Module 17: Mechanism Design & Optimal Auctions

Information Economics (Ec 515)  ·  George Georgiadis

Examples:

- Auctions
- Bilateral trade
- Production and distribution in society

General Setup

- $N$ agents
- Each agent has private information $\theta_i; \; \theta = \{\theta_i\}_{i=1}^N$.
- Outcomes $y \in Y$; often allocation plus transfers: $y = \{k, t_1, .., t_N\}$.
- Utility $u_i = u_i(y, \theta)$
  - Quasi-linear utility: $u_i = u^k_i(\theta) - t_i$.
- Mechanism designer’s objective: “Implement” a choice rule $\psi : \Theta \rightarrow Y$ to maximize objective; e.g.,
  - Efficiency: maximize $\sum_i u^k_i(\theta)$
  - Revenue: maximize $E_\theta[\sum_i t_i(\theta)]$

Definition. A choice rule $\psi : \Theta \rightarrow Y$ is incentive compatible with respect to an equilibrium concept “X” if each agent revealing his type truthfully (i.e., reporting $\hat{\theta}_i = \theta_i$) is an “X”-equilibrium.
Equilibrium Concepts

1. Dominant-strategy (strategy-proof) implementation: For all $i$, $\theta_i$, $\tilde{\theta}_i$, $\theta_{-i}$ and $\tilde{\theta}_{-i}$

$$u_i\left(\psi\left(\theta_i, \tilde{\theta}_{-i}\right); \theta\right) \geq u_i\left(\psi\left(\tilde{\theta}_i, \tilde{\theta}_{-i}\right); \theta\right)$$

- Reporting truthfully is an optimal strategy for each agent irrespective of the others’ strategies.
- Quite restrictive.

2. Bayesian Nash implementation:

- There is a common prior $\pi$ over $\theta$, and the agents’ beliefs $\pi_i(\cdot|\theta_i)$ over $\Theta_{-i}$ are given by Bayesian updating.
- For all $i$, $\theta_i$ and $\tilde{\theta}_i$

$$\mathbb{E}_{\pi_i(\cdot|\theta_i)}u_i\left(\psi\left(\theta_i, \tilde{\theta}_{-i}\right); \theta\right) \geq \mathbb{E}_{\pi_i(\cdot|\theta_i)}u_i\left(\psi\left(\tilde{\theta}_i, \tilde{\theta}_{-i}\right); \theta\right)$$

- Reporting truthfully is an optimal strategy on expectation, given beliefs $\pi_i(\cdot|\theta_i)$.

3. Ex-post implementation: For all $i$, $\theta_i$, $\tilde{\theta}_i$ and $\theta_{-i}$

$$u_i\left(\psi\left(\theta_i, \theta_{-i}\right); \theta\right) \geq u_i\left(\psi\left(\tilde{\theta}_i, \theta_{-i}\right); \theta\right)$$

- Each agent finds it optimal to report truthfully given that others also report truthfully - after others’ types are revealed (“no regret”).
- Advantage: Robust against different priors and higher order beliefs.

Revelation Principle

- Set of all mechanisms has little structure.
- Focus on a particular class of mechanism: Revelation mechanism $S_i = \Theta_i$; i.e., strategy is to state a type $\tilde{\theta}$.

Theorem. (Revelation Principle for Bayesian Nash implementation) A choice rule $\psi$ is (partially) implementable by any mechanism if and only if it is incentive compatible.

- Proof: Skipped.
Very robust result.

- Holds for all standard implementation concepts.

If agents control actions $a_i$ on top of common decision $\psi$, then one can replace any mechanism with a centralized mechanism where

- Each agent reports his type $\tilde{\theta}_i$; and
- the mechanism designer recommends actions $\tilde{a}_i$.
- In equilibrium, the agents are truthful $\tilde{\theta}_i = \theta_i$ and obedient ($a_i = \tilde{a}_i$).

i.e., Moral hazard together with adverse selection (Myerson, Ecta ’82)

If agents can act sequentially and acquire further information, then one can replace any mechanism with a centralized mechanism where

- Agents report everything they have learned so far; and
- the mechanism designer recommends actions $\tilde{a}_i$.
- In equilibrium, the agents are truthful and obedient.

Not robust to:

- Communication costs
- Bounded rationality.

