Problem 1. Static Moral Hazard

Consider an agency relationship in which the principal contracts with the agent. The monetary result of the relationship depends on both agent’s effort and state of nature as follows:

<table>
<thead>
<tr>
<th>states:</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>result when $e = 6$</td>
<td>$x = 60,000$</td>
<td>$x = 60,000$</td>
<td>$x = 30,000$</td>
</tr>
<tr>
<td>result when $e = 4$</td>
<td>$x = 30,000$</td>
<td>$x = 60,000$</td>
<td>$x = 30,000$</td>
</tr>
</tbody>
</table>

Both parties believe that the probability of each state is one third. The payoff functions of the principal and the agent are given by

\[
B(x, w) = x - w \\
U(w, e) = \sqrt{w} - e^2,
\]

where $x$ is the monetary result of the relationship and $w$ is the wage received by the agent. Suppose that the agent will only accept the contract if he obtains an expected utility level of at least 114.

1. What would be the effort and the wage if effort is contractible?

2. What is the optimal contract when effort is not contractible? What wage scheme induces $e = 4$ at the minimum cost for the principal? What wage scheme induces $e = 6$ at the minimum cost for the principal?

3. Which effort level does the principal prefer when effort is not observable? Discuss.

Solution of Problem 1:

Part 1: For $e = 6$, $w_6 = 22500$ and $B_6 = 27500$. For $e = 4$, $w_4 = 16900$ and $B_4 = 23100$. Thus the principal will induce $e = 6$ with the wage offer $w_6 = 22500$.

Part 2: To induce $e = 4$, the principal can just offer the previous wage scheme. To induce $e = 6$, the optimal contract becomes $w(30000) = 12100$ and $w(60000) = 28900$. This contract returns $B_6^* = 26700$ in expectation.

Part 3: The principal gets a higher payoff by inducing $e = 6$ when effort level is not observable.

Problem 2: Normal-Linear Model

The following normal-linear model is regularly used in applied models. Given action $a$, output is $q = a + x$, where $x \sim N(0, \sigma^2)$. The cost of effort is $c(a)$, where $c(\cdot)$ is increasing and convex. The agent’s utility equals $u(w(q) - c(a))$, while the principal’s profit is $q - w(q)$. Suppose that the agent’s reservation utility is $u(0)$. 

Assume that the principal uses a linear contract of the form \( w(q) = \alpha + \beta q \), and that the agent’s utility is CARA: \( u(w) = -e^{-w} \).

1. Suppose that \( w \sim N(\mu, \sigma^2) \). Shows that \( E[u(w)] = u(\bar{w}) \), where \( \bar{w} = \mu - \frac{\sigma^2}{2} \).

2. Suppose that effort is unobservable. The principal’s problem is

\[
\begin{align*}
\max_{w(\cdot), a} & \quad E[q - w(q)] \\
\text{s.t.} & \quad E[u(w(q) - c(a))] | a \geq u(0)
\end{align*}
\]

\[
a \in \arg\max_{a' \in \mathbb{R}} E[u(w(q) - c(a')) | a']
\]

Using the first order approach, characterize the optimal contract \((a, \beta, a)\). [Hint: write utilities in terms of their certainty equivalent.]

3. How would the solution change if the agent knows \( x \) before choosing his action (but after signing the contract)? What if \( u(\cdot) \) is an arbitrary strictly increasing and concave function?

Solution for Problem 2:

Part 1: Certainty Equivalence Suppose that \( w \sim N(\mu, \sigma^2) \). Then

\[
\begin{align*}
-\mathbb{E}[u(w)] & = \frac{1}{\sqrt{2\pi\sigma^2}} \int \exp(-w) \exp\left[-\frac{(w - \mu)^2}{2\sigma^2}\right] dw \\
& = \frac{1}{\sqrt{2\pi\sigma^2}} \int \exp\left[-\frac{2w\sigma^2 + (w - \mu)^2}{2\sigma^2}\right] dw \\
& = \frac{1}{\sqrt{2\pi\sigma^2}} \int \exp\left[-\frac{(w - \mu + \sigma^2)^2 - \sigma^4 + 2\mu\sigma^2}{2\sigma^2}\right] dw \\
& = \exp(-\mu + \frac{\sigma^2}{2}) \frac{1}{\sqrt{2\pi\sigma^2}} \int \exp\left[-\frac{(w - \mu + \sigma^2)^2}{2\sigma^2}\right] dw \\
& = -u(-\mu + \frac{\sigma^2}{2})
\end{align*}
\]

since the latter term integrates to 1.

Part 2: Optimal Contract The principal’s problem can be written as

\[
\begin{align*}
\max_{w(\cdot), a} & \quad a - (\alpha + \beta a) \\
\text{s.t.} & \quad \alpha + \beta a - \frac{\beta^2}{2} \sigma^2 - c(a) \geq 0
\end{align*}
\]

\[
a \in \arg\max_{a' \in \mathbb{R}} \left\{ \alpha + \beta a' - \frac{\beta^2}{2} \sigma^2 - c(a') \right\}
\]

---

1 We call \( \bar{w} \) the certainty equivalent of \( w \).

2 In other words, the agent observes the noise \( x \) and chooses his effort \( a \) as a function of \( x \).
First, we can use (IR) to substitute for $a$. Second, using the first order approach, (IC) can be written as $\beta = c'(a)$.

