 Kellogg  
Finance 440  

Problem Set #2: Risk

This project is designed to provide you with experience implementing the principles of mean-variance portfolio analysis using just a few securities, and to show you that a computer can easily extract and identify efficient portfolios from estimates of means and covariances that you provide.

You can download the Excel file called “optimize.xls” from the course Web page. This file has two spreadsheets. The one titled “Returns Data” provides you with daily returns data on five stocks for the entire year of 1992, broken into the first and second halves of the year. The market values of the firms that these stocks represent are contained in the top row of the spreadsheet. Below the market values are dates in the far left column, and daily returns in the remaining five columns. The last column contains the daily returns to a market index (the S&P 500). This is your proxy for the returns on the market.

You are going to compare the returns from several portfolios composed of varying proportions of the five stocks above: (a) a portfolio with an equal dollar amount invested in each stock (equal weighted portfolio), (b) a portfolio with an initial investment in each stock proportional to its market value (value weighted portfolio), and (c) an optimal portfolio. The computer will calculate the optimal portfolio by choosing portfolio weights that minimize the variance of the portfolio for a given level of expected returns. That is, it will solve the mean-variance portfolio problem.

To implement this, the spreadsheet titled “Optimizer” has the solution to the mean-variance portfolio problem for five securities programmed into it. In its upper-left is a list named “means” and a table named “covariance matrix”. If you provide inputs for these fields, the spreadsheet will automatically construct the efficient frontier of investments in those five securities. A graph of the frontier is provided on the top right of the spreadsheet. In the center of the spreadsheet is a table that contains three things. The first row is a list of expected returns (that you can alter as desired); the second row is the variance of the efficient portfolio that has the expected return that is given in the row just above it. The third through eighth rows are the weights you must apply to each of the five stocks in order to form the efficient portfolio with the expected return given in the top row of the table.

Part 1 of the assignment involves three main steps. The first step is to estimate the means and the covariance matrix of the returns to the five stocks using data from the first half of 1992; and also to estimate the mean return to the equal- and value-weighted portfolios of the five stocks during this time period. The second step is to plug the means and covariance estimates for the five stocks into the second spreadsheet, and identify the weights associated with the efficient portfolios of those five stocks that have the same mean return as the equal- and value-weighted portfolios. (The equal- and value-weighted portfolios will not in general be efficient since their weights are not chosen to minimize variance.) The final step is to use the two sets of optimal weights, the equal weights, and the value weights, to form four portfolios that will be held during the second half of 1992. Using these weights, construct the daily returns series for each of the four portfolios. Then compare the means and variances of these four portfolios’ returns during the second half of 1992.
After you’ve done this, answer the following questions:

1. What are the means and variances (or standard deviations) of the four portfolios during the second-half of 1992? Did the optimal portfolios outperform the equal- and value-weighted portfolios? Explain.

2. What did you expect based on the class discussion? Are you surprised by your findings, either because they are different from what you expected or because they are so similar to what you expected? What might explain these findings?

Part 2 of the assignment requires you to estimate the beta of each stock using its data from the first half of 1992 and the market index return provided on the first spreadsheet.

3. What are the betas for the stocks computed from the first six-months’ returns data? You can find the betas in either of two ways:
   a) Regress the returns of the individual stocks on the market returns (if you haven't already, you can refer to Class Notes #3, which has a section on "Estimating Beta Using Regression Analysis").
   b) Plug the variances and covariances computed in part 1 into the statistical formula for beta. (The resulting betas should be the same in either case!)

4. Are the average returns to the stocks during the second six months ranked in accordance with the ranking of their betas computed from the first six months of data? What did you expect based on the class discussion? Are you surprised? What might explain what you found?

5. Re-compute the stock betas from the second six months and check whether they have changed much. Does the ranking of the stocks’ average returns correspond to the ranking of these betas? With reference to the standard errors, how precise are these beta estimates?

Some Hints:

Formulas for computing the daily returns to the equal- and value-weighted portfolios during both halves of the year are already programmed for you; all you need to do is copy them down the column. Formulas for computing the optimal portfolios’ returns are also programmed for you; you have to insert the optimal weights from the second spreadsheet into the table that appears in the middle of the first spreadsheet.

