the (core) property is satisfied, it suffices to check that there is no deviation from $a+1$ to $[b, a]$, but this follows from $g(a) = r$. The other two properties, (feasibility) and (persitence), hold by construction, and thus $\phi(s)$ is MVE. The Appendix fills in the details for this argument.

We next study the uniqueness of monotone MVE. We first introduce the following definitions.

Definition 4 *Individual preferences are* single-peaked *if for every $i \in N$ there is $x \in S$ such that for states $y, z \in S$ such that $z < y < x$ or $z > y > x$, $u_i(z) < u_i(y) < u_i(z)$.*

This definition is standard. The next definition defines precisely what we mean by one-step transitions.

Definition 5 *We say that* only one-step transitions are possible *if for any $x, y \in S$ with $|x-y| > 1$, $y \notin F(x)$.*

The next examples shows that equilibrium is not always unique.

Example 2 (Example with two MVE) There are three states A, B, C, and two players 1 and 2. The decision-making rule is unanimity in all states. Payoffs are given by

id	A	B	C
1	20	5	10
2	10	5	20

Suppose that β is sufficiently close to 1, e.g., $\beta = 0.9$. Then there are two MVE. In one, $\phi_1(A) = \phi_2(B) = A$ and $\phi_1(C) = C$. In another, $\phi_2(A) = A$, $\phi_2(B) = \phi_2(C) = C$. This is possible because preferences are not single-peaked, and there are more than one quasi-median voters in all states.

However, single-peakedness alone is not enough, as the next example shows.

Example 3 (Example with single-peaked preferences and two MVE) There are three states A, B, C, and two players 1 and 2. The decision-making rule is unanimity in state A and dictatorship of player 2 in states B and C. Payoffs are given by

id	A	B	C
1	2	25	20
2	1	20	25

Then ϕ_1 given by $\phi_1(A, B, C) = (B, C, C)$ and ϕ_2 given by $\phi_2(A, B, C) = (C, C, C)$ are both MVE when the discount factor is any $\beta \in [0, 1)$.

The following theorem presents cases where equilibrium is (generically) unique.

Theorem 2 ***(Uniqueness)*** *The monotone MVE is (generically) unique if at least one of the following conditions holds:*

1. *for every $s \in S$, M_s is a singleton;*

2. *only one-step transitions are possible and preferences are single-peaked.*

In other words, we can prove uniqueness essentially for the same set of assumptions for which we can establish that any MVE is monotone (Theorem 9 below), and in the second case we require, in addition, that preferences are single-peaked. This means that if either of the conditions in Theorem 2 holds, then there is a unique MVE, and this MVE is monotone.

3.2 Stochastic environments

We now extend our analysis to the case in which there are stochastic shocks. As our analysis will clarify, this also enables us to deal with potentially "nonstationary" problems where the distribution of political power or economic preferences will change in a specific direction in the future. By Assumption 1, environments are ordered as $E^1, E^2, \ldots, E^h$ so that $\pi(E^x, E^y) = 0$ if $x > y$. This means that if we reached environment E^h, there are no further shocks, and the analysis from Section 3.1 is applicable. In particular, we get the same conditions for existence and uniqueness of MVE. We now use backward induction to find equilibrium transition mappings in earlier environments.

The following Lemma is crucial for the analysis.

Lemma 5 *Suppose ϕ is a monotone MVE in a stationary environment. Then continuation payoff vector $\{V_i(s)\}_{i\in N}^{s\in S}$ satisfies the ID condition.*

Lemma 5 is the cornerstone of our study of stochastic environments. It suggests that if utility functions satisfy ID, then, for any monotone MVE ϕ, so do continuation utilities. This results will enable us to apply backward induction arguments as in the non-stochastic case.

To proceed by backward induction, let us take MVE ϕ_{E^h} in the environment E^h; its existence is guaranteed by Theorem 1. Suppose that we have found MVE $\{\phi_E\}_{E\in\{E^{k+1},\ldots,E^h\}}$ for $k = 1,\ldots,h-1$; let us construct ϕ_{E^k} which would make $\{\phi_E\}_{E\in\{E^h,\ldots E^h\}}$ MVE in the environments $\{E^k,\ldots,E^h\}$. Suppose that as long as the environment is E^k, transition mappings are given by ϕ_{E^k}. Then continuation utilities of agent i are given by

$$\begin{aligned}
V^{\phi}_{E^k,i}(s) &= u_{E^k,i}(s) + \beta \sum_{E'\in\{E^k,\ldots,E^h\}} \pi\left(E^k, E'\right) V^{\phi}_{E',i}(\phi_{E'}(s)) \\
&= u_{E^k,i}(s) + \beta \sum_{E'\in\{E^{k+1},\ldots E^h\}} \pi\left(E^k, E'\right) V^{\phi}_{E',i}(\phi_{E'}(s)) \qquad (15)\\
&+ \beta\pi\left(E^k, E^k\right) V^{\phi}_{E^k,i}(\phi_{E^k}(s)).
\end{aligned}$$

By induction, we know $\phi_{E'}$ and $V^{\phi}_{E'}(\phi_{E'}(s))$ for $E' \in \{E^{k+1},\ldots E^h\}$. If suffices, therefore, to show that there exists mapping ϕ_{E^k} such that continuation values $\left\{V^{\phi}_{E^k,i}(s)\right\}_{s\in S}$, determined from (15), would make ϕ_{E^k} a MVE. Denote

$$\tilde{u}_{E^k,i}(s) = u_{E^k,i}(s) + \beta_{E^k} \sum_{E' \in \{E^{k+1},\ldots E^h\}} \pi\left(E^k, E'\right) V^{\phi}_{E',j}\left(\phi_{E'}(s)\right), \tag{16}$$

$$\tilde{\beta} = \beta\pi\left(E^k, E^k\right) \tag{17}$$

Then equation (15) may be rewritten as

$$V^{\phi}_{E^k,i}(s) = \tilde{u}_{E^k,i}(s) + \tilde{\beta} V^{\phi}_{E^k,i}\left(\phi_{E^k}(s)\right). \tag{18}$$

Notice that $\left\{\tilde{u}_{E^k,i}(s)\right\}_{i\in N}^{s\in S}$ satisfy ID; this follows from Lemma 5 and from the additivity of the ID property. We also have that $\tilde{\beta} \in [0,1)$. This means that if we consider a game without shocks, with the environment given by

$$E = \left(N, S, \tilde{\beta}, \left\{\tilde{u}_{E^k,i}(s)\right\}_{i\in N}^{s\in S}, \left\{W_{E^k,s}\right\}_{s\in S}, \left\{F_{E^k}(s)\right\}_{s\in S}\right), \tag{19}$$

then the recursive equation for continuation values under the transition mapping ϕ_{E^k} would be given precisely by (18). This makes Theorem 1 applicable; therefore, there is a transition mapping ϕ_{E^k} which constitutes a MVE in the environment E. But then by definition of MVE, since $\{\phi_E\}_{E\in\{E^{k+1},\ldots,E^h\}}$ was MVE, we have that $\{\phi_E\}_{E\in\{E^k,\ldots E^h\}}$ is MVE in the environments $\left\{E^k,\ldots,E^h\right\}$. This proves the induction step, and proceeding likewise, we can obtain the entire MVE $\phi = \{\phi_E\}_{E\in\{E^1,\ldots E^h\}}$.

Notice that this reasoning used backward induction, and thus Assumption 1 was indispensable. We have proved the following result.

Theorem 3 ***(Existence)*** *Suppose that all environments $E \in \mathcal{E}$ satisfy the assumptions of the paper, and Assumption 1 holds. Then there is a MVE $\phi = \{\phi_E\}_{E\in\mathcal{E}}$. The evolution $s_0, s_1 = \phi_{E^1}(s_0), s_2 = \phi_{E^2}(s_1), \ldots$ results in a limit state $s_\tau = s_{\tau+1} = \ldots = s_\infty$, but need not be monotone. The limit state may depend on the time of arrival of shocks.*

Now that we have established existence of MVE in a stochastic environment, a natural question is whether or not it is unique. Our approach to this question is similar: using backward induction, we reduce the problem to studying uniqueness of MVE in the environment E given by (19), where the utilities are given by (16) and the discount factor is given by (17). However, this is not straightforward, because single-crossing condition need not be preserved for continuation utilities, as the next example shows (and it also need not be additive).

Example 4 (Continuation utilities need not satisfy single-peakedness) There are four states and three players, player 1 is the dictator in state A, player 2 is the dictator in state B, and player 3 is the dictator in states C and D. The payoffs are given by the following matrix:

$$\begin{array}{ccccc} id & A & B & C & D \\ 1 & 20 & 30 & 90 & 30 \\ 2 & 5 & 20 & 85 & 90 \\ 3 & 5 & 25 & 92 & 99 \end{array}.$$

All payoffs are single-peaked. Suppose $\beta = 0.5$; then the unique equilibrium has $\phi(A) = C$, $\phi(B) = \phi(C) = \phi(D) = D$. Let us compute the continuation payoffs of player 1. We have: $V_1(A) = 40$, $V_1(B) = 30$, $V_1(C) = 50$, $V_1(D) = 30$; the continuation utility of player 1 is thus not single-peaked.

Nevertheless, we can establish uniqueness under one of the following conditions.

Theorem 4 ***(Uniqueness)*** *The monotone MVE is (generically) unique if at least one of the following conditions holds:*

1. *for every environment $E \in \mathcal{E}$ and any state $s \in S$, M_s is a singleton;*
2. *in each environment, only one-step transitions are possible; each player's preferences are single-peaked; and, moreover, for each state s there is a player i such that $i \in M_{E,s}$ for all $E \in \mathcal{E}$ and the peaks (for all $E \in \mathcal{E}$) of i's preferences do not lie on different sides on s.*

The first case is the same as in the stationary environment studied above. The second is more demanding, but nevertheless worth stating. The last complex condition holds automatically if political rules do not change as a result of shocks, and neither do players' ideal states under each environment.

3.3 Noncooperative game

So far, we have not specified a noncooperative game which would substantiate MVE. We do so in this section, and first we describe the game fully. For all environments $E \in \mathcal{E}$ and states $s \in S$, we introduce a protocol $\theta_{E,s}$, which is a finite sequence of all states in $S \setminus \{s\}$.

1. The environment E_{t-1} and state s_{t-1} are inherited from period $t-1$.

2. Shock which determines E_t may occur: $E_t = E \in \mathcal{E}$ with probability $\pi(E_{t-1}, E)$,.

3. Let $b_1 = s_{t-1}$. In the subsequent stages, alternative b_j, $j = 1, \dots, m-1$, is voted against $\theta_{E_t, s_{t-1}}(j)$. That is, all agents are ordered in a sequence and must support either b_j or $\theta_{E_t, s_{t-1}}(j)$. If the set of those who supported $\theta_{E_t, s_{t-1}}(j)$ is a winning coalition, i.e., is in $W_{E_t, s_{t-1}}$, then $b_{j+1} = \theta_{E_t, s_{t-1}}(j)$; otherwise, $b_{j+1} = b_j$. When all alternatives have been voted, the new state is $s_t = b_m$.

4. Each individual gets instantaneous payoff given by (5).

We study Markov Perfect equilibria of this game.[13] Naturally, with every MPE, a set of transition mappings $\phi = \{\phi_E\}_{E \in \mathcal{E}}$ is associated: $\phi_E(s)$ is the state with which period which started with state s_{t-1} and where there was a shock leading to state E ends. We can get the following results.

Theorem 5 *The following is true:*

1. *For any MVE ϕ (monotone or not) there exists a set of protocols $\{\theta_{E,s}\}_{E \in \mathcal{E}}^{s \in S}$ such that there exists a Markov Perfect equilibrium of the game above which implements ϕ. Moreover, if ϕ is the unique MVE, then protocol*

$$\{\theta_{E,s}(j)\}_{j=1}^{m-1} = (s+1, s+2, \dots, m, s-1, s-2, \dots, 1) \tag{20}$$

   may be used;

2. *Conversely, if for some set of protocols $\{\theta_{E,s}\}_{E \in \mathcal{E}}^{s \in S}$ and some MPE σ, the corresponding transition mapping $\phi = \{\phi_E\}_{E \in \mathcal{E}}$ is monotone, then it is MVE.*

3. *Under either of the assumptions of Theorem 9, a nonmonotone MPE cannot exist for any set of protocols.*

This theorem establishes the relation between the cooperative and noncooperative approaches. On the one hand, any MVE may be made an MPE of the game, if a protocol is taken appropriately. If the equilibrium is unique, such protocol is easy to describe, and one

[13] To avoid the usual problems with equilibria in voting games, we assume sequential voting for some fixed sequence of players. See Acemoglu, Egorov, and Sonin (2009) for a solution concept which would refine out unnatural equilibria in voting games with simultaneous voting.

possible variant is given by (20). In fact, a stronger result is true: the protocol given by (20) always has a monotone MPE. On the other hand, an MPE gives rise to an MVE, provided that the transition mapping is monotone. Part 3 of Theorem 5 gives sufficient conditions which ensure that any MPE is monotone.

3.4 Comparative statics

In this section, we compare different environments, and study properties that must hold. Comparative statics results are strongest when equilibrium is unique; hence, throughout this section, we assume that either of Theorem 4, which guarantee uniqueness of MVE, holds. We also assume that parameter values are generic.

Definition 6 *We say that environments E^1 and E^2, defined for the same set of players and set of states, coincide on $S' \subset S$, if for each $i \in N$ and for any states $x \in S'$, $u_{E^1,i}(x) = u_{E^2,i}(x)$, $W_{E^1,x} = W_{E^2,x}$, and also $F_{E^1}|_{S'} = F_{E^2}|_{S'}$ (in the sense that for $x, y \in S'$, $y \in F_{E^1}(x) \Leftrightarrow y \in F_{E^2}(x)$) and $\beta_{E^1} = \beta_{E^2}$.*

Our next result shows that in two environments, E^1 and E^2 that coincide on a subset of states (and differ arbitrarily on other states), there is a simple way of characterizing the transition mapping of one environment at the steady state of the other.