Hence the problem becomes

$$\max_a \quad a - c(a) - \frac{\sigma^2}{2} \left[ c'(a) \right]^2$$

The optimal $a^*$ then solves the FOC

$$1 - c'(a^*) - c'(a^*) c''(a^*) \sigma^2 = 0$$

Re-arranging, and using (IC), we obtain

$$\beta^* = \frac{1}{1 + c''(a^*) \sigma^2}$$

**Part 3: Information Structure**  
The solution would not change if the agent knows $x$. This can be seen from the (IC) constraint, which is independent of $x$. Intuitively, this is because the contract is linear: incentives are unchanged if output increases by a fixed amount $x$. This is a key insight underlying Holmstrom and Milgrom (1987).

However, if $u(\cdot)$ is an arbitrary strictly increasing and concave function, then the agent’s effort depends on $x$, and he will exert a higher effort level, the smaller $x$ is.

**Problem 3: Insurance**

Consider a risk-averse agent, with increasing and concave utility $u(\cdot)$ and initial wealth $W_0$, who faces the risk of having an accident and losing an amount $x$ of her wealth. The agent has access to a perfectly competitive market of risk-neutral insurers who can offer schedules $R(x)$ of repayments net of any insurance premium.\(^3\) Assume that the distribution of $x$, which depends on accident-prevention effort $a$, is as follows:

$$f(0, a) = 1 - p(a)$$
$$f(x, a) = p(a)g(x) \quad \text{for } x > 0$$

where $\int g(x) \, dx = 1$ and $p(\cdot)$ is strictly decreasing and convex. The individual’s (increasing and convex) cost of effort, separable from her utility of money, is $c(\cdot)$. Therefore, the agent’s utility is given by

$$u(W_0 - x + R(x)) - c(a)$$

1. Suppose there is no insurance market. What action $\hat{a}$ does the agent take?
2. Suppose that $a$ is contractible. Describe the first best payment schedule $R(x)$ and the effort choice $a^*$.
3. Suppose $a$ is not contractible. Describe the second best payment schedule $R(x)$.
4. Interpret the second best payment schedule. Would the agent ever have an incentive to hide an accident?  
\(i.e., \text{report } x = 0 \text{ when } x > 0.\)

**Solution for Problem 3:**

**First-Best Contract**  
In a perfectly competitive market, all the identical risk averse agents will go to the insurers who offer an insurance contract that gives them maximal utility. As long as profits are greater than zero, a competing insurance company can offer a preferred contract and attract all of the agents. Hence, the first-best contract

\(^3\)This assumption implies that the expected profit of each insurer will be equal to 0.
with repayment schedule $R(x)$ solves

$$\max_{R(x),a} (1 - p(a)) u(W_0 + R(0)) + p(a) \int_0^\infty u(W_0 + R(x) - x) g(x) dx - c(a)$$

subject to

$$(1 - p(a)) R(0) + p(a) \int_0^\infty R(x) g(x) dx = 0.$$  \hspace{1cm} \text{(IR)}$$

Denote by $\lambda$ the multiplier of the individual rationality constraint. The first order conditions for this problem are

$$u'(W_0 + R^*(x) - x) p(a^*) g(x) - \lambda p(a^*) g(x) = 0 \text{ for } \forall x > 0$$

$$u'(W_0 + R^*(0)) (1 - p(a^*)) - \lambda (1 - p(a^*)) = 0 \text{ for } x = 0$$

$$\Leftrightarrow$$

$$u'(W_0 + R^*(x) - x) = \lambda \text{ for } \forall x \geq 0$$

and

$$p'(a^*) \left[ \int_0^\infty u(W_0 + R^*(x) - x) g(x) dx - u(W_0 + R^*(0)) \right] - \lambda \left( \int_0^\infty R^*(x) g(x) dx - R^*(0) \right) = \psi'(a^*).$$

Given strict risk aversion ($u'' < 0$), the first-best contract will guarantee a constant payoff for the agent across all states

$$R(x) - x = R(0) \text{ for } \forall x \geq 0.$$  

**Second-Best Contract**  
In order to find the second-best contract, the insurance company solves

$$\max_{R(x),a} (1 - p(a)) u(W_0 + R(0)) + p(a) \int_0^\infty u(W_0 + R(x) - x) g(x) dx - c(a)$$

subject to

$$\int_0^\infty R(x) p(a) g(x) dx + R(0) (1 - p(a)) = 0$$

$$a \in \arg\max_{\hat{a}} (1 - p(\hat{a})) u(W_0 + R(0)) + p(\hat{a}) \int_0^\infty u(W_0 + R(x) - x) g(x) dx - c(\hat{a}).$$

We can apply the first-order approach here and replace therefore the incentive compatibility constraint by the first order condition

$$p'(a) \left[ \int u(W_0 + R(x) - x) g(x) dx - u(W_0 + R(0)) \right] - c'(a) = 0.$$
The Lagrangian for this problem is

\[ \mathcal{L} = (1 - p(a)) u(W_0 + R(0)) + p(a) \int_0^\infty u(W_0 + R(x) - x) g(x) \, dx - c(a) \\
- \lambda \left[ (1 - p(a)) R(0) + p(a) \int R(x) g(x) \, dx \right] \\
+ \mu \left\{ p'(a) \int u(W_0 + R(x) - x) g(x) \, dx - u(W_0 + R(0)) \right\} - c'(a). \]

