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Group knowledge and individual introspection*

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Abstract

I study distributed knowledge, which is what a group of privately informed agents can possibly know if they communicate freely with one another. Contrary to the extant literature, I study differently introspective agents. Three categories of agents are considered: non-introspective, positively introspective, and fully introspective. When a non-introspective agent knows something, she may fail to know that she knows it. On the contrary, when a fully introspective agent knows something, she always knows that she knows it. A fully introspective agent is positively introspective and, when she does not know something, she also knows that she does not know it. I give two equivalent characterizations of distributed knowledge: one in terms of knowledge operators and the other in terms of possibility relations, i.e., binary relations. I study distributed knowledge by modelling explicitly the communication and inference making process behind it. I show that there are two significantly different cases to consider. In the first, distributed knowledge is driven by the group member who is sophisticated enough to replicate all the inferences that anyone else in the group can make. In the second case, no member is sophisticated enough to replicate what anyone else in the group can infer. As a result, distributed knowledge is determined by a two-person subgroup who can jointly replicate what others infer. The latter case depicts a wisdom-of-the-crowd effect, in which the group knows more than what any of its members could possibly know by having access to all the information available within the group.

JEL CLASSIFICATION: C70, D82, D83

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1 Introduction

The goal of this paper is to explore the relationship between individual and aggregate knowledge. Broadly speaking, the question we are interested in is: What do groups know? As it is often the case with group properties, the answer depends on the criterion by which individual knowledge is aggregated. There are two limiting cases. On the one hand, we have *common knowledge*. A fact is common knowledge when it is public: Everyone in the group knows that fact, everyone knows that everyone knows it, and so on. On the other hand, we have *distributed knowledge*. A fact is distributed knowledge if it can be known by merging what each group member knows. For example, if a group member, say A, knows a fact p and another member B knows that p implies q, then knowledge of q is distributed between A and B. As Fagin et al. (1995) incisively put it, common knowledge is "what every fool knows"; distributed knowledge is "what a wise man would know". Taken together, these two notions circumscribe all feasible configurations of group knowledge. To wit, a group cannot know less than what is common knowledge, and cannot know more than what is distributed knowledge. While much effort has been devoted to understanding common knowledge, both in economics and neighboring disciplines, less attention has been paid to distributed knowledge. In this paper, we concentrate on the latter concept.

Why focusing on distributed knowledge? First, since we know a great deal about common knowledge, we want to get an equally clear picture of the other limiting case. In more practical terms, a better understanding of distributed knowledge would enable us to assess what groups such as juries, parliaments, and committees, can *potentially* know and, as a consequence, what decisions they could or could not make. Second, and more substantially, the very concept of distributed knowledge is not as straightforward as it may seem, and the extant literature has overlooked a crucial aspect of it: individual inference making. More specifically, distributed knowledge presupposes some form of communication in order for group members to share what they know, and knowledge sharing calls for individual inference making. If B says to A that p implies q, then A has to infer that knowledge of q follows by combining her knowledge of p with the new piece of knowledge learned from B. Without the latter inference, we cannot truly say that knowledge of q is distributed between A and B. The point we want to make here is that distributed knowledge crucially depends on how people assess what they learn from others. We now give an elementary, motivating example to show how the standard treatment of distributed knowledge may fail to accommodate individual inference making, so leading to unsatisfactory results.

Consider a situation of incomplete information with two possible states of the world, ω_1

and ω_2. There are two persons, Ann (a) and Bob (b), whose knowledge is represented by the possibility relations and, equivalently, by the knowledge operators in the table below.

Events	K_a	K_b
$\emptyset$	$\emptyset$	$\emptyset$
$\{\omega_1\}$	$\emptyset$	$\{\omega_1\}$
$\{\omega_2\}$	$\emptyset$	$\emptyset$
$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$

States	P_a	P_b
ω_1	$\{\omega_1, \omega_2\}$	$\{\omega_1\}$
ω_2	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$

Table 1: Knowledge operators (left) and possibility relations (right)

The knowledge we are referring to is about events, which are subsets of the set of possible states $\{\omega_1, \omega_2\}$. For each state ω, a person's possibility relation tells us the states that that person considers possible when ω occurs. For a given event, a person's knowledge operator tells us the states at which that person knows that the given event holds. A person knows an event at a state ω if that event holds at all the states that he or she considers possible at ω. Thus we can see that Bob knows the following: that $\{\omega_1\}$ holds only when ω_1 is true, and that $\{\omega_1, \omega_2\}$ always holds. As for Ann, she just knows that $\{\omega_1, \omega_2\}$ holds in both states. Now, what is distributed knowledge between Ann and Bob? As per the standard characterization (see, e.g., Fagin et al. (1995)), distributed knowledge can be found by taking the intersection of the individual possibility relations. In our case, distributed knowledge would coincide with what Bob knows. We claim that this is not always satisfactory, especially when people are not equally rational. In the example, Bob is not fully rational because he fails to make the following inference: "There are two states. In one of them I know that $\{\omega_1\}$ holds, in the other I do not. Therefore, when I do not know whether $\{\omega_1\}$ holds, the true state must be ω_2. In sum, I always know the true state." In other words, Bob is not negatively introspective and, as a consequence, his possibility relation does not induce a partition of the set of possible states. As for Ann, let us assume she is fully rational, which is consistent with her having a partitional possibility relation. If B shares all his knowledge with Ann, the latter will make the inference that Bob failed to make on his own. Thus Ann will always know the true state. Subsequently, if Ann shares her revised, perfect knowledge with Bob, he will be able to learn the true state also at ω_2 in spite of his lack of negative introspection. In sum, perfect knowledge of the prevailing state is distributed between Ann and Bob. The standard characterization is of no avail in our case because it implicitly assumes that all group members are equally rational.

The main questions we ask in this paper are: How to find distributed knowledge in groups where people need not be equally rational? How do individual reasoning skills affect what a group can possibly know? More specifically, we want to characterize distributed knowledge in groups where people are more or less rational depending on how introspective they are. We consider three kinds of persons. Fully rational people are fully introspective. Not only do they know everything they know, they also know everything they do not know. Less rational are people who are just positively introspective. As Bob in the example above, they know everything they know, but they may fail to know something they do not know. Finally, we have non-introspective people, who may fail to know something they do or do not know.

Contrary to existing models, we choose to formalize explicitly the communication and inference making processes that lead to distributed knowledge. As (Rubinstein, 1998, p. 43) aptly remarks in his discussion of the standard set-theoretic model of interactive knowledge, "the model has to be thought of as a reduced form derived from a more complete model, one that captures the decision maker's inference process." Thus we can say that our strategy in this paper is to develop such a more complete model in order to characterize distributed knowledge. We assume that communication is honest, non-strategic, and truthful. As for inference making, we introduce three revision operators to pinpoint what inferences a person can make upon learning a new piece of knowledge. Each revision operator maps any knowledge operator to a revised knowledge operator. One revision operator captures positive introspection. In particular, it encapsulates the cognitive process by which one infers higher-order knowledge of an event ("I know that I know p") from first-order knowledge of it ("I know p"). Similarly, the second revision operator describes full introspection. This is a process by which not only can one infer higher-order knowledge of an event, but one can also infer higher-order knowledge about the lack of knowledge of an event: "If I do not know p, then I know that I do not know p." The third and final revision operator, called distributive closure, is a logical closure and is not related to introspection. It captures the fact that one is able to make all inferences based on logical deductions, but not necessarily those based on introspection about one's knowledge. The three revision operators enable us to keep track of how people at different levels of rationality process what they learn from others. A fully introspective person makes inferences that are represented by the full introspection operator and the distributive closure; a positively introspection person bases all her inferences on the positive introspection operator and the distributive closure; a non-introspective person uses only the distributive closure.

We make the following contributions. First, we provide a general, set-theoretic framework for the study of knowledge revision. The framework is general enough to accommodate agents who are fully introspective, positively introspective, or not introspective at all. Our main innovation is the introduction of revision operators, which enable us to link knowledge revision to inference making skills with relative ease. While our overarching goal is to characterize distributed knowledge, the framework we offer is of independent interest and can be thought of as a foundation for standard models of interactive knowledge. As the quote from A. Rubinstein above emphasizes, knowledge is usually taken as given in those models. With our framework, it is possible to examine how people reason and how they arrive at knowing what they do. Second, we give a full characterization of distributed knowledge for groups whose members may have different levels of introspection. The characterization relies on the revision operators mentioned above, so linking individual reasoning skills to aggregate knowledge. The standard treatment of distributed knowledge reduces to a particular instance of our model, in which all group members share the same level of introspection. Third, we offer an equivalent characterization of distributed knowledge in terms of possibility relations. In order to establish a part of this characterization, we introduce a new generalization of the left trace for binary relations, which we call left n-ary trace. All in all, finding distributed knowledge through revision operators is intuitive but might be rather laborious; using possibility relations is perhaps less intuitive but computationally less demanding.

Here are our main findings. First, inference making is order dependent. Consider a fully or positively introspective agent who learns what another person knows. Being introspective, our agent processes this influx of knowledge by making inferences based on the appropriate introspection revision operator and the distributive closure. We find that, in general, each of the introspection revision operators and the distributive closure do not commute. Consequently, what an agent ultimately knows depends both on her degree of introspection and the order in which she makes inferences. It may well be the case that two equally introspective agents process the very same body of knowledge but reach different conclusions just because of the order in which they reason. We keep track of such order dependence by introducing types, each of which tells us how introspective an agent is and what order she follows in making inferences. Second, distributed knowledge is type dependent. For a given profile of knowledge operators or possibility relations, different profiles of individual types give rise to possibly different configurations of distributed knowledge. The latter result highlights how distributed knowledge depends on the distribution of inference skills within a group. Finally, our main result is contained in Proposition 7. There are two conceptually different cases. In the first, there is a group member who is able to make all the inferences that everyone else in the group can make. We can think of such a member as the "wise man" mentioned in the first paragraph above. Distributed knowledge can easily be found by letting everyone communicate everything they know to the wise man. This result is essentially analogous to the standard characterizations of distributed knowledge for homogeneous groups. In the second case, there is not a wise man in the group. That is, no group member can replicate all the inferences that everyone else can make. However, there must be at least two agents who can jointly infer what everyone else in group can. We find that these two agents can jointly act as the "wise man". As a consequence, distributed knowledge can be found in a two-step procedure. First, everyone communicates her knowledge operator to everyone else. Then, distributed knowledge can be attained in the second step by letting just the two agents who act as wise man talk to each other. Remarkably, distributed knowledge in this case coincides with what we would get if we added a single wise man to the group, i.e., someone who can replicate all the inferences in the group. In sum, the second case depicts a wisdom-of-the-crowd effect, in which the group knows more than what each of its members could possibly do alone.

The rest of the paper is organized as follows. We review the related literature in the next section. In Section 3, we introduce the baseline model, which consists of knowledge operators and possibility relations on a fixed state space. Then in Section 4 we introduce revision operators in order to formalize inference making and study their properties. We put revision operators into use in Section 5, where we define types and see how differently introspective agents revise knowledge. Finally, we give a full characterization of distributed knowledge in Section 6.

2 Related Literature

This paper touches upon different literatures on the epistemic foundations of game theory and learning. The concept of distributed knowledge was introduced by Halpern and Moses (1990). Since then, it has been studied mostly in computer science and epistemic logic. Relevant contributions include van der Hoek et al. (1999), from which we borrow the principle of full communication, Roelofsen (2012), and Ågotnes and Wáng (2017). There are at least two crucial differences between this literature, which is rooted in computer science and logic, and our paper. First, we do not use any formalism from epistemic logic. As is standard in economics and game theory, we study our problem within the framework of knowledge operators and possibility relations *à la* Aumann. The classical sources are Aumann (1976) and Aumann (1999). Differently put, we study distributed knowledge in a purely semantic model. Second, and more substantially, while distributed knowledge is commonly studied for a fixed modal system, we cover cases where different agents in the same group may satisfy different modal systems. Our three categories of agents, namely fully introspective, positively introspective, and non-introspective, correspond to the three modal systems **S5**, **S4**, and **T**, respectively.

To the best of our knowledge, the closest paper is Fukuda (2019). He studies distributed knowledge in a set-theoretic, semantic framework. He allows agents to hold false beliefs. In other words, he dispenses with the veridicality axiom. In this regard, his analysis is more general than ours. On the other hand, Fukuda (2019) does not model the communication and learning process that leads to distributed knowledge. We do provide such a model and show that it is crucial in order to obtain the wisdom-of-the-crowd effect in part (b) of Proposition 7; namely, the fact that the group can know more than what any of its members can possibly know by herself. Thus the two characterizations of distributed knowledge are not equivalent: in Fukuda (2019), the characterization is axiomatic, whereas ours is based on communication and learning. All in all, the two papers overlap but are not nested into each other.