Theorem 6 *Suppose that environments E^1 and E^2 coincide on $S' = [1, s] \subset S$. Suppose that for some MVE ϕ_1 in E^1, $\phi_1(x) = x$. Then there exists MVE ϕ_2 in E^2 such that $\phi_2(x) \geq x$. Moreover, if $\phi_1(x) = x$ holds for any MVE ϕ_1 in E^1, then $\phi_2(x) \geq x$ for every MVE ϕ_2 in E^2.*

Intuitively, the theorem says that if x is a steady state (limit state) in environment E^1 and environments E^1 and E^2 coincide on a subset of states $[1, s]$ that includes x, then in an environment the MVE in E^2 will necessarily involve a transition to a higher stakes than x.

The reason why this is so (and the idea of the proof) is simple. To explain it, let us introduce the notation $\phi|_{S'}$ to represent the transition function ϕ restricted to the subset of states S'. Now, if we had that $\phi_2(x) < x$, then $\phi_1|_{S'}$ and $\phi_2|_{S'}$ would be two different mappings both of which would be MVEs on S'. But this would contradict the uniqueness of MVE. Of course, if $y \in S'$ is such that $\phi_2(y) = y$, then $\phi_1(y) \geq y$. This proposition does not say anything about the existence of a steady state in S' for either of the mappings; nevertheless, if such a steady state does exist, it must either be a steady state also for the other mapping, or the other mapping should move it right. Obviously, these results generalize for the case where $S' = [s, m]$ rather than $[1, s]$.

Theorem 6 compares MVEs in two distinct environments. In this sense, we can think of it as a comparative static with respect to an unanticipated shock (taking us from one to the other environment). We can also derives a similar result when there is a stochastic transition from one environment to another. This is done in the next corollary.

Corollary 1 *Suppose that $\mathcal{E} = \left\{E^1, E^2\right\}$ and, furthermore, that E^1 and E^2 coincide on $S' = [1, s] \subset S$. Suppose that, for some MVE ϕ_{E^1} in E^1 and some $x \in S'$, $\phi_{E^1}(x) = x$. Suppose also that this state x is reached before the shock arrives at time t. Then under some MVE, for all $\tau \geq t$, $s_\tau \geq x$. Moreover, if $\phi_{E^1}(x) = x$ for all MVEs ϕ_{E^1} in E^1, then $s_\tau \geq x$ for all $\tau \geq t$ in any MVE.*

Suppose that before the shock, the society had found itself in a steady state, and as a result of the shock, only higher states were affected (agents' utilities, sets of winning coalitions, or feasibility of transitions could change). Corollary 1 implies that this could only make the society move towards the direction where shock happened or stay where it was. In other words, the only possibility for the society to stay in the region $[1, x-1]$ is not to leave it before the shock arrives.

To understand the logic of this corollary, suppose that preferences change in such a way that utilities of all agents become higher in some state on the right of where the society is before the shock. Then it would be intuitive that transitions take place towards that state. But in contrast, the corollary implies that even when the utilities of all agents become lower in one of the states on the right, the society may still decide to move to the right. Intuitively, it is possible that some transition to the right would benefit the current decision-makers, but the possibility of further transitions to the right made them prefer the status quo. The shock removed this last threat by making it empty, and now the society may be willing to make a transition to the right. Of course, it is possible that the society will stay where it was; this would be the case, for example, if the shock was minor.

One implication of this corollary can be derived by considering x as a stable democracy and states to the left of x as less democratic states (states to the right might correspond to further developments of democracy or other refinements). Then the caller implies that certain types of shocks may disrupt the emergence of a stable democracy if they arrive early. But if they arrive late, after this stable democratic state has already been reached, they would not create a reversal (though they may act as an impetus for additional transitions in a further democratic direction).

Corollary 1 was formulated under the assumption that stable state x was reached before the shock occurred. The next result removes this constraint, but only under the assumption that the discount factor is low enough, i.e., that players are sufficiently myopic.

Theorem 7 *Suppose that $\mathcal{E} = \{E^1, E^2\}$, $0 < \pi\left(E^1, E^2\right) < 1$, $\pi\left(E^2, E^1\right) = 0$, and E^1 and E^2 coincide on $S' = [1, s] \subset S$. Then there exists $\beta_0 > 0$ such that if $\beta_{E^1} = \beta_{E^2} < \beta_0$, then in the unique MVE ϕ, if the initial state is $s_0 \in S'$ such that $\phi_{E^1}(s_0) \geq s_0$, then the entire path $s_0, s_1, s_2, \ldots$ (induced both under environment E^1 and after the switch to E^2) is monotone. Moreover, if the shock arrives at time t, then for all $\tau \geq t$, $s_\tau \geq \tilde{s}_\tau$, where $\tilde{s}_\tau$ is the hypothetical path if the shock never arrives.*

In a monotone MVE, equilibrium paths are monotone without shocks. But with shocks, this is no longer true, because the arrival of the shock can change the direction of the path. This theorem shows that when the discount factor is sufficiently low and two environments coincide on a subset of states, then the equilibrium path is monotone even with shocks. Moreover, we can also establish a ranking between the equilibrium path with and without the shock.

In addition, if we specify what the shock changes, we can also derive additional results on the dynamics of equilibrium paths, which is done in the next theorem for the case in which shocks change the set of quasi-median voters — i.e., it changes the distribution of political power in a specific way.

Theorem 8 *Suppose that environments E^1 and E^2 have the same payoffs, $u_{E^1,i}(x) = u_{E^2,i}(x)$, the same discount factors $\beta_{E^1} = \beta_{E^2}$, that the same transitions are feasible ($F_{E^1} = F_{E^2}$) and suppose that $M_{E^1,x} = M_{E^2,x}$ for $x \in [1, s]$ and $\min M_{E^1,x} = \min M_{E^2,x}$ for $x \in [s+1, m]$. Then for any MVE ϕ_1 in E^1 there is MVE ϕ_2 in E^2 such that $\phi_1(x) = \phi_2(x)$ for any $x \in [1, s]$.*

This result suggests that if in some right states the sets of winning coalition change in a way that the sets of quasi-median voters change on the right without changes on the left (for example, because some additional players on the right to veto players), then the mapping is unaffected for states on the left (i.e., those states that are not directly affected by the change). For example, applied to the dynamics of democratization, this theorem implies that an absolute monarch's decision of whether to move to a constitutional monarchy is not affected by the power that the poor will able to secure—provided that the middle class is still a powerful player.

This theorem implies the following corollary.

Corollary 2 *Let* $\mathcal{E} = \left\{E^1, \ldots, E^h\right\}$*. Suppose that:*

1. *for all environments* $E, E' \in \mathcal{E}$ *and for all states* $x \in S$ *and individuals* $i \in N$*, we have* $u_{E,i}(x) = u_{E',i}(x)$*,* $F_E = F_{E'}$*,* $\beta_E = \beta_{E'}$ *and* $\min M_{E,x} = \min M_{E',x}$*;*

2. *if* $x \in [1, s]$*, then* $\max M_{E,x} = \max M_{E',x}$*.*

Then there is an MVE $\phi = \{\phi_E\}_{E \in \mathcal{E}}$ *such that* $\phi_{E^1}(x) = \cdots = \phi_{E^h}(x)$ *for all* $x \in [1, s]$*. In this MVE, if* $s_0 \in [1, s]$ *and there is a stable* $x \in [s_0, s]$*, then arrival of shocks does not alter the equilibrium paths.*

We thus have identified a class of shocks which do not change the evolution of the game. A priori, one could imagine, for example, that if the poor lose the ability to protect democracy, and instead the elite will be able to stage a coup, then this consideration may affect the desire of the elite to extend the franchise and move to democracy, or that it may affect the desire of the monarch to grant more power to the broad elite in the first place. Corollary 2 suggests, however, that unless something else is going on, these shocks and these considerations alone are not sufficient to change the equilibrium path (unless, of course, the society starts the game in democracy with the elite dreaming of staging a coup).

4 Application: Repression and Radicalism

In this section, we apply our general framework and the results derived so far to the study of repression and radicalism.

4.1 Formal setup of a model of radical politics.

In what follows, we use the language and formalism of Section 2 and show how the model may be applied to study radical politics as in Example 1 in the introduction. We first describe the initial environment, E_1. There is a fixed set of n players $N = \{-l, \ldots, r\}$ (so $n = l+r+1$), which we interpret as groups of (potentially large numbers of) individuals with the same preferences (e.g., ethnicities or classes). We interpret the order of groups as representing some economic interests (e.g., from poor on the left to rich on the right). We assume that each group solve the collective action problem as all the members of the group have the same preferences.

The weight of each group $i \in N$ is denoted by γ_i and represents the number of people within the group, and thus the group's political power. Throughout this exercise, we assume genericity, in the sense that there are no two disjoint combinations of groups with exactly equal total number of individuals (see Acemoglu et al., 2008, for a discussion of this assumption). We assume that group 0 contains the median voter (we can always enumerate the groups to ensure this). Individuals in group i have an ideal bliss point b_i, where $\{b_i\}$ is increasing in i, and have preferences over policy given by

$$u_i(p) = -(p - b_i)^2 . \tag{21}$$

The set of states is $S = \{-l-r, \ldots, l+r\}$ (so the total number of states is $m = 2l+2r+1 = 2n-1$), and they correspond to different combinations of political rights. We think of repression as a way of reducing the political rights of certain groups. In particular, the set of players who are *not* repressed in state s is H_s, where $H_s = \{-l, \ldots, r+s\}$ for $s \leq 0$ and $H_s = \{-l+s, \ldots, r\}$ for $s > 0$[14]; those and only those who are not repressed are eligible to vote. This specification implies that states below 0 correspond to repressing the rich (in the leftmost state $s = -l-r$ only the group $-l$ participates in decision-making); similarly, states above 0 correspond to repressing the poor (again, the rightmost state $s = l + r$ on the group r has vote and state 1 involves repressing just the most extreme left group, $-l$). The middle state $s = 0$ involves no repression and corresponds to full democracy (where again notation is chosen in such a way that group 0 contains the median voter and rules in state 0).

All decisions in the society are made by a simple majority of those currently empowered. This applies to transitions between states as well as to policy decisions within a given state

[14]We can allow for "partial" repression, where the votes of players who are repressed are discounted with some factor. This would correspond to, say, repressing only a certain fraction of some population group. Ultimately, this would complicate the notation without delivering major insights.

We can also allow for repressing any subsets of groups, thus having to consider $2^n - 1$ rather than $2n - 1$ states (see Acemoglu et al., 2012 [AER PAPER] for a model of repression of religious or secular groups where this is allowed). To save on notation and focus on substantive issues related to shocks, we drop this possibility here.

(in the latter case, we do not model it explicitly, as there is a Condorcet-winning policy b_{M_s}, where M_s is the group in state s where the effective median voter is located; hence, a reasonable decision-making mechanism would deliver this policy b_{M_s}). Furthermore, we can assume that repression is costly: repressing agents of type j for one period costs C_j, and that this cost is incurred by all agents within society (e.g., economic efficiency is reduced or taxes have to be raised to support repression). For simplicity, we assume that the radical group $-l$ is smaller than the next group: $\gamma_{-l} < \gamma_{-l+1}$. In this case radicals cannot ensure their preferred policy unless all other groups are repressed. (This is not an important assumption, but it is realistic and simplifies the statements of results below.)

More formally, in state s, coalition X is winning if and only if

$$\sum\nolimits_{i \in H_s \cap X} \gamma_i > \frac{1}{2} \sum\nolimits_{i \in H_s} \gamma_i. \tag{22}$$

For generic sequences of proportions across groups, $\{\gamma_i\}$, there is one group, which is the quasi-median voter as defined above. In this section, we will, interchangeably, refer to a quasi-median voter (QMV), median voter or quasi-median group.

Given the above description, the stage payoff of individuals in group i in state s can be written as:

$$u_i(s) = -\left(b_{M_s} - b_i\right)^2 - \sum\nolimits_{j \notin H_s} \gamma_j C_j. \tag{23}$$

Until the last result we present in this section, the reader may focus on the case in which $C_j = C$ for all $j \in N$.

One can easily check that all assumptions are satisfied in this environment. In particular, the single-crossing assumption holds, as the sequence $\{b_i\}$ is increasing in i.

It is possible that a radical will come to power. We model this by assuming that there is a set of h "radical" environments $R_{-l-r}, \ldots, R_{-l-r+h-1}$, and probability $\lambda_j \in [0,1]$ that society will transition to radical environment $j = 1, \ldots, m$ starting from E. Environment R_j is the same as E, except that in environment R_j, if the current state is one of $-l-r, \ldots, j$, the decision-making rule comes into the hands of the most radical group $-l$. In other words, in radical environment R_j, radicals can grab power momentarily in space to the left of j. In the context of the Bolshevik Revolution, this would correspond to assuming that in some possible environments, Bolsheviks would have been able to grab control when Kerensky was in power but not when some other further right government was in power. This formulation implies that the probability of a radical coming to power if the current state is s is $\mu_s = \sum_{j=-l-r}^{s} \lambda_j$, and it is (weakly) increasing in s. Note also that the radical environment creates an opportunity for the radicals to grab power. To do so, they will also need to repress the rest of the population (which is potentially costly for them as specified above).

We can also contrast between permanent and transitory shocks: let us assume that in each period in any of the environments R_j, there is a chance ν of returning back, which we model as transition to the final environment L identical to E in terms of payoffs and winning coalitions. In this way, $\nu = 0$ will correspond to a permanent shock, and as ν increases, the tenure of the radical will become shorter.

4.2 Characterization and comparative statics

We first study the benchmark case if there are no radicals, and then see the difference that radicals bring.

Proposition 1 ***(Equilibria without radicals)*** *In the absence of shocks (i.e., environment L never changes), there exists a unique MVE given by a function $\phi_L : S \to S$. In this equilibrium:*

1. *Democracy is stable: $\phi_L(0) = 0$.*
2. *For any costs of repression $\{C_j\}_{j \in N}$, the equilibrium state/institution never features more repression than the initial one: if $s < 0$ then $\phi_L(s) \in [-s, 0]$, and if $s > 0$, then $\phi_L(s) \in [0, s]$.*
3. *Consider repression costs parametrized by k: $C_j = kC_j^*$, where $\left\{C_j^*\right\}$ are positive constants. Then there is $k^* > 0$ such that: if $k > k^*$, then $\phi_L(s) = 0$ for all s, and if $k < k^*$, then $\phi_L(s) \neq 0$ for some s.*

This proposition implies that if radicals are never a concern, democracy is stable. Moreover, repression will not arise if it was not present from the beginning, and if it was, then there will not be more repression and, perhaps, there will be less. Intuitively, the quasi-median voter can choose the preferred policy anyway, and since repression is costly, choosing more of it makes no sense. On the other hand, having less repression may change the balance of power, but it may be worth it if the savings are sufficient.