Maximizing with respect to \( R(x) \) for every positive \( x \) leads to the following first order conditions

\[ \frac{\partial \mathcal{L}}{\partial R(x)} = 0 \Leftrightarrow \frac{1}{u'(W_0 + R(x) - x)} = \frac{1}{\lambda} + \frac{\mu p'(a)}{p(a)} \text{ for } \forall x > 0 \]

\[ \frac{\partial \mathcal{L}}{\partial R(0)} = 0 \Leftrightarrow \frac{1}{u'(W_0 + R(0))} = \frac{1}{\lambda} - \frac{\mu p'(a)}{1 - p(a)} \]

where we assumed that \( p(a) \) never equals to 0 or 1. In other words, we assume that it is extremely costly for the agent to ensure there will be no loss whatsoever, but extremely cheap to make sure that the loss does not occur with probability 1. Given the positive Lagrange multipliers and \( p'(a) < 0 \), we find for \( \forall x > 0 \)

\[ \frac{1}{u'(W_0 + R(0))} > \frac{1}{\lambda} > \frac{1}{u'(W_0 + R(x) - x)} \]

\[ \Leftrightarrow \]

\[ R(0) > R(x) - x. \]

So the optimal contract is such that the insuree is punished when a loss occurs, but the punishment is not related to the size of the loss \( x \) since the probability of a larger loss is independent of the agent’s effort. Therefore, the second-best contract will induce variation in the insuree’s payoff only between 0 and \( x > 0 \). The exact shape of the contract will be fully determined by the first order condition with respect to effort and the zero-profit condition.

**Problem 4: Private Evaluations with Limited Liability**

A principal employs an agent. The agent privately chooses an action \( a \in \{L, H\} \) at cost \( c(a) \), where \( c(H) > c(L) \). The principal privately observes output \( q \sim f(q \mid a) \) on \( [\underline{q}, \bar{q}] \). Assume that this distribution function satisfies strict MLRP; i.e., \( f(q \mid H) \) is strictly increasing in \( q \). Suppose the principal reports that output is \( \bar{q} \). The principal then pays out \( t(\bar{q}) \), the agent receives \( w(\bar{q}) \), where \( w(\bar{q}) \leq t(\bar{q}) \), and the difference is burned. The payments \( \{t, q\} \) are contractible.

Payoffs are as follows. The principal receives \( q - t \). The agent receives \( u(w) - c(a) \) where \( u(\cdot) \) is strictly increasing and concave. The agent has no (IR) constraint, but does have limited liability; i.e., \( w(q) \geq 0 \) for all \( q \).

1. First, suppose that the principal wishes to implement \( a = L \). Characterize the optimal contract.

2. Second, suppose that the principal wishes to implement \( a = H \).

   (a) Write down the principal’s problem as maximizing expected profits subject to the agent’s (IC) constraint, the principal’s (IC) constraint, the limited liability constraint and the constraint that \( w(q) \leq t(q) \).

   (b) Argue that \( t(q) \) is independent of \( q \).

   (c) Characterize the optimal contract. How does the wage vary with \( q \)?
Solution for Problem 4:

Part 1: Implementing $a = L$: Set $w(q) = t(q) = c(L)$.

Part 2: Implementing $a = H$: Part (a): The principal’s problem is

$$\max_{w(\cdot), t(\cdot)} \mathbb{E}[q - t(q) \mid H]$$

s.t. $\mathbb{E}[u(w(q)) \mid H] - c(H) \geq \bar{u}$

$$\mathbb{E}[u(w(q)) \mid H] - c(H) \geq \mathbb{E}[u(w(q)) \mid L] - c(L)$$

$q - t(q) \geq q - t(\tilde{q}) \quad \forall \tilde{q}$

$t(q) \geq w(q)$

$w(q) \geq 0 \quad \forall q$

Part (b): This follows from (ICP). (Same proof as we did in class.)

Part (c): We can rewrite (ICA) as

$$\mu u'(w(q)) \left[ 1 - \frac{f(q)}{f(q|H)} \right] f(q|H) dq - c(H) + c(L) \geq 0$$

Let the multipliers on (ICA), (FE) and (LL) be $\mu, \phi(q)f(q|H)$, and $\lambda(q)f(q|H)$. The FOC with respect to $w(q)$ is

$$\mu u'(w(q)) \left[ 1 - \frac{f(q)}{f(q|H)} \right] f(q|H) - \phi(q) + \lambda(q) = 0$$

Suppose $w \in (0, t)$. Then we must have

$$\frac{f(q)}{f(q|H)} = 1$$

This uniquely determines some output $q^*$. When $q < q^*$, then $w = 0$. When $q \geq q^*$, then $w = t$. Finally, we choose $t$ so that (ICA) holds with equality.

Problem 5: Moral Hazard with Persistent Effort (25 points)

An agent chooses effort $e \in \{e_L, e_H\}$ at time 0 at cost $c(e) \in \{0,c\}$. At time $t \in \{1,2\}$, output is binomial; i.e., $q_t \in \{q_L, q_H\}$ is realized according to the i.i.d distribution $\Pr\{q_t = q_H|e_L\} = \pi_L$ and $\Pr\{q_t = q_H|e_H\} = \pi_H$. A contract is a pair of wages $(w_1(q_1), w_2(q_1,q_2))$. The agent’s utility is then

$$u(w_1(q_1)) + u(w_2(q_1,q_2)) - c(e),$$

where $u(\cdot)$ is increasing and concave, while the firm’s profits are

$$q_1 + q_2 - w_1(q_1) - w_2(q_1,q_2),$$

where we ignore discounting. The agent has outside option $2u_0$. Also, assume that the principal wishes to implement effort $e_H$.