You will find the Excel functions called “=Average( )” and “=Stdev( )” useful. Also, you will need to use the “Covariance” and “Regression” functions found by clicking on the “Tools” menu then selecting the “Data Analysis...” item. If the “Data Analysis...” item is not there, choose “Add-Ins” from the “Tools” menu, and add the “Analysis ToolPak - VBA” selection. The “Covariance” function will compute the covariance matrix for you, and the “Regression” function will estimate a regression for you.
The covariance function will give you the “lower-triangle” of the covariance matrix (which is symmetric), so you have to fill in the empty spaces before copying it to the second spreadsheet. When you copy these numbers (and the means, the weights, or any other numbers that are the result of a calculation done by the spreadsheet), use the “Paste Special” item from the “Edit” menu in order to copy “Values”.

In order to identify the weights associated with the efficient portfolio whose expected return is the same as that of an equal- or value-weighted portfolio, plug the expected return to the equal- or value-weighted portfolio into the first row of the “wide” table in the second spreadsheet labeled “Portfolio Expected Return”. **Do not change any entries in rows below this, however.**

The return numbers (and the means and standard deviations) will be small because they are daily returns. Carry out the analysis using daily returns; but feel free to annualize them if this helps you to interpret your final results.
THE OPTIMIZATION METHOD

Some of you may be curious about the formulas behind the calculations in the optimize.xls spreadsheet. The solution uses linear algebra and calculus; it is described in this note.

Analytics of the solution to the Mean-Variance Portfolio Problem for n risky assets:

Let:  
\( V = n \times n \) variance-covariance matrix of returns to assets  
\( m = n \times 1 \) vector of expected returns to assets  
\( x = n \times 1 \) vector of portfolio weights  
\( i = n \times 1 \) vector of ones  
\( r = \) portfolio expected return

The problem is to select the set of weights that minimizes the portfolio’s variance subject to the constraints that (a) the portfolio’s expected return equals \( r \), and (b) the portfolio weights add to one. Analytically:

\[
\begin{align*}
\text{min}_x & \quad x' V x \\
\text{s.t.} & \quad x' m = r \\
& \quad x' i = 1 
\end{align*}
\]

Taking partial derivatives with respect to each element of \( x \), and given the value of \( r \), the solution to this problem is

\[
x^*_r = k V^{-1} \{ x_0 + r x_1 \},
\]

where

\[
k = 1 / \left\{ (m' V^{-1} m) (i' V^{-1} i) - (i' V^{-1} m)^2 \right\}
\]

\[
x_0 = (m' V^{-1} m) i - (m' V^{-1} i) m
\]

\[
x_1 = (i' V^{-1} i) m - (m' V^{-1} i i)
\]

The variance of the least risky portfolio having expected return \( r \) is \((x^*_r)' V x^*_r\). Plotting points defined as \( \{ r, (x^*_r)' V x^*_r \} \) traces out the efficient set of portfolios.
**AN ADDITIONAL PRACTICE PROBLEM**

In this problem we will compute the value of investing in an Individual Retirement Account (IRA). Suppose that your tax rate is now and will forever be 30%. The before-tax risk-free interest rate is 10%. The facts about an IRA are as follows:

- Contributions to an IRA are tax-deductible. Thus, tax is not paid currently on the money which is put into an IRA.
- Interest earned on an IRA account is tax-free in the year in which it is earned.
- Tax is paid on all money taken out of an IRA in the year in which it is taken out.

Suppose that you are now 30, and that next year (age 31) and every year thereafter you will put $2,000 into an IRA. You will make your last contribution at age 65, and will withdraw the entire amount at age 67, two years after the last contribution is made. Assume that retirement funds not invested in an IRA will be invested in T-bills.

What is the NPV of investing in an IRA?

This is a difficult problem (from an old D30 exam) to help clarify how to work with personal taxes. Again, the key is to find after-tax cashflows and to discount by the after-tax opportunity cost.

At age 65, you will have accumulated $2,000(F/A,10%,35) = $542,049

At age 67, this will have increased to $542,049(F/P,10%,2) = $655,879

At age 67 you pay 30% of this in taxes, so your after-tax cashflow is $655,879(1-.3) = $459,115

In year 31, 32, ..., 65, your after-tax cash outflow is -$2,000(1 -.3) = -$1,400

(Your are paying the $2,000 into the IRA, but reducing your current taxes by 30% of this amount since taxes on the income is tax-deferred.)

You are told that your next best opportunity is to invest in taxable t-bills, so your after-tax opportunity cost is 10%(1-.3) = 7%. Again, this is because if you invested in the t-bills you would pay tax on the interest received each year, and effectively earn 7% after tax.

Then the NPV is -$1,400(P/A,7%,35) + $459,115(P/F,7%,37) = -$18,127 + $37,559.7 = $19,432.7

The positive NPV is due to the time value of deferring taxes.