Topic 2
Risk and Return I: Diversification and Portfolio Choice

Turbo Finance
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Goals for Topic 2

• Develop a measure of risk, and then develop a theory telling us what a financial security with a certain level of risk should earn.
• Knowing these risk measures and their earning implications, we can then
  – derive a discount rate for valuing both financial assets and real investments with that same level of risk, and
  – evaluate alternative investment strategies.
Outline of Topics

• Review of Probability
  – Random variables & Summary Statistics
• The Economics of Portfolio Choice
  – Expectation & Variance
  – Constructing portfolios
• Portfolio Theory
  – Diversification! Many risky assets plus a riskless asset
• Summary

I. Review of Probability

A. Random Variables and Probability Distributions

Random variables are variables that take on different values in different states of nature.

A random variable can be described entirely by its probability distribution.

Example 2-1. The face that shows when you toss a coin is a random variable.

The possible realizations are “head” and “tail”.

If “heads” is twice as likely as “tails”, then the probability distribution is:

<table>
<thead>
<tr>
<th>Coin Toss</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Head</td>
<td>2/3</td>
</tr>
<tr>
<td>Tail</td>
<td>1/3</td>
</tr>
</tbody>
</table>
It is common to associate numbers with the outcomes of random variables:

Say Gambler 1 bets $1 on heads and Gambler 2 bets $1 on tails.

The random variable that characterizes the winnings and losses of Gambler 1 has the probability distribution:

\[ W_1 = \text{Gambler 1's Winnings} \]

<table>
<thead>
<tr>
<th>Realization</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>+$1</td>
<td>2/3</td>
</tr>
<tr>
<td>-$1</td>
<td>1/3</td>
</tr>
</tbody>
</table>

1. The return to a share of common stock can also be described as a random variable.

Price is \( P_0 \).
Return after one period is \( \text{DIV}_1 + P_1 \).

The security's return over the period is the value of "R" that solves:

\[ \text{DIV}_1 + P_1 = (1+R)P_0 \]

or

\[ R = \frac{\text{DIV}_1 + P_1 - P_0}{P_0} \]

R is a random variable because \( \text{DIV}_1 \) and \( P_1 \) are not known.

Note that:

a) R can take on any value from -1 to infinity

b) In principle it can be completely described by its probability distribution

c) Describing the entire probability distribution is not feasible, so we focus on a few summary statistics
A. Expectation and Standard Deviation

A general probability distribution for the return to a stock, \( R \), is given by

\[
R = \text{Return to a Stock} \\
\begin{array}{|c|c|}
\hline
\text{Realization} & \text{Probability} \\
\hline
R_1 & p_1 \\
R_2 & p_2 \\
\vdots & \vdots \\
R_n & p_n \\
\hline
\end{array}
\]

Only restriction: the probabilities must add to one.

The **expectation** of \( R \) is

\[
\mu_R = E[R] = p_1 r_1 + p_2 r_2 + \ldots + p_n r_n.
\]

*We need to weight outcomes by their probabilities since more likely realizations would occur more often in many trials.*

To characterize how far \( R \) departs from its expectation, we need a measure of variation. *Variation can be interpreted as measure of risk.*

The **variance** of \( R \) is defined by

\[
\sigma_R^2 = \text{Var}[R] = p_1 (r_1 - \mu_R)^2 + p_2 (r_2 - \mu_R)^2 + \ldots + p_n (r_n - \mu_R)^2.
\]

This is the average deviation of \( R \) from its expectation in squared units.

To make the units the same as those of \( R \), we take the square-root to get the standard deviation:

\[
\sigma_R = \text{StdDev}[R] = \sqrt{\text{Var}[R]}.
\]
A. Covariance and Correlation

The covariance describes the degree to which two random variables move in the same direction.

*This is important for finance because the risk of a portfolio depends on the covariance between the securities in the portfolio.*

1. Systematic and Idiosyncratic risks
   
a) Returns on different stocks tend to move together due to economy-wide risks.
   