1. What is the first best contract, assuming effort is observable?

2. Suppose the firm cannot observe the agent’s effort. Set up the firm’s problem.
3. Characterize the optimal first-period and second-period wages.

4. How do wages vary over time? In particular, can you provide a full ranking of wages across the different states and time periods?

Solution for Problem 5:

Part 1: In this case, we write down the principal’s problem as
- \( \max_{w(q_1, q_2)} E[q_1 + q_2 - w_1(q_1) - w(q_1, q_2)|e_H] \)
- \( E[u(w_1(q_1)) + u(w_2(q_1, q_2)) - c|e_H] \geq 2u_0 \) (IR)

Note that since \( e \) is observable, the principal sets a constant wage \( \bar{w} \) for \( e = e_H \); for \( e = e_L \), just set the wage as the negative infinity. As we did in the last problem set, we can set a simple Lagrangian function, and find the first order condition as

\[
u'(\bar{w}) = \frac{1}{\lambda'}
\]

where \( \lambda \) is the Lagrangian multiplier for (IR). Thus, we find that \( \bar{w} = (u')^{-1}(\frac{1}{\lambda'}) \).

Part 2:
- \( \max_{w(q_1, q_2)} E[q_1 + q_2 - w_1(q_1) - w(q_1, q_2)|e_H] \)

subject to
- \( E[u(w_1(q_1)) + u(w_2(q_1, q_2)) - c|e_H] \geq 2u_0 \) (IR)
- \( E[u(w_1(q_1)) + u(w_2(q_1, q_2)) - c|e_H] \geq E[u(w_1(q_1)) + u(w_2(q_1, q_2))|e_L] \) (IC)

Part 3: As we did in the last problem set, we set a Lagrangian function with multipliers \( \lambda \) and \( \mu \) that correspond to (IR) and (IC), respectively.

To avoid technical difficulties, we assume that the optimization problem does have a unique interior solution; that is, a set of first order conditions characterizes the optimal choice of \( w(\cdot) \). Moreover, observe that \( w(q, q') = w(q', q) \).

The first order conditions are the following:

\[
w(q_L) = (u')^{-1}\left(\frac{\pi_H}{\lambda\pi_H + \mu(\pi_H - \pi_L)}\right)
\]

\[
w(q_H) = (u')^{-1}\left(\frac{1 - \pi_H}{\lambda(1 - \pi_H) - \mu(\pi_H - \pi_L)}\right)
\]

\[
w(q_{L}, q_L) = (u')^{-1}\left(\frac{(1 - \pi_H)^2}{\lambda(1 - \pi_H)^2 + \mu((1 - \pi_L)^2 - (1 - \pi_H)^2)}\right)
\]

\[
w(q_{L}, q_H) = (u')^{-1}\left(\frac{(1 - \pi_H)\pi_H}{\lambda(1 - \pi_H)\pi_H + \mu(1 - \pi_H)\pi_H - (1 - \pi_L)\pi_L}\right)
\]

\[
w(q_H, q_H) = (u')^{-1}\left(\frac{\pi_H^2}{\lambda\pi_H^2 + \mu(\pi_H^2 - \pi_L^2)}\right).
\]

Part 4: By the assumption that \( u \) is strictly concave, we know that both \( u' \) and \( (u')^{-1} \) are decreasing. With this observation and the previous first order conditions, we have that

\[
w(q_H, q_H) \geq w(q_H) \geq w(q_L, q_H) = w(q_H, q_L) \geq w(q_L) \geq w(q_L, q_L).
\]
Problem 6: Screening

Consider the monopoly problem analyzed in section 2.1, but assume that the monopolist has one unit of the good for sale, at zero cost, while the buyer can have the following utility:

\[ \theta_L - T \]

or:

\[ \log(\theta_H - T). \]

The buyer’s risk aversion thus rises with her valuation. Show that the seller can implement the first-best outcome (that is, sell the good for sure, leave no rents to either type of buyer, and avoid any cost of risk in equilibrium) by using a random scheme.

Solution of Problem 6:

First Best  We use the additional condition \( \theta_H - 1 > \theta_L \) to answer this question. When the monopolist knows it is facing the low-type buyer, it will sell the good at

\[ T_L = \theta_L. \]

Similarly, when it knows that it is facing a high-type buyer, it sets the price at

\[ T_H = \theta_H - 1, \]

again leaving the buyer no rents.

The assumption \( \theta_H - 1 > \theta_L \) implies that the monopolist prefers to face a high-type to a low-type.

Second Best  Under asymmetric information, the monopolist can design randomized contracts for the risk-neutral low-value type to extract rents from the risk-averse high-value type. The monopolist can even achieve the first-best profit by offering the following choice of contracts:

- \( q = 1 \) for sure with a price \( T = \theta_H - 1 \)
- \( q = 1 \) for sure with a price \( T = \theta_L + \gamma \) with probability \( \frac{1}{2} \) and \( T = \theta_L - \gamma \) with probability \( \frac{1}{2} \).

Notice that this contract indeed implements the first-best outcome: the good is sold for sure, no rents are left to either type of buyer and no risk premium must be paid. Moreover, the contract can be made incentive compatible by setting \( \gamma \) large enough so that the high type will have no incentive to choose the contract designed for the low type (of course the latter always prefers the random contract, since we assume \( \theta_H - 1 > \theta_L \)):

\[ \log(1) = 0 \geq \frac{1}{2} \log(\theta_H - \theta_L - \gamma) + \frac{1}{2} \log(\theta_H - \theta_L + \gamma). \]

Problem 7: Costly State Verification

Consider a financial contracting problem between a wealth-constrained, risk-neutral entrepreneur and a wealthy risk-neutral investor. The cost of investment at date \( t = 0 \) is \( I \). The project generates a random return on investment at date \( t = 1 \) of \( \pi(\theta, I) = 2 \min\{\theta, I\} \), where \( \theta \) is the state of nature, uniformly distributed on \([0, 1]\).