Full vs. Partial implementation:

- Partial: $\psi(\theta)$ is an equilibrium.
- Full: $\psi(\theta)$ is the only equilibrium.
Optimal Auctions

- $N$ bidders.
- $\theta \in [\underline{\theta}, \bar{\theta}]$ with pdf $f$.
- Mechanism specifies:
  1. Allocation function $p_i : [\underline{\theta}, \bar{\theta}]^N \rightarrow [0,1]$ for each agent $i$ such that $p_i \geq 0$ and $\sum_i p_i \leq 1$.
     - If the seller has $n$ objects for sale, then $\sum_i p_i \leq n$.
  2. Transfer function $t_i : [\underline{\theta}, \bar{\theta}] \rightarrow \mathbb{R}$ for each agent $i$.

- Independent private values (IPV) model: $u_i (\theta_i) = \theta_i p_i - t_i$
- Revenue: $\sum_i t_i + (1 - \sum_i p_i) \theta_0$
  - $\theta_0$: seller’s value. Can be shown that the seller can disclose $\theta_0$ wolog.

Examples of Auctions

1. First-Price Auction: $p_i (\theta) = 1$ if $\theta_i > \theta_{-i}$, and $t_i (\theta_i) = p_i (\theta) b (\theta_i)$.
   - $b (\theta_i)$ is the bid of type $\theta_i$.
   - Under symmetry assumptions.
   - Otherwise: Maskin and Riley (REStud, 2000)

2. Second-Price Auction: $p_i (\theta) = 1$ if $\theta_i > \theta_{-i}$, and $t_i (\theta_i) = p_i (\theta) b (\theta_{(2)})$.
   - $b (\theta_{(2)})$ is the second-highest bid.

3. All-pay Auction: $p_i (\theta) = 1$ if $\theta_i > \theta_{-i}$, and $t_i (\theta_i) = b (\theta_i)$.

4. Raffle: $n (\theta_i) = \#$ of tickets, $p (\theta) = \frac{n (\theta_i)}{\sum_j n (\theta_j)}$, and $t_i (\theta_i) = c n (\theta_i)$. 
Revenue Maximization

\[
\begin{align*}
\text{max} & \quad \mathbb{E}_\theta \left[ \sum_i t_i (\theta_i) + \left[ 1 - \sum_i p_i (\theta) \right] \theta_0 \right] \\
\text{s.t.} & \quad u_i (\theta_i; \theta_i) \geq 0 \\
& \quad u_i (\theta_i; \theta_i) \geq u_i (\theta_i; \tilde{\theta}_i)
\end{align*}
\]

where \( u_i (\theta_i; \tilde{\theta}_i) = \mathbb{E}_{\theta-i} \left[ p_i (\theta_i, \theta_i) \theta_i - t (\theta_i, \theta_i) \right] \).

**Proposition.** is IC if and only if

1. \( u_i (\theta_i; \theta_i) = u_i (\theta_i; \hat{\theta}_i) + \int_{\theta_i}^{\tilde{\theta}_i} \mathbb{E}_{\theta-i} [p_i (s, \theta_i)] \, ds \) (IC-FOC)

2. \( \mathbb{E}_{\theta-i} [p_i (\theta_i, \theta_i)] \) increases in \( \theta_i \) (Monotonicity)

\( (IR) \) can be replaced by \( u (\hat{\theta}_i; \theta_i) = 0 \).

○ **Proof:** Similar to the single-agent case.

○ Re-write objective function:

\[
\text{Revenue} = \mathbb{E}_\theta \left[ \sum_i p_i (\theta) \theta_i + \left[ 1 - \sum_i p_i (\theta) \right] \theta_0 - \sum_i u_i (\theta_i; \theta_i) \right]
\]

○ Calculate expected rent:

\[
\mathbb{E}_{\theta_i} [u_i (\theta_i; \theta_i)] = u_i (\theta_i; \hat{\theta}_i) + \int_{\theta_i}^{\tilde{\theta}_i} \mathbb{E}_{\theta-i} [p_i (s, \theta_i)] \, ds \, \mathbb{E}_{\theta-i} [1 - F (\theta_i)] \mathbb{E}_{\theta-i} (\theta_i) \quad \text{(IR)}
\]

\[
= - \left[ \mathbb{E}_{\theta-i} [p_i (\theta_i, \theta_i)] [1 - F (\theta_i)] \right] + \int_{\theta_i}^{\tilde{\theta}_i} \mathbb{E}_{\theta-i} [p_i (\theta_i, \theta_i)] [1 - F (\theta_i)] \, d\theta_i
\]

\[
= \mathbb{E}_\theta \left[ p_i (\theta) \frac{1 - F (\theta_i)}{f (\theta_i)} \right]
\]