In addition, high cost of repression makes the society immediately transit to full democracy, but if repressions are not costly, this will not happen (at least not immediately).

Now suppose the radicals are a concern. In what follows, ϕ denotes the initial environment.

Proposition 2 ***(Radicals)*** *There exists a unique MVE. Suppose when the society is at state s, there is a transition to environment R_z happens (where $z \geq s$) so that radicals can grab power. Then the radicals are more likely to move to their preferred state $-l-r$ if: (a) they are more radical (meaning their ideal point b_{-l} is lower, i.e., further from 0); (b) they are "weaker" (i.e., z is smaller) in the sense that there is a smaller set of states in which they are able to control power.*

This proposition implies that when the opportunity arises, radicals are more likely to move to a situation in which they repress all other groups in society, thus moving to state $-l-r$, when their preferences are more extreme (because in that case the outcome without such repression would be unsatisfactory for them). The same is also true when they are "weaker". This is because when z is lower and they are weaker in this sense, there is a greater range of states such that once there is a transition to one of these states, even if the environment stays at R_z, they will lose the ability to repress other groups and secure outcome that they prefer; this in turn encourages them to repress all other groups in society and move to state $-l-r$.

The next proposition discusses the implications of the possibility of radicals coming to power on the political equilibrium before it happens. To formulate it, denote the expected continuation utility of group $i \in N$ from staying in state $s \in S$ until the shock, and then following the equilibrium play:

$$W_i(s) = u_i(s) + \beta \sum_{z=-l-r}^{-l-r+h-1} \lambda_z V_{R_z,i}(s).$$

Proposition 3 ***(Repression by moderates anticipating radicals)*** *The transition mapping before radicals come to power, ϕ_E, satisfies the following properties.*

1. *If $s \leq 0$, then $\phi_E(s) \geq s$.*

2. *If $W_0(0) < W_0(s)$ for some $s > 0$, then there is a state $x \geq 0$ such that $\phi_E(s) > s$. In other words, in some state there is an increase in repression in order to decrease the chance of radical coming to power.*[15]

[15] Note that we cannot say that this state is 0, because 0 may be made stable by slippery slope.

3. *If for all states $y > x \geq 0$, $W_{M_x}(y) < W_{M_x}(x)$, then for all $s \geq 0$, $\phi_E(s) \leq s$. In other words, in this case an increase in repression of the left does not happen. The condition is more likely to be satisfied if costs C are high enough (k is high enough).*

The first statement of this Proposition is quite strong. It says that if the median voter would prefer a more repressive state *if he could ensure no further repression unless radicals come to power* (which he cannot do because he is not in control in that state), then there is some state starting from which there will be an increase in repressions against the left. This is a non-trivial result: e.g., slippery slope considerations may make state 0 stable. Nevertheless, starting from some state the society will see an increase in repressions against the left. The second part gives a sufficient condition for the opposite (and thus a necessary condition for an increase in repression). Finally, the third part says that if radicals move to their preferred state immediately, it does not matter whether the shock is permanent or transitory.

The next result compares the transition in anticipation of radicals (environment E) and in the case where radicals are gone—or, equivalently, if they are impossible (environment L).

Proposition 4 *Either $\phi_E|_{[0,l+r]} = \phi_L|_{[0,l+r]}$, or there is $s \geq 0$ such that $\phi_E(x) = \phi_L(x)$ for $0 \leq x < s$ and $\phi_E(x) > \phi_L(x)$.*

The first part implies that anticipation of radicals leads to weakly more repression, at least starting from a sufficiently "democratic" state. It says that the lowest states $s \geq 0$ where the society's decision before and after the radicals would differ necessarily implies a relative increase in repression in anticipation of radicals. This result need not generalize to all states $s > 0$ due to slippery slope considerations: if for some s the society becomes afraid of radicals enough so that this prevents further democratization, this state may become more attractive to right-wing radical groups, who may then decrease repression. Of course, the first option (that the two mappings are identical) is also a possibility; this would happen, e.g., if radicals are sufficiently unlikely.

The next result deals with stability of democracy.

Proposition 5 ***(Stability of democracy without a threat of radical)*** *Suppose that full democracy $s = 0$ (or more generally, any state which is most favorite to the decision-maker there) does not allow for radicals coming to power (has $\mu_s = 0$). Then this state is stable in* *all environments, and any state $s > 0$ will lead to (weakly) less repression, in the sense that $\phi_E(s) \in [0, s]$ for $s > 0$.*

Proposition 5 shows that if full democracy does not allow for radicals, then it is stable regardless of possibility of radicals in other, less democratic states.

What if radicals can come to power in any state — or, more generally, what can we say about a state which is stable in the environment E while nevertheless facing the possibility of a radical? With the help of Theorem 6, we get the following results.

Proposition 6 ***(Less repressions)*** *Suppose that there is a state $s \geq 0$ (i.e., full democracy or some state favoring the right), which is stable in E for some set of probabilities $\{\mu_j\}$. Let us change $\{\mu_j\}$ to $\left\{\mu_j'\right\}$ such that $\mu_j' = \mu_j$ for $j \geq s$. In this case, there be (weakly) less repression of the left: $\phi_E'(s) \geq \phi_E(s) = s$.*

This is a corollary of a general comparative statics results, but it is an interesting result on its own. It says that if the society manages to change the probability of a radical coming to power in elite-controlled states (e.g., through better repression technologies), although not in the current state, then there may only be an increase, but never a decrease in repression of radicals. This will happen regardless of the direction of the change. The intuition is simple: if there is a lower chance of a radical coming to power, then these states become more attractive to groups richer than the median. But even if this chance increases, it is possible that repression may increase: while the pro-elite states do not become more attractive, it is possible that some of them become stable. Hence, such a change may alleviate the concerns for a slippery slope.

Proposition 7 ***(Role of radicals in history)*** *Suppose the society was in a stable state $x \geq 0$ (in environment E) before the radical came to power. Then the ultimate state, after the radical comes and possibly goes, will be some $y \leq x$.*

In other words, the history of having radicals radicals can only contribute to dismantling of pro-elite institutions. This may happen in two ways. First, the radical may lock in power forever. Second, he might decide not to do that (e.g., because it is costly, and leave the power eventually). The proposition says that after he leaves, i.e., in the absence of a threat from radical, the society will not increase repression of the left (by moving from $x \geq 0$ to some $y > x$) than it was. (Another way to say it: if the society wanted to move to the right of x before after the threat is gone, it would do so before the radical a fortiori.) (Notice that the previous result (Proposition 6) showed that the change in the *threat* of radicals can actually make the regime more pro-elite.)

In the next result, we apply Throem 8 to show that all the results come from radicals grabbing power rather than just becoming influential enough to become veto players. Namely, suppose that shocks merely make radicals veto players rather than give all the power to them.

Proposition 8 ***(Radicals as veto players)*** *If shocks make radicals veto players while preserving democratic decision-making, then mapping $\phi(s)|_{s\geq 0}$ is the same as in the benchmark case where the initial environment is stable.*

Notice that this result is not 100% trivial. Indeed, the mappings may differ for $s < 0$, as the radical may prevent against the right groups from going away. This change in the mapping $\phi_E(s)|_{s<0}$ could in principle affect the incentives of decision-makers in states $s \geq 0$. However, we show that this does not happen. The possibility of radicals becoming veto players never leads to repression against left groups.

Our last result deals with strategic complementarity of repressions. Granted, if repressing left-wing groups becomes easier, then precautionary repression become more likely. But it turns out that even if repression by left-wing group becomes more likely, so does precaurionary repression. Intuitively, the more the median voter is afraid that a radical will seize power forever, the more he wants to prevent it. In other words, repressions feature the following strategic complementarity: the ease of repressing one part of political spectrum leads to more repression of the other part.

Formally, we have the following result. Consider the same environment, but suppose that the cost of repressing right-wing groups decreases for the most radical group $-l$ in all environments. More precisely, in all environments, the utility function of radicals in states where right-wing groups are repressed changes: For $s < 0$, we have

$$u_{-l}(s) = -(b_{M_s} - b_{-l})^2 - \rho \sum_{j \notin H_s} \gamma_j C_j,$$

where $\rho \in [0,1]$ ($\rho = 1$ corresponds to the previous case, and a lower ρ means that the radicals are more tolerant to repressing right-wing groups. It is easy to check that such environments satisfy the properties we require, including the increasing differences one. We have the following result.

Proposition 9 ***(Strategic Complementarity)*** *Suppose ρ decreases. Then it becomes more likely that $\phi_E(s) > s$ for at least one $s \geq 0$.*

5 Extensions

In this section, we relax some of the assumption made in Section 2. First, in Subsection 5.1, we formulate the (simple and relatively mild) conditions under which all MVE are monotone. This justifies our focus on monotone MVE in the first place. Then, in Subsection 5.2, we show how this paper generalizes Roberts (1999) on voting in clubs, which would suggest that this framework may be useful for club theory with dynamic collective decision-making and stochastic changes in power and/or preferences. After that, in Subsection 5.3, we study the possibility that there is an infinite number (a continuum) of states and/or agents and establish an existence of MVE.[16] Finally, we show that our approach to stochastic shocks allows for studying situations where the probabilities of transitions, $\pi(E, E')$, may depend on the state of the world. This would be realistic, for example, when studying political experimentation.

5.1 Monotone vs nonmonotone MVE

So far, we focused on monotone MVE. In many interesting cases this is without loss of generatlity, as the following theorem establishes.

Theorem 9 *The following are sufficient conditions for any MVE ϕ to be (generically) monotone:*

1. *In all environments, the sets of quasi-median voters in two different states have either none or exactly one individual in common: $\forall E \in \mathcal{E}, x, y \in S : x \neq y \Rightarrow |M_{E,x} \cap M_{E,y}| \leq 1$.*
2. *In all environments, only one-step transitions are possible.*

This theorem is quite general. Part 1 covers, in particular, all situations where the sets of quasi-median voters are singletons in all states. This implies that whenever in each state there is

[16] An earlier version also allowed for the possibility of an infinte number of shocks; we proved the existence of a "mixed-strategy" MVE and provided an example where MVE, as defined in Definition 3, does not exist. The earlier version also had a brief treatment of other possible shocks, like strategic political experimentation. This goes beyond the scope of the paper and is omitted.

a dictator (which may be the same for several states), or there is majority voting among sets of odd numbers of players, any MVE is monotone, and thus all results in the paper are applicable to all MVEs. The second part suggests that if only one-step transitions, i.e., transitions to adjacent states are possible, then again any MVE is monotone. This means that our focus on monotone MVE is without any loss of generality for many interesting and relevant cases. Also, coupled with the result that monotone MVE always exist, this justifies our focus on monotone MVE even if the conditions of Theorem 9 fail.

In addition, inspection of the conditions in Theorem 9 reveals that they are weaker than conditions in Theorem 2 and 4. Consequently, when these theorems guarantee the uniqueness of a monotone MVE, they, in fact, guarantee that it is unique in the class of all possible MVEs. Moreover, if MVE is unique, it is monotone.

The next example shows that nonmonotone MVE are possible, if both conditions in Theorem 9 fail.

Example 5 There are three states A, B, C, and two players 1 and 2. The decision-making rule is unanimity in all states. Payoffs are given by

id	A	B	C
1	30	50	40
2	10	40	50

Suppose β is relatively close to 1, e.g., $\beta = 0.9$. Then there is a nonmonotone MVE $\phi(A) = \phi(C) = C$, $\phi(B) = B$. (There is also a monotone equilibrium with $\phi(A) = \phi(B) = B$, $\phi(C) = C$.)

The next example shows that genericity is also an important requirement.

Example 6 (Example with nonmonotone equilibrium due to nongeneric preferences.) There are two states A and B and two players 1 and 2. Player 1 is the dictator in both stattes. Payoffs are given by

id	A	B
1	20	20
2	15	25

Take any discount factor β, e.g., $\beta = 0.5$, and any protocol. Then ϕ given by $\phi(A) = B$ and $\phi(B) = A$ is nonmonotone (in fact, cyclic). This equilibrium is only possible for measure 0 of preferences.

However, even if nonmonotone MVE exist, they still possess a certain degree of monotonicity, namely, "monotone paths", as the next results show.

Definition 7 *A mapping $\phi = \{\phi_E\}_{E\in\mathcal{E}}$ has* monotone paths *if for any $E \in \mathcal{E}$ and $x \in S$, $\phi(x) \geq x$ implies $\phi_E^2(x) \geq \phi_E(x)$.*

In other words, all equilibrium paths that this mapping generates, as long as the environment does not change, are weakly monotone. We have the following result:

Theorem 10 *Any MVE ϕ (not necessarily monotone) has, generically, monotone paths.*

5.2 Relationship to Roberts's model

As discussed in the Introduction, our paper is most closely related to Roberts (1999). Our notion of MVE extends that of Roberts, who also looks at a dynamic equilibrium in an environment that satisfies single-crossing type restrictions. More specifically, in Roberts's model, the society consists of n individuals, and there are n possible states $s_k = \{1, \ldots, k\}$, $1 \leq k \leq n$. Each state s_k describes the situation where individuals $\{1, \ldots, k\}$ are members of the club, while others are not. There is the following condition on individual payoffs:

$$\text{for all } l > k \text{ and } j > i, \;\; u_j(s_l) - u_j(s_k) > u_i(s_l) - u_i(s_k), \tag{24}$$

which is the same as the strict increasing differences condition we imposed above (Definition 1).

Roberts (1999) focuses on deterministic environments with majoritarian voting among club members. He then looks at a notion of Markov Voting Equilibrium (defined as an equilibrium path where there is a transition to a new club whenever there is an absolute majority in favor of it) and a median voter rule (defined as an equilibrium path where at each point current median voter chooses the transition for the next step). Roberts proves existence for mixed-strategy equilibria for each of the voting rules; they define the same set of clubs that are stable under these rules.