1. Characterize the first-best level of investment, \( I^{FB} \).
2. Suppose that the realized return at $t = 1$ is freely observable only to the entrepreneur. A cost $K > 0$ must be paid for the investor to observe $\pi(\theta, I)$. Derive the second-best contract under the assumptions of (a) deterministic verification and (b) zero expected profit for the investor, taking into account that repayments cannot exceed realized returns (net of inspection costs).

3. Show that the second-best optimal investment level is lower than $I^{FB}$.

Solution of Problem 7:

Part 1: Characterization of First-Best Level of Investment  

The first-best level of investment $I^{FB}$ is found by maximizing the expected return of the investment. That is, $I^{FB}$ solves

$$\max_I E [\pi(\theta, I) - I]$$

$$\max_I E [2 \min \{\theta, I\}] - I$$

which can be simplified to

$$\max_I 2 \left( \int_0^1 \theta f(\theta) d\theta + I (1 - I) \right) - I$$

$$\max_I 2 \left( \frac{I^2}{2} + I (1 - I) \right) - I$$

$$\max_I I - I^2.$$ 

Hence, differentiating with respect to $I$ gives the first-best level of investment

$$I^{FB} = \frac{1}{2}.$$ 

Part 2: Second-Best Contract  

Appealing to the revelation principle we can reduce the set of relevant contracts to those where the entrepreneur truthfully reveals $\theta$. When there is no verification, the contract can only specify a repayment $r(\theta)$. However when there is verification the contract can specify a repayment contingent on both the announced return $\hat{\theta}$ and the verified return $\theta, r(\hat{\theta}, \theta)$. Application of the revelation principle ensures that in practice the entrepreneur always announces $\hat{\theta} = \theta$. The contract will specify a different repayment when $\hat{\theta} \neq \theta$ to ensure the entrepreneur has a sufficient incentive to truthfully reveal $\theta$. Since this repayment never occurs in equilibrium it is efficient to provide the maximum incentive by setting $r(\hat{\theta}, \theta) = \theta$ whenever $\hat{\theta} \neq \theta$ so there is no benefit from reporting a false return if it will be subsequently verified for sure. Returns are always truthfully revealed so we will denote the repayment contingent on the return $\theta$ and announced return $\hat{\theta}$ as simply $r_v(\theta)$.

In this question we are asked to restrict our attention to contracts with deterministic verification, that is the probability of a verification $p(\theta) \in \{0, 1\}$ for all $\theta$. We can now reduce the problem to one of minimizing the expected costs of verification subject to meeting the entrepreneur’s incentive constraints for truth telling and the financier’s participation constraint

$$\min_{p(\theta), r(\theta)} K \int_0^1 p(\theta) d\theta$$

subject to the financier’s participation constraint

$$\int_0^1 p(\theta) [r_v(\theta) - K] d\theta + \int_0^1 (1 - p(\theta)) r(\theta) d\theta = I,$$  (IR)
the set of incentive constraints
\[
\begin{align*}
  r_v (\theta_1) & \leq r (\theta_2) \text{ for all } \theta_1 \neq \theta_2 \text{ such that } p (\theta_1) = 1, p (\theta_2) = 0 \\
  r (\theta_1) & = r (\theta_2) = r \text{ for all } \theta_1 \neq \theta_2 \text{ such that } p (\theta_1) = 0 = p (\theta_2)
\end{align*}
\]
and the set of limited wealth constraints
\[
\begin{align*}
  r_v (\theta) & \leq \theta \text{ for all } \theta \text{ such that } p (\theta) = 1 \\
  r (\theta) & \leq \theta \text{ for all } \theta \text{ such that } p (\theta) = 0.
\end{align*}
\]

The incentive constraints require that for any two returns that do not require verification the repayment \( r (\cdot) \) to the financier is the same, and the repayment for any return requiring a verification is less than or equal to the repayment \( r (\cdot) \) when no verification is required. If either of these cases were violated the entrepreneur could avoid a high repayment by reporting a return which did not require verification.

There are only two dimensions along which the contract can be varied, the subset of returns to be verified and the size of the repayment within that subset. The solution to this problem realizes on three observations:

First, observe that \( \theta = 0 \) must be included in the verification subset since otherwise the entrepreneur would always claim \( \theta = 0 \) and never make a repayment;

Second, note that any contract that minimizes the expected costs of verification must have \( r_v (\theta) = \min \{ \theta, r \} \).

To see this note that when \( r < \theta \) incentive compatibility precludes \( r_v (\theta) > r \), but if \( r_v (\theta) < r \) the contract would involve inefficiently high verification costs since \( r_v (\theta) \) could be increased to \( r \) without violating any wealth or incentive constraint. Doing so relaxes the financier’s (IR) constraint and allows no verification to be carried out for this level of return, reducing the expected costs of verification. If \( r > \theta \) any contract with \( r_v (\theta) < \theta \) would be inefficient since \( r_v (\theta) \) could be increased to \( \theta \) generating greater cash flows to the financier relaxing her (IR) constraint and allowing the verification subset and associated costs to be reduced.