   Also called: systematic risk, undiversifiable risk, or \( \beta \) risk
   
b) Differences in returns are due to firm-specific risks.
   
   Also called: idiosyncratic risk, diversifiable risk, or non-systematic risk

Understanding the implications of systematic vs. idiosyncratic risk is one of the most important topics in finance.

**Example 2-2: Rolls Royce vs. Generic Foods**

Rolls profitable in boom, unprofitable in bust: “procyclical”

Generics unprofitable in boom, profitable in bust: “countercyclical”

Suppose there are two possibilities for the economy-- boom and bust

The return to the Rolls Royce and Generic Food can be described by the joint probability distribution:

<table>
<thead>
<tr>
<th>Generic</th>
<th>Rolls Royce</th>
<th>(boom) +50%</th>
<th>(bust) -20%</th>
</tr>
</thead>
<tbody>
<tr>
<td>(boom) -10%</td>
<td>%</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>(bust) +25%</td>
<td>0</td>
<td>1/4</td>
<td></td>
</tr>
</tbody>
</table>

Note that the probabilities are for “joint” events, and that these probabilities add to one.
In general, the joint probability distribution between two random variables (e.g., the random returns to two different stocks, \( R \) and \( S \)) looks like:

\[
\begin{array}{c|cccc}
R \backslash S & s_1 & s_2 & \ldots & s_n \\
\hline
r_1 & p_{11} & p_{12} & \ldots & p_{1n} \\
r_2 & p_{21} & p_{22} & \ldots & p_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
r_n & p_{n1} & p_{n2} & \ldots & p_{nn}
\end{array}
\]

1. Like expectation and standard deviation, the covariance is a summary statistic.

The covariance between \( R \) and \( S \) is defined as

\[
\text{Cov}[R, S] = \sigma_{R,S} = p_{11} (r_1 - \mu_R) (s_1 - \mu_S) + p_{12} (r_1 - \mu_R) (s_2 - \mu_S) + \ldots + p_{nm} (r_n - \mu_R) (s_n - \mu_S).
\]

This is the probability-weighted average of all possible joint deviations of the random variables from their respective means.

- If \( R \) and \( S \) tend to move together Cov will be positive.
- If \( R \) and \( S \) tend to move opposite Cov will be negative.
- Covariance is stated in squared units.

1. Correlation is a normalized measure of covariance.

The correlation between \( R \) and \( S \) is

\[
\text{Corr}[R, S] = \rho_{R,S} = \frac{\text{Cov}[R, S]}{\text{StdDev}[R] \times \text{StdDev}[S]}
\]

The correlation is unitless: it only takes values in the interval \([-1, 1]\).

- Random variables that have a correlation of 1 are said to be “perfectly positively correlated”
- Those that have a correlation of -1 are said to be “perfectly negatively correlated”
- If their correlation is zero, they are said to be “uncorrelated”
**Example (continued):** Statistics for Generic and Rolls Royce.

E[r_{Rolls}] = 
E[r_{Generic}] = 
Var[r_{Rolls}] = 
StdDev[r_{Rolls}] = 
Var[r_{Generic}] = 
StdDev[r_{Generic}] = 
Cov[r_{Rolls}, r_{Generic}] = 
Corr[r_{Rolls}, r_{Generic}] =

---

1. Estimators of expectation, variance, and covariance

In practice, we cannot observe joint distributions of stock returns directly, so we cannot calculate the “true” values of expectation, variance, and covariance.

**Solution:** estimate statistics from (recent) historical data.
Suppose we have a random sample of observations of the returns $R$ and $S$ across time, denoted $R_1, R_2, ..., R_T$ and $S_1, S_2, ..., S_T$.

The estimators of expectation, variance and covariance are given by:

$$
\hat{\mu}_R = \frac{1}{T} \sum_{t=1}^{T} R_t
$$

$$
\hat{\sigma}_R^2 = \frac{1}{T-1} \sum_{t=1}^{T} (R_t - \hat{\mu}_R)^2
$$

$$
\hat{\sigma}_{RS} = \frac{1}{T-2} \sum_{t=1}^{T} (R_t - \hat{\mu}_R)(S_t - \hat{\mu}_S)
$$

Notice that since the sample is random, we can weight each observation equally because more likely realizations will appear more frequently.