Roberts's notion of Markov Voting Equilibrium is closely related to ours, only differing from ours in the way that he treats clubs that have even numbers of members and thus might have ties. In any case, the two notions coincide for generic preferences.

This description clarifies that Roberts's setup is essentially a special case of what we have considered so far. The dimensions in which our framework is more general are several: First, Roberts focuses on the deterministic and stationary environment, whereas we allow for nonstationary elements and arbitrary stochastic shocks. Second, we allow for a much richer set of states and richer distributions of political power across states (e.g., instead of majority rule, we allow weighted supermajority rule, which could be different across states, or dictatorial rule; we also allow for limited transition rules such as one-step transitions whereas Roberts does not). Third, we prove existence of pure strategy equilibria and provide conditions for uniqueness. Fourth, we provide general conditions for equilibria to be characterized by monotone transition maps and to exhibit monotone sample paths. Fifth, we show the relationship between this equilibrium concept and MPE of a fully specified dynamic game.

Most importantly, however, we provide a fairly complete characterization of the structure of transitions and steady states, and show that at this level of generality, the range of comparative static results can be derived. Such comparative statics are not only new but, thanks to their generality, quite widely applicable. We also show how the framework can be applied in a somewhat more specific but still general analysis of dynamics of political rights and repression, and derive additional results in this context.

5.3 Continuous space

Here, we assume that states, and perhaps individuals, are taken from a continuous set. We study Markov Voting equilibria in such environments. Namely, we study Markov Voting equilibria in discrete environments obtained by sampling a sufficiently dense but finite set of points.

More precisely, assume that the set of states is $S = [s_l, s_h]$, and the set of individuals is given by a unit continuum $N = [i_l, i_h]$. (The construction and reasoning below are easily extendable to the case where the are a finite number of individuals but a continuum of states, or vice versa.) We assume that each individual has a utility function $u_i(s) : S \to \mathbb{R}$, which is continuous as a function of $(i, s) \in N \times S$ and satisfies the SID condition: for all $i > j$, $x > y$,

$$u_i(x) - u_i(y) > u_j(x) - u_j(y). \tag{25}$$

The mapping F is assumed to be upper-hemicontinuous on S and to satisfy Assumption 5. Finally, for each state s there is a set of winning coalitions W_s, which are assumed to satisfy Assumption 3. As before, for each state s, we have a non-empty set of quasi-median voters M_s (which may nevertheless be a singleton). We make the following monotonicity of quasi-median voters assumption: functions $\inf M_s$ and $\sup M_s$ are continuous and increasing functions of s.

For simplicity, we assume there are no shocks, so the environment is fixed. Time is discrete as before. We are interested in monotone transition functions $\phi : S \to S$; however, we do not impose additional restrictions, e.g., continuity (it may be possible that there is no equilibrium with a continuous transition function). The notions of MVE is the same as before, i.e., it is given by Definition 3.

In this environment, we can establish the following existence result.

Theorem 11 ***(Existence)*** *In the environment with continous set of states and/or continuous set of individuals, there exists a MVE ϕ. Moreover, take any sequence of sets of states $S_1 \subset S_2 \subset \cdots$ and any sequence of individuals $N_1 \subset N_2 \subset \cdots$ such that $\bigcup_{j=1}^{\infty} S_j$ is dense in S and $\bigcup_{j=1}^{\infty} N_j$ is dense in N. Consider any sequence of monotone functions $\left\{\phi_j : S_j \to S_j\right\}_{j=1}^{\infty}$ which are MVE (not necessarily unique) in the environment*

$$E_j = \left(N, S, \beta, \{u_i(s)\}_{i \in N_j}^{s \in S_j}, \{W_s\}_{s \in S_j}, \{F_j(s)\}_{s \in S_j}\right) \tag{26}$$

(existence of such MVE is guaranteed by Theorem 1, as all assumptions are satisfied). Then there is a subsequence $\{j_k\}_{k=1}^{\infty}$ such that $\left\{\phi_{j_k}\right\}_{k=1}^{\infty}$ converges, pointwisely on $\bigcup_{j=1}^{\infty} S_j$, to some MVE $\phi : S \to S$.

While this existence result does not characterize the set of equilibria in full, it guarantees existence, and also shows that a MVE may be found as a limit of equilibria for finite sets of states and individuals. The idea of the proof is simple. Take an increasing sequence of sets of states , $S_1 \subset S_2 \subset \cdots$ and an increasing sequence of sets of individuals $N_1 \subset N_2 \subset \cdots$ such that $S_\infty = \bigcup_{j=1}^{\infty} S_j$ is dense in S and $N_\infty = \bigcup_{j=1}^{\infty} N_j$ is dense in N. For each S_j, take MVE ϕ_j. We know that ϕ_i is a monotone function on S_i; Since let us complement it to a monotone (not necessarily continuous) function on S which we denote by $\tilde{\phi}_i$ for each i. Since S_∞ and N_∞

are countable, there is a subsequence ϕ_{j_k} which converges to some $\phi : S_\infty \to S_\infty$ pointwisely. (Indeed, we can pick a subsequence which converges on S_1, then a subsequence converging on S_2 etc; then use a diagonal process.) We then complete it to a function $\phi : S \to S$ by demanding that ϕ is either left-continuous or right-continuous at any point; in the Appendix, we show that we can do that so that the continuation values are either left-continuous or right-continuous as well). Then this continuity of continuation values will ensure that ϕ is MVE.

6 Conclusion

This paper has provided a general framework for the analysis of dynamics of institutional change (e.g., democratization, extension of political rights or repression), and how this interacts with (anticipated and unanticipated) changes in the distribution of political power and changes in economic structure (e.g., social mobility or other changes affecting individuals' preferences over different types of economic policies and allocations). We have focused on the Markov voting equilibria, which require that economic and political changes should take place if there exists a subset of players with the power to implement such changes and who will obtain higher expected continuation utility by doing so. Under the assumption that different economic and social institutions/policies as well as individual types can be ordered, and preferences and the distribution of political power satisfy "single crossing," we prove the existence of pure-strategy equilibria and provide conditions for their uniqueness.

Despite its generality, we have shown that the framework yields a number of comparative static results. For example, if there is a change from one environment to another (with different economic payoffs and distribution of political power) but the two environments coincide up to a certain state s' and before the change the steady state of equilibrium was that some state $x \leq s'$, then the new steady state that emerges after the change in environment can be no smaller than x. Another comparative static result is the following: again consider a change leaving preferences and the distribution of the power the same in states $s \leq s'$, but now arriving before the steady state $x \leq s'$ is reached. Then when all agents in society have discount factor sufficiently small (smaller than some threshold $\bar{\beta}$), the direction of changes states will remain the same as before (i.e., if there were transitions towards higher states before, this will continue, and vice versa). Finally, we have also shown that a change in environment makes extreme states "sticky" takes place away from these extreme states, then the equilibrium trajectory is not affected.

We have also shown how this framework can be applied to the study of radical politics and repression, and derived a range of additional comparative statics for this more specific application.

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Appendix

Proof of Lemma 1. "If": Suppose $M_s \subset P$, so for each $i \in M_s$, $w_i(y) > w_i(x)$. Consider two cases. If $y > x$, then ID implies that $w_j(y) > w_j(x)$ for all $j \geq \min M_s$, and such players j form a winning coalition by definition of QMV. If $y < x$, then, similarly, $w_j(y) > w_j(x)$ for all $j \leq \max M_s$, and thus for a winning coalition. In either case, P contains a subset (either $[\min M_s, n]$ or $[1, \max M_s]$) which is a winning coalition, and thus $P \in W_s$.

"Only if": Suppose $P \in W_s$. First, consider the case $y > x$. Let $i = \min P$; then ID implies that for all $j \geq i$, $w_j(y) > w_j(x)$. This means that $P = [i, n]$, and is thus a connected coalition. Since P is winning, we must have $j \in P$ for any $j \in M_s$, so $M_s \subset P$. The case where $y < x$ is similar, so $M_s \subset P$.

The proofs for relations $\geq, <, \leq$ are similar and are omitted. ■

Proof of Lemma 2. Part 1. Take $y > x$ and any $i \in N$. We have:

$$\begin{aligned} V_i^{\phi}(y) - V_i^{\phi}(x) &= u_i(y) + \sum\nolimits_{k=1}^{\infty} \beta^k u_i\left(\phi^k(y)\right) - u_i(x) - \sum\nolimits_{k=1}^{\infty} \beta^k u_i\left(\phi^k(x)\right) \\ &= (u_i(y) - u_i(x)) + \sum\nolimits_{k=1}^{\infty} \beta^k \left(u_i\left(\phi^k(y)\right) - u_i\left(\phi^k(x)\right)\right). \end{aligned} \tag{27}$$

The first term is (weakly) increasing in i if $\{u_i(s)\}_{i \in N}^{s \in S}$ satisfies ID, and the second is (weakly) increasing in i as $\phi^k(y) \geq \phi^k(x)$ for $k \geq 1$ due to monotonicity of ϕ. Consequently, (10) is (weakly) increasing in i.

Part 2. If ϕ is monotone, then Part 1 applies. Otherwise, for some $x < y$ we have $\phi(x) > \phi(y)$, and this means that $y = x + 1$; there may be one or more such pairs. Notice that for such x and y, we have

$$\begin{aligned} V_i^{\phi}(y) - V_i^{\phi}(x) &= \left(u_i(y) + \sum\nolimits_{k=1}^{\infty} \beta^{2k-1} u_i(x) + \sum\nolimits_{k=1}^{\infty} \beta^{2k} u_i(y)\right) \\ &\quad - \left(u_i(x) + \sum\nolimits_{k=1}^{\infty} \beta^{2k-1} u_i(y) + \sum\nolimits_{k=1}^{\infty} \beta^{2k} u_i(x)\right) \\ &= \frac{1}{1+\beta}(u_i(y) - u_i(x)). \end{aligned} \tag{28}$$

This is (weakly) increasing in i.

Let us now modify instantaneous payoffs and define

$$\tilde{u}_i(x) = \begin{cases} u_i(x) & \text{if } \phi(x) = x \text{ or } \phi^2(x) \neq x; \\ (1-\beta) V_i(x) & \text{if } \phi(x) \neq x = \phi^2(x). \end{cases} \tag{29}$$

Consider mapping $\tilde{\phi}$ given by

$$\tilde{\phi}(s) = \begin{cases} \phi(x) & \text{if } \phi(x) = x \text{ or } \phi^2(x) \neq x; \\ x & \text{if } \phi(x) \neq x = \phi^2(x). \end{cases} \tag{30}$$

Then $\tilde{\phi}$ is monotone and $\{\tilde{u}_i(x)\}_{i\in N}^{x\in S}$ satisfies ID. By Part 1, the continuation values $\left\{\tilde{V}_i^{\tilde{\phi}}(x)\right\}_{i\in N}^{x\in S}$ computed for $\tilde{\phi}$ and $\{\tilde{u}_i(x)\}_{i\in N}^{x\in S}$ using (10) satisfy ID as well. But by construction, $\tilde{V}_i^{\tilde{\phi}}(x) = V_i^{\phi}(x)$ for each i and s, and thus $\left\{V_i^{\phi}(x)\right\}_{i\in N}^{x\in S}$ satisfies ID. ■

Proof of Lemma 3. Suppose, to obtain a contradiction, that for each $x, y \in S$ such that $y \in F(x)$ and (11) holds, ϕ' given by (12) is not monotone.

Take $x, y \in S$ such that $|y - \phi(x)|$ is minimal among all such pairs $x, y \in S$ (informally, we consider the shortest deviation). By our assertion, ϕ' is not monotone. Since ϕ is monotone and ϕ and ϕ' differ by the value at x only, there are two possibilities: either for some $z < x$, $y = \phi'(x) < \phi(z) \leq \phi(x)$ or for some $z > x$, $\phi(x) \leq \phi(z) < \phi'(x) = y$. Assume the former (the latter case may be considered similarly). Let s be defined by

$$s = \min(z \in S : \phi(z) > y); \tag{31}$$

in the case under consideration, the set of such z is nonempty (e.g., x is its member, and z found earlier is one as well), and hence state s is well-defined. We have $s < x$; since ϕ is monotone, $\phi(s) \leq \phi(x)$.

Notice that a deviation in state s from $\phi(s)$ to y is monotone: indeed, there is no state $\tilde{z}$ such that $\tilde{z} < s$ and $y < \phi(\tilde{z}) \leq \phi(s)$ by construction of s, and there is no state $\tilde{z} > s$ such that $\phi(s) \leq \phi(\tilde{z}) < y$ as this would contradict $\phi(s) > y$. Moreover, it is feasible, so $y \in F(s)$: this is automatically true if $y = s$, if $y > s$, this follows from $s < y < \phi(s)$, and if $y < s$, this follows from $y = \phi'(x)$ and $y < s \leq x$. By assertion, this deviation cannot be profitable, i.e., $V^{\phi}(y) \not>_s V^{\phi}(\phi(s))$. By Lemma 2, since $y < \phi(s)$, $V_{\max M_s}^{\phi}(y) \leq V_{\max M_s}^{\phi}(\phi(s))$.Since $s < x$, Assumption 4 implies (for $i = \max M_x$) $V_i^{\phi}(y) \leq V_i^{\phi}(\phi(s))$.

On the other hand, (11) implies $V_i^{\phi}(y) > V_i^{\phi}(\phi(x))$. We therefore have

$$V_i^{\phi}(\phi(s)) \geq V_i^{\phi}(y) > V_i^{\phi}(\phi(x)) \tag{32}$$

and thus, by Lemma 2, since $\phi(s) < \phi(x)$ (we know $\phi(s) \leq \phi(x)$, but $\phi(s) = \phi(s)$ would contradict (32)),

$$V^{\phi}(\phi(s)) >_x V^{\phi}(\phi(x)). \tag{33}$$

Notice, however, that $y < \phi(s) < \phi(x)$ implies that $|\phi(s) - \phi(x)| < |y - \phi(x)|$. This contradicts the choice of y such that $|y - \phi(x)|$ is minimal among pairs $x, y \in S$ such that $y \in F(x)$ and (11) is satisfied. This contradiction proves that our initial assertion was wrong, and this proves the lemma. ■

Proof of Lemma 4. We show first that if [1] is the case, then [2] and [3] are not satisfied. We then show that if [1] does not hold, then either [2] or (3) are satisfied, and finish the proof by showing that [2] and [3] are mutually exclusive.