Our work is also related to the AGM belief revision theory of Alchourrón et al. (1985). As in their framework, we model how agents revise knowledge upon receiving new information. There is a crucial difference though. In AGM theory, new information is in the form of a single proposition; in our paper, new information is given by a knowledge operator, which describes what an agent knows at every possible state and not just at the prevailing one. This means that agents in our model revise both their actual and their counterfactual knowledge. The learning process in our model is also close to Basu (2019) and Sadler (2021). While Basu (2019) studies probabilistic beliefs in an AGM framework, we study non-probabilistic beliefs and allow for revision of counterfactual knowledge. As for Sadler (2021), his model is set-theoretic as it is ours. However, he does not consider counterfactual knowledge. In addition, his learning axioms are rooted in behavioral economics while our revision operators mirror the introspection axioms from standard models of interactive knowledge and beliefs.

Some of our results are analogous to those of Mueller-Frank (2014). In the context of observational learning, and in a population in which not everyone is Bayesian, Mueller-Frank (2014) shows that belief aggregation is driven by the Bayesian members of the group. Even though the context is different, this result is conceptually close to the first part of our Proposition 7, in which we show that distributed knowledge is determined by the highest type in the group, provided that such highest type exists.

3 Knowledge operators and possibility relations

Fix a non-empty, finite set of states of the world Ω and a finite set of agents $I = \{1, 2, \ldots, N\}$, with $N \geq 2$. Subsets of Ω are called events. Agents' knowledge about events can be represented by knowledge operators (Subsection 3.1) or possibility relations (Subsection 3.2). I confine myself to the case in which the two representations are equivalent (Subsection 3.3).

3.1 Knowledge operators

A knowledge operator K is a function $K : 2^{\Omega} \to 2^{\Omega}$. An event $A \subseteq \Omega$ holds at a state $\omega \in \Omega$ if $\omega \in A$. If K_i is agent i's knowledge operator, then $K_i(A)$ is the event "i knows that A". To wit, $K_i(A)$ is the set of states in which i knows that A holds.

A knowledge operator K satisfies:

- *distributivity* if for all $A, B \subseteq \Omega$,

$$K(A \cap B) = K(A) \cap K(B); \tag{K0}$$

- *necessitation* if

$$K(\Omega) = \Omega; \tag{K0$'$}$$

- *veridicality* if for all $A \subseteq \Omega$,

$$K(A) \subseteq A; \tag{K1}$$

- *positive introspection* if for all $A \subseteq \Omega$,

$$K(A) \subseteq K(K(A)); \tag{K2}$$

- *negative introspection* if for all $A \subseteq \Omega$,

$$\neg K(A) \subseteq K(\neg K(A)), \tag{K3}$$

  where $\neg$ denotes the set complement operation.

The five axioms above are standard and have been studied extensively—see, among many others, Fagin et al. (1995), Morris (1996), Rubinstein (1998), and Samet (1990). Recall that negative introspection implies positive introspection and that distributivity implies monotonicity, which is defined as follows. An operator K is *monotone* if for all $A,B \subseteq \Omega$,

$$A \subseteq B \implies K(A) \subseteq K(B). \tag{1}$$

Call $\mathscr{K}$ the set of all knowledge operators on Ω. The following subsets of $\mathscr{K}$ will be used later on:

- $\mathscr{K}^v$ is the set of all knowledge operators that satisfy (K1);
- $\mathscr{K}^0$ is the set of all knowledge operators that satisfy (K0) and (K0′);
- $\mathscr{K}^1$ is the set of all knowledge operators that satisfy (K0), (K0′), and (K1);
- $\mathscr{K}^2$ is the set of all knowledge operators that satisfy (K0), (K0′), (K1), and (K2);
- $\mathscr{K}^3$ is the set of all knowledge operators that satisfy (K0), (K0′), (K1), (K2), and (K3).

It is clear that $\mathscr{K}^3 \subseteq \mathscr{K}^2 \subseteq \mathscr{K}^1 \subseteq \mathscr{K}^0 \subseteq \mathscr{K}$.

3.2 Possibility relations

A possibility relation P is a binary relation over Ω, i.e., $P \subseteq \Omega \times \Omega$. If (ω, ω') is in agent i's possibility relation, then i considers ω' possible when ω occurs. It is often convenient to describe a possibility relation with its lower contour sets. Formally, the lower contour set of P at ω is the set $P(\omega) := \{\omega' \in \Omega : (\omega, \omega') \in P\}$. In words, $P(\omega)$ is the set of states that, according to P, are considered possible when ω occurs.

A possibility relation P is:

- *reflexive* if for all $\omega \in \Omega$,
$$(\omega, \omega) \in P;$$
- *transitive* if for all $\omega, \omega', \omega'' \in \Omega$,
$$(\omega, \omega') \in P \text{ and } (\omega', \omega'') \in P \implies (\omega, \omega'') \in P;$$
- *Euclidean* if for all $\omega, \omega', \omega'' \in \Omega$,
$$(\omega, \omega') \in P \text{ and } (\omega, \omega'') \in P \implies (\omega', \omega'') \in P;$$

- *symmetric* if for all $\omega, \omega' \in \Omega$,

$$(\omega, \omega') \in P \implies (\omega', \omega) \in P.$$

Call $\mathscr{P}$ the set of all possibility relations on Ω. The following subsets of $\mathscr{P}$ will be used later on:

- $\mathscr{P}^1$ is the set of all *reflexive* relations on Ω;
- $\mathscr{P}^2$ is the set of all *reflexive* and *transitive* relations on Ω;
- $\mathscr{P}^3$ is the set of all *reflexive*, *transitive* and *Euclidean* relations on Ω.

It is clear that $\mathscr{P}^3 \subseteq \mathscr{P}^2 \subseteq \mathscr{P}^1 \subseteq \mathscr{P}$. It is also easy to check that $P \in \mathscr{P}^3$ if and only if P is an equivalence relation, i.e., a reflexive, symmetric and transitive relation. A similar taxonomy of possibility relations with a discussion of their properties can be found in Geanakoplos (2021).

Some of the results in Section 5 involve two specific types of binary relations: the left trace and the left n-ary trace. Following (Aleskerov et al., 2007, p. 69), the *left trace* of $P \in \mathscr{P}$ is the binary relation T_P constructed as follows:

$$\text{For all } \omega, \omega' \in \Omega, \quad (\omega, \omega') \in T_P \iff P(\omega') \subseteq P(\omega).$$

It is easy to check that T_P is reflexive and transitive. The symmetric part of T_P is the binary relation E_P obtained as:

$$\text{For all } \omega, \omega' \in \Omega, \quad (\omega, \omega') \in E_P \iff (\omega, \omega') \in T_P \text{ and } (\omega', \omega) \in T_P.$$

It is clear that $(\omega, \omega') \in E_P$ if and only if $P(\omega') = P(\omega)$.

The left n-ary trace is a generalization of the left trace. To the best of my knowledge, this is the first paper where the left n-ary trace is introduced. For a given n-tuple $\mathbf{P} = (P_1, \ldots, P_n)$ of relations in $\mathscr{P}$, the *left n-ary trace* of $\mathbf{P}$ is the binary relation $T_{\mathbf{P}}$ constructed as follows:

$$\text{For all } \omega, \omega' \in \Omega, \quad (\omega, \omega') \in T_{\mathbf{P}} \iff \begin{array}{l} \text{for all } i \in \{1, \ldots, n\} \text{ there exists a } j \in \{1, \ldots, n\} \\ \text{such that } P_j(\omega') \subseteq P_i(\omega). \end{array}$$

The left n-ary trace $T_{\mathbf{P}}$ is reflexive and transitive. Clearly, $n = 1$ and $T_{\mathbf{P}} = T_P$ when

$\mathbf{P} = (P)$. The symmetric part of $T_{\mathbf{P}}$ is denoted $E_{\mathbf{P}}$ and defined in the obvious way:

$$\text{For all } \omega, \omega' \in \Omega, \quad (\omega, \omega') \in E_{\mathbf{P}} \iff (\omega, \omega') \in T_{\mathbf{P}} \text{ and } (\omega', \omega) \in T_{\mathbf{P}}.$$

Table 2 contains an example with two possibility relations P_1 and P_2 on $\Omega = \{\omega_1, \ldots, \omega_4\}$, the corresponding left traces T_{P_1} and T_{P_2}, and the left binary trace $T_{\mathbf{P}}$, where $\mathbf{P} = (P_1, P_2)$.

ω	$P_1(\omega)$	$P_2(\omega)$	$T_{P_1}(\omega)$	$T_{P_2}(\omega)$	$T_{\mathbf{P}}(\omega)$
ω_1	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$
ω_2	$\{\omega_2, \omega_3\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$
ω_3	$\{\omega_2, \omega_3, \omega_4\}$	$\{\omega_2, \omega_3, \omega_4\}$	$\{\omega_2, \omega_3, \omega_4\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3, \omega_4\}$
ω_4	$\{\omega_4\}$	$\{\omega_1, \omega_4\}$	$\{\omega_4\}$	$\{\omega_4\}$	$\{\omega_4\}$

Table 2: The left traces and the left binary trace of P_1 and P_2.

This subsection concludes with a characterization of $E_{\mathbf{P}}$ in terms of minimal sets. Such a characterization is often helpful because it enables us to find $E_{\mathbf{P}}$ without having to compute $T_{\mathbf{P}}$ before. Given $\mathbf{P}$, the minimal set of $\mathbf{P}$ at ω is the set $\mathsf{Min}(\omega)$ of minimal elements of $\{P_i(\omega) : i \in \{1, \ldots, n\}\}$. More formally,

$$\mathsf{Min}(\omega) := \{P_i(\omega) : i \in \{1, \ldots, n\} \text{ and } P_j(\omega) \subseteq P_i(\omega) \implies P_j(\omega) = P_i(\omega) \text{ for all } j \in \{1, \ldots, n\}\}.$$

Claim 1. *Given an n-tuple $\mathbf{P} = (P_1, \ldots, P_n)$ of binary relations in $\mathscr{P}$, we have that for all $\omega, \omega' \in \Omega$,*

$$(\omega, \omega') \in E_{\mathbf{P}} \iff \mathsf{Min}(\omega) = \mathsf{Min}(\omega').$$

Proof. It is immediate that $\mathsf{Min}(\omega) = \mathsf{Min}(\omega')$ implies $(\omega, \omega') \in E_{\mathbf{P}}$. To show the other direction, suppose $(\omega, \omega') \in E_{\mathbf{P}}$. Take any $P_i(\omega) \in \mathsf{Min}(\omega)$. Since $(\omega, \omega') \in T_{\mathbf{P}}$, there is a j such that $P_j(\omega') \subseteq P_i(\omega)$. Furthermore, there exists a k such that $P_k(\omega') \in \mathsf{Min}(\omega')$ and $P_k(\omega') \subseteq P_j(\omega')$. Thus $P_k(\omega') \subseteq P_i(\omega)$. Analogously, since $(\omega', \omega) \in T_{\mathbf{P}}$ we can find a $P_\ell(\omega) \in \mathsf{Min}(\omega)$ such that $P_\ell(\omega) \subseteq P_k(\omega')$. Since $P_i(\omega)$ is a minimal element, we must have $P_\ell(\omega) = P_k(\omega') = P_i(\omega)$. Therefore, $\mathsf{Min}(\omega) \subseteq \mathsf{Min}(\omega')$. The reverse set inclusion can be proved analogously. □

3.3 The duality between knowledge operators and possibility relations

Throughout this paper, I confine myself to knowledge operators that can be represented by a possibility relation. More generally, it is well-known that every possibility relation represents a knowledge operator, but only certain knowledge operators can be represented by a possibility relation.

A possibility relation P represents a knowledge operator K if for all events $A \subseteq \Omega$,

$$K(A) = \{\omega \in \Omega : P(\omega) \subseteq A\}. \tag{2}$$

In words, an agent knows A at ω if A holds at every state that the agent considers possible when ω occurs. Notice that the operator K in (2) always satisfies both distributivity (K0) and necessitation (K0′).

Consider the function $f : \mathscr{P} \to \mathscr{K}^0$ that associates to each relation P the knowledge operator $f(P)$ defined as per (2). Proposition 1 below shows that f is a dual isomorphism[1], i.e., an order-reversing bijection. As for the underlying order structure, $\mathscr{P}$ is partially ordered by set inclusion. In addition, $\mathscr{K}$, and consequently its subset $\mathscr{K}^0$, is partially ordered by pointwise set inclusion. Specifically, for any $K, K' \in \mathscr{K}$,

$$K \leq K' \iff K(A) \subseteq K'(A) \text{ for all } A \subseteq \Omega.$$

In words, when $K \leq K'$ a person with knowledge K' knows at least as much as someone with knowledge K does.

Proposition 1. *The function $f : \mathscr{P} \to \mathscr{K}^0$ is a dual isomorphism. That is, f is a bijection such that*

$$P \subseteq P' \iff f(P') \leq f(P)$$

for all $P, P' \in \mathscr{P}$.

Proof. I split the argument in four independent parts.

