Suppose an investor has initial wealth \( w \) and utility \( u(\cdot) \).
- Assume that \( u' > 0 > u'' \).

Investor chooses between two assets:
- safe asset, which yields a return \( 1 + r \) for every dollar invested with probability 1 \((r > 0)\).
- risky asset, which yields a return of \( 1 + R \) for every dollar invested
- \( R \) takes values \( R_1 < R_2 < \ldots < R_n \), with probabilities \( p_1, \ldots, p_n \).

Assume that: \( R_1 < r \) and \( \mathbb{E}[R] = \sum_{i=1}^{n} p_i R_i > r \).

**Investor’s choice:** how much to invest in each asset.

Let \( A \) be the amount of money she invests in the risky asset.

Then, the investor’s expected return is given by
\[
A \sum_{i=1}^{n} p_i (1 + R_i) + (w - A) (1 + r) .
\]

**Investor’s problem:**
\[
\max_A \sum_{i=1}^{n} p_i u(A(1 + R_i) + (w - A) (1 + r))
\]

**First order conditions:**
\[
\sum_{i=1}^{n} p_i u'(A(1 + R_i) + (w - A) (1 + r)) (R_i - r) = 0.
\]
○ Second order conditions?
  - Always satisfied! (Explain.)

○ There is a unique $A^*$ that solves this equation.
  - If $\mathbb{E} [R] = \sum_{i=1}^{n} p_i R_i > r$, then $A^* > 0$.
  - Proof by contradiction.

○ If $u$ has DARA, then $A$ is increasing in $w$.

**An Application: Insurance Problem**

○ A consumer buying insurance.

○ Two possible outcomes (states of nature): good (G) and bad (B).

○ *Example:*
  - Good outcome: your house doesn’t burn down.
  - Bad outcome: your house burns down.

○ Consumer’s income:
  - Income at state G: $y$.
  - Income at state B: $y - L$.
  - Assume $y > L$.

○ G occurs with probability $p \in (0, 1)$ (B with prob. $1 - p$)

○ Consumer can buy coverage (insurance).

○ If consumers buys coverage $C$:
  - He receives $C$ from insurance company if outcome is $B$.
  - He receives 0 from insurance company if outcome is $G$.

○ Getting coverage $C$ costs $\pi C$ ($\pi =$ cost of coverage).

○ Consumer chooses coverage level $C$. 
Consumer’s income if he purchases coverage \( C \):

- Income at state G: \( y_G = y - \pi C \)
- Income at state B: \( y_B = y - L - \pi C + C = y - L + (1 - \pi)C \)

Note that \( \pi < 1 \): otherwise, the consumer will not purchase insurance.

What effect does a marginal increase in \( C \) have on \( y_G \) and \( y_B \)?

\[
\begin{align*}
\frac{\partial y_G}{\partial C} &= -\pi \\
\frac{\partial y_B}{\partial C} &= 1 - \pi \Rightarrow \\
\frac{\partial y_B}{\partial C} / \frac{\partial y_G}{\partial C} &= \frac{1 - \pi}{\pi}
\end{align*}
\]

- By marginally increasing \( C \), you trade off \( 1 - \pi \) units of consumption in bad state by \( \pi \) units of consumption in good state.
- \( \frac{1 - \pi}{\pi} \) is called the “price-ratio”.

Given \((y_G, y_B)\), the consumer’s utility is:

\[
U(y_G, y_B) = pu(y_G) + (1 - p) u(y_B).
\]

Assume that \( u' > 0 \) and \( u'' < 0 \): The consumer likes more income and he is risk averse.

Consumer’s problem

\[
\max_C \{ pu(y - \pi C) + (1 - p) u(y - L - \pi C + C) \}.
\]

First order conditions:

\[
p u'(y - \pi C) \bigg|_{y_G} (-\pi) + (1 - p) u'(y - L + (1 - \pi)C) \bigg|_{y_B} (1 - \pi) = 0.
\]

This implies that:

\[
(1 - p) u'(y_B) (1 - \pi) = pu'(y_G) \pi \Rightarrow \frac{1 - \pi}{\pi} = \frac{pu'(y_G)}{(1 - p) u'(y_B)}.
\]
Solution: MRS = Price ratio

Second order conditions?

Suppose $1 - p \geq \pi$. Then, $C^* > 0$.

If price of insurance is lower than probability of bad outcome, consumer will buy coverage.

Example: Suppose $u(y) = \ln y$, so $u'(y) = \frac{1}{y}$.

In this case:

$$\frac{1 - \pi}{\pi} = \frac{pu'(y_G)}{(1 - p)u'(y_B)} = \frac{py}{py_B \pi}.$$

Since $y_G = y - \pi C$ and $y_B = y - L - \pi C + C$,

$$\frac{1 - \pi}{\pi} = \frac{p}{1 - p} \frac{y - L - \pi C + C}{y - \pi C}.$$

This equation pins down the level of coverage $C$. Solving for $C$ yields:

$$C = \frac{y(1 - \pi - p) + pL}{1 - \pi}.$$

Assuming $1 - \pi - p > 0$, level of coverage is increasing in $L$ and $y$.

What about changes in $p$?

$$\frac{\partial C}{\partial p} = \frac{L - \frac{y}{\pi}}{1 - \pi} = \frac{\pi L - y}{\pi(1 - \pi)}.$$

Since $y > L$, agent buys less coverage as $p$ increases.

As $p$ increases, bad state becomes less likely.

What about changes in $\pi$?

$$\frac{\partial C}{\partial \pi} = \frac{\partial}{\partial \pi} \left( \frac{y(1 - \pi - p) + pL}{1 - \pi} \right).$$
This derivative is negative:
As price of coverage increases, agent buys less coverage.

**Perfect Competition in the Insurance Sector**

- What are the profits of the insurance company?
  - Insurance company earns $\pi C$ on the consumer.
  - Insurance company pays consumer an amount $C$ with probability $1 - p$.

- Total expected profits: $\pi C - (1 - p)C = (\pi + p - 1)C$.
  - Suppose that there is perfect competition in the insurance sector.
  - Profits of the insurance company are zero, so $\pi = 1 - p$.

- Recall FOCs:
  \[
  \frac{1 - \pi}{\pi} = \frac{pu'(y_G^*)}{(1 - p)u'(y_B^*)}
  \]
  - Under perfect competition in the insurance sector, $\pi = 1 - p$, so
  \[
  1 = \frac{u'(y_G^*)}{u'(y_B^*)} \Rightarrow u'(y_G^*) = u'(y_B^*) \Rightarrow y_G^* = y_B^*.
  \]
  - Consumer insures perfectly!
  \[
  y_G = y - \pi C = y - L - \pi C + C = y_B \Rightarrow \\
  C = L.
  \]

**References**