First, suppose, to obtain a contradiction, that both [1] and [2] hold. Then [2] implies that for some $z \in [a+1, \phi(a+1)]$ such that $z \in F(a)$, $V^{\phi}(z) >_a V^{\phi}(\phi(a))$, but this contradicts that ϕ is MVE, so [1] cannot hold. We can similarly prove that if (1) holds, then [3] is not satisfied.

Second, suppose that [1] does not hold. Notice that for any $x \in S$, $\phi(x) \in F(x)$ and $V^{\phi}(\phi(x)) \geq_x V^{\phi}(x)$, because these properties hold for ϕ_1 if $x \in [1, a]$ and for ϕ_2 if $x \in [a+1, m]$. Consequently, if ϕ is not MVE, then it is because the (core) condition in Definition 3 is violated. Lemma 3 then implies existence of a monotone deviation, i.e., $x, y \in S$ such that $y \in F(x)$ and $V^{\phi}(y) >_x V^{\phi}(\phi(x))$. Since ϕ_1 and ϕ_2 are MVEs on their respective domains, we must have that either $x \in [1, a]$ and $y \in [a+1, m]$ or $x \in [a+1, m]$ and $y \in [1, m]$. Assume the former; since the deviation is monotone, we must have $x = a$ and $a + 1 \leq y \leq \phi(a+1)$. Hence, we have $V^{\phi}(y) >_a V^{\phi}(\phi(a))$, and this shows that [2] holds. If we assumed the latter, we would similarly get that [3] holds. Hence, if [1] does not hold, then either [2] or [3] does.

Third, suppose that both (2) and (3) hold. Let

$$x \in \arg \max_{z \in [\phi(a), \phi(a+1)] \cap F(a)} V^{\phi}_{\min M_a}(z), \tag{34}$$

$$y \in \arg \max_{z \in [\phi(a), \phi(a+1)] \cap F(a+1)} V^{\phi}_{\max M_{a+1}}(z); \tag{35}$$

then $x \geq a+1 > a \geq y$. By construction, $V^{\phi}_{\min M_a}(x) > V^{\phi}_{\min M_a}(y)$ and $V^{\phi}_{\max M_{a+1}}(y) > V^{\phi}_{\max M_{a+1}}(x)$ (the inequalities are strict because they are strict in [2] and [3]). But this violates the ID property that $\left\{V^{\phi}_i(s)\right\}_{i \in N}^{s \in S}$ satisfies as ϕ is monotone (indeed, $\min M_a \leq \max M_{a+1}$ by Assumption 4). This contradiction proves that (2) and (3) are mutually exclusive, which completes the proof. ■

For the proof of Theorem 1, the following auxiliary result (which is itself a corollary of Lemma 4) is helpful.

Lemma 6 ***(Extension of Equilibrium)*** *Let $S_1 = [1, m-1]$ and $S_2 = \{m\}$. Suppose that $\phi : S_1 \to S_1$ is a monotone MVE. Let*

$$a = \max \left(\arg \max_{b \in [\phi(m-1), m-1] \cap F(a)} V^{\phi}_{\max M_m} (b) \right). \tag{36}$$

If

$$V^{\phi} (a) >_m u (m) / (1 - \beta), \tag{37}$$

then mapping $\phi' : S \to S$ defined by

$$\phi' (s) = \begin{cases} \phi (s) & \text{if } s < m \\ a & \text{if } s = m \end{cases} \tag{38}$$

is MVE. A similar statement, mutatis mutandis, applies for $S_1 = \{1\}$ and $S_2 = [2, m]$.

Proof of Lemma 6. Mapping ϕ' satisfies property (1) 3 by construction. Let us show that it satisfies property (2). Suppose, to obtain a contradiction, that this is not the case. By Lemma 3, there are states $x, y \in S$ such that

$$V^{\phi'} (y) >_x V^{\phi'} \left(\phi' (x) \right). \tag{39}$$

If $x < m$ then, since deviation is monotone, $y \leq \phi (m) = a \leq m - 1$. For any $z \leq m - 1$, $\left(\phi' \right)^k (z) = \phi^k (z)$ for any $k \geq 0$, and thus $V^{\phi'} (z) = V^{\phi} (z)$; therefore,

$$V^{\phi} (y) >_x V^{\phi} (\phi (x)). \tag{40}$$

However, this would contradict that ϕ is a MVE on S_1. Consequently, $x = m$. If $y < m$, then (39) implies, given $a = \phi' (m)$,

$$V^{\phi} (y) >_m V^{\phi} (a). \tag{41}$$

Since the deviation is monotone, $y \in [\phi (m-1), m-1]$, but then (41) contradicts the choice of a in (36). This implies that $x = y = m$, so (39) may be rewritten as

$$V^{\phi'} (m) >_m V^{\phi} (a). \tag{42}$$

But since

$$V^{\phi'} (m) = u (m) + \beta V^{\phi} (a), \tag{43}$$

(42) implies

$$u (m) >_m (1 - \beta) V^{\phi} (a). \tag{44}$$

This, however, contradicts (37), which proves that ϕ' satisfies property (2) of Definition 3.

To prove that ϕ' is MVE, we need to establish that it satisfies property (3) of Definition 3, i.e.,

$$V^{\phi'}\left(\phi'(x)\right) \geq_x V^{\phi'}(x) \tag{45}$$

for each $x \in S'$. If $x \in S$ (i.e., $x < m$), then $\left(\phi'\right)^k(x) = \phi^k(x)$ for any $k \geq 0$, so (45) is equivalent to

$$V^{\phi}(\phi(x)) \geq_x V^{\phi}(x). \tag{46}$$

Since ϕ is MVE on S, (46) holds for $x < m$. It remains to prove that (45) is satisfied for $x = m$. In this case, (45) may be rewritten as

$$V^{\phi}(a) \geq_m V^{\phi'}(m). \tag{47}$$

Taking (43) into account, (47) is equivalent to

$$(1-\beta) V^{\phi}(a) \geq_m u(m), \tag{48}$$

which is true, provided that (37) is satisfied. We have thus proved that ϕ' is MVE on S', which completes the proof. ■

Proof of Theorem 1. We prove this result by induction by the number of states. For any set X, let Φ^X be the set of monotone MVE, so we have to prove that $\Phi^X \neq \varnothing$.

Base: If $m = 1$, then $\phi : S \to S$ given by $\phi(1) = 1$ is MVE for trivial reasons.

Step: Suppose that if $|S| < m$, then MVE exists. Let us prove this if $|S| = m$. Consider the set $A = [1, m-1]$, and for any $a \in A$, consider two monotone MVE $\phi_1^a : [1, a] \to [1, a]$ and $\phi_2^a : [a+1, m] \to [a+1, m]$. Without loss of generality, we may assume that

$$\begin{aligned} \phi_1^a &\in \arg \max_{\phi \in \Phi^{[1,a]}, z \in [\phi(a), a] \cap F(a+1)} V^{\phi}_{\max M_{a+1}}(z), & (49) \\ \phi_2^a &\in \arg \max_{\phi \in \Phi^{[a+1,m]}, z \in [a+1, \phi(a+1)] \cap F(a)} V^{\phi}_{\min M_a}(z). & (50) \end{aligned}$$

Define $\phi^a : S \to S$ by

$$\phi^a(s) = \begin{cases} \phi_1^a(s) & \text{if } s \leq a \\ \phi_2^a(s) & \text{if } s > a \end{cases}. \tag{51}$$

Let us define function $f : A \to \{1, 2, 3\}$ as follows. By Lemma 4, for every split $S = [1, a] \cup [a+1, m]$ given by $a \in A$ and for MVE ϕ_1^a and ϕ_2^a, exactly one of three properties hold;

let $f(a)$ be the number of the property. Then, clearly, if for some $a \in A$, $f(a) = 1$, then ϕ^a is a monotone MVE by construction of function f.

Now let us consider the case where for every $a \in A$, $f(a) \in \{2, 3\}$. We have the following possibilities.

First, suppose that $f(1) = 2$. This means that (since $\phi_1^a(1) = 1$ for $a = 1$)

$$\arg \max_{z \in [1, \phi(2)] \cap F(1)} V_{\min M_1}^{\phi^1}(z) \subset \left[2, \phi^1(2)\right]. \tag{52}$$

Let

$$b \in \arg \max_{z \in [2, \phi(2)]} V_{\min M_1}^{\phi^1}(z) \tag{53}$$

and define $\phi' : S \to S$ by

$$\phi'(s) = \begin{cases} b & \text{if } s = 1 \\ \phi^1(s) & \text{if } s > 1 \end{cases}; \tag{54}$$

let us prove that ϕ' is a MVE. Notice that (52) and (53) imply

$$V_{\min M_1}^{\phi^1}(b) > V_{\min M_1}^{\phi^1}(1). \tag{55}$$

By Lemma 2, since $b > 1$,

$$V^{\phi^1}(b) >_1 V^{\phi^1}(1). \tag{56}$$

Notice, however, that

$$V^{\phi^1}(1) = u(1) / (1 - \beta), \tag{57}$$

and also $V^{\phi^1}(b) = V^{\phi_2^1}(b)$; therefore, (56) may be rewritten as

$$V^{\phi_2^1}(b) >_1 u(1) / (1 - \beta). \tag{58}$$

By Lemma 6, $\phi' : S \to S$ defined by (54), is a MVE.

Second, suppose that $f(m-1) = 3$. In this case, using the first part of Lemma 6, we can prove that there is a MVE similarly to the previous case.

Finally, suppose that $f(1) = 3$ and $f(m-1) = 2$ (this already implies $m \geq 3$), then there is $a \in [2, m-1]$ such that $f(a-1) = 3$ and $f(a) = 2$. Define, for $s \in S \setminus \{a\}$ and $i \in N$,

$$V_i^*(s) = \begin{cases} V_i^{\phi_1^{a-1}}(s) & \text{if } s < a \\ V_i^{\phi_2^a}(s) & \text{if } s > a \end{cases}. \tag{59}$$

Let us first prove that there exists $b \in \left(\left[\phi_1^{a-1}(a-1), a-1\right] \cup [a+1, \phi_2^a(a+1)]\right) \cap F(a)$ such that

$$V^*(b) >_a u(a) / (1-\beta), \tag{60}$$

and let B be the set of such b (so $B \subset \left(\left[\phi_1^{a-1}(a-1), a-1\right] \cup [a+1, \phi_2^a(a+1)]\right) \cap F(a)$). Indeed, since $f(a-1) = 3$,

$$\arg \max_{z \in \left[\phi^{a-1}(a-1), \phi^{a-1}(a)\right] \cap F(a)} V_{\max M_a}^{\phi^{a-1}}(z) \subset \left[\phi^{a-1}(a-1), a-1\right]. \tag{61}$$

Let

$$b \in \arg \max_{z \in \left[\phi^{a-1}(a-1), a-1\right] \cap F(a)} \left(V_{\max M_a}^{\phi^{a-1}}(z)\right), \tag{62}$$

then (61) and (62) imply

$$V_{\max M_a}^{\phi^{a-1}}(b) > V_{\max M_a}^{\phi^{a-1}}(a). \tag{63}$$

By Lemma 2, since $b < a$,

$$V^{\phi^{a-1}}(b) >_a V^{\phi^{a-1}}(a). \tag{64}$$

We have, however,

$$V^{\phi^{a-1}}(a) = V^{\phi_2^{a-1}}(a) = u(a) + \beta V^{\phi_2^{a-1}}\left(\phi_2^{a-1}(a)\right) \geq_a u(a) + \beta V^{\phi_2^{a-1}}(a) = u(a) + \beta V^{\phi^{a-1}}(a) \tag{65}$$

($V^{\phi^{a-1}}(a) = V^{\phi_2^{a-1}}(a)$ by definition of ϕ^{a-1}, and the inequality holds because ϕ_2^{a-1} is MVE on $[a, m]$). Consequently, (63) and (64) imply (60). (Notice that using $f(a) = 2$, we could similarly prove that there is $b \in [a+1, \phi^a(a+1)]$ such that (60) holds.)

Let us now take state some quasi-median voter in state a, $j \in M_a$, and state $d \in \left[\phi_1^{a-1}(a-1), a-1\right] \cup [a+1, \phi_2^a(a+1)]$ such that

$$d = \arg \max_{b \in B} V_j^*(b), \tag{66}$$

and define monotone mapping $\phi : S \to S$ as

$$\phi(s) = \begin{cases} \phi_1^{a-1}(s) & \text{if } s < a \\ d & \text{if } s = a \\ \phi_2^a(s) & \text{if } s > a \end{cases} \tag{67}$$

(note that $V^\phi(s) = V^*(s)$ for $x \neq a$). Let us prove that $\phi(s)$ is MVE.

By construction of d (66), we have that $b \in \left[\phi_1^{a-1}(a-1), \phi_2^a(a+1)\right] \cap F(a)$,

$$V^\phi(b) \not>_a V^\phi(d). \tag{68}$$

This is automatically true for $b \in B$, whereas if $b \notin F(a) \setminus B$ and $b \neq a$, the opposite would imply

$$V^\phi(b) >_a u(a) / (1-\beta), \tag{69}$$

which would contradict $b \notin B$; finally, if $b = a$,

$$V^\phi(a) >_a V^\phi(d) \tag{70}$$

is impossible, as this would imply

$$u(a) >_a (1-\beta) V^\phi(d) \tag{71}$$

contradicting (60), given the definition of d (66). Now, Lemma 6 implies that $\phi' = \phi|_{[1,a]}$ is a MVE on $[1, a]$.