I. The Economics of Portfolio Choice

**Big question**: How do individuals evaluate the value of risky investments?

We need to answer this question in order to consider:

- Asset prices
- Portfolio choice
A. Expectation and Variance as Objects of Choice

1. Modeling assumptions (these are standard in finance and economics):

   a) Individual investment decisions depend only on expected returns, standard deviations, and covariances

   b) Non-Satiation: All else equal, people prefer higher expected returns.

   c) Risk Aversion: All else equal, people will prefer an investment with a smaller standard deviation.

Example 2-3. Pensions
Your employer gives you a choice among six mutually-exclusive retirement funds.

Based on the funds’ historical performance, you compute the expected return and standard deviation for each choice:

<table>
<thead>
<tr>
<th>Fund</th>
<th>μ</th>
<th>σ</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.30</td>
<td>0.20</td>
</tr>
<tr>
<td>2</td>
<td>0.25</td>
<td>0.19</td>
</tr>
<tr>
<td>3</td>
<td>0.10</td>
<td>0.05</td>
</tr>
<tr>
<td>4</td>
<td>0.28</td>
<td>0.17</td>
</tr>
<tr>
<td>5</td>
<td>0.19</td>
<td>0.07</td>
</tr>
<tr>
<td>6</td>
<td>0.15</td>
<td>0.10</td>
</tr>
</tbody>
</table>

If you are non-satiated and risk averse, which ones would you surely not choose?
The fact that investors demand a higher expected return on higher risk investments is borne out by historical data on returns over the 1926-1989 period:

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Annual Yield (Nominal)</th>
<th>Annual Std Dev</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small Stocks</td>
<td>12.2</td>
<td>35.3</td>
</tr>
<tr>
<td>S&amp;P 500 Stocks</td>
<td>10.3</td>
<td>20.9</td>
</tr>
<tr>
<td>Corporate Bonds</td>
<td>5.2</td>
<td>8.5</td>
</tr>
<tr>
<td>LT Governmt Bonds</td>
<td>4.5</td>
<td>8.6</td>
</tr>
<tr>
<td>T-Bills</td>
<td>3.6</td>
<td>3.4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Annual Yield (Real)</th>
<th>Annual Std Dev</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small Stocks</td>
<td>8.9</td>
<td>34.7</td>
</tr>
<tr>
<td>S&amp;P 500 Stocks</td>
<td>7.0</td>
<td>21.0</td>
</tr>
<tr>
<td>Corporate Bonds</td>
<td>2.0</td>
<td>9.9</td>
</tr>
<tr>
<td>LT Governmt Bonds</td>
<td>1.4</td>
<td>10.1</td>
</tr>
<tr>
<td>T-Bills</td>
<td>0.5</td>
<td>4.4</td>
</tr>
</tbody>
</table>
A. Constructing Portfolios

Individual assets --

(whether they are financial securities like stocks and bonds, or projects that are selected by firms)

-- are not held in isolation but are held in portfolios with other assets.

It follows that…

* A security's value reflects its contribution to the expected return and standard deviation of a portfolio.

1. Evaluating Returns on a Portfolio

A portfolio is just a basket or combination of assets.

**Example 2-4:** Suppose you have $100 to invest.

You invest $20 in asset 1 \((r_1=10\%)\)

$50 in asset 2 \((r_2=15\%)\)

$30 in asset 3 \((r_3=12\%)\)

The proportional weighting are 2/10, 5/10 and 3/10, respectively.

What will be the return to this portfolio?

Portfolio Return =

The return to any portfolio is simply the weighted sum of the returns to the individual assets.

