Two Applications of the principal-agent model to credit markets

○ An entrepreneur (E - borrower) has a project.
  - Project requires investment \( I > 0 \).
  - Entrepreneur has assets \( A \in [0, I) \).
  - Requires to borrow \( I - A \) from a Lender (L).

○ If undertaken, project either succeeds and yields profits \( \pi = R > 0 \), or it fails and yields \( \pi = 0 \).

○ Both E and L are risk-neutral.

○ E privately chooses effort \( e \in \{e_H, e_L\} \)
  - Assume \( c(e_H) = B > 0 \) and \( c(e_L) = 0 \).
  - Let \( p(e) \) be the probability that project succeeds, where \( \Delta = p(e_H) - p(e_L) > 0 \).

○ Assumptions: The project has
  - positive NPV if E works: \( p(e_H) R - I - B > 0 \)
  - negative NPV if E shirks: \( p(e_L) R - I < 0 \)

○ L offers E a contract to lend him \( I - A \):
  - Contract specifies repayment \( z \) from E to L, as a function of the realized profits.
  - There is a competitive lending market, so lender earns zero expected profits.

○ Assume E is protected by limited liability, so \( z \leq \pi \).
  - If \( \pi = 0 \), then repayment is zero \( \implies \) both E and L get zero profits.
If \( \pi = R \), then repayment is \( z \in [0, R] \) \( \Rightarrow \) E gets \( R - z \) and L gets \( z \).

- If E puts high effort:
  - Lender’s expected profits are: \( p(e_H) z - (I - A) \).
  - Entrepreneur’s expected profits are: \( p(e_H) (R - z) - A - B \).

- If Entrepreneur puts low effort:
  - L’s expected profits are \( p(e_L) z - (I - A) \).
  - E’s expected profits are \( p(e_L) (R - z) - A \).

- Recall that project has positive NPV only if E puts effort:
  - Suppose L offers a contract that induces E to put low effort. Then:
    \[
    \frac{[p(e_L) z - (I - A)] + [p(e_L) (R - z) - A]}{\text{Profits to Lender}} < 0.
    \]
  - No loan that induces E to put low effort will ever be given out - such a loan would give a negative payoff either to E or to L.

- Suppose that L offers a contract that induces E to put high effort.
  - If E puts high effort, L’s expected profits are \( p(e_H) z - (I - A) \).
  - Perfect competition among lenders implies that
    \[
    z = \frac{I - A}{p(e_H)}.
    \]

- L must provide incentives for E to put high effort.
  - Incentive compatibility constraint:
    \[
    p(e_H) (R - z) - B - A \geq p(e_L) (R - z) - A
    \Rightarrow \Delta (R - z) \geq B
    \Rightarrow R - \frac{B}{\Delta} \geq z
    \]
These two equations imply that

\[
R - \frac{B}{\Delta} \geq \frac{I - A}{p(e_H)}
\]

\[
\implies A \geq I - p(e_H) \left( R - \frac{B}{\Delta} \right) = \bar{A}
\]

- E will only get financing if \( A \geq \bar{A} \).
- To provide incentives, E must have a high stake in the project (i.e., enough “skin in the game”).
- If the principal cannot provide incentives, then he will not finance the project.

Case 1: \( A \geq \bar{A} \)

- E will get financing, and his repayment scheme is \( z = \frac{I - A}{p(e_H)} \).
- L earns zero profits (competitive lending market).
- E’s stake in the firm:

\[
R - z = R - \frac{I - A}{p(e_H)} \geq R - \frac{I - \bar{A}}{p(e_H)} = \frac{B}{\Delta}.
\]

- E has incentives to put effort.

Case 2: \( A < \bar{A} \)

- E must borrow a large amount, and hence repay a large amount to L.
- This reduces his stake in the project, so he doesn’t have incentives to put effort.
- There is no loan agreement that induces effort and allows L to recover the investment.
- There is credit rationing!

Determinants of credit rationing:

- Level of assets that E owns A.
- How costly it is to provide incentives: how large B is relative to \( \Delta \).
- How costly the investment is (i.e., how large I is).