- *f is injective*. Suppose $P \neq P'$. This means that there exists a state ω such that $P(\omega) \neq P'(\omega)$. The latter is equivalent to saying that $P'(\omega) \setminus P(\omega) \neq \emptyset$ or $P(\omega) \setminus P'(\omega) \neq \emptyset$. Suppose $P'(\omega) \setminus P(\omega) \neq \emptyset$. Call $K := f(P)$ and $K' := f(P')$. Take the event $A = P(\omega)$. We have $\omega \in K(A)$ but $\omega \notin K'(A)$. The case $P(\omega) \setminus P'(\omega) \neq \emptyset$ is analogous.

- *f is surjective*. For every $K \in \mathscr{K}^0$, one needs to find a $P \in \mathscr{P}$ such that K is represented by P according to (2). Suppose $K \in \mathscr{K}^0$. Construct the binary relation $P \in \mathscr{P}$ as follows. For all $\omega \in \Omega$,

$$P(\omega) := \bigcap \{A \subseteq \Omega : \omega \in K(A)\}. \tag{3}$$

[1] The terminology is borrowed from Monjardet (2007).

One can show that K is represented by P. Take any $B \subseteq \Omega$. Suppose $\omega \in K(B)$. It follows immediately by (3) that $P(\omega) \subseteq B$. Now suppose that $P(\omega) \subseteq B$ for some $\omega \in \Omega$. Since $\omega \in \Omega = K(\Omega)$ by necessitation, it follows by (3) that $P(\omega)$ is always the intersection of a *non-empty* family of events $A_1, \ldots, A_m$, with $m \geq 1$. By definition, $\omega \in \cap_{i=1}^m K(A_i)$. By distributivity, we have that $\cap_{i=1}^m K(A_i) = K(\cap_{i=1}^m A_i)$. Since $\cap_{i=1}^m A_i = P(\omega) \subseteq B$, it follows by monotonicity (1) that $K(\cap_{i=1}^m A_i) \subseteq K(B)$. Therefore, $\omega \in K(B)$.

- $P \subseteq P' \implies f(P') \leq f(P)$. Suppose $P \subseteq P'$, which is equivalent to $P(\omega) \subseteq P'(\omega)$ for all states ω. Call $K := f(P)$ and $K' := f(P')$. Thus for all events A we have

$$K'(A) = \{\omega : P'(\omega) \subseteq A\} \subseteq \{\omega : P(\omega) \subseteq A\} = K(A).$$

- $f(P') \leq f(P) \implies P \subseteq P'$. Call $K := f(P)$ and $K' := f(P')$. By way of contradiction, suppose $K' \leq K$ but $P \not\subseteq P'$. The latter is equivalent to $P(\omega) \setminus P'(\omega) \neq \emptyset$ for some ω. Choose $B = P'(\omega)$. Clearly, $\omega \in K'(B)$. Since $P(\omega) \not\subseteq P'(\omega)$, we also have $\omega \notin K(B)$. This contradicts the assumption that $K'(A) \subseteq K(A)$ for all events A.

□

In light of Proposition 1, knowledge operators and possibility relations are interchangeable provided that the former satisfy distributivity and necessitation. This implies that one can use (2) to find the knowledge operator represented by a given possibility relation P. Conversely, one can use (3) to retrieve the possibility relation that represents a given knowledge operator K. Equivalently, one can also use the relation $\hat{P}$ employed by Morris (1996) in the proof of his Theorem 1 and defined as follows. For all $\omega \in \Omega$,

$$\hat{P}(\omega) := \left\{\omega' \in \Omega : \omega \notin K(\neg\{\omega'\})\right\}. \tag{4}$$

In words, ω' is in $\hat{P}(\omega)$ if the agent does not know at ω that $\neg\{\omega'\}$ holds; thus, she thinks that ω' is possible. One can show that $\hat{P}$ in (4) is equal to P in (3).

Finally, the dual isomorphism between K and P implies that each property of the former corresponds to a specific property of the latter, and vice versa.

Proposition 2. *Let K be a knowledge operator in $\mathscr{K}^0$ and let $P \in \mathscr{P}$ be the possibility relation that represents K. Then the following hold.*

(i) $K \in \mathscr{K}^1$ *if and only if* $P \in \mathscr{P}^1$.

(ii) $K \in \mathscr{K}^2$ *if and only if* $P \in \mathscr{P}^2$.

(iii) $K \in \mathscr{K}^3$ *if and only if* $P \in \mathscr{P}^3$.

Proof. The proposition is an immediate corollary of Lemma 2 in Morris (1996). $\square$

4 Revision operators

The notion of distributed knowledge I adopt in this paper presupposes a process of communication and inference making. I introduce three revision operators in order to model inference making. A revision operator captures a specific set of inferences that an agent may possibly make upon learning another agent's knowledge operator. The first revision operator describes inferences based on positive introspection (K2). The second operator is based on both positive (K2) and negative introspection (K3), and the last one captures distributivity (K0). The exact set of inferences that an agent actually makes depends on the agent's level of introspection and is represented by revision types, which are introduced in Section 5.

A few definitions before getting to revision operators. Given a knowledge operator $K \in \mathscr{K}$, the image of K is

$$\mathsf{Img}(K) := \{K(A) : A \subseteq \Omega\},$$

and the set of complements of the elements of $\mathsf{Img}(K)$ is

$$\mathsf{Img}(\neg K) := \{\neg K(A) : A \subseteq \Omega\}.$$

Finally, the set of fixed points of K is

$$\mathsf{Fix}(K) := \{A \subseteq \Omega : A = K(A)\}.$$

4.1 Positive Introspection

The positive introspection operator represents inferences based on the positive introspection axiom (K2). Formally, the *positive introspection operator* is a function $(.)^+ : \mathscr{K}^v \to \mathscr{K}^v$ that maps each K to the revised knowledge operator K^+ constructed as follows. For all $A \subseteq \Omega$,

$$K^+(A) = \begin{cases} A & \text{if } A \in \mathsf{Img}(K) \\ K(A) & \text{if } A \notin \mathsf{Img}(K). \end{cases}$$

In words, if $A = K(B)$ for some event B, then knowledge of B is equivalent to A. Thus, a positively introspective agent realizes that her knowing B is equivalent to *knowing* A. As a result, her revised knowledge about A is $K^+(A) = A$. If $A \neq K(B)$ for all $B \subseteq \Omega$, then there is no event $K(B)$ equivalent to A. Hence the agent's knowledge of A is left unchanged and $K^+(A) = K(A)$.

As an example of the reasoning behind the positive introspection operator, consider a non-introspective agent whose knowledge at a certain point in time is described by K in Table 3. The state space is $\Omega = \{\omega_1, \omega_2, \omega_3\}$. If the agent were positively introspective, she would pause a moment to reflect upon her knowledge and reason along the following lines. "I never know when $\{\omega_2\}$ holds. But I know that $\{\omega_1, \omega_2\}$ holds at ω_2, and only at ω_2. Therefore, when I am at a state in which I know that $\{\omega_1, \omega_2\}$ holds, I can conclude that the state must be ω_2. Thus I know when $\{\omega_2\}$ holds." A positively introspective agent could follow such a line of reasoning either before or after the true state of the world has occurred. In the former case, the agent ponders what she would know at every possible state of the world, so thinking about her future knowledge. In the other case, the agent reflects both on her factual and counterfactual knowledge. Assuming ω_2 has occurred, the realization that $K^+(\{\omega_2\}) = \{\omega_2\}$ is a piece of factual knowledge. On the contrary, the realization that $K^+(\{\omega_3\}) = \{\omega_3\}$ refers to what the agent would have known if ω_3 had occurred.

A	$K(A)$	$K^+(A)$	$K^\pm(A)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\omega_1\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\omega_2\}$	$\emptyset$	$\{\omega_2\}$	$\{\omega_2\}$
$\{\omega_3\}$	$\emptyset$	$\{\omega_3\}$	$\{\omega_3\}$
$\{\omega_1, \omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_1, \omega_2\}$
$\{\omega_1, \omega_3\}$	$\emptyset$	$\emptyset$	$\{\omega_1, \omega_3\}$
$\{\omega_2, \omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$
$\{\omega_1, \omega_2, \omega_3\}$	$\{\omega_1, \omega_2, \omega_3\}$	$\{\omega_1, \omega_2, \omega_3\}$	$\{\omega_1, \omega_2, \omega_3\}$

Table 3: A knowledge operator K and the corresponding revised operators K^+ and $K^\pm$.

One can also interpret the positive introspection operator in terms of self-evident events. Recall that an event is self-evident if $A \subseteq K(A)$. Assuming veridicality, self-evident events are nothing other than the fixed points of the knowledge operator in hand. It is easy to check that K^+ preserves the fixed points of K. That is, if A is self-evident according to K, then it is self-evident according to K^+ as well. More generally, we have that that $\mathsf{Img}(K) = \mathsf{Fix}(K^+)$. This means that $(.)^+$ turns all the elements in the image of K into self-evident events. The following claim says that K^+ is the least knowledge operator to accomplish that.

Claim 2. *Let $K \in \mathscr{K}^v$. For all $J \in \mathscr{K}$, if $K \leq J$ and $\mathsf{Img}(K) \subseteq \mathsf{Fix}(J)$, then $K^+ \leq J$.*

Proof. Suppose $K \leq J$ and $\mathsf{Img}(K) \subseteq \mathsf{Fix}(J)$. If $A \in \mathsf{Img}(K)$, then $K^+(A) = A = J(A)$. Otherwise, $K^+(A) = K(A) \subseteq J(A)$. □

The following proposition lists the main properties of the positive introspection operator.

Proposition 3. *For all $K \in \mathscr{K}^v$, the following are true.*

(i) K^+ *satisfies (a) veridicality and (b) positive introspection.*

(ii) $K^+ = K$ *if and only if* K *satisfies positive introspection.*

(iii) $K \leq K^+$*, i.e., the positive introspection operator* $(.)^+$ *is extensive.*

(iv) $K^{++} = K^+$*, i.e., the positive introspection operator* $(.)^+$ *is idempotent.*

Proof. See Appendix A.1. □

The extensivity of $(.)^+$ says that no piece of knowledge is lost or forgotten when changing K into K^+. Idempotence says that K^+ cannot be further refined through inferences rooted in the positive introspection axiom.

The positive introspection operator $(.)^+$ is not a closure operator in that it fails to be monotone[2], i.e., it is not the case that $K \leq J$ implies $K^+ \leq J^+$ for all $K, J \in \mathscr{K}^v$. An example is given in Table 4, where the underlying state space is $\Omega = \{\omega_1, \omega_2\}$.

A	$K(A)$	$J(A)$	$K^+(A)$	$J^+(A)$	$K^\pm(A)$	$J^\pm(A)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\omega_1\}$	$\emptyset$	$\emptyset$	$\{\omega_1\}$	$\emptyset$	$\{\omega_1\}$	$\emptyset$
$\{\omega_2\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{\omega_2\}$	$\emptyset$
$\{\omega_1, \omega_2\}$	$\{\omega_1\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$

Table 4: An example where $K \leq J$ but $K^+ \not\leq J^+$ and $K^\pm \not\leq J^\pm$.

[2] The monotonicity mentioned here should not be confused with the monotonicity in (1). The former refers to an operator that maps knowledge operators to knowledge operators. The latter refers to a single knowledge operator that maps events to events.

4.2 Full Introspection

The full introspection operator describes inferences rooted in negative introspection (K3) and, consequently, positive introspection (K2). Formally, the *full introspection operator* is a function $(.)^{\pm} : \mathscr{K}^v \to \mathscr{K}^v$ that maps each K to the revised knowledge operator $K^{\pm}$ constructed as follows. For all $A \subseteq \Omega$,

$$K^{\pm}(A) = \begin{cases} A & \text{if } A \in \mathsf{Img}(K) \cup \mathsf{Img}(\neg K) \\ K(A) & \text{otherwise.} \end{cases}$$

It is clear that $(.)^{\pm}$ subsumes $(.)^{+}$. The difference between the two is that the full introspection operator also takes the following into account. If $A = \neg K(B)$ for some event B, then lack of knowledge of B is equivalent to A. Thus, a fully introspective agent realizes that her not knowing B is equivalent to *knowing* A. Hence, her revised knowledge about A is $K^{\pm}(A) = A$.

As an example, consider again an agent with knowledge K in Table 3. If the agent were fully introspective, she would think through her knowledge as follows. "I know that $\{\omega_1, \omega_2\}$ holds at ω_2, and only at ω_2. This means that I do not know $\{\omega_1, \omega_2\}$ when either ω_1 or ω_3 occurs. Therefore, when I am at a state in which I do not know $\{\omega_1, \omega_2\}$, I can conclude that the state must be either ω_1 or ω_3. So I know when $\{\omega_1, \omega_3\}$ occurs."

In terms of self-evident events, it is easy to check that $\mathsf{Img}(K) \cup \mathsf{Img}(\neg K) = \mathsf{Fix}(K^{\pm})$. Thus $(.)^{\pm}$ turns all the events in the image of K and their complements into self-evident events. In addition, $K^{\pm}$ is the least knowledge operator to perform such a transformation.