The weights (sum to one and) represent the proportion of your wealth invested in each asset.
The expression for the random return to a general portfolio with three assets is:

\[ R_p = w_1 R_1 + w_2 R_2 + w_3 R_3. \]

Summary statistics for portfolio returns are:

\[
E[R_p] = w_1 E[R_1] + w_2 E[R_2] + w_3 E[R_3]
\]

\[
\text{Var}[R_p] = w_1^2 \text{Var}[R_1] + w_2^2 \text{Var}[R_2] + w_3^2 \text{Var}[R_3] \\
+ 2w_1w_2 \text{Cov}[R_1,R_2] + 2w_1w_3 \text{Cov}[R_1,R_3] \\
+ 2w_2w_3 \text{Cov}[R_2,R_3]
\]

1. Portfolio Variance

The covariance terms in the expression for portfolio variance are the mathematical counterpart to the idea of diversification.

a) With three assets, the portfolio’s variance depends on three variance terms and six covariance terms.

b) With N assets, the portfolio’s variance depends on N variance terms and N(N-1) covariance terms.

c) With many assets, the portfolio’s variance depends more on the average covariance---i.e., how the assets are related---than on their average variance.

The contribution from each stock to a portfolio’s variance can be represented in tabular form:

<table>
<thead>
<tr>
<th>Stk 1</th>
<th>Stk 2</th>
<th>Stk 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stk 1</td>
<td>(w_1^2 \text{Var}[R_1])</td>
<td>(w_1w_2 \text{Cov}[R_1,R_2])</td>
</tr>
<tr>
<td>Stk 2</td>
<td>(w_1w_2 \text{Cov}[R_1,R_2])</td>
<td>(w_2^2 \text{Var}[R_2])</td>
</tr>
<tr>
<td>Stk 3</td>
<td>(w_1w_3 \text{Cov}[R_3,R_1])</td>
<td>(w_2w_3 \text{Cov}[R_3,R_2])</td>
</tr>
</tbody>
</table>
When forming portfolios, people will choose portfolio weights optimally, i.e., to take full advantage of diversification.

When people do this, the price they are willing to pay for an asset does not depend on the asset’s own risk, but instead on how much risk the asset contributes to an optimally-weighted portfolio.

Before pursuing this insight further, we compute the expectation and standard deviation of some very simple portfolios....

1. Examples: Two-Asset Portfolios

Example 2-5. Suppose there are only two assets in the economy. Based on historical data, you estimate statistics on returns:

\[ \mu_1 = 0.10 \text{ and } \sigma_1 = 0.04; \]
\[ \mu_2 = 0.15 \text{ and } \sigma_2 = 0.09. \]

The correlation between the assets’ returns is \( \rho_{12} = 0.25. \)

Identify the combinations of portfolio expected return and standard deviation that you can obtain by combining these portfolios. Are there possibilities that you would surely never choose?

The set of portfolios you can choose have as their random return:

\[ R_p = w R_1 + (1-w) R_2. \]

For a given portfolio choice (i.e., choice of \( w \)), the portfolio’s expected return is

\[ E[R_p] = w E[R_1] + (1-w) E[R_2] = \]

and its standard deviation is the square-root of its variance,

\[ \text{Var}[R_p] = \]
We can plot these on a graph like in the pension example for various choices of "w" between -1 and 2:

Two Asset Portfolios

![Graph showing two asset portfolios with different weights and their corresponding expectations and standard deviations.]

a) Short Selling

To short sell an asset:

i) Borrow the asset from someone who owns it, then sell it into the market at its current price, say $P_0$.

ii) Later close out the short position by buying the security for $P_1$ and returning it to the lender.

Cashflows:

0                                    1
+P_0                                  -P_1

The short-seller pays the asset’s random return.