Third, note that any contract with a disconnected verification set \([0, \bar{\theta}] \cup [\theta, \bar{\theta}]\) would be inefficient since an obvious improvement can be made by shifting to a connected set with the same probability mass. Such a change would allow \( r \) to be raised, increasing expected repayments to the financier and relaxing the (IR) constraint. Once again, the higher expected repayments thus generated would relax the participation constraints and enables small saving of expected audit costs (see section 5.3 for more detail).

The contract which minimizes verification costs has a single connected verification region \( \Pi_v = [0, \bar{\theta}] \) with \( \bar{\theta} < 1 \) and \( r_v (\theta) = \theta \), and a non-verification region \( \bar{\theta} < \theta \leq 1 \) where \( r (\theta) = r = \bar{\theta}. \bar{\theta} \) is given by the financier’s (IR) constraint, that is \( \bar{\theta} \) solves
\[
\int_0^{\bar{\theta}} (r - K) d\theta + (1 - \bar{\theta})\bar{\theta} = I
\]

\( \iff \)
\[
(1 - K)\bar{\theta} - \frac{\bar{\theta}^2}{2} = I.
\]

Part 3: Second-Best Optimal Investment Level  When verification is costly, the optimal level of investment is found by maximizing the return, net of verification costs. The optimization problem is therefore
\[
\max_I I - I^2 - K\bar{\theta}(I)
\]

Hence, differentiating and assuming an interior solution
\[
I_{\text{SB}} = \frac{1}{2} - \frac{K \bar{\theta}}{2 \partial I}.
\]
Since $\frac{\partial q}{\partial I} > 0$ the second-best level of optimal investment $I_{SB} < \frac{1}{2} = I_{FB}$.

**Problem 8. Auctions**

Consider a two-person, independent private-value auction with valuations uniformly distributed on $[0, 1]$. Compare the following assumptions on utilities: (a) bidder $i$ ($i = 1, 2$) has utility $v_i - P$ when she wins the object and has to pay $P$, while her outside option is normalized to zero; (b) bidder $i$ ($i = 1, 2$) has utility $\sqrt{v_i - P}$ when she wins the object and has to pay $P$, while her outside option is normalized to zero.

1. Compare the seller’s expected revenue in cases (a) and (b) for the Vickrey auction.
2. Compare the seller’s expected revenue in cases (a) and (b) for the (linear) symmetric bidding equilibrium of the first-price, sealed-bid auction.
3. Discuss.

**Solution of Problem 8:**

**Part 1: Second-price Auction**

**Case (a):** $u_i(v_i) = v_i - P$  As is standard in a Vickrey auction the dominant strategy for each bidder is to bid her valuation. The seller’s expected revenue is the expected value of the lower of the two bids. The cumulative distribution function of the lower bid $b_{(1)}$ is

$$
\Pr(b_{(1)} < X \cap b_{(2)} < X) = 1 - \Pr(b_{(1)} > X \cap b_{(2)} > X) = 1 - (1 - x)^2
$$

$$
\Rightarrow F\left(b_{(1)}\right) = 2b_{(1)} - b_{(1)}^2.
$$

The expected value of the lower bid is

$$
E\left(b_{(1)}\right) = \int_0^1 x (2 - 2x) dx = \frac{1}{3}
$$

**Case (b):** $u_i(v_i) = \sqrt{v_i - P}$  The dominant strategy and revenue is the same as in Case 1 because although the form of the utility function has changed the bidder still only wants to acquire the good when the price is below her valuation so the equilibrium bidding strategies are unchanged and subsequently so is the seller’s revenue.

**Part 2: First Price Auction**

**Case (a):** $u_i(v_i) = v_i - P$  Let $g_i(b_i)$ be the probability that buyer $i$ expects to obtain the object if she bids $b_i$. If her valuation is $v_i$ she solves

$$
\max_{b_i} (v_i - b_i) g_i(b_i).
$$

She trades off between the probability of winning and the surplus when winning the auction. The first order condition is

$$
(v_i - b_i) g_i'(b_i) - g_i(b_i) = 0.
$$

We are looking for a symmetric equilibrium where bids are increasing in the bidder’s valuation, that is $\frac{\partial b_i}{\partial v_i} > 0$. When $v_i$ is distributed uniformly between 0 and 1, in equilibrium, the probability of winning $g_i(b_i)$ for an
individual with valuation \( v_i \) is \( v_j \). Replacing \( v_i \) by \( g (b_i) \) gives

\[
[g (b_i) - b_i]g' (b_i) - g (b_i) = 0.
\]

The solution to this differential equation is \( g (b_i) = 2b_i \) which implies that \( b_i = \frac{v_i}{2} \).