Claim 3. *Let $K \in \mathscr{K}^v$. For all $J \in \mathscr{K}$, if $K \leq J$ and $\mathsf{Img}(K) \cup \mathsf{Img}(\neg K) \subseteq \mathsf{Fix}(J)$, then $K^{\pm} \leq J$.*

Proof. Suppose $K \leq J$ and $\mathsf{Img}(K) \cup \mathsf{Img}(\neg K) \subseteq \mathsf{Fix}(J)$. If $A \in \mathsf{Img}(K) \cup \mathsf{Img}(\neg K)$, then $K^{\pm}(A) = A = J(A)$. Otherwise, $K^{\pm}(A) = K(A) \subseteq J(A)$. □

The following proposition lists the main properties of the positive introspection operator. Notice that $(.)^{\pm}$ is not a closure operator in that it fails to be monotone.

Proposition 4. *For all $K \in \mathscr{K}^v$, the following are true.*

(i) *$K^{\pm}$ satisfies (a) veridicality, (b) necessitation, (c) positive introspection, and (d) negative introspection.*

(ii) *$K^{\pm} = K$ if and only if K satisfies negative introspection.*

(iii) $K \leq K^{\pm}$*, i.e., the full introspection operator* $(.)^{\pm}$ *is extensive.*

(iv) $K^{\pm\pm} = K^{\pm}$*, i.e., the full introspection operator* $(.)^{\pm}$ *is idempotent.*

Proof. See Appendix A.2. □

Notice that, just like $(.)^{+}$, the full introspection operator $(.)^{\pm}$ is not a closure operator in that it fails to be monotone (see Table 4).

At this juncture, one might wonder why I do not consider a revision operator for negative introspection alone. The reason is that such an operator would eventually coincide with the full introspection operator. More specifically, one can show the following. Suppose $(.)^{-}$ is the revision operator that maps each K in $\mathscr{K}^{v}$ to the knowledge operator K^{-} constructed as follows. For all $A \subseteq \Omega$,

$$K^{-}(A) = \begin{cases} A & \text{if } A \in \mathsf{Img}(\neg K) \\ K(A) & \text{otherwise.} \end{cases}$$

It is easy to check that $(.)^{-}$ is not idempotent. However, applying $(.)^{-}$ twice gives exactly the full introspection operator, i.e., $K^{--} = K^{\pm}$.

4.3 Distributive closure

The distributive closure represents inferences based on the distributivity axiom (K0). Formally, the *distributive closure* is a function $(.)^{\mathbf{d}} : \mathscr{K}^{v} \to \mathscr{K}^{v}$ that maps each K to the revised knowledge operator $K^{\mathbf{d}}$ constructed as follows. For all $A \subseteq \Omega$,

$$K^{\mathbf{d}}(A) = \bigcup \{\cap_{i=1}^{n} K(B_i) : \, B_1, \ldots, B_n \subseteq \Omega, \, \cap_{i=1}^{n} B_i \subseteq A, \, n \geq 1\}.$$

In words, if A is jointly implied by a sequence of events $B_1, \ldots, B_n$, then simultaneous knowledge of all those n events implies knowledge of A.

As an example of the reasoning behind the distributive closure, consider a case with five states of the world $\omega_1, \ldots, \omega_5$. Suppose an agent's knowledge is represented by K in Table 5. The agent always knows the state space but never knows any other event that is not reported in the first column of the table. If her knowledge satisfied distributivity, the agent would be able to draw the following inferences. "I never know the event $\{\omega_2, \omega_5\}$. But this event holds if and only if the three events $\{\omega_1, \omega_2, \omega_3, \omega_5\}$, $\{\omega_2, \omega_3, \omega_4, \omega_5\}$ and $\{\omega_1, \omega_2, \omega_4, \omega_5\}$ simultaneously hold. Now, when ω_2 occurs, and only in that case, I know that all the three events above hold. Therefore, I can conclude that also $\{\omega_2, \omega_5\}$ holds at ω_2. And if I know that $\{\omega_2, \omega_5\}$ holds, I can infer that every superset of it holds as well."

A	$K(A)$	$K^{\mathbf{d}}(A)$
$\{\omega_2, \omega_5\}$	$\emptyset$	$\{\omega_2\}$
$\{\omega_1, \omega_2, \omega_3, \omega_5\}$	$\{\omega_1, \omega_2, \omega_3\}$	$\{\omega_1, \omega_2, \omega_3\}$
$\{\omega_2, \omega_3, \omega_4, \omega_5\}$	$\{\omega_2, \omega_3, \omega_4\}$	$\{\omega_2, \omega_3, \omega_4\}$
$\{\omega_1, \omega_2, \omega_4, \omega_5\}$	$\{\omega_1, \omega_2, \omega_4\}$	$\{\omega_1, \omega_2, \omega_4\}$
...	...	...

Table 5: A portion of a knowledge operator K and of its distributive closure $K^{\mathbf{d}}$.

The following proposition lists the main properties of the distributive closure.

Proposition 5. *For all $K, J \in \mathscr{K}^v$, the following are true.*

(i) $K^{\mathbf{d}}$ *satisfies (a) veridicality and (b) distributivity.*

(ii) $K^{\mathbf{d}} = K$ *if and only if K satisfies distributivity.*

(iii) $K \leq K^{\mathbf{d}}$*, i.e., the distributive closure* $(.)^{\mathbf{d}}$ *is extensive.*

(iv) $K^{\mathbf{dd}} = K^{\mathbf{d}}$*, i.e., the distributive closure* $(.)^{\mathbf{d}}$ *is idempotent.*

(v) $K \leq J$ *implies* $K^{\mathbf{d}} \leq J^{\mathbf{d}}$*, i.e., the distributive closure* $(.)^{\mathbf{d}}$ *is monotone.*

Proof. See Appendix A.3. □

Parts (3)-(5) above say that the distributive closure $(.)^{\mathbf{d}}$ is a closure operator. As the lemma below shows, the distributive closure also preserves positive and negative introspection. On the other hand, it is easy to check that neither of the two introspection operators preserves distributivity. This difference between the distributive closure and the two introspection operators plays an important role in defining revision types in Section 5.

Lemma 1. *Let $K \in \mathscr{K}^v$.*

(i) If K satisfies positive introspection, so does $K^{\mathbf{d}}$.

(ii) If K satisfies negative introspection, so does $K^{\mathbf{d}}$.

Proof. See Appendix A.3. □

5 Knowledge revision

In this section I formalize how differently introspective agents revise their knowledge upon learning what others know. Consider an agent, say 1, whose knowledge at a certain point in time is represented by an operator $K_1 \in \mathscr{K}^1$. One can think of K_1 as knowledge acquired by 1 on her own. Suppose 1 learns the knowledge operators $K_2, \ldots, K_n \in \mathscr{K}^1$ of $n-1$ other agents. For example, those $n-1$ agents communicate their operators directly to agent 1, as they are assumed to do in Section 6. How would 1 process the influx of knowledge she faces? I assume that agent 1 carries out her revision process in two steps. First, she merges K_1 with all the other $n-1$ knowledge operators. Formally, this amounts to forming a new knowledge operator $\cup_{i=1}^{n} K_i$ through pointwise union. For all events A, the value of $\cup_{i=1}^{n} K_i$ at A is

$$\cup_{i=1}^{n} K_i(A) := K_1(A) \cup \cdots \cup K_n(A).$$

To simplify notation, from now on I write $\cup_i$ and $\cap_i$ when the index set is clear from the context. In the second stage, agent 1 thinks through $\cup_i K_i$ using some of the revision operators introduced in the preceding section, depending on how introspective she is. More specifically, I model levels of introspection by introducing revision types. An agent's revision type indicates (a) which revision operators the agent uses, and (b) the order in which she uses them.

5.1 Revision types

For any given agent, there are three possible levels of introspection: non-introspection, positive introspection, and full introspection. A *non-introspective* agent revises knowledge only through the distributive closure $(.)^{\mathbf{d}}$. If agent 1 in the scenario above is non-introspective, she will complete her revision process by forming the knowledge operator $(\cup_i K_i)^{\mathbf{d}}$. Since the distributive closure is idempotent, the agent cannot refine her knowledge by further iterations of $(.)^{\mathbf{d}}$. Notice that $(\cup_i K_i)^{\mathbf{d}} \in \mathscr{K}^1$. Clearly, it may well be the case that new inferences based on positive or negative introspection can be drawn from $(\cup_i K_i)^{\mathbf{d}}$. A case in point is when $(\cup_i K_i)^{\mathbf{d}}$ equals K in Table 3. However, I stipulate by definition that a non-introspective agent only draws inferences that are represented by the distributive closure. The revision type of a non-introspective agent is denoted by $(\mathbf{d})$ so as to emphasize that the distributive closure fully identifies the way in which she revises knowledge.

A *positively introspective* agent uses both $(.)^{+}$ and $(.)^{\mathbf{d}}$ to revise her knowledge. Here one needs to consider two distinct cases depending on the order in which the revision operators are employed. In the first case, suppose the agent revises $(\cup_i K_i)$ by applying $(.)^+$ first, and $(.)^{\mathbf{d}}$ afterwards. That is, the agent starts out by forming $(\cup_i K_i)^+$, and then applies the distributive closure to the latter. Abusing notation, I denote $(\cup_i K_i)^{+\mathbf{d}}$ the resulting knowledge operator. It follows immediately from Propositions 3 and 5 and Lemma 1 that $(\cup_i K_i)^{+\mathbf{d}}$ satisfies both positive introspection and distributivity. Thus no further inferences can be drawn from $(\cup_i K_i)^{+\mathbf{d}}$ by subsequent iterations of the positive introspection operator or of the distributive closure. An agent's revision type is $(+\mathbf{d})$ when she revises her knowledge by applying first the positive introspection operator and then the distributive closure.

In the second case, suppose the agent uses the same two revision operators as above but in reverse order. First she forms $(\cup_i K_i)^{\mathbf{d}}$, and then applies the positive introspection operator to the latter so as to obtain $(\cup_i K_i)^{\mathbf{d}+}$. Importantly, the agent can further refine $(\cup_i K_i)^{\mathbf{d}+}$ by applying the distributive closure to it. In fact, while $(\cup_i K_i)^{\mathbf{d}+}$ always satisfies positive introspection, it may fail to satisfy distributivity. Thus I assume that the agent completes her revision process by applying again the distributive closure so as to form $(\cup_i K_i)^{\mathbf{d}+\mathbf{d}}$. Since the distributive closure preserves positive introspection, there is no further inference that can be drawn from $(\cup_i K_i)^{\mathbf{d}+\mathbf{d}}$ through the distributive closure or the positive introspection operator. An agent's revision type is $(\mathbf{d}+\mathbf{d})$ when she revises her knowledge by applying the distributive closure and the positive introspection operator in the exact order just illustrated. Notice that both $(\cup_i K_i)^{+\mathbf{d}}$ and $(\cup_i K_i)^{\mathbf{d}+\mathbf{d}}$ belong to $\mathscr{K}^2$ yet they need not be equal. This means that types $(+\mathbf{d})$ and $(\mathbf{d}+\mathbf{d})$ may reach different conclusions when revising the very same knowledge operator.

The definition of fully introspective types mirrors the definition of positively introspective ones. A *fully introspective* agent uses both $(.)^{\pm}$ and $(.)^{\mathbf{d}}$ to revise her knowledge. Two cases are possible. In the first, the agent applies the full introspection operator first, and then the distributive closure. The resulting operator is denoted by $(\cup_i K_i)^{\pm\mathbf{d}}$ and the agent's revision type is $(\pm\mathbf{d})$. In the remaining case, the agent starts out with $(.)^{\mathbf{d}}$, then applies $(.)^{\pm}$, and finally $(.)^{\mathbf{d}}$. The resulting operator is indicated as $(\cup_i K_i)^{\mathbf{d}\pm\mathbf{d}}$ and the agent's revision type is $(\mathbf{d}\pm\mathbf{d})$. By Propositions 4 and 5 and Lemma 1, there is no further inference that either type can draw through the full introspection operator or the distributive closure. Both $(\cup_i K_i)^{\pm\mathbf{d}}$ and $(\cup_i K_i)^{\mathbf{d}\pm\mathbf{d}}$ belong to $\mathscr{K}^3$ but they need not be equal. This means that two fully introspective agents may reach different conclusions depending on the order in which they process the very same knowledge operator.

In sum, there are five revision types that describe how agents with different levels of introspection revise knowledge. All types are collected in the following set:

$$\Theta = \{(\mathbf{d}), (+\mathbf{d}), (\mathbf{d}+\mathbf{d}), (\pm\mathbf{d}), (\mathbf{d}\pm\mathbf{d})\}.$$

When an agent of type $\theta \in \Theta$ revises a knowledge operator K, I write K^θ to indicate the resulting revised operator.