Suppose, to obtain a contradiction, that ϕ is not MVE. Since ϕ is made from MVE ϕ' on $[1, a]$ and MVE ϕ_2^a on $[a+1, m]$, properties (1) and (3) of Definition 3 are satisfied, and thus there are only two possible monotone deviations that may prevent ϕ from being MVE. First, suppose that for some $y \in [a+1, \phi_2^a(a+1)] \cap F(a)$,

$$V^\phi(y) >_a V^\phi(d). \tag{72}$$

However, this would contradict (66) (and if $y \notin B$, then (72) is impossible as $d \in B$). The second possibility is that for some $y \in [d, a]$,

$$V^\phi(y) >_{a+1} V^\phi(\phi_2^a(a+1)). \tag{73}$$

This means that

$$V^\phi_{\max M_{a+1}}(y) > V^\phi_{\max M_{a+1}}(\phi_2^a(a+1)). \tag{74}$$

At the same time, for any $x \in [a+1, \phi_2^a(a+1)] \cap F(a)$, we have

$$V^\phi_{\max M_{a+1}}(x) \leq V^\phi_{\max M_{a+1}}(\phi_2^a(a+1)) \tag{75}$$

(otherwise Lemma 2 would imply a profitable deviation to x). This implies that for any such x,

$$V^\phi_{\max M_{a+1}}(y) > V^\phi_{\max M_{a+1}}(x). \tag{76}$$

Now, recall that

$$\phi_1^a \in \arg \max_{\phi \in \Phi^{[1,a]}, z \in [\phi(a), a] \cap F(a)} V^{\phi}_{\max M_{a+1}}(z) . \tag{77}$$

This means that there is $z \in [\phi_1^a(a), a] \cap F(a)$ such that

$$V^{\phi_1^a}_{\max M_{a+1}}(z) \geq V^{\phi}_{\max M_{a+1}}(y), \tag{78}$$

and thus for any $x \in [a+1, \phi_2^a(a+1)] \cap F(a)$,

$$V^{\phi_1^a}_{\max M_{a+1}}(z) > V^{\phi}_{\max M_{a+1}}(x) . \tag{79}$$

But $\phi_1^a = \phi^a$ on the left-hand side, and $\phi = \phi^a$ on the right-hand side. We therefore have that the following maximum is achieved on $[\phi^a(a), a]$:

$$\arg \max_{z \in [\phi^a(a), \phi^a(a+1)] \cap F(a)} V^{\phi^a}_{\max M_{a+1}}(z) \subset [\phi^a(a), a], \tag{80}$$

i.e., that (3) in Lemma 4 holds. But this contradicts that $f(a) = 2$. This contradiction completes the induction step, which proves existence of MVE.

The last statement follows from that any MVE has monotone paths, and any monotone sequence converges. ■

Proof of Theorem 2. Part 1. Suppose that there are two MVEs ϕ_1 and ϕ_2. Without loss of generality, assume that m is the minimal number of states for which this is possible, i.e., if $|S| < m$, then transition mapping is unique. Obviously, $m \geq 2$.

Consider the set $Z = \{x \in S \mid \phi_1(x) \neq \phi_2(x)\}$, and denote $a = \min Z$, $b = \max Z$. Without loss of generality, assume that ϕ_1 and ϕ_2 are enumerated such that $\phi_1(a) < \phi_2(a)$.

Let us first prove the following auxiliary result: that $a < m$ and $b > 1$ and if $x \in [\max\{2, a\}, b]$, then $\phi_1(x) < x \leq \phi_2(x)$, and if $x \in [a, \min\{b, m-1\}]$, then $\phi_1(x) \leq x < \phi_2(x)$.

Let us first prove that if $\phi_1(x) = x$, then $x = 1$ or $x = m$. Indeed, assume the opposite and consider $\phi_2(x)$. If $\phi_2(x) < x$, then $\phi_1|_{[1,x]} \neq \phi_2|_{[1,x]}$ are two MVEs for the set of states $[1, x]$, which contradicts the choice of m. If $\phi_2(x) > x$, we get a similar contradiction for $[x, m]$, and if $\phi_2(x) = x$, we get a contradiction by considering $[1, x]$ if $a < x$ and $[x, m]$ if $a > x$. Similarly, if $\phi_2(x) = x$, then either $x = 1$ or $x = m$.

Assume, to obtain a contradiction, that $a = m$. Then $Z = \{m\}$, so $\phi_1|_{[1,m-1]} = \phi_2|_{[1,m-1]}$, and then having $\phi_1(m) \neq \phi_2(m)$ is impossible for generic parameter values. We would get a similar contradiction if $b = 1$, which proves that $a < m$ and $b > 1$, thus proving the first part of Lemma.

Let us now show that for $x \in [a, b] \setminus \{1, m\}$, we have that either $\phi_1(x) < x < \phi_2(x)$ or $\phi_2(x) < x < \phi_1(x)$. Indeed, neither $\phi_1(x) = x$ nor $\phi_2(x) = x$ is possible. If $\phi_1(x) < x$ and $\phi_2(x) < x$, then $\phi_1|_{[1,x]}$ and $\phi_2|_{[1,x]}$ are two different MVEs on $[1, x]$, which is impossible; we get a similar contradiction if $\phi_1(x) > x$ and $\phi_2(x) > x$. This also implies that if $a < x < b$, then $x \in Z$.

We now prove that for any $x \in Z$, $\phi_1(x) < \phi_2(x)$. Indeed, suppose that $\phi_2(x) > \phi_1(x)$ (equality is impossible as $x \in Z$); then $x > a \geq 1$. If $x < m$, then, as we proved, we must have $\phi_2(x) < x < \phi_1(x)$, and if $x = m$, then $\phi_2(x) < \phi_1(x) \leq m = x$. In either case, $\phi_2(x) < x$, and since $\phi_2(a) > \phi_1(a) \geq 1$, then by monotonicity of ϕ_2 there must be $y : 1 \leq a < y < x \leq m$ such that $\phi_2(y) = y$, but we proved that this is impossible. Hence, $\phi_1(x) < \phi_2(x)$ for any $x \in Z$, and using the earlier result, we have $\phi_1(x) < x < \phi_2(x)$ for any $x \in Z \setminus \{1, m\}$.

To finish the proof, it suffices to show that $\phi_1(1) = 1$ and $\phi_2(m) = m$. Suppose, to obtain a contradiction, that $\phi_1(1) > 1$. We then have $\phi_2(1) > 1$, then $\phi_1(2) \geq 2$ and $\phi_2(2) \geq 2$ and thus $\phi_1|_{[2,m]}$ and $\phi_2|_{[2,m]}$ are MVEs on $[2, m]$, and since $b \neq 1$, they must be different, which would again contradict the choice of m. We would get a similar contradiction if $\phi_2(m) = m$. This completes the proof of the auxiliary result.

To finish the proof of the Theorem, notice that the auxiliary result implies, in particular, that $Z = [a, b] \cap S$, so Z does not have "gaps". We define function $g : Z \to \{1, 2\}$ as follows. If $V_{M_x}^{\phi_1}(x) > V_{M_x}^{\phi_2}(x)$, then $g(x) = 1$, and if $V_{M_x}^{\phi_2}(x) > V_{M_x}^{\phi_1}(x)$, then $g(x) = 2$; the exact equality cannot hold generically. Intuitively g picks the equilibrium (left or right) that agent M_x prefers).

Let us prove that $g(a) = 2$ and $g(b) = 1$. Indeed, suppose that $g(a) = 1$; since $a < m$, we must have $\phi_1(a) \leq a < \phi_2(a)$ (with equality if $a = 1$ and strict inequality otherwise). Consider two cases. If $a > 1$, then for $x < a$, $\phi_1(x) = \phi_2(x)$, and since $\phi_1(a) < a$, then $V_{M_a}^{\phi_1}(\phi_1(a) \mid a) = V_{M_a}^{\phi_2}(\phi_1(a) \mid a)$. But $V_{M_x}^{\phi_1}(x) > V_{M_x}^{\phi_2}(x)$ implies that $V_{M_a}^{\phi_1}(\phi_1(a) \mid a) > V_{M_a}^{\phi_2}(\phi_2(a) \mid a)$ (provided that $\beta \neq 0$), and thus $V_{M_a}^{\phi_2}(\phi_1(a) \mid a) > V_{M_a}^{\phi_2}(\phi_2(a) \mid a)$, which contradicts that ϕ_2 is MVE. If $a = 1$, then $g(a) = 1$ would imply that $V_{M_1}^{\phi_1}(1) > V_{M_1}^{\phi_2}(1)$. But $\phi_1(1) = 1$, which means $\frac{u_{M_1}(1)}{1-\beta} > V_{M_1}^{\phi_2}(1)$, thus $u_{M_1}(1) + \beta V_{M_1}^{\phi_2}(1) > V_{M_1}^{\phi_2}(1)$. But $V_{M_1}^{\phi_2}(1) = u_{M_1}(1) + \beta V_{M_1}^{\phi_2}(\phi_2(1) \mid 1)$, and thus, provided that $\beta \neq 0$, we have $V_{M_1}^{\phi_2}(1 \mid 1) > V_{M_1}^{\phi_2}(\phi_2(1) \mid 1)$. This contradicts that ϕ_2 is an MVE, thus proving that $g(a) = 2$. We can similarly prove that $g(b) = 1$.

Clearly, there must be two states $s, s+1 \in Z$ such that $g(s) = 2$ and $g(s+1) = 1$. For such s, let us construct mapping ϕ as follows:

$$\phi(x) = \begin{cases} \phi_1(x) & \text{if } x \leq s \\ \phi_2(x) & \text{if } x > s \end{cases}; \tag{81}$$

then $\phi(s) \leq s < \phi_2(s)$ (inequality is strict unless $s = 1$) and $\phi(s+1) \geq s+1 > \phi(s)$ (inequality is strict unless $s+1 = m$), which means that mapping ϕ is monotone. Now, $g(s) = 2$ implies that $u_{M_s}(x) + \beta V_{M_s}^{\phi_2}(\phi_2(s) \mid s) = V_{M_s}^{\phi_2}(s) > V_{M_s}^{\phi_1}(s) = u_{M_s}(x) + \beta V_{M_s}^{\phi_1}(\phi_1(s) \mid s)$. But $V_{M_s}^{\phi_2}(\phi_2(s) \mid s) = V_{M_s}^{\phi}(\phi_2(s) \mid s)$ and $V_{M_s}^{\phi_1}(\phi_1(s) \mid s) = V_{M_s}^{\phi}(\phi_1(s) \mid s)$, and thus $V_{M_s}^{\phi}(\phi_2(s) \mid s) > V_{M_s}^{\phi}(\phi_1(s) \mid s)$ (note also that $s+1 \leq \phi_2(s) \leq \phi_2(s+1)$). Similarly, $g(s+1) = 1$ implies $V_{M_{s+1}}^{\phi}(\phi_1(s+1) \mid s+1) > V_{M_{s+1}}^{\phi}(\phi_2(s+1) \mid s+1)$. But this contradicts Lemma 4 for mapping ϕ. This contradiction completes the proof.

Part 2. As in Part 1, we can assume that m is the minimal number of states for which this is possible. We can then establish that if $\phi_1(x) = x$, then $x = 1$ or $x = m$. If $\phi_1(x) < x < \phi_2(x)$ or vice versa, then for all $i \in M_x$, there must be both a state $x_1 < x$ and a state $x_2 > x$ such that $u_i(x_1) > u_i(x)$ and $u_i(x_2) > u_i(x)$, which contradicts the assumption in this case. Since for $1 < x < m$, $\phi(x) \neq x$, we get that $\phi_1(x) = \phi_2(x)$ for such x. Let us prove that $\phi_1(1) = \phi_2(1)$. If this is not the case, then $\phi_1(1) = 1$ and $\phi_2(1) = 2$ (or vice versa). If $m = 2$, then monotonicity implies $\phi_2(2) = 2$, and if $m > 2$, then, as proved earlier, we must have $\phi_2(x) = x+1$ for $1 < x < m$ and $\phi_2(m) = m$. In both cases, we have $\phi_1(x) = \phi_2(x) > 1$ for $1 < x \leq m$. Hence, $V_i^{\phi_1}(2) = V_i^{\phi_2}(2)$ for all $i \in N$. Since ϕ_1 is MVE, we must have $u_i(1) / (1-\beta) \geq V_i^1(2)$ for $i \in M_1$, and since ϕ_2 is MVE, we must have $V_i^2(2) \geq u_i(1) / (1-\beta)$. Generically, this cannot hold, and this proves that $\phi_1(1) = \phi_2(1)$. We can likewise prove that $\phi_1(m) = \phi_2(m)$, which implies that $\phi_1 = \phi_2$. This contradicts the hypothesis of non-uniqueness. ■

Proof of Lemma 5. This result immediately follows from 2. ■

Proof of Theorem 3. The existence is proved in the text. Since, on equilibrium path, there is only a finite number of shocks, then from some period t on, the environment will be the same, some E^x. Since ϕ_{E^x} is monotone, the sequence $\{s_t\}$ has a limit by Theorem 1. The fact that this limit may depend on the sequence of shocks realization may be shown by a simple example. ■

Proof of Theorem 4. Part 1. Without loss of generality, suppose that h is the minimal number for which two monotone MVE $\phi = \{\phi_E\}_{E \in \mathcal{E}}$ and $\phi' = \{\phi'_E\}_{E \in \mathcal{E}}$ exist. If we take $\tilde{\mathcal{E}} = \{E^2, \dots, E^h\}$ with the same environments $E^2, \dots, E$ and the same transition probabilities, we will have a unique monotone MVE $\tilde{\phi} = \{\phi_E\}_{E \in \mathcal{E}'} = \{\phi'_E\}_{E \in \mathcal{E}'}$ by assumption. Now, with the help of transformation used in the proof of 3 we get that ϕ_{E^1} and ϕ'_{E^1} must be MVE in a certain stationary environment E'. However, by Theorem 2 such MVE is unique, which leads to a contradiction.