The seller’s expected revenue is the expected value of the higher of the two bids. The bidding strategy is linear in the buyer’s valuation. Thus, we can find the expected value of the highest bid as half the expected value of the highest valuation. The probability distribution of the highest valuation \( v(2) \) is \( f (v(2)) = 2v(2) \). The expected value of the highest valuation is

\[
E \left( v(2) \right) = \int_0^1 x \left( 2x \right) dx = \frac{2}{3}
\]

and therefore

\[
E \left( b(2) \right) = \frac{1}{3}.
\]

Case (b): \( u_i (v_i) = \sqrt{v_i - P} \) Let \( g_i (b_i) \) be the probability that buyer \( i \) expects to obtain the object if she bids \( b_i \). If her valuation is \( v_i \) she solves

\[
\max_{b_i} (\sqrt{v_i - b_i}) g_i (b_i) = \frac{v_i}{2} \sqrt{v_i - b_i} \cdot g_i (b_i).
\]

She trades off between the probability of winning and the surplus upon winning. The first order condition is then

\[
(\sqrt{v_i - b_i}) g_i' (b_i) - \frac{g_i (b_i)}{2\sqrt{v_i - b_i}} = 0.
\]

We are looking for a symmetric equilibrium where bids are increasing with the valuation. As before the probability of winning \( g (b_i) \) for an individual with valuation \( v_i \) is \( v_i \). Replacing \( v_i \) by \( g (b_i) \) gives

\[
2g (b_i) - b_i g' (b_i) - g (b_i) = 0.
\]

The solution to this differential equation is \( g (b_i) = \frac{3}{2} b_i \) which implies that \( b_i = \frac{2v_i}{3} \).

The seller’s expected revenue is the expected value of the highest of the two bids. Since the bidding strategy is linear in the buyer’s valuation we can simply find the expected value of the highest valuation and multiply by \( \frac{2}{3} \) . The probability distribution of the highest valuation \( v(2) \) is \( f (v(2)) = 2v(2) \). The expected value of the lowest bid is therefore

\[
E \left( v(2) \right) = \int_0^1 x \left( 2x \right) dx = \frac{2}{3}
\]

and

\[
E \left( b(2) \right) = \frac{4}{9}.
\]

**Part 3: Discussion** The idea of this question is to illustrate that revenue equivalence breaks down once the assumption of risk neutrality, case (a) in this question, is relaxed. First, in both the Vickrey and first price sealed bid auctions the risk neutral utility representation results in the same expected payoff to the seller of \( \frac{1}{4} \). Second, in the Vickrey auction bidders’ behavior is unaffected by their attitude toward risk. Regardless of their level of risk aversion the dominant strategy for each bidder is to always bid her valuation. Thus, the seller’s payoff is unchanged (equal to \( \frac{1}{4} \)). Third, in the first price sealed bid auction we see that risk aversion changes the equilibrium bidding strategies of the buyers. They now bid \( \frac{2}{3} \) of their valuation compared to \( \frac{1}{2} \) under risk neutrality. In the first price sealed bid auction the buyer trades off between the surplus they gain in case they win the auction and the probability of winning when they choose their strategy. A risk averse buyer will therefore prefer to insure herself against
losing the auction when her valuation is above the selling price. She will do this by making a higher bid. Under these auction rules we see that buyers make higher bids than under risk neutrality and subsequently the payoff to the seller is also higher. If buyers are risk averse the first priced sealed bid auction generates more revenue for the seller than the Vickrey auction.

**Problem 9: Public Goods Provision**

A firm is considering building a public good (e.g., a swimming pool). There are \( n \) agents in the economy, each with i.i.d private value \( \theta_i \sim U[0, 1] \). The cost of the swimming pool is \( cn \), where \( c > 0 \).

First suppose the government passes a law that says the firm cannot exclude people from entering the swimming pool. A mechanism thus consists of a build decision \( P(\theta_1, .., \theta_n) \in [0, 1] \) and a payment by each agent \( t_i \in (\theta_1, .., \theta_n) \in \mathbb{R} \). The mechanism must be individually rational and incentive compatible.

1. Consider an agent with type \( i \), whose utility is given by 

\[
\theta_i P - t_i
\]

Derive her utility in a Bayesian incentive compatible mechanism.

2. Given a build decision \( P(\cdot) \), derive the firm’s profits.

3. What is the firm’s optimal build decision?

4. Show that as \( n \to \infty \), the probability of provision \( P(\theta_1, .., \theta_n) \) goes to 0 for all \( \{\theta_1, .., \theta_n\} \).

**Solution of Problem 9:**

**Part 1:** This is very similar to the technique used in the revenue maximizing auction. We can think of the public good provision as an auction in which every citizen must pay and the good is provided iff total transfers exceed some threshold. As in the lecture notes, the mechanism \( \{P, t\} \) is IC iff

\[
u_i(\theta_i | \theta_i) = u_i(0|0) + \int_0^{\theta_i} \mathbb{E}_{\theta_{-i}}[P(s, \theta_{-i})]ds.
\]

We also know that IR implies \( u_i(0|0) = 0 \). Lastly, we determine what form the transfers must take by equating the function form and characterization derived above:

\[
\theta_i \mathbb{E}_{\theta_{-i}}[P(\theta_i, \theta_{-i})] - \mathbb{E}_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})] = u_i(\theta_i|\theta_i) = \int_0^{\theta_i} \mathbb{E}_{\theta_{-i}}[P(s, \theta_{-i})]ds \Rightarrow
\]

\[
\mathbb{E}_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})] = \theta_i \mathbb{E}_{\theta_{-i}}[P(\theta_i, \theta_{-i})] - \int_0^{\theta_i} \mathbb{E}_{\theta_{-i}}[P(s, \theta_{-i})]ds.
\]