At this juncture, one might wonder why every type uses the distributive closure and why I ignore the two possible types $(\mathbf{d}+)$ and $(\mathbf{d}\pm)$. First, I examine a specific form of bounded rationality: introspection on one's own knowledge. More introspective agents are more rational. Being more or less introspective depends on using, or not using, the full or positive introspection operator in knowledge revision; it does not depend on the distributive closure. Second, I take introspection skills as independent of computational skills. I assume that agents may have limited introspection skills but they do not face any limit on the computational burden of what they can potentially do. Notice that the distributive closure calls for the processing of potentially long and non-trivial sequences of events. Finally, I assume that each agent revises knowledge as much as possible, provided that she only adopts the revision operators that form her type. The two candidate types $(\mathbf{d}+)$ and $(\mathbf{d}\pm)$ are not included in Θ because $(\cup_i K_i)^{\mathbf{d}+}$ and $(\cup_i K_i)^{\mathbf{d}\pm}$ can be further refined through the distributive closure.

5.2 Revision in terms of possibility relations

Revision types are defined in terms of revision operators, which are transformations of knowledge operators. The dual isomorphism established in Proposition 1 implies that, for each revision type, there is a unique possibility relation that represents the revised knowledge of that type. As the following proposition shows, the representation of revised knowledge for introspective types hinges on the left trace and the left n-ary trace introduced in Subsection 3.2.

Proposition 6. *Let $K_1, \ldots, K_n$ be $n \geq 1$ knowledge operators in $\mathscr{K}^1$, and let $\mathbf{P} = (P_1, \ldots, P_n)$ be the profile of possibility relations in $\mathscr{P}^1$ that represent those n knowledge operators. Then the following hold.*

(i) $(\cup_i K_i)^{\mathbf{d}}$ *is represented by* $\cap_i P_i$*;*

(ii) $(\cup_i K_i)^{+\mathbf{d}}$ *is represented by* $T_{\mathbf{P}}$*, i.e., the left n-ary trace of* $\mathbf{P}$*;*

(iii) $(\cup_i K_i)^{\mathbf{d}+\mathbf{d}}$ *is represented by* $T_{\cap_i P_i}$*, i.e., the left trace of* $\cap_i P_i$*;*

(iv) $(\cup_i K_i)^{\pm\mathbf{d}}$ *is represented by* $E_{\mathbf{P}}$*, i.e., the symmetric part of the left n-ary trace of* $\mathbf{P}$*;*

(v) $(\cup_i K_i)^{\mathbf{d}\pm\mathbf{d}}$ *is represented by* $E_{\cap_i P_i}$*, i.e., the symmetric part of the left trace of* $\cap_i P_i$*.*

Proof. See Appendix A.5. □

Besides its theoretical interest, the proposition above has "practical" relevance in the sense that it is often easier to compute revised knowledge by possibility relations than by revision operators. A case in point is the example in Subsection 6.1.

5.3 A preorder on the set of revision types

There is a natural preorder on the set of revision types. Formally, consider the binary relation $\preceq$ on the type set Θ. For all types θ and θ', we have that $\theta \preceq \theta'$ if, for all finite sequences of $n \geq 1$ operators $K_1, \ldots, K_n$ in $\mathscr{K}^1$, the following holds:

$$(\cup_{i=1}^n K_i)^{\theta} \leq (\cup_{i=1}^n K_i)^{\theta'}.$$

In words, $\theta \preceq \theta'$ if revision according to θ' always yields as much knowledge as revision according to θ. It is easy to check that $\preceq$ is reflexive and transitive but not necessarily antisymmetric. The graph of $\preceq$ is an immediate consequence of the claim below, and is represented in Figure 1.

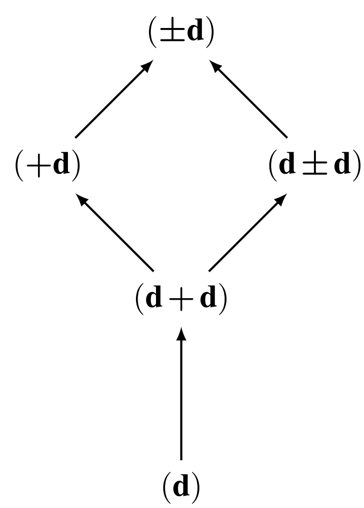

Figure 1: The preorder $\preceq$ on the set of revision types. Loops and arcs implied by transitivity are omitted.

Claim 4. *Let $K_1,\dots,K_n$ be $n \geq 1$ knowledge operators in $\mathscr{K}^1$, and let $\mathbf{P} = (P_1,\dots,P_n)$ be the profile of relations in $\mathscr{P}^1$ that represent those n knowledge operators. Then the following are true:*

(i) $(\cup_i K_i)^{\mathbf{d}} \leq (\cup_i K_i)^{\mathbf{d+d}} \leq (\cup_i K_i)^{\mathbf{d\pm d}} \leq (\cup_i K_i)^{\pm\mathbf{d}}$*;*

(ii) $(\cup_i K_i)^{\mathbf{d}} \leq (\cup_i K_i)^{\mathbf{d+d}} \leq (\cup_i K_i)^{+\mathbf{d}} \leq (\cup_i K_i)^{\pm\mathbf{d}}$.

Dually, also the following are true:

(i') $\cap_i P_i \supseteq T_{\cap_i P_i} \supseteq E_{\cap_i P_i} \supseteq E_{\mathbf{P}}$*;*

(ii') $\cap_i P_i \supseteq T_{\cap_i P_i} \supseteq T_{\mathbf{P}} \supseteq E_{\mathbf{P}}$.

Proof. I show the dual statements (i') and (ii'). It is immediate that $\cap_i P_i \supseteq T_{\cap_i P_i} \supseteq E_{\cap_i P_i}$ and that $T_{\mathbf{P}} \supseteq E_{\mathbf{P}}$. In order to show that $T_{\cap_i P_i} \supseteq T_{\mathbf{P}}$, notice that $(\omega,\omega') \in T_{\mathbf{P}}$ implies $\cap_j P_j(\omega') \subseteq P_i(\omega)$ for all $i \in \{1,\dots,n\}$. Therefore, $\cap_j P_j(\omega') \subseteq \cap_i P_i(\omega)$ and $(\omega,\omega') \in T_{\cap_i P_i}$. Finally, $E_{\cap_i P_i} \supseteq E_{\mathbf{P}}$ follows easily from $T_{\cap_i P_i} \supseteq T_{\mathbf{P}}$. □

6 Distributed knowledge

This section gives a formal definition of distributed knowledge and then shows how one can calculate distributed knowledge in groups of differently introspective agents. The formal definition follows the principle of full communication of van der Hoek et al. (1999), which says that distributed knowledge is the knowledge that group members can attain if they were fully able to communicate with one another and share everything they know. Communication is non-strategic and truthful. Such a notion of distributed knowledge is deliberately permissive since the ultimate goal is to understand what a group can know *potentially*.

Agents are allowed to communicate for as long as they want. Time is discrete and indexed by $t = 0,1,2,\dots$. Let $(\theta_1,\dots,\theta_N)$ be a profile of revision types, which are held fixed throughout the entire communication process. Let $(K_1,\dots,K_N)$ be a profile of initial knowledge operators in $\mathscr{K}^1$. At each stage t, everyone communicates her knowledge operator to everyone else. Then every agent revises her knowledge in light of what she has learned from others and according to her own revision type. At stage $t+1$, the same process of communication and revision takes place, and so on in all subsequent stages. The resulting knowledge of each agent i is represented by the sequence $(K_i^t : t = 0,1,2,\dots)$ defined recursively as follows. For each $i \in I$,

$$K_i^0 = K_i$$
$$K_i^t = (\cup_{j \in I} K_j^{t-1})^{\theta_i} \quad \text{for } t = 1, 2, \ldots$$

Distributed knowledge is what anyone knows "at the end" of the communication process just described. Formally, distributed knowledge is the knowledge operator $K_{\mathbf{D}}$ obtained as

$$K_{\mathbf{D}} = \cup_{i \in I} \left(\cup_{t=0}^{\infty} K_i^t \right). \tag{5}$$

The next proposition shows how (5) reduces to rather simple expressions that depend only on initial knowledge operators and on some of the revision types in the group.

Proposition 7. *Let $(K_1, \ldots, K_N)$ be a profile of N initial knowledge operators in $\mathscr{K}^1$, let $\mathbf{P} = (P_1, \ldots, P_N)$ be the profile of possibility relations representing those N knowledge operators, and let $(\theta_1, \ldots, \theta_N)$ be a profile of revision types.*

(i) If there is a type $\bar{\theta} \in \{\theta_i : i \in I\}$ such that $\theta_i \preceq \bar{\theta}$ for all $i \in I$, then

$$K_{\mathbf{D}} = (\cup_i K_i)^{\bar{\theta}}.$$

(ii) In the complementary case, in which $\theta_i \neq (\pm\mathbf{d})$ for all $i \in I$ and there are distinct $j, k \in I$ such that $\theta_j = (+\mathbf{d})$ and $\theta_k = (\mathbf{d} \pm \mathbf{d})$, we have

$$K_{\mathbf{D}} = \left((\cup_i K_i)^{+\mathbf{d}} \cup (\cup_i K_i)^{\mathbf{d}\pm\mathbf{d}} \right)^{\mathbf{d}\pm\mathbf{d}} = (\cup_i K_i)^{\pm\mathbf{d}}.$$

Proof. (i) Without loss of generality, suppose $\theta_N = \bar{\theta}$. By the definition of $\preceq$, for all $j \in I$,

$$K_j^1 = (\cup_i K_i)^{\theta_j} \leq (\cup_i K_i)^{\bar{\theta}} = K_N^1.$$

Thus $\cup_i K_i^1 = K_N^1$. Then it is easy to check that for all $j \in I$,

$$K_j^2 = (\cup_i K_i^1)^{\theta_j} = (K_N^1)^{\theta_j} = K_N^1,$$

from which the result follows.

(ii) I use the following claim to prove part (ii).

Claim 5. *Let $K_1, \dots, K_n$ be $n \geq 1$ knowledge operators in $\mathscr{K}^2$, and let $\mathbf{P} = (P_1, \dots, P_n)$ be the profile of relations in $\mathscr{P}^2$ that represent those n knowledge operators. Then we have:*

$$(\cup_i K_i)^{\mathbf{d}} = (\cup_i K_i)^{\mathbf{d+d}} = (\cup_i K_i)^{+\mathbf{d}} \leq (\cup_i K_i)^{\mathbf{d}\pm\mathbf{d}} = (\cup_i K_i)^{\pm\mathbf{d}}$$

and, dually,

$$\cap_i P_i = T_{\cap_i P_i} = T_{\mathbf{P}} \supseteq E_{\cap_i P_i} = E_{\mathbf{P}}. \tag{6}$$

Proof of Claim 5. I only prove the dual statement (6). Suppose $(\omega, \omega') \in \cap_i P_i$. Then $\omega' \in \cap_i P_i(\omega) \subseteq P_j(\omega)$ for all $j \in \{1, \dots, n\}$. Since each P_j is transitive, $\omega' \in P_j(\omega)$ implies $P_j(\omega') \subseteq P_j(\omega)$. Thus $(\omega, \omega') \in T_{\mathbf{P}}$ and $\cap_i P_i \subseteq T_{\mathbf{P}}$. By Claim 4, we can conclude $\cap_i P_i = T_{\cap_i P_i} = T_{\mathbf{P}} \supseteq E_{\cap_i P_i}$. Since $T_{\cap_i P_i} = T_{\mathbf{P}}$, we also have $E_{\cap_i P_i} = E_{\mathbf{P}}$. $\square$

To prove part (ii) of the proposition, suppose no one is of type $(\pm\mathbf{d})$. Without loss of generality, suppose $\theta_{N-1} = (+\mathbf{d})$ and $\theta_N = (\mathbf{d} \pm \mathbf{d})$. By Claim 4, for all $j \in I$, either

$$K_j^1 = (\cup_i K_i)^{\theta_j} \leq (\cup_i K_i)^{+\mathbf{d}} = (\cup_i K_i)^{\theta_{N-1}}$$

or

$$K_j^1 = (\cup_i K_i)^{\theta_j} \leq (\cup_i K_i)^{\mathbf{d}\pm\mathbf{d}} = (\cup_i K_i)^{\theta_N},$$

or both. Therefore,

$$\cup_i K_i^1 = (\cup_i K_i)^{+\mathbf{d}} \cup (\cup_i K_i)^{\mathbf{d}\pm\mathbf{d}}.$$

Since both $(\cup_i K_i)^{+\mathbf{d}}$ and $(\cup_i K_i)^{\mathbf{d}\pm\mathbf{d}}$ are in $\mathscr{K}^2$, we can use Claim 5 to conclude that for all $j \in I$,

$$K_j^2 = (\cup_i K_i^1)^{\theta_j} \leq \left((\cup_i K_i)^{+\mathbf{d}} \cup (\cup_i K_i)^{\mathbf{d}\pm\mathbf{d}}\right)^{\mathbf{d}\pm\mathbf{d}} = K_N^2,$$

from which it follows that $K_{\mathbf{D}} = K_N^2$. It remains to show that $K_N^2 = (\cup_i K_i)^{\pm\mathbf{d}}$. By Proposition 6, the revised knowledge operator $(\cup_i K_i)^{+\mathbf{d}}$ is represented by $T_{\mathbf{P}}$, the operator $(\cup_i K_i)^{\mathbf{d}\pm\mathbf{d}}$ is represented by $E_{\cap_i P_i}$, and K_N^2 is represented by the symmetric part of the left trace of $T_{\mathbf{P}} \cap E_{\cap_i P_i}$. Now, the relation $T_{\mathbf{P}} \cap E_{\cap_i P_i}$ is reflexive and transitive because it is the intersection of two reflexive and transitive relations. Moreover, it is easy to check that the left trace of a reflexive and transitive binary relation is equal to the relation itself. Therefore, the relation $T_{\mathbf{P}} \cap E_{\cap_i P_i}$ coincides with its left trace, and its symmetric part is $E_{\mathbf{P}} \cap E_{\cap_i P_i}$. By Claim 4, we have $E_{\mathbf{P}} \cap E_{\cap_i P_i} = E_{\mathbf{P}}$. Finally, by

Proposition 6, the relation $E_{\mathbf{P}}$ represents $(\cup_i K_i)^{\pm\mathbf{d}}$, so ending the proof.