Part 2. The proof is similar to that of Part 1. The only step is that we need to verify that we can apply Part 2 of Theorem 2 to the stationary environment E'. In general, this will not be the case. However, it is easy to notice (by examining the proof of Part 2 of Theorem 2) that instead of single-peakedness, we could require a weaker condition: that for each $s \in S$ there is $i \in M_s$ such that there do not exist $x < s$ and $y > s$ such that $u_i(x) \geq u_i(s)$ and $u_i(y) \geq u_i(s)$. We can then prove that if $\{u_i(s)\}_{i \in N}^{s \in S}$ satisfy this property and ϕ is MVE, then $\{V_i^\phi(s)\}_{i \in N}^{s \in S}$ also does. The rest of the proof follows. ■

Proof of Theorem 5. Part 1. It suffices to prove this result for stationary case. For each $s \in S$ take any protocol such that if $\phi(s) \neq s$, then $\theta_s(m-1) = \phi(s)$ (i.e., the desired transition is considered last). We claim that there is a strategy profile σ such that if for state s, $\phi(s) = s$, then no proposal is accepted in periods where $s_t = s$, and if $\phi(s) \neq s$, then no proposal but the last one, $\phi(s)$, is accepted, and the last one is accepted. Indeed, under such profile, the continuation strategies are given by (10). Hence, if the state is s such that $\phi(s) = s$, there is no winning coalition that wants to have any other alternative $x \neq s$ accepted. If the state is s such that $\phi(s) \neq s$, then, anticipating that $\phi(s)$ will be accepted over s in the last voting round, no winning coalition has an incentive to deviate and potentially support another state $x \neq \phi(s)$; at the last round, however, $\phi(s)$ would be supported because of property (2) of Definition 3.

To prove that protocol (20) will suffice if the equilibrium is unique, we make the following observation. Close inspection of the proof of Theorem 1 reveals that we could actually prove existence of monotone MVE which satisfies an additional requirement: If $x < y < \phi(x)$ or $x > y > \phi(x)$, then

$$\left\{i \in N : V_i^\phi(\phi(x)) \geq V_i^\phi(y)\right\} \in W_{E,x}. \tag{82}$$

If the equilibrium is unique, it satisfoes this additional constraint. It is then straightforward to prove that protocol (20) would suffice.

Part 2. If the transition mapping is monotone, then continuation utilities $\left\{V_{E,i}^{\phi}(s)\right\}_{i\in N}^{s\in S} = \left\{V_{E,i}^{\sigma}(s)\right\}_{i\in N}^{s\in S}$ satisfy ID for any $E \in \mathcal{E}$. Again, the proof that ϕ is MVE reduces to the stationary case. For each state s, let us define a resolute irreflexive binary relation $\succ_s$ on S as follows: $x \succ_s y$ if either $V^{\phi}(x) >_s V^{\phi}(y)$ or ($V^{\phi}(y) \not>_s V^{\phi}(x)$ and for some $a < b$, $\theta_s(a) = x$ and $\theta_s(b) = y$), with the convention that $\theta_s(0) = s$. In other words, $\succ_s$ resolves $>_s$ for continuation values by giving precedence to states which are voted earlier in the protocol θ_s. The theory of amendment agendas (see Shepsle and Weingast, 1984, and Austen-Smith and Banks, 1999) implies that $\phi(s)$ must be the state that satisfies both properties of Definition 3. The details are omitted to save space.

Part 3. The proof uses theory of amendment agendas, but otherwise similar to the proof of Theorem 9 and is omitted. ■

Proof of Theorem 6. Suppose, to obtain a contradiction, that $\phi_2(x) < x$. Then $\phi_1|_{S'}$ and $\phi_2|_{S'}$ are mappings from S' to S' such that both are MVE. Moreover, they are different, as $\phi_1(x) = x > \phi_2(x)$. However, this would violate the assumed uniqueness (either assumption needed for Theorem 2 continues to hold if the domain is restricted), which completes the proof. ■

Proof of Corollary 1. Consider an alternative set of environments $\mathcal{E}' = \left\{E^0, E^2\right\}$, where E^0 coincides with E^2 on S, but the transition probabilities are the same as in $\mathcal{E}$. Clearly, ϕ' such that $\phi'_{E^0} = \phi'_{E^2} = \phi_{E^2}$ is a MVE in $\mathcal{E}'$. Let us now consider stationary environments $\tilde{E}^0$ and $\tilde{E}^1$ obtained from $\mathcal{E}'$ and $\mathcal{E}$, respectively, using the procedure from the proof of Theorem 3. Suppose, to obtain a contradiction, that $\phi_{E^2}(x) < x$, then environments $\tilde{E}^0$ and $\tilde{E}^1$ coincide on $[1.s]$ by construction. Theorem 6 then implies that, since $\phi_{E^1}(x) = x$, then $\phi'_{E^0}(x) \geq x$ (since ϕ'_{E^0} and ϕ_{E^1} are the unique MVE in $\tilde{E}^0$ and $\tilde{E}^1$, respectively). But by definition of ϕ', $x < \phi'_{E^0}(x) = \phi_{E^2}(x) \leq x$, a contradiction. This contradiction completes the proof. ■

Proof of Theorem 7. Let us first prove this result for the case where each QMV is a singleton. Both before and after the shock, the mapping that would map any state x to a state which maximizes the instantaneous payoff $u_{M_x}(y)$ would be a monotone MVE for $\beta < \beta_0$. By uniqueness, ϕ_{E^1} and ϕ_{E^2} would be these mappings under E^1 and E^2, respectively. Now it is clear that if the shock arrives at period t, and the state at the time of shock is $x = s_{t-1}$, then $\phi_{E^2}(x)$ must be either the same as $\phi_{E^1}(x)$ or must satisfy $\phi_{E^2}(x) > s$. In either case, we get a monotone sequence after the shock. Moreover, the sequence is the same if $s_\tau \leq s$, and if $s_\tau > s$, then we have $s_\tau > s \geq \tilde{s}_\tau$ automatically.

The general case may be proved by observing that a mapping that maps each state x to an alternative which maximizes by $u_{\min M_x}(y)$ among the states such that $u_i(y) \geq u_i(x)$ for all $i \in M_x$ is a monotone MVE. Such mapping is generically unique, and by the assumption of uniqueness it coincides with the mapping ϕ_{E^1} if the environment is E^1 and it coincides with ϕ_{E^2} if the environment is E^2. The remainder of the proof is analogous. ■

Proof of Theorem 8. It is sufficient, by transitivity, to prove this Theorem for the case where $\max M_{E^1,x} \neq \max M_{E^2,x}$ for only one state $x \in [s+1, m]$. Moreover, without loss of generality, we can assume that $\max M_{E^1,x} < \max M_{E^2,x}$. Notice that if $\phi_1(x) \geq x$, then ϕ_1 is MVE in environment E^2, and by uniqueness must coincide with ϕ_2.

Consider the remaining case $\phi_1(x) < x$; it implies $\phi_1(x-1) \leq x-1$. Consequently, $\phi_1|_{[1,x-1]}$ is MVE under either environment restricted on $[1, x-1]$ (they coincide on this interval). Suppose, to obtain a contradiction, that $\phi_1|_{[1,s]} \neq \phi_2|_{[1,s]}$; since $x > s$, we have $\phi_1|_{[1,x-1]} \neq \phi_2|_{[1,x-1]}$. We must then have $\phi_2(x-1) > x-1$ (otherwise there would be two MVEs $\phi_1|_{[1,x-1]}$ and $\phi_2|_{[1,x-1]}$ on $[1, x-1]$, and therefore $\phi_2(x) \geq x$. Consequently, $\phi_2|_{[x,m]}$ is MVE on $[x, m]$ under environment E^2 restricted on $[x, m]$. Let us prove that $\phi_2|_{[x,m]}$ is MVE on $[x, m]$ under environment E^1 restricted on $[x, m]$ as well. Indeed, if it were not the case, then there must be a monotone deviation, as fewer QMV (in state x) imply that only property (1) of Definition 3 may be violated. Since under E^1, state x has fewer quasi-median voters than under E^2, it is only possible if $\phi_2(x) > x$, in which case $\phi_2(x+1) \geq x+1$. Then $\phi_2|_{[x+1,m]}$ would be MVE on $[x+1, m]$, and by Lemma 6 we could get MVE $\tilde{\phi}_2$ on $[x, m]$ under environment E^1. This MVE $\tilde{\phi}_2$ would be MVE on $[x, m]$ under environment E^2. But then under environment E^2 we have two MVE, $\tilde{\phi}_2$ and $\phi_2|_{[x,m]}$ on $[x, m]$, which is impossible.

We have thus shown that $\phi_1|_{[1,x-1]}$ is MVE on $[1, x-1]$ under both E^1 and E^2, and the same is true for $\phi_2|_{[x,m]}$ on $[x, m]$. Take mapping ϕ given by

$$\phi(y) = \begin{cases} \phi_1(y) & \text{if } y < x \\ \phi_2(y) & \text{if } y > x \end{cases}. \tag{83}$$

Since $\phi_1|_{[1,x-1]} \neq \phi_2|_{[1,x-1]}$ and $\phi_1|_{[x,m]} \neq \phi_2|_{[x,m]}$ ($\phi_1(x-1) \leq x-1$, $\phi_2(x-1) > x-1$, $\phi_1(x) < x$, $\phi_2(x) \geq x$), ϕ is not MVE in E^1 nor it is in E^2. By Lemma 4, in both E^1 and E^2 only one type of monotone deviation (at $x-1$ to some $z \in [x, \phi_2(x)]$ or at x to some $z \in [\phi_1(x-1), x]$) is possible. But the payoffs under the first deviation is the same under both E^1 and E^2; hence, in both environments it is the same type of deviation.

Suppose that it is the former deviation, at $x-1$ to some $z \in [x, \phi_2(x)]$. Consider the following restriction on feasible transitions:

$$\tilde{F}(a) = \begin{cases} F(a) & \text{if } a \geq x; \\ F(a) \cap [1, x-1] & \text{if } a < x; \end{cases} \tag{84}$$

denote the resulting environments by $\tilde{E}^1$ and $\tilde{E}^2$. This makes the deviation impossible, and thus ϕ is MVE in $\tilde{E}^1$ (in $\tilde{E}^2$ as well). However, ϕ_1 is also MVE in $\tilde{E}^1$, as it is not affected by the increase in cost, and this contradicts uniqueness. Finally, suppose that the deviation is at x to some $z \in [\phi_1(x-1), x]$. Then consider the costs

$$\bar{F}(a) = \begin{cases} F(a) & \text{if } a < x; \\ F(a) \cap [x, m] & \text{if } a \geq x; \end{cases} \tag{85}$$

denote the resulting environments by $\bar{E}^1$ and $\bar{E}^2$. This makes the deviation impossible, and thus ϕ is MVE in $\tilde{E}^2$. However, ϕ_2 is also MVE in $\tilde{E}^1$, as it is not affected by the increase in cost. Again, this contradicts uniqueness, which completes the proof. ■

Proof of Corollary 2. The proof is similar to the proof of Corollary 1 and is omitted. ■

Proof of Theorem 9. Part 1. It suffices to prove this result in stationary environments. Suppose MVE ϕ is nonmonotone, which means there are states $x, y \in S$ such that $x < y$ and $\phi(x) > \phi(y)$. By property (1) of Definition 3 applied to state x, we get

$$V_{\max M_x}(\phi(x)) \geq V_{\max M_x}(\phi(y)), \tag{86}$$

and if we apply it to state y,

$$V_{\min M_y}(\phi(y)) \geq V_{\min M_y}(\phi(x)). \tag{87}$$

Since $\max M_x \leq \min M_y$ by assumption, (86) implies

$$V_{\min M_y}(\phi(x)) \geq V_{\min M_x}(\phi(y)). \tag{88}$$

Since in the generic case inequalities are strict, this contradicts (87).

Part 2. Again, consider stationary environments only. If ϕ is monotone, then for some $x, y \in S$ we have $x < y$ and $\phi(x) > \phi(y)$, which in this case implies $\phi(x) = y = x + 1$ and $\phi(y) = x$. Property 2 of Definition 3, when applied to state x, implies that for all $i \in M_x$,

$$V_i(y) \geq V_i(x). \tag{89}$$

This means that generically, for all $i \in M_y$,

$$V_i(y) > V_i(x). \tag{90}$$

The same property 2, when applied to state y, implies that for all $i \in M_y$,

$$V_i(x) \geq V_i(y). \tag{91}$$

But (91) contradicts (90) as costs are nonnegative. This contradiction completes the proof. ■

Proof of Theorem 10. Take any MVE ϕ. Suppose, to obtain a contradiction, that for some x, $\phi(x) > x$, but $\phi^2(x) < \phi(x)$ (the other case is considered similarly). Denote $y = \phi(x)$ and $z = \phi(y)$. By property (2) of Definition 3 applied to state y, for all $i \in M_y$,

$$V_i(z) \geq V_i(y). \tag{92}$$

The means that (92) holds for all $i \in M_x$. However, property (1) of Definition 3, applied to state x, implies that, generically, at least for one $i \in M_x$,

$$V_i(y) > V_i(z). \tag{93}$$

But this contradicts (92), and this contradiction completes the proof. ■

Proof of Theorem 11. Take an increasing sequence of sets of points, $S_1 \subset S_2 \subset S_3 \subset \cdots$, so that $\bigcup_{i=1}^{\infty} S_i$ is dense. For each S_i, take MVE ϕ_i. We know that ϕ_i is a monotone function on S_i; let us complement it to a monotone (not necessarily continuous) function on S which we denote by $\tilde{\phi}_i$ for each i. Since $\tilde{\phi}_i$ are monotone functions from a bounded set to a bounded set, there is a subsequence $\tilde{\phi}_{i_k}$ which converges to some $\tilde{\phi}$ pointwisely. (Indeed, we can pick a subsequence which converges on S_1, then a subsequence converging on S_2 etc; then use a diagonal process. After it ends, the set of points where convergence was not achieved is at most countable, so we can repeat the diagonal procedure.) To show that $\tilde{\phi}$ is a MVE, suppose not, then there are two points x and y such that y is preferred to $\tilde{\phi}(x)$ by all members of M_x. Here, we need to apply a continuity argument and say that it means that the same is true for some points in some S_i. But this would yield a contradiction. ■

Proof of Proposition 1. Part 1. We start by proving that there exists a unique monotone MVE. To show this, we need to establish that all requirements for existence and generic uniqueness are satisfied.