○ Compile:

\[
\text{Revenue} = \mathbb{E}_\theta \left[ \sum_i p_i (\theta) \left[ \theta_i - \frac{1 - F (\theta_i)}{f (\theta_i)} \theta_0 \right] \right] + \theta_0
\]
**Proposition.** (Revenue Equivalence): *Any auction that has the same allocation function, generates the same revenue.*

**Proof.**

- Revenue depends on \( p(\cdot) \), but not on \( t(\cdot) \).

- **Implication:** What matters is allocations; not “how you get there”.

- **Optimal Auction:**
  - Award good to agent \( i \) if \( MR(\theta_i) > \max\{\theta_0, MR(\theta_{-i})\} \).
  - If \( MR(\theta) \) increases in \( \theta \), then (Monotonicity) is satisfied, and we have an optimal auction. Otherwise, we need to “iron it”.

**Implementation:**

- First-price auction with reserve price \( r = MR^{-1}(\theta_0) \).

- Second-price auction with entry fee \( e = MR^{-1}(\theta_0) F^{-1}(MR^{-1}(\theta_0)) \).

**Example:**

- \( N \) bidders, \( \theta_i \sim U[0,1], \theta_0 = 0 \).

- \( MR(\theta) = 2\theta - 1 \).

- Award good to agent with highest value if \( \theta \geq \frac{1}{2} \); *i.e.*, reserve price \( r = \frac{1}{2} \).

- Note: \( r > \theta_0 \). Why? (By increasing the reserve price, the seller can reduce information rents.)

**Deriving bidding strategies:**

- Assume that bidding functions are (i) monotone, and (ii) symmetric.

- First-price auction:

\[
\begin{align*}
 u_i(\theta_i, \theta_i) &= \mathbb{E}_{\theta_{-i}} [(\theta_i - b(\theta_i)) p_i(\theta)] = F^{-1}(\theta_i) [\theta_i - b(\theta_i)] \\
 u_i(\theta_i, \theta_i) &= \int_{\theta_i}^{\theta_0} \mathbb{E}_{\theta_{-i}} [p(s, \theta_{-i})] ds = \int_{\theta_i}^{\theta_0} F_{N-1}(s) ds
\end{align*}
\]
Equating the two expressions, we obtain

\[ b(\theta) = \theta - \frac{\int_{\theta}^{\theta} F^{-1}(s) \, ds}{F^{-1}(\theta)} \]

**Asymmetries:**

- Suppose \( \theta_i \sim F_1(\cdot) \) (i.e., valuations come from different distributions).
- Define: \( MR_i(\theta_i) = \theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)} \)
- Revenue = \( \mathbb{E}_\theta [\sum_i p_i(\theta) [MR(\theta_i) - \theta_0]] + \theta_0 \)
- If bidder \( j \) has ex-ante higher valuation than bidder \( i \) (i.e., if \( \frac{1-F_j(\theta)}{f_j(\theta)} > \frac{1-F_i(\theta)}{f_i(\theta)} \)), then bias auction in favor of \( \theta_i \). (Formally, we say that \( \theta_j > HRO \theta_i \)).
  - If \( \theta_i = \theta_j - \epsilon \), then still allocate good to bidder \( i \).
  - Favor weak bidders to induce the stronger bidders to bid higher.

**Welfare Maximization (First Best)**

\[
\max_{p_i(\cdot)} \left\{ \mathbb{E}_\theta \left[ \sum_i p_i(\theta) \theta_i + \left[ 1 - \sum_i p_i(\theta) \right] \theta_0 \right] \right\}
\]

- **Solution:** Allocate the good to the agent with the highest valuation (incl. seller)
  - \( p_i(\theta) = 1 \) if and only if \( \theta_i > \theta_j \) for all \( j \neq i \) (otherwise 0).

- **Implementation:**
  1. First-price auction with reserve price \( \theta_0 \).
  2. Second-price auction with reserve price \( \theta_0 \).
  3. All-pay auction with reserve price \( \theta_0 \).

**References**