□

Proposition 7 shows that there are two different cases to consider. In the first, there is a highest type $\bar{\theta}$ in the group. An agent of type $\bar{\theta}$ makes all the inferences that anyone else in group can make (This is the interpretation, i.e., how to interpret the highest type). As a result, distributed knowledge is nothing other than what anyone of type $\bar{\theta}$ knows after having learned what others know. We mentioned in the Introduction that distributed knowledge is essentially "what a wise man would know". Thus we can say that any agent of type $\bar{\theta}$ acts as the "wise man" who processes all the knowledge in the group and whose inferences determine distributed knowledge. In the second case, we do not have a highest type. Who is the "wise man" now? It turns out that the two maximal types $(+\mathbf{d})$ and $(\mathbf{d}\pm\mathbf{d})$ *jointly* act as the "wise man". More specifically, distributed knowledge can be attained in two steps. First, everyone communicates her knowledge operator to everyone else. Then distributed knowledge can be attained in the second step by letting just the two types $(+\mathbf{d})$ and $(\mathbf{d}\pm\mathbf{d})$ talk to each other. Remarkably, distributed knowledge in this case coincides with what it would have been if someone of type $(\pm\mathbf{d})$ had been a group member. In sum, part (2) of the proposition depicts a wisdom-of-the-crowd effect, in which the group knows more than what each of its members could possibly do alone. Differently put, distributed knowledge in part (i) can be attained by a single group member who has access to everyone's knowledge. In part (ii), two agents are necessary. We emphasize that part (ii) only holds in heterogeneous groups.

6.1 Example

This is an example of how to calculate distributed knowledge. There are three agents and four states of the world. The initial knowledge operators are K_1, K_2 and K_3 in Table 6, and the corresponding possibility relations are reported in Table 7. Suppose revision types are $(\theta_1, \theta_2, \theta_3) = ((\mathbf{d}), (\mathbf{d}+\mathbf{d}), (\mathbf{d}\pm\mathbf{d}))$. The highest type in the group is $\theta_3 = (\mathbf{d}\pm\mathbf{d})$. By part (i) of Proposition 7, distributed knowledge is determined by the highest type and is equal to $(\cup_i K_i)^{\mathbf{d}\pm\mathbf{d}}$ in Table 8. This means that distributed knowledge is attained just by transferring each agent's initial knowledge to agent 3. Now suppose the profile of types is $(\theta'_1, \theta'_2, \theta'_3) = ((\mathbf{d}), (+\mathbf{d}), (\mathbf{d}\pm\mathbf{d}))$. The two types $\theta'_2 = (+\mathbf{d})$ and $\theta'_3 = (\mathbf{d}\pm\mathbf{d})$ are maximal but none of them is the highest type in the group. By part (ii) of Proposition 7, distributed knowledge is $(\cup_i K_i)^{\pm\mathbf{d}}$ in Table 8. Notice that no one in the group is of type $(\pm\mathbf{d})$. In this case, one cannot attain distributed knowledge just by transferring initial knowledge to a single group member. In fact, at the end of the first round of communication, the knowledge operators of the three agents are $K_1^1 = (\cup_i K_i)^{\mathbf{d}}$, $K_2^1 = (\cup_i K_i)^{+\mathbf{d}}$ and $K_3^1 = (\cup_i K_i)^{\mathbf{d}\pm\mathbf{d}}$, all of which can be found in Table 8. It takes another round of communication to finally get $K_3^2 = (\cup_i K_i^1)^{\mathbf{d}\pm\mathbf{d}} = (\cup_i K_i)^{\pm\mathbf{d}}$, which gives us distributed knowledge. Notice that $(\cup_i K_i)^{\pm\mathbf{d}}$ is strictly finer than all the three knowledge operators K_1^1, K_2^1 and K_3^1 formed at the end of the first round of communication.

A	$K_1(A)$	$K_2(A)$	$K_3(A)$	$\cup_i K_i(A)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\omega_1\}$	$\emptyset$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$
$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\emptyset$	$\{\omega_2\}$
$\{\omega_3\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\omega_4\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1\}$	$\{\omega_1,\omega_2\}$
$\{\omega_1,\omega_3\}$	$\emptyset$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$
$\{\omega_1,\omega_4\}$	$\emptyset$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$
$\{\omega_2,\omega_3\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\emptyset$	$\{\omega_2\}$
$\{\omega_2,\omega_4\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\emptyset$	$\{\omega_2\}$
$\{\omega_3,\omega_4\}$	$\{\omega_3\}$	$\emptyset$	$\emptyset$	$\{\omega_3\}$
$\{\omega_1,\omega_2,\omega_3\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1\}$	$\{\omega_1,\omega_2\}$
$\{\omega_1,\omega_2,\omega_4\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1\}$	$\{\omega_1,\omega_2\}$
$\{\omega_1,\omega_3,\omega_4\}$	$\{\omega_3,\omega_4\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1,\omega_3,\omega_4\}$
$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_2,\omega_3\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_2,\omega_3,\omega_4\}$
Ω	Ω	Ω	Ω	Ω

Table 6: Initial knowledge operators and their union.

More generally, Table 8 collects all the five knowledge operators that can possibly describe distributed knowledge for the group under consideration when their initial knowledge is given by K_1, K_2 and K_3 in Table 6. The prevailing profile of revision types determines the unique operator that represents distributed knowledge. One can calculate each operator in Table 8 by applying the relevant revision operators to $\cup_i K_i$. The calculations are straightforward yet laborious; it is quicker to leverage on the duality between knowledge operators and possibility relations. For instance, suppose one wants to calculate $(\cup_i K_i)^{\mathbf{d}\pm\mathbf{d}}$. Proposition 6 says that the latter is represented by $E_{\cap_i P_i}$, i.e., the symmetric part of the left trace of $\cap_i P_i$. One can easily calculate $E_{\cap_i P_i}$ just by following the definitions in Subsection 3.2, and the result is in Table 7. Then one can form $(\cup_i K_i)^{\mathbf{d}\pm\mathbf{d}}$ via (2), which defines the knowledge operator represented by any given possibility relation. More generally, Table 7 collects the possibility relations that represent the knowledge operators in Table 8. To conclude, notice that $E_{\mathbf{P}}$, which is the symmetric part of the left ternary trace of $\mathbf{P} = (P_1, P_2, P_3)$, can be formed directly through minimal sets as per Claim 1: it is not necessary to calculate first the left ternary trace $T_{\mathbf{P}}$.

ω	$P_1(\omega)$	$P_2(\omega)$	$P_3(\omega)$	$\cap_i P_i(\omega)$, $T_{\cap_i P_i}(\omega)$, $E_{\cap_i P_i}(\omega)$	$T_{\mathbf{P}}(\omega)$	$E_{\mathbf{P}}(\omega)$
ω_1	$\{\omega_1, \omega_2\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$
ω_2	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2, \omega_3, \omega_4\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$
ω_3	$\{\omega_3, \omega_4\}$	$\{\omega_2, \omega_3, \omega_4\}$	$\{\omega_2, \omega_3, \omega_4\}$	$\{\omega_3, \omega_4\}$	$\{\omega_3\}$	$\{\omega_3\}$
ω_4	$\{\omega_1, \omega_3, \omega_4\}$	$\{\omega_2, \omega_3, \omega_4\}$	$\{\omega_2, \omega_3, \omega_4\}$	$\{\omega_3, \omega_4\}$	$\{\omega_3, \omega_4\}$	$\{\omega_4\}$

Table 7: Initial possibility relations, their intersection, and their traces.

A	$(\cup_i K_i)^{\mathbf{d}}(A)$, $(\cup_i K_i)^{\mathbf{d}+\mathbf{d}}(A)$, $(\cup_i K_i)^{\mathbf{d}\pm\mathbf{d}}(A)$	$(\cup_i K_i)^{+\mathbf{d}}(A)$	$(\cup_i K_i)^{\pm\mathbf{d}}(A)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$
$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$
$\{\omega_3\}$	$\emptyset$	$\{\omega_3\}$	$\{\omega_3\}$
$\{\omega_4\}$	$\emptyset$	$\emptyset$	$\{\omega_4\}$
$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$
$\{\omega_1, \omega_3\}$	$\{\omega_1\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$
$\{\omega_1, \omega_4\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1, \omega_4\}$
$\{\omega_2, \omega_3\}$	$\{\omega_2\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\{\omega_2, \omega_4\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2, \omega_4\}$
$\{\omega_3, \omega_4\}$	$\{\omega_3, \omega_4\}$	$\{\omega_3, \omega_4\}$	$\{\omega_3, \omega_4\}$
$\{\omega_1, \omega_2, \omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2, \omega_3\}$	$\{\omega_1, \omega_2, \omega_3\}$
$\{\omega_1, \omega_2, \omega_4\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2, \omega_4\}$
$\{\omega_1, \omega_3, \omega_4\}$	$\{\omega_1, \omega_3, \omega_4\}$	$\{\omega_1, \omega_3, \omega_4\}$	$\{\omega_1, \omega_3, \omega_4\}$
$\{\omega_2, \omega_3, \omega_4\}$	$\{\omega_2, \omega_3, \omega_4\}$	$\{\omega_2, \omega_3, \omega_4\}$	$\{\omega_2, \omega_3, \omega_4\}$
Ω	Ω	Ω	Ω

Table 8: Revised knowledge operators.

7 Conclusion

We have provided a characterization of distributed knowledge for groups in which members might not be equally rational. We have modelled explicitly the communication process that leads to distributed knowledge as well as the inferences that agents make along the way. Our main result shows that there are two fundamental cases. If there is a highest type in the group, i.e., a member who can replicate all the inferences that anyone else in group makes, then such highest type determines what is distributed knowledge. In more practical terms, distributed knowledge can be attained just by transferring each member's knowledge to the highest type. This means that unilateral communication is sufficient. In the case in which no highest type is in the group, reciprocal communication becomes necessary in order to attain distributed knowledge. In addition, distributed knowledge coincides with what it would have been had the highest type been in the group. This means that the group can replicate what the highest type would be able to accomplish by herself. All in all, our results show the power of communication to generate new knowledge. On a normative level, our results suggest that heterogeneous groups can know more than groups in which everyone processes information in the same way.

A Appendix

A.1 Proof of Proposition 3

(i) *Veridicality* of K^+ follows from the veridicality of K and the definition of K^+. To show that K^+ satisfies *positive introspection*, notice that if $A \in \mathsf{Img}(K)$, then $K^+(A) = A$. Consequently, $K^+(K^+(A)) = K^+(A) = A$. If $A \notin \mathsf{Img}(K)$, then $K^+(A) = K(A)$. Thus $K^+(K^+(A)) = K^+(K(A)) = K(A)$ since $K(A) \in \mathsf{Img}(K)$.

(ii) If $K^+ = K$, then part (i) above implies that K satisfies positive introspection. To show the other direction, suppose K satisfies positive introspection. Take any event $A \subseteq \Omega$. If $K(A) = A$, it follows immediately by the definition of K^+ that $K^+(A) = A$. Now suppose $K(A) \neq A$. There are two cases. If $A \notin \mathsf{Img}(K)$, then $K^+(A) = K(A)$. In the remaining case, $A \in \mathsf{Img}(K)$. This means that there exists an event $B \neq A$ such that $K(B) = A$. By positive introspection, $K(B) = K(K(B))$. Since $K(B) = A$, we get $K(A) = A$, which contradicts the assumption that $K(A) \neq A$.

(iii) This follows immediately by the definition of K^+ and the axiom of veridicality.

(iv) By part (i), the revised knowledge operator K^+ satisfies positive introspection. Thus $K^{++} = K^+$ by part (ii).