(ID) Consider group i and take two states x, y with $x > y$. The policy in state x is b_{M_x} and the policy in state b_{M_y}. Since $M_x \geq M_y$ and b is increasing in the identity of the group, we have $b_{M_x} \geq b_{M_y}$. Take the difference

$$\begin{aligned} u_i(x) - u_i(y) &= -(b_{M_x} - b_i)^2 - \sum\nolimits_{j \notin H_x} \gamma_j C_j - \left(-(b_{M_y} - b_i)^2 - \sum\nolimits_{j \notin H_y} \gamma_j C_j \right) \\ &= (b_{M_x} - b_{M_y})(2b_i - b_{M_x} - b_{M_y}) - \sum\nolimits_{j \notin H_x} \gamma_j C_j + \sum\nolimits_{j \notin H_y} \gamma_j C_j. \end{aligned}$$

This only depends on i through b_i, which is increasing in b_i. Hence, ID is satisfied.

(Monotone QMV) The QMV in state s is M_s. If $s \leq 0$, then an increase in s implies that groups on the right get more power, and $s \leq 0$, then an increase in s implies that groups on the left get more power.

(Costs) There are no costs of transition, and thus the assumption holds trivially.

(QMV are singletons) This holds generically, when no two disjoint sets of groups have the same power.

This establishes that there is a unique monotone MVE. To show that $\phi(0) = 0$, suppose not. Without loss of generality, $\phi(0) > 0$. Then if $s_1 = 0$, monotonicity implies that $s_t > 0$ for all $t > 1$. But $M_0 = 0$, thus $b_{M_0} = b_0$ and $u_{M_0}(0) = 0$, while $u_{M_0}(s) < 0$ for $s \neq 0$. This shows that if $\phi(0) > 0$, there is a profitable deviation to 0. This contradiction completes the proof.

Part 2. Consider the case $s < 0$ (the case $s > 0$ is considered similarly). Since $\phi(0) = 0$, monotonicity implies that $\phi(s) \leq 0$. To show that $\phi(s) \geq s$, suppose, to obtain a contradiction, that $\phi(s) < s$. Then, starting from the initial state $s_1 = s$, the equilibrium path will involve

$s_t < s$ for all $t > 1$. Notice, however, that for the QMV M_s, $u_{M_s}(s) = -\sum_{j \notin H_s} \gamma_j C_j$, and for $x < s$, $u_{M_s}(x) = -(b_{M_x} - b_{M_s}) - \sum_{j \notin H_x} \gamma_j C_j < u_{M_s}(s)$, as H_x is a strict superset of H_s. Again, there is a profitable deviation, which completes the proof.

Part 3. Consider the mapping ϕ such that $\phi(s) = 0$ for all s. Under this mapping, continuation utilities are given by

$$V_i^{\phi}(s) = -(b_{M_s} - b_i)^2 - k \sum_{j \notin H_s} \gamma_j C_j^* - \frac{\beta}{1-\beta}(b_0 - b_i)^2,$$

and in the absense of costs of transtitions, $V_i^{\phi}(s \mid x) = V_i^{\phi}(s)$ for all agents i and all states s, x. Now, the two conditions required to hold for ϕ to be an MVE simplify to:

$$\begin{aligned} \text{for any } s, x \quad &: \quad V_{M_s}^{\phi}(0) \geq V_{M_s}^{\phi}(x); \\ \text{for any } s \quad &: \quad V_{M_s}^{\phi}(0) \geq V_{M_s}^{\phi}(s); \end{aligned}$$

clearly, the second line of inequalities is a subset of the first. This simplifies to

$$\text{for any } s, x\text{: } k \sum_{j \notin H_x} \gamma_j C_j^* \geq (b_{M_s})^2 - (b_{M_x} - b_{M_s})^2.$$

Clearly, as k increases, the number of equations that are true weakly increases. Furthermore, for k high enough, the left-hand side becomes arbitrarily large for all x except for $x = 0$ where it remains zero, but for $x = 0$, $b_{M_x} = 0$ and thus the right-hand side is zero as well. Finally, if k is small enough, the left-hand side is arbitrarily close to 0 for all s and x, and thus the inequality will be violated, e.g., for $s = x = 1$. This proves that there is a unique positive k^* with the required property. ■

Proof of Proposition 2. Part 1. The equilibrium exists and is unique because the required properties hold in each of the environments, and thus Theorems 3 and 4 are applicable.

Let ϕ_L be the mapping after radicals have left. Since the environment L is static, ϕ_L coincides with ϕ from Proposition 1 (i.e., if radicals are impossible). Now take any radical environment R_z (so states $x \leq z$ are controlled by radicals). Notice that $\phi_{R_z}(s)$ is the same for all $s \leq z$ (otherwise, setting $\phi_{R_z}(s) = \phi_{R_z}(z)$ for all $s < z$ would yield another MVE since there is not cost of transition, thus violating uniqueness). Consider two situations: $z < 0$ and $z \geq 0$.

Suppose first that $z < 0$. Then $\phi_{R_z}(0) = 0$ (similar to the proof of Part 1 of Proposition 1), and thus by monotonicity $\phi_{R_z}(s) \in [-l-r, 0]$. For any x such that $z < x < 0$, $\phi_{R_z}(x) \geq x$ (again, similar to that proof). Notice that as b_{-l} varies, the mapping $\phi_{R_z}|_{[z+1,l+r]}$ does not change. Indeed, equilibrium paths starting from $x \geq z+1$ remain within that range, and thus continuation utilities of M_x for any $x \geq z+1$ do not depend on b_{-l}; moreover, a deviation from $x \geq z+1$ to some $y \leq z$ cannot be profitable for obvious reasons. The state $\phi_{R_z}(z)$ is such that it maximizes the continuation utility of the radical $-l$ among the following alternaties: moving to some state $y \leq z$, staying there until transition to environment L and moving according to ϕ_L, and moving to some state $y > z$, moving according to ϕ_{R_z} until the transition to L and according to ϕ_L after the transition. Notice that as b_{-l} decreases, the continuation utilities of the radical $-l$ under all these options, except of moving to state $y = -l-r$, strictly decrease, while the payoff of that option remains unchanged (and equal to $-\frac{1}{1-\beta} k \sum_{j>-l} \gamma_j C_j^*$). Hence, an decrease in b_{-l} makes this transition more likely starting from state z, and thus for all $s \leq z$.

Now suppose that $z \geq 0$. Trivially, we must have $\phi_{R_z}(z) \leq 0$. In this case, $\phi_{R_z}|_{[z+1,l+r]}$ may depend on b_{-l}, moving to $y \in [z+1, l+r]$ is suboptimal for the radical anyway. So in this case, the equilibrium $\phi_{R_z}(z)$ maximizes the radical's continuation utility among the options of moving to some $y \leq 0$, staying there until transition to L, and then moving according to ϕ_L. Again, only for $y = -l-r$ the continuation payoff remains unchanged as b_{-l} decreases, and for all other options it decreases. Hence, in this case, too, a lower b_{-l} makes $\phi_{R_z}(z) = -l-r$ more likely. Moreover, since the equilibrium path starting from any $y \leq 0$ will only feature states $s \leq 0$, and for all possible $y \leq 0$, the path for lower y is first-order stochastically dominated by the path for higher y, an increase in k makes $\phi_{R_z}(z) = -l-r$ less likely.

It remains to prove that an increase in z decreases the chance of transition to $-l-r$ for any given $s \leq z$. This equivalent to saying that a higher z decreases the chance that $\phi_{R_z}(-l-r) = -l-r$. Suppose that z increases by one. If $z \geq 0$ (thus increasing to $z+1 \geq 1$), then $\phi_{R_z}(-l-r)$ does not change as moving to $y \geq 1$ was dominated anyway. If $z < 0$ (thus increasing to $z+1 \leq 0$), then this increase does not change $\phi_{R_z}|_{[z+2,l+r]}$, and thus the only change is the option to stay in $z+1$ as long as the shock leading to L does not arrive. This makes staying in $-l-r$ weakly less attractive for the radical, and for some parameter values may make him switch.

Part 2. Suppose, to obtain a contradiction, that for some $s \leq 0$, $\phi_E(s) < s$. Without loss of generality we may assume that this is the lowest such s, meaning $\phi_E(s)$ is ϕ_E-stable. Consider a deviation at s from $\phi_E(s)$ to s. This deviation has the following effect on continuation utility. First, in the period of deviation, the QMV M_s gets a higher instantaneous payoff. Second, the continuation utilities if a transition to R_z for some z takes place immediately after that may differ (if there is no shock, then both paths will converge at $\phi_E(s)$ thus yielding the same continuation utilities). Consider possible case. If $z \geq s$, then the radicals are in power in both s and $\phi_E(s)$. As showed in the proof of Part 1, the radicals will transit to the same state, thus resulting in the same path and continuation utilities. If, however, $z < s$, then the transition in R_z will be chosen by M_s if he stayed in s, hence, this transition will maximize his continuation payoff under R_z, and this need not be true if he moved to $\phi_E(s)$ (regardless of whether radicals rule in this state or they don't). In all cases, the continuation utility after the current period is weakly higher if he stayed in s than if he moved to $\phi_E(s) < s$, and taking into account the first effect, we have a strictly profitable deviation. This contradicts the definition of MVE, which completes the proof. ■

Proof of Proposition 3. Part 1. Suppose, to obtain a contradiction, that $\phi_E(s) \leq x$ for all $x \geq 0$. By Part 2 of Proposition 2, $\phi_E(s) \geq s$ for $s \leq 0$, which now implies $\phi_E(0) = 0$.

As in Theorem 3, we may treat the environment E as static, with $W_i(s)$ as quasi-utilities and $\tilde{\beta} = \beta(1-\mu)$ as the discount factor. The payoff from staying in 0 for player $M_0 = 0$ is $V_0(0) = \frac{W_0(0)}{1-\tilde{\beta}}$. By definition of MVE, $V_{M_s}(\phi_E(s)) \geq V_{M_s}(s)$, and since continuation utilities satisfy ID, $\phi_E(s) \leq s$, and $M_0 \leq M_s$, it must be that $V_0(\phi_E(s)) \geq V_0(s)$. Since $V_0(s) = W_0(s) + \tilde{\beta}V_0(\phi_E(s))$, we have $V_0(\phi_E(s)) \geq \frac{W_0(s)}{(1-\tilde{\beta})}$. Consequently, it must be that $V_0(\phi_E(s)) > V_0(0)$. This is impossible if $\phi_E(s) = 0$, and it suggests a profitable deviation at 0 from 0 to s otherwise. This contradiction proves that such x exists.

Part 2. Suppose, to obtain a contradiction, that for some $s > 0$, $\phi_E(s) > s$. Without loss of generality, assume that $\phi_E(s)$ is itself ϕ_E-stable. By definition of MVE, $V_{M_s}(\phi_E(s)) \geq V_{M_s}(s)$. This is equivalent to $\frac{W_{M_s}(\phi_E(s))}{1-\tilde{\beta}} \geq W_{M_s}(s) + \frac{\tilde{\beta}W_{M_s}(\phi_E(s))}{1-\tilde{\beta}}$, thus implying $W_{M_s}(\phi_E(s)) \geq W_{M_s}(s)$. Setting $y = \phi_E(s)$ and $x = s$, we have $y > x \geq 0$ and $W_{M_x}(y) \geq W_{M_x}(x)$, a contradiction. This completes the proof. ■

Proof of Proposition 5. Proposition 1 proved this result for environment L. For any of the radical environments R_z ($z < 0$), the quasi-utility of the QMV of state 0, group 0, is $\tilde{u}_{R_z,0}(0) = 0$, and for $s \neq 0$, $\tilde{u}_{R_z,0}(s) < 0$. This means that continuation utility $\tilde{V}_{R_z,0}(s) < 0$. Hence, if $\phi_{R_z}(0) = s \neq 0$, there would be a profitable deviation at 0 from s to 0; this proves that $\phi_{R_z}(0) = 0$. Now, monotonicity yields that $\phi_{R_z}(s) \geq s$ for all $s \geq 0$. This tells us that if we consider $R_z|_{[0,l+r]}$ to be a static environment with quasi-utilities $\tilde{u}_{R_z,i}(s)$ and the quasi-discount factor $\tilde{\beta} = \beta(1-\nu)$, then $\phi_{R_z}|_{[0,l+r]}$ is an MVE. But notice that $\phi_L|_{[0,l+r]}$ is also an MVE in this environment, because continuation utilities $\tilde{V}_{R_z,i}(s)$ would equal the corresponding continuation utilities in the environment L, where it is an MVE: $\tilde{V}_{R_z,i}(s) = V_{L,i}(s)$. Since the MVE must be unique, we have $\phi_{R_z}|_{[0,l+r]} = \phi_L|_{[0,l+r]}$, and thus $\phi_{R_z}(s) \in [0,s]$ for $s \geq 0$, because this property holds for ϕ_L. Another iteration of this argument would establish the same for the initial environment E, which completes the proof. ■

Proof of Proposition 6. This is an immediate corollary of Theorem 6. ■

Proof of Proposition 7. Let us first prove that for any R_z and any $x \geq 0$, $\phi_{R_z}(x) \leq x$. Suppose, to obtain a contradiction, that $\phi_{R_z}(x) > x \geq 0$. Consider two cases. If $z \geq x$ (so radicals are in power), then at x they have a profitable deviation from $\phi_{R_z}(x)$ to x, since the path starting at x is first-order stochastically dominated by one starting at $\phi_{R_z}(x) > x$, both are contained in $[0, l+r]$, and on this set the preferences are radicals are monotone. Consequently, in this case, $\phi_{R_z}(x) > x$ is impossible. The second case is $z < x$, meaning that M_x is the QMV. In that case deviation to x is again profitable: indeed, $V_{L,M_x}(x)$ is maximal among all $V_{L,M_x}(y)$ for $y \geq x$, and the path $\phi_{R_z}(x), \phi_{R_z}^2(x), \ldots$ yields, pointwisely, lower utility than the path $x, \phi_{R_z}(x), \phi_{R_z}^2(x), \ldots$. This shows that $\phi_{R_z}(x) \leq x$.

Now suppose that $x \geq 0$ is stable in E. Then it does not change if a shock never arrives, and the result holds trivially. Once a transition to R_z has taken place, we have $\phi_{R_z}(x) \leq x$, implying that the entire path satisfies this property. If there is never a transition to L, then the statement again holds; otherwise, suppose that this shock arrives when the society is at $s \leq x$. Since $\phi_L(x) \leq x$, we must have that $\phi_L(s) \leq x$, and so the entire path lies below x. Convergence follows from finiteness of S, and the ultimate state y satisfies $y \leq x$. ■