Thus, the effect of a short sale on a portfolio return is the same as “holding” the shorted asset in the portfolio with a negative weight.
Example 2-6. The Effect of Correlation on Available Risk/Return Opportunities

Consider again two stocks. Assume each has an expected rate of return and standard deviation of the rate of return given by:

\[ E[R_a] = E[R_b] = .20 \]
\[ \text{StdDev}[R_a] = \text{StdDev}[R_b] = .22 \]

The available risk/return combinations will depend on the correlation between the two stocks:

<table>
<thead>
<tr>
<th>Corr[R_a,R_b]</th>
<th>1.0</th>
<th>0.5</th>
<th>0.0</th>
<th>-0.5</th>
<th>-1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>StdDev[.5R_a+.5R_b]</td>
<td>22%</td>
<td>19%</td>
<td>16%</td>
<td>11%</td>
<td>0%</td>
</tr>
</tbody>
</table>

- Note that the expected rate of return on the portfolio will always be 20%.
- Diversification will wipe out risk unless assets are perfectly correlated.
- Even with a positive correlation, there is less risk in holding a portfolio than in holding individual stocks.

Questions to think about:

What component of the portfolio variance formula assures that holding additional uncorrelated assets reduces portfolio variance?

How can holding positively correlated assets be less risky than holding any individual asset?
I. Portfolio Theory

We now look at portfolio choice with many available assets.

There are two main cases:

(A) All assets are risky
(B) There is also a risk-free security (T-bills)

We’ll start with the first case, and then include the second.

A. Many risky assets

1. The expected return-risk hyperbola

The risk-return trade-off of portfolios with many risky assets can be diagrammed as

The defining feature of each point on the upper boundary is that it represents a portfolio with the lowest standard deviation for a given level of expected return.

The boundary point marked “A” plots the expected return and standard deviation of the lowest-risk portfolio that has expected return $r_a$

This means that the list of weights corresponding to portfolio “A” solves the following problem:

$$\min_{w_1, \ldots, w_n} \sigma_p(w_1, \ldots, w_n)$$

s.t. $\mu_p(w_1, \ldots, w_n) = r_a$

$$w_1 + \ldots + w_n = 1.$$ 

The solution is a set of weights $(w_1^*, w_2^*, \ldots, w_n^*)$ associated with the minimum-standard-deviation portfolio that has expected return $r_e$.

If we solve this minimization problem for all levels of the expected return, we get the boundary hyperbola.

[These are the calculations performed by the OPTIMIZER spreadsheet.]
Having constructed the hyperbola, an individual can choose the expected return and standard deviation that best match his attitude toward risk.

This portfolio strategy is easy to implement:

Since we know the weights associated with each boundary portfolio, one need only purchase (or short-sell) the assets in accordance with that list of weights.

---

B. Riskless and Risky Assets

How does our portfolio analysis change if there is also a risk-free asset?

Consider a portfolio, “*”, constructed from a riskless asset with fixed return $r_f$, and a risky portfolio whose random return is $R_p$.

Then

$$R_* = w r_f + (1-w) R_p$$

Compute the expectation and standard deviation of “*”, as functions of “w”:

$$\mu_*(w) = w r_f + (1-w) \mu_p$$

$$\sigma^2_*(w) = w^2 \sigma^2_f + (1-w)^2 \sigma^2_p + 2w(1-w) \sigma_f \sigma_p$$

or

$$\sigma_*(w) = (1-w) \sigma_p$$

Since the riskless return is non-random, it has neither variance nor covariance with other random returns.
For a given value of $w$, these expressions give us values for $\sigma_*(w)$ and $\mu_*(w)$. If we plot pairs $\{\sigma_*(w), \mu_*(w)\}$ for different values of $w$, we see

![Diagram showing a straight-line relationship between $\mu$ and $\sigma$](image)

This is a straight-line since both expressions above are linear functions of “$w$”.

Points on the line between $r_f$ and “$p$” are portfolios that take long positions in both the riskless asset and portfolio “$p$”.

Points to the northeast of “$p$” involve short-sales of the riskless asset (borrowing by making a riskless promise to repay) and investing the proceeds in portfolio “$p$”.

Note that this portfolio “$p$” could have been any portfolio of risky assets, either to the interior of the hyperbola, or on the boundary.