$\square$

A.2 Proof of Proposition 4

(i) *Veridicality* of $K^\pm$ follows from the veridicality of K and the definition of $K^\pm$. As for *necessitation*, it is clear that $K(\emptyset) = \emptyset$ since K satisfies veridicality. Thus $\Omega = \neg K(\emptyset) \in \mathsf{Img}(\neg K)$ and $K^\pm(\Omega) = \Omega$. To show that $K^\pm$ satisfies *positive introspection*, notice that if $A \in \mathsf{Img}(K) \cup \mathsf{Img}(\neg K)$, then $K^\pm(A) = A$. Consequently, $K^\pm(K^\pm(A)) = K^\pm(A) = A$. If $A \notin \mathsf{Img}(K) \cup \mathsf{Img}(\neg K)$, then $K^\pm(A) = K(A)$. Therefore, $K^\pm(K^\pm(A)) = K^\pm(K(A)) = K(A)$ since $K(A) \in \mathsf{Img}(K)$. Finally, to show that $K^\pm$ satisfies *negative introspection*, suppose $A \in \mathsf{Img}(K) \cup \mathsf{Img}(\neg K)$. Then $K^\pm(A) = A$ and $\neg K^\pm(A) = \neg A$. Moreover, $K^\pm(\neg K^\pm(A)) = K^\pm(\neg A)$. If $A \in \mathsf{Img}(K)$, then $\neg A \in \mathsf{Img}(\neg K)$ and $K^\pm(\neg A) = \neg A$. Analogously, if $A \in \mathsf{Img}(\neg K)$, then $\neg A \in \mathsf{Img}(K)$ and $K^\pm(\neg A) = \neg A$. Now suppose $A \notin \mathsf{Img}(K) \cup \mathsf{Img}(\neg K)$. By definition, $K^\pm(A) = K(A)$ and, consequently, $\neg K^\pm(A) = \neg K(A)$. Furthermore, $K^\pm(\neg K^\pm(A)) = K^\pm(\neg K(A)) = \neg K(A)$, since $\neg K(A) \in \mathsf{Img}(\neg K)$.

(ii) If $K^{\pm} = K$, then part (i) above implies that K satisfies negative introspection. To show the other direction, suppose K satisfies negative introspection. Take any $A \subseteq \Omega$. If $K(A) = A$, then $K^{\pm}(A) = A$ by the definition of $K^{\pm}$. Now suppose $K(A) \neq A$. If $A \notin \mathsf{Img}(K) \cup \mathsf{Img}(\neg K)$, then $K^{\pm}(A) = K(A)$. If $A \in \mathsf{Img}(K) \cup \mathsf{Img}(\neg K)$, there are two subcases. First, $A \in \mathsf{Img}(\neg K)$. This means that there exists an event B such that $\neg K(B) = A$. By negative introspection, $\neg K(B) = K(\neg K(B))$. Since $A = \neg K(B)$, we get $K(A) = A$, which contradicts the assumption $K(A) \neq A$. In the remaining subcase, $A \in \mathsf{Img}(K)$. Hence there is an event B such that $A = K(B)$, which is equivalent to $\neg A = \neg K(B)$. By negative introspection, $\neg K(B) = K(\neg K(B))$ and, consequently, $\neg A = K(\neg A)$, which is equivalent to $A = \neg K(\neg A)$. By negative introspection again, $\neg K(\neg A) = K(\neg K(\neg A))$ and, consequently, $K(A) = A$, so reaching a contradiction.

(iii) This follows by the definition of $K^{\pm}$ and the fact that K satisfies veridicality.

(iv) By part (i), the revised knowledge operator $K^{\pm}$ satisfies negative introspection. Thus $K^{\pm\pm} = K^{\pm}$ by part (ii).

□

A.3 Proof of Proposition 5

(i) *Veridicality* of $K^{\mathbf{d}}$ follows from the veridicality of K and the definition of $K^{\mathbf{d}}$. To show that $K^{\mathbf{d}}$ satisfies *distributivity*, notice that $K^{\mathbf{d}}$ satisfies monotonicity (1). Hence, $K^{\mathbf{d}}(A \cap B) \subseteq K^{\mathbf{d}}(A) \cap K^{\mathbf{d}}(B)$ for all $A, B \subseteq \Omega$. To show the reverse inclusion, take any $A, B \subseteq \Omega$. The case where $K^{\mathbf{d}}(A) \cap K^{\mathbf{d}}(B) = \emptyset$ is trivial. So suppose $\omega \in K^{\mathbf{d}}(A) \cap K^{\mathbf{d}}(B) \neq \emptyset$. By the definition of $K^{\mathbf{d}}$, there exist events $C_1, \ldots, C_n, D_1, \ldots, D_m \subseteq \Omega$ such that $\cap_{i=1}^{n} C_i \subseteq A$, $\cap_{j=1}^{m} D_j \subseteq B$, and

$$\begin{aligned} \omega &\in (\cap_{i=1}^{n} K(C_i)) \cap \left(\cap_{j=1}^{m} K(D_j)\right) \\ &= K(C_1) \cap \cdots \cap K(C_n) \cap K(D_1) \cap \cdots \cap K(D_m). \end{aligned} \tag{7}$$

Since $\cap_{i=1}^{n} C_i \subseteq A$ and $\cap_{j=1}^{m} D_j \subseteq B$, we have

$$C_1 \cap \cdots \cap C_n \cap D_1 \cap \cdots \cap D_m \subseteq A \cap B.$$

Hence it follows by the definition of $K^{\mathbf{d}}$ that the set (7) is included in $K^{\mathbf{d}}(A \cap B)$, so proving that $K^{\mathbf{d}}$ satisfies distributivity.

(ii) If $K^{\mathbf{d}} = K$, then part (i) above implies that K satisfies distributivity. To show the other direction, suppose K satisfies distributivity. Fix $A \subseteq \Omega$ and take any sequence of events $B_1, \dots, B_n$ such that $\cap_{i=1}^n B_i \subseteq A$. Since K satisfies distributivity, and consequently monotonicity, we have

$$\cap_{i=1}^n K(B_i) = K(\cap_{i=1}^n B_i) \subseteq K(A).$$

This means that $K^{\mathbf{d}}(A)$ is the union of subsets of $K(A)$. Therefore, $K^{\mathbf{d}}(A) \subseteq K(A)$. In addition, it follows easily from the definition of $K^{\mathbf{d}}$ that $K(A) \subseteq K^{\mathbf{d}}(A)$. Thus we can conclude that $K(A) = K^{\mathbf{d}}(A)$ for all $A \subseteq \Omega$.

(iii) This follows immediately from the definition of $K^{\mathbf{d}}$.

(iv) The revised knowledge operator $K^{\mathbf{d}}$ satisfies distributivity by part (i). Thus $K^{\mathbf{dd}} = K^{\mathbf{d}}$ by part (ii).

(v) Suppose $K \leq J$, i.e., $K(A) \subseteq J(A)$ for all $A \subseteq \Omega$. It follows that $\cap_{i=1}^n K(B_i) \subseteq \cap_{i=1}^n J(B_i)$ for all sequences $B_1, \dots, B_n \subseteq \Omega$. Therefore, $K^{\mathbf{d}} \leq J^{\mathbf{d}}$.

$\square$

A.4 Proof of Lemma 1

(i) Suppose $K \in \mathscr{K}^v$ satisfies positive introspection. Fix $A \subseteq \Omega$. If $K^{\mathbf{d}}(A)$ is the empty set, it is obvious that $K^{\mathbf{d}}(A) \subseteq K^{\mathbf{d}}(K^{\mathbf{d}}(A))$. Now suppose $K^{\mathbf{d}}(A) \neq \emptyset$ and take $\omega \in K^{\mathbf{d}}(A)$. By the definition of $K^{\mathbf{d}}$, there exist events $B_1, \dots, B_n \subseteq \Omega$ such that $\cap_{i=1}^n B_i \subseteq A$ and $\omega \in \cap_{i=1}^n K(B_i) \subseteq K^{\mathbf{d}}(A)$. By positive introspection and veridicality, we have $\cap_{i=1}^n K(B_i) = \cap_{i=1}^n K(K(B_i))$. By using the fact that $\cap_{i=1}^n K(B_i) \subseteq K^{\mathbf{d}}(A)$ and the definition of $K^{\mathbf{d}}$, one can conclude that

$$\omega \in \cap_{i=1}^n K(K(B_i)) \subseteq K^{\mathbf{d}}(K^{\mathbf{d}}(A)).$$

(ii) Suppose $K \in \mathscr{K}^v$ satisfies negative introspection. Fix $A \subseteq \Omega$. If $\neg K^{\mathbf{d}}(A)$ is the empty set, then it is obvious that $\neg K^{\mathbf{d}}(A) \subseteq K^{\mathbf{d}}(\neg K^{\mathbf{d}}(A))$. Now suppose $\neg K^{\mathbf{d}}(A) \neq \emptyset$ and take $\omega \in \neg K^{\mathbf{d}}(A)$. By the definition of $K^{\mathbf{d}}$ and De Morgan's laws, the event $\neg K^{\mathbf{d}}(A)$ is

$$\neg K^{\mathbf{d}}(A) = \bigcap \{\cup_{i=1}^n \neg K(B_i) : B_1, \dots, B_n \subseteq \Omega,\ \cap_{i=1}^n B_i \subseteq A,\ n \geq 1\}. \tag{8}$$

As per (8), the event $\neg K^{\mathbf{d}}(A)$ is the intersection of infinitely many events of the form $\cup_{i=1}^{n} \neg K(B_i)$. Notice that $\neg K^{\mathbf{d}}(A)$ is fully determined by sequences $B_1, \ldots, B_n$ in Ω such that $B_i \neq B_j$ for all $i \neq j$. Since the state space Ω is finite, there are only finitely many such sequences. Thus one can write $\neg K^{\mathbf{d}}(A)$ as the intersection of *finitely* many events of the form $\cup_{i=1}^{n} \neg K(B_i)$. In addition, recall that set intersection distributes over union and that for any two finite families $\{A_j\}_{j \in J}$ and $\{A_k\}_{k \in K}$ we have

$$\left(\cup_{j \in J} A_j\right) \cap \left(\cup_{k \in K} A_k\right) = \bigcup_{(j,k) \in J \times K} (A_j \cap A_k). \tag{9}$$

Now, by repeated application of (9), the event $\neg K^{\mathbf{d}}(A)$ can be written as the union of finitely many events of the form $\cap_{i=1}^{m} \neg K(C_i)$. Since $\omega \in \neg K^{\mathbf{d}}(A)$ by assumption, there must exist a sequence $C_1, \ldots, C_m$ in Ω for which

$$\omega \in \cap_{i=1}^{m} \neg K(C_i) \subseteq \neg K^{\mathbf{d}}(A).$$

By negative introspection and veridicality, we have $\cap_{i=1}^{m} \neg K(C_i) = \cap_{i=1}^{m} K(\neg K(C_i))$. By using the fact that $\cap_{i=1}^{m} \neg K(C_i) \subseteq \neg K^{\mathbf{d}}(A)$ and by the definition of $K^{\mathbf{d}}$ one can conclude that

$$\omega \in \cap_{i=1}^{m} \neg K(C_i) = \cap_{i=1}^{m} K(\neg K(C_i)) \subseteq K^{\mathbf{d}}(\neg K^{\mathbf{d}}(A)).$$

$\square$

A.5 Proof of Proposition 6

The proof relies on the following two lemmata.

Lemma 2. *Let $K_1, \ldots, K_n$ be $n \geq 1$ knowledge operators in $\mathscr{K}^1$, and let $\mathbf{P} = (P_1, \ldots, P_n)$ be the profile of relations in $\mathscr{P}^1$ that represent those n knowledge operators. In addition, let Q be the relation that represents $(\cup_{i=1}^{n} K_i)^{+\mathbf{d}}$. For all $\omega, \omega' \in \Omega$, the following are equivalent:*

(i) $(\omega, \omega') \in T_{\mathbf{P}}$;

(ii) *for all* $A \subseteq \Omega$, $\omega \in \cup_{i=1}^{n} K_i(A) \implies \omega' \in \cup_{i=1}^{n} K_i(A)$;

(iii) *for all* $A \subseteq \Omega$, $\omega \in (\cup_{i=1}^{n} K_i)^{+}(A) \implies \omega' \in (\cup_{i=1}^{n} K_i)^{+}(A)$;

(iv) *for all* $A \subseteq \Omega$*,* $\omega \in (\cup_{i=1}^n K_i)^{+\mathbf{d}}(A) \implies \omega' \in (\cup_{i=1}^n K_i)^{+\mathbf{d}}(A)$*;*

(v) $(\omega, \omega') \in Q$*.*

Proof of Lemma 2. It is easier to show pairwise equivalence as follows.