Crucial constraint for these results: limited liability constraint.
Recall that in the general principal-agent problem, we could implement the optimal solution when the agent was risk-neutral.

* In that case, the optimal contract was to “sell the firm” to the agent.
* But this doesn’t satisfy limited liability!

In this problem, credit rationing wouldn’t matter without limited liability.

* If we drop the limited liability constraint, we are assuming that E has enough money to fund the project herself!

**Motivating Debt Contracts**

* Debt contract: First $D$ of profits go to investors.

**Model:**

* Risk-neutral entrepreneur seeks funding from risk-neutral investor
* Output $q \sim f(q | a)$ satisfies MLR
* Investor puts in funds $I$
* Entrepreneur makes a TIOLI offer to repay $r(q) \in [0, q]$ in state $q$.
* Entrepreneur’s utility: $w(q) - c(a)$, where $w(q) = q - r(q)$.

* Entrepreneur’s Problem:

$$\max_{r(q), a} \mathbb{E}[q - r(q) | a] - c(a)$$

s.t. $\mathbb{E}[r(q) | a] \geq I$ (IR)

$$a \in \arg \max_{a'} \mathbb{E}[q - r(q) | a'] - c(a')$$ (IC)

$$0 \leq r(q) \leq q$$ (feasibility)

* Straightforward that IR should bind.
Ignore (feasibility) and write the Lagrangian:

\[ L = \int_{\mathbb{R}} [q - r(q)] dF(q|a) - c(a) + \lambda \left[ \int_{\mathbb{R}} r(q) dF(q|a) - I \right] + \mu \left\{ \int_{\mathbb{R}} [q - r(q)] \frac{f_a(q|a)}{f(q|a)} dF(q|a) - c'(a) \right\} \]

\[ = \int_{\mathbb{R}} q \left[ 1 + \mu \frac{f_a(q|a)}{f(q|a)} \right] dF(q|a) + \int_{\mathbb{R}} r(q) \left[ -1 + \lambda - \mu \frac{f_a(q|a)}{f(q|a)} \right] dF(q|a) - \lambda I - \mu c'(a) \]

- Second line follows from FOC approach.

- Take FOC with respect to \( r \):

\[ \frac{dL}{dr} = -1 + \lambda - \mu \frac{f_a(q|a)}{f(q|a)} \]

- \( r \) does not appear anywhere \( \implies \) solution will be “bang-bang”.

- Optimal contract:

\[ r(q) = \begin{cases} q & \text{if } \lambda \geq 1 + \mu \frac{f_a(q|a)}{f(q|a)} \\ 0 & \text{otherwise.} \end{cases} \]

- Optimal \( \lambda \) and \( \mu \) will be such that (IR) binds.

- MLR \( \implies \frac{f_a(q|a)}{f(q|a)} \) increases in \( q \). Therefore (assuming \( \mu > 0 \)), \( \exists q^* \) such that \( r(q) = q \) for \( q \leq q^* \).

(Can show that \( \mu > 0 \) using a similar approach as in standard principal-agent problem.)

\[ r(q) \quad q^* \]

- Intuition:

  - Incentive problem: induce the agent to exert high effort.
  - Must be rewarded when \( q \) is large.
- The entrepreneur’s reward = $q - r(q)$.

- Problem: Because $r(q)$ decreases in $q$,
  1. the entrepreneur can borrow money (without the investor knowing), reduce pay-
     ment, and repay the borrowed money later; and
  2. the investor has incentives to sabotage the project if $q$ is “large”.

- Solution: Add the constraint $r'(q) \geq 0$.
  - Then the optimal contract becomes a debt contract:

$$r(q) = \begin{cases} q & \text{if } q \leq D \\ D & \text{otherwise.} \end{cases}$$

- $D$ is chosen such that the investor’s IR constraint binds:

$$\mathbb{E}[r(q) \mid a^*] = I$$

where $a^* \in \arg \max_{a'} \mathbb{E}[q - r(q) \mid a'] - c(a')$.

References

Board S., (2011), Lecture Notes.


Review.