![Diagram showing a hyperbola with a point “p”](image)

It is clear from this graph, however, that “$p$” is not the best portfolio of risky assets that an individual could choose to combine with the riskless asset.
In fact, there is only one risky portfolio that individuals will combine with the riskless asset---the tangency portfolio marked “m”:

Conclusion: All individuals should choose portfolios that combine the riskless asset and “m”!

1. The capital market line, or CML, describes all portfolios combining the riskless asset and the market portfolio

The selection along this line (the heavy line above) that a particular individual makes depends on his or her subjective taste for risk.

But everyone holds a share of the same risky portfolio, “m”.

For this reason, the tangency portfolio is called the market portfolio.

a) In principle the market portfolio represents the value-weighted index of all traded risky assets in the economy.

b) In practice a broad index of stocks (such as the S&P 500) is used as a proxy for the market portfolio.
Example 2.7 Looking at the CML Menu

Assume: \( r_f = 2.5\% \), \( E[r_m] = 11.3\% \), \( \sigma_m = 22.4\% \)

a) All Funds in T-Bills:

b) Half in T-Bills, Half in Market Portfolio:

c) All Funds in Market Portfolio:

d) Levering Up Strategy:

I. Summary

Key Concepts:

Variation can be interpreted as measure of risk.

Understanding the implications of systematic vs. idiosyncratic risk is one of the most important topics in finance.

A security’s value reflects its contribution to the expected return and standard deviation of a portfolio.

The covariance terms in the expression for portfolio variance are the mathematical counterpart to the idea of diversification.

Definitions:

Random variables are variables that take on different values in different states of nature.
Notation:

The expectation of \( R \) is \( \mu_R = E[R] = \mu_1 P_1 + \mu_2 P_2 + \ldots + \mu_r P_r \).

The variance of \( R \) is defined by

\[
\sigma_R^2 = \text{Var}[R] = P_1 (\mu_1 - \mu_R)^2 + P_2 (\mu_2 - \mu_R)^2 + \ldots + P_r (\mu_r - \mu_R)^2.
\]

To make the units the same as those of \( R \), we take the square-root to get the standard deviation:

\[
\sigma_R = \text{StdDev}[R] = \sqrt{\text{Var}[R]}.
\]

The covariance between \( R \) and \( S \) is defined as

\[
\text{Cov}[R, S] = \sigma_{R,S} = P_{11} (\mu_1 - \mu_R) (\mu_1 - \mu_S) + P_{12} (\mu_1 - \mu_R) (\mu_2 - \mu_S) + \ldots
\]

\[
+ P_{NM} (\mu_N - \mu_R) (\mu_N - \mu_S).
\]

The correlation between \( R \) and \( S \) is

\[
\text{Corr}[R, S] = \rho_{R,S} = \frac{\text{Cov}[R, S]}{\sqrt{\text{Var}[R]} \sqrt{\text{Var}[S]}}.
\]

The estimators of expectation, variance and covariance are given by:

\[
\hat{\mu}_R = \frac{1}{T} \sum_{t=1}^{T} R_t, \quad \hat{\sigma}_R^2 = \frac{1}{T-1} \sum_{t=1}^{T} (R_t - \hat{\mu}_R)^2, \quad \hat{\sigma}_{RS} = \frac{1}{T-2} \sum_{t=1}^{T} (R_t - \hat{\mu}_R) (S_t - \hat{\mu}_S).
\]

The expression for the random return to a general portfolio with three assets is:

\[
R_P = w_1 R_1 + w_2 R_2 + w_3 R_3.
\]

Summary statistics for portfolio returns are:

\[
E[R_P] = w_1 E[R_1] + w_2 E[R_2] + w_3 E[R_3]
\]

\[
\text{Var}[R_P] = w_1^2 \text{Var}[R_1] + w_2^2 \text{Var}[R_2] + w_3^2 \text{Var}[R_3] + 2 w_1 w_2 \text{Cov}[R_1, R_2] + 2 w_1 w_3 \text{Cov}[R_1, R_3] + 2 w_2 w_3 \text{Cov}[R_2, R_3].
\]