- $(i) \implies (ii)$**.** Suppose $(\omega, \omega') \in T_{\mathbf{P}}$. If $\omega \in \cup_{i=1}^n K_i(A)$ for some $A \subseteq \Omega$, then $\omega \in K_h(A)$ for some $h \in \{1, \dots, n\}$. This means that $P_h(\omega) \subseteq A$. By the definition of the left n-ary trace, there exists a $j \in \{1, \dots, n\}$ such that $P_j(\omega') \subseteq P_h(\omega)$. Therefore, $\omega' \in K_j(A) \subseteq \cup_{i=1}^n K_i(A)$.

- $(ii) \implies (i)$**.** Suppose $b)$ holds. It is clear that, for all $i \in \{1, \dots, n\}$, $\omega \in K_i(P_i(\omega)) \subseteq \cup_{j=1}^n K_j(P_i(\omega))$. It follows by $b)$ that $\omega' \in \cup_{j=1}^n K_j(P_i(\omega))$ too. The latter implies that $\omega' \in K_h(P_i(\omega))$ for some $h \in \{1, \dots, n\}$, which is equivalent to $P_h(\omega') \subseteq P_i(\omega)$. Thus $(\omega, \omega') \in T_{\mathbf{P}}$.

- $(ii) \implies (iii)$**.** This follows from the definition of $(\cup_{i=1}^n K_i)^{+}$.

- $(iii) \implies (ii)$**.** This follows from the fact that, for all $A \subseteq \Omega$, the event $\cup_{i=1}^n K_i(A)$ is a fixed point of $(\cup_{i=1}^n K_i)^{+}$.

- $(iii) \implies (iv)$**.** This follows from the definition of $(\cup_{i=1}^n K_i)^{+\mathbf{d}}$.

- $(iv) \implies (iii)$**.** Suppose $d)$ holds. For all $A \subseteq \Omega$, the event $(\cup_{i=1}^n K_i)^{+}(A)$ is a fixed point of $(\cup_{i=1}^n K_i)^{+}$. In addition, since $(.)^{\mathbf{d}}$ is extensive, and since $(\cup_{i=1}^n K_i)^{+\mathbf{d}}$ satisfies veridicality, we can conclude that

  $$(\cup_{i=1}^n K_i)^{+}(A) = (\cup_{i=1}^n K_i)^{+\mathbf{d}}\left((\cup_{i=1}^n K_i)^{+}(A)\right),$$

  from which $c)$ follows.

- $(iv) \implies (v)$**.** Suppose $d)$ holds. Since Q represents $\left(\cup_{i=1}^n K_i\right)^{+\mathbf{d}}$ by assumption,

  $$Q(\omega) = \bigcap\{A : \omega \in (\cup_{i=1}^n K_i)^{+\mathbf{d}}(A)\} \supseteq \bigcap\{A : \omega' \in (\cup_{i=1}^n K_i)^{+\mathbf{d}}(A)\} = Q(\omega').$$

  Since $Q \in \mathscr{P}^2$, Q is reflexive. Thus we can conclude that $\omega' \in Q(\omega)$, i.e., $(\omega, \omega') \in Q$.

- $(v) \implies (iv)$**.** Suppose $(\omega, \omega') \in Q$. Since Q is transitive, $\omega' \in Q(\omega)$ implies $Q(\omega') \subseteq Q(\omega)$, which in turn implies $d)$.

$\square$

Lemma 3. *Let $K_1, \dots, K_n$ be $n \geq 1$ knowledge operators in $\mathscr{K}^1$, and let $\mathbf{P} = (P_1, \dots, P_n)$ be the profile of relations in $\mathscr{P}^1$ that represent those n knowledge operators. In addition, let Q be the relation that represents $(\cup_{i=1}^n K_i)^{\pm\mathbf{d}}$. For all $\omega, \omega' \in \Omega$, the following are equivalent:*

(i) $(\omega, \omega') \in E_{\mathbf{P}}$;

(ii) for all $A \subseteq \Omega$, $\omega \in \cup_{i=1}^n K_i(A) \iff \omega' \in \cup_{i=1}^n K_i(A)$;

(iii) for all $A \subseteq \Omega$, $\omega \in (\cup_{i=1}^n K_i)^{\pm}(A) \iff \omega' \in (\cup_{i=1}^n K_i)^{\pm}(A)$;

(iv) for all $A \subseteq \Omega$, $\omega \in (\cup_{i=1}^n K_i)^{\pm\mathbf{d}}(A) \iff \omega' \in (\cup_{i=1}^n K_i)^{\pm\mathbf{d}}(A)$;

(v) $(\omega, \omega') \in Q$.

Proof of Lemma 3. It is easier to show pairwise equivalence as follows.

- $(i) \iff (ii)$**.** This follows immediately from the equivalence between (i) and (ii) in Lemma 2.
- $(ii) \implies (iii)$**.** This follows from the definition of $(\cup_{i=1}^n K_i)^{\pm}$.
- $(iii) \implies (ii)$**.** This follow from the fact that, for all $A \subseteq \Omega$, the event $\cup_{i=1}^n K_i(A)$ is a fixed point of $(\cup_{i=1}^n K_i)^{\pm}$.
- $(iii) \implies (iv)$**.** This follows from the definition of $(\cup_{i=1}^n K_i)^{\pm\mathbf{d}}$.
- $(iv) \implies (iii)$**.** Suppose $d)$ holds. For all $A \subseteq \Omega$, the event $(\cup_{i=1}^n K_i)^{\pm}(A)$ is a fixed point of $(\cup_{i=1}^n K_i)^{\pm}$. In addition, since $(.)^{\mathbf{d}}$ is extensive, and since $(\cup_{i=1}^n K_i)^{\pm\mathbf{d}}$ satisfies veridicality, we can conclude that

$$(\cup_{i=1}^n K_i)^{\pm}(A) = (\cup_{i=1}^n K_i)^{\pm\mathbf{d}}\left((\cup_{i=1}^n K_i)^{\pm}(A)\right),$$

  from which $c)$ follows.
- $(iv) \implies (v)$**.** Suppose $d)$ holds. Since Q represents $\left(\cup_{i=1}^n K_i\right)^{\pm\mathbf{d}}$ by assumption,

$$Q(\omega) = \bigcap\{A : \omega \in (\cup_{i=1}^n K_i)^{\pm\mathbf{d}}(A)\} = \bigcap\{A : \omega' \in (\cup_{i=1}^n K_i)^{\pm\mathbf{d}}(A)\} = Q(\omega').$$

  Since $Q \in \mathscr{P}^3$, Q is reflexive. Thus we can conclude that $\omega' \in Q(\omega)$, i.e., $(\omega, \omega') \in Q$.
- $(v) \implies (iv)$**.** Suppose $(\omega, \omega') \in Q$. Since Q is an equivalence relation, $\omega' \in Q(\omega)$ implies $Q(\omega') = Q(\omega)$, which in turn implies $d)$.

$\square$

Now we can prove Proposition 6 as follows.

(i) It is clear that $(\cup_{i=1}^n K_i)^{\mathbf{d}} \in \mathscr{K}^1$ and $\cap_{i=1}^n P_i \in \mathscr{P}^1$. Call L the knowledge operator represented by $\cap_{i=1}^n P_i$. By (3), for all $A \subseteq \Omega$,

$$L(A) = \{\omega : \cap_{i=1}^n P_i(\omega) \subseteq A\}.$$

We need to show that $(\cup_{i=1}^n K_i)^{\mathbf{d}} = L$. Take any $A \subseteq \Omega$. First we show that $(\cup_{i=1}^n K_i)^{\mathbf{d}}(A) \subseteq L(A)$. Suppose $\omega \in (\cup_{i=1}^n K_i)^{\mathbf{d}}(A)$. This means that there exist events $B_1, \dots, B_m$ such that $\cap_{j=1}^m B_j \subseteq A$ and

$$\omega \in \bigcap_{j=1}^m \left(\cup_{i=1}^n K_i(B_j)\right). \tag{10}$$

Since each P_i represents K_i, (10) is equivalent to

$$\omega \in \bigcap_{j=1}^m \left(\cup_{i=1}^n \left\{\omega' : P_i(\omega') \subseteq B_j\right\}\right),$$

from which we see that, for all $j \in \{1, \dots, m\}$, there exists a $i \in \{1, \dots, n\}$ such that $P_i(\omega) \subseteq B_j$. Therefore,

$$\cap_{i=1}^n P_i(\omega) \subseteq \cap_{j=1}^m B_j \subseteq A,$$

so proving that $\omega \in L(A)$.
To show the reverse set inclusion, suppose $\omega \in L(A)$, so that $\cap_{i=1}^n P_i(\omega) \subseteq A$. It is clear that $\omega \in K_i(P_i(\omega))$ for all $i \in \{1, \dots, n\}$. Therefore, we have

$$\omega \in \cap_{i=1}^n K_i(P_i(\omega)) \subseteq \bigcap_{i=1}^n \left(\cup_{j=1}^n K_j(P_i(\omega))\right) \subseteq (\cup_{i=1}^n K_i)^{\mathbf{d}}(A).$$

(ii) The knowledge operator $(\cup_{i=1}^n K_i)^{+\mathbf{d}}$ belongs to $\mathscr{K}^2$ and is represented by the binary relation $Q \in \mathscr{P}^2$ defined by (3). By Lemma 2, $(\omega, \omega') \in Q$ if and only if $(\omega, \omega') \in T_{\mathbf{P}}$ for all $\omega, \omega' \in \Omega$. Thus $Q = T_{\mathbf{P}}$.

(iii) It is clear that $(\cup_{i=1}^n K_i)^{\mathbf{d}} \in \mathscr{K}^1$. By part (i) of Proposition 6, $(\cup_{i=1}^n K_i)^{\mathbf{d}}$ is represented by $\cap_{i=1}^n P_i$. Then the result follows immediately from part (ii) of the same proposition.

(iv) The knowledge operator $(\cup_{i=1}^{n} K_i)^{\pm\mathbf{d}}$ belongs to $\mathscr{K}^3$ and is represented by the binary relation $Q \in \mathscr{P}^3$ defined by (3). By Lemma 3, $(\omega, \omega') \in Q$ if and only if $(\omega, \omega') \in E_{\mathbf{P}}$ for all $\omega, \omega' \in \Omega$. Thus $Q = E_{\mathbf{P}}$.

(v) It is clear that $(\cup_{i=1}^{n} K_i)^{\mathbf{d}} \in \mathscr{K}^1$. By part (i), $(\cup_{i=1}^{n} K_i)^{\mathbf{d}}$ is represented by $\cap_{i=1}^{n} P_i$. Then the result follows immediately from part (iv) of Proposition 6.

$\square$

References

Thomas Ågotnes and Yì N Wáng. Resolving distributed knowledge. *Artificial Intelligence*, 252:1–21, 2017.

Carlos E Alchourrón, Peter Gärdenfors, and David Makinson. On the logic of theory change: Partial meet contraction and revision functions. *The journal of symbolic logic*, 50(2):510–530, 1985.

Fuad Aleskerov, Denis Bouyssou, and Bernard Monjardet. *Utility maximization, choice and preference*. Springer, 2nd edition, 2007.

Robert J Aumann. Agreeing to disagree. *The annals of statistics*, pages 1236–1239, 1976.

Robert J Aumann. Interactive epistemology I: Knowledge. *International Journal of Game Theory*, 28(3):263–300, 1999.

Pathikrit Basu. Bayesian updating rules and AGM belief revision. *Journal of Economic Theory*, 179:455–475, 2019.

Ronald Fagin, Joseph Y Halpern, Yoram Moses, and Moshe Vardi. *Reasoning about knowledge*. MIT press, 1995.

Satoshi Fukuda. Can the crowd be introspective? Modeling distributed knowledge from collective information through inference. Working paper, Bocconi University, 2019.

John Geanakoplos. Game theory without partitions, and applications to speculation and consensus. *The B.E. Journal of Theoretical Economics*, 21(2):361–394, 2021.

J. Y. Halpern and Y. Moses. Knowledge and common knowledge in a distributed environment. *Journal of the* ACM, 37(3):549–587, 1990.

Bernard Monjardet. Some order dualities in logic, games and choices. *International Game Theory Review*, 9(01):1–12, 2007.

Stephen Morris. The logic of belief and belief change: A decision theoretic approach. *Journal of economic theory*, 69(1):1–23, 1996.

Manuel Mueller-Frank. Does one Bayesian make a difference? *Journal of Economic Theory*, 154:423–452, 2014.

Floris Roelofsen. Distributed knowledge. *Journal of Applied Non-Classical Logics*, 17(2): 255–273, 2012.

Ariel Rubinstein. *Modeling bounded rationality*. MIT press, 1998.

Evan Sadler. A practical guide to updating beliefs from contradictory evidence. *Econometrica*, 89(1):415–436, 2021.

D. Samet. Ignoring ignorance and agreeing to disagree. *Journal of Economic Theory*, 52(1): 190–207, 1990.

Wiebe van der Hoek, Bernd van Linder, and John-Jules Meyer. Group knowledge is not always distributed (neither is it always implicit). *Mathematical Social Sciences*, 38(2): 215–240, 1999.