**Part 2:** The firm’s profits is the total transfers received minus the cost of production (if the item is produced). The expected transfer from person \( i \) is simply

\[
\int_0^1 \mathbb{E}_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})]d\theta_i = \int_0^1 \left[ \theta_i \mathbb{E}_{\theta_{-i}}[P(\theta_i, \theta_{-i})] - \int_0^{\theta_i} \mathbb{E}_{\theta_{-i}}[P(s, \theta_{-i})]ds \right]d\theta_i
\]

\[
= \int_0^1 \theta_i \mathbb{E}_{\theta_{-i}}[P(\theta_i, \theta_{-i})] - \int_0^1 \int_0^{\theta_i} \mathbb{E}_{\theta_{-i}}[P(s, \theta_{-i})]dsd\theta_i.
\]
This should look familiar. The first part is the expected welfare to person \( i \) and the second part is the expected rent left to person \( i \) required by IC. This reduces to

\[
\int_0^1 \mathbb{E}_{\theta_{-i}} [P(\theta_i, \theta_{-i})] \left( \theta_i - \frac{1 - F_i(\theta_i)}{f(\theta_i)} \right) f(\theta_i) d\theta_i
\]

\[
= \int_0^1 \mathbb{E}_{\theta_{-i}} [P(\theta_i, \theta_{-i})] (2\theta_i - 1) d\theta_i,
\]

where we use the properties of the uniform distribution. Summing across \( i \) gives total expected revenue of:

\[
\sum_i \left[ \int_0^1 \mathbb{E}_{\theta_{-i}} [P(\theta_i, \theta_{-i})] (2\theta_i - 1) d\theta_i \right]
\]

\[
= \mathbb{E}_\theta \left[ P(\theta) \left( \sum_i (2\theta_i - 1) \right) \right]
\]

Since expected costs is given by \( \mathbb{E}_\theta [P(\theta)cn] \), then expected profit is given by:

\[
\mathbb{E}_\theta \left[ P(\theta) \left( \sum_i (2\theta_i - 1) - cn \right) \right]
\]

**Part 3:** The firm should build whenever the term

\[
\left[ \sum_i (2\theta_i - 1) - cn \right]
\]

is positive. Thus we have the optimal build decision as:

\[
P(\theta) = \begin{cases} 
1 & \sum_i (2\theta_i - 1) - cn > 0 \\
0 & \text{otherwise}
\end{cases}
\]

**Part 4:** Probability of building the good is equal to the probability that \( \sum_i (2\theta_i - 1) - cn > 0 \). This is equivalent to

\[
\frac{1}{n} \sum_i \left( \theta_i - \frac{1}{2} \right) > \frac{c}{2}.
\]

However, we know that since each \( \theta_i \) is iid and \( \mathbb{E}(\theta_i) = \frac{1}{2} \), this probability must go to zero.

**Problem 10. VCG Mechanism**

Suppose Bill and Linda are buying a new car. They refuse to drive anything other than Italian sports cars, Ferrari (\( F \)), Lamborghini (\( L \)), or Maserati (\( M \)). Bill’s type, \( \theta_B \), is well known, while Linda is mysterious and can be of two types, \( \theta_1^L \) or \( \theta_2^L \). Their utility functions are as follows.

\[
u_B(x) = \begin{cases} 
9 & \text{if } x = F \\
7 & \text{if } x = M \\
-1 & \text{if } x = \mathcal{L}
\end{cases},
\]

\[
u_{L1}(x|\theta_1^L) = \begin{cases} 
2 & \text{if } x = F \\
5 & \text{if } x = M \\
7 & \text{if } x = \mathcal{L}
\end{cases},
\]

\[
u_{L2}(x|\theta_2^L) = \begin{cases} 
6 & \text{if } x = F \\
8 & \text{if } x = M \\
1 & \text{if } x = \mathcal{L}
\end{cases}
\]

1. What is the efficient decision rule (i.e., first best?)
2. Is there a mechanism that implements the decision rule? If so, fully write out the mechanism and prove that it implements \(d\), as well as satisfies the agent’s IR and IC constraints. Assume payoff is linear, \(u_i - t_i\), and each agent’s outside option is 0 (i.e., they must both agree, or no car is purchased). Is this mechanism budget balanced?

3. Suppose Linda now has an outside option of 4. Can the above still be implemented? What can be implemented?

**Solution of Problem 10:**

**Part 1:**

\[
d(\theta_B, \theta^k_L) = \begin{cases} 
\mathcal{M} & \text{if } k = 1 \\
\mathcal{F} & \text{if } k = 2 
\end{cases}
\]

**Part 2:** We can set a mechanism that transfers 2 from Linda to Bill when she reports \(\theta^1_L\) and 0 otherwise. Formally, \(t_L(\theta_B, \theta^1_L) = 2\) and \(t_B(\theta_B, \theta^1_L) = -2\), while \(t_L(\theta_B, \theta^2_L) = 0 = t_B(\theta_B, \theta^1_L)\). Thus when her true type is \(\theta^2_L\) her utility from deviation is now equal to her utility from truth-telling, hence IC holds. Yes, it is balanced.

**Part 3:** No it cannot, since if Linda is of type \(\theta^1_L\) her final payoff under the mechanism is \(u(d(\theta_B, \theta^1_L)|\theta^1_L) - t_L(\theta_B, \theta^1_L) = 3 < 4\). The only decision rule that can be implemented now is \(d(\theta_B, \theta^k_L) = \mathcal{M}\) for all \(k\). This is clear because \(\mathcal{L}\) can never be implemented due to Bill’s IR constraint. \(\mathcal{F}\) cannot be implemented for when Linda is type \(\theta^1_L\) because of her IR constraint, so it must be that \(d(\theta_B, \theta^1_L) = \mathcal{M}\). Now, since IR also implies \(t_L(\theta_B, \theta^1_L)) \leq 1\), it must be that \(d(\theta_B, \theta^2_L) = \mathcal{M}\) as well, otherwise we violate IC.