Finance 440 - Turbo Finance

Topic 7a
The Yield Curve and
the Term Structure of Interest Rates

Goals for Topic 7a

• Construct the "Yield Curve" (a.k.a. The Term Structure of Interest Rates) from pure discount bonds, and from coupon-bearing bonds.

• Use the yield curve to price other bonds, or more generally, any package of cash flows with similar characteristics in terms of maturity, risk and liquidity.

• Understand and interpret implied forward rates
Outline of Topics

• Pure Discount Bonds
• Constructing Spot Yield Curves
• Implied Forward Rates
• Duration and Convexity
• Summary

I. Pure Discount Bonds

Pure Discount Bonds pay only a single cash flow – the face value, at maturity

The prices of these bonds (relative to their face value) are direct indicators of the discounting – or the “pure discount”
B_o(T) is the time-0 equilibrium price of a pure discount bond with maturity T, per dollar of face value

\[ B_o(T) = \frac{1}{(1+r_0(T))^T} \]

Where \( r_0(T) \) is the time zero, per-period, rate of return on a T-period investment,

… or the “spot rate” on the T-period investment.

If we know the prices of pure discount bonds with maturities 1, 2, 3,... then we also know what the price of a riskless bond with any complicated stream of future payments has to be.

Example: Suppose that \( B_o(1) = $0.91 \), \( B_o(2) = $0.79 \) and \( B_o(3) = $0.65 \).

What is the price of a “New” riskless bond with a face value of $1000 that makes coupon payments each period at a rate of 8%, and matures in 3 periods?
The same stream of payments is made by a portfolio that includes:

- 

So the price of this bond must be the same as the price of the portfolio:

Portfolio Value =

An equivalent approach is to solve for each interest rate first, using the definition of an interest rate

\[ B_0(t) = \frac{1}{1+r_0(t)}^t \text{ for } t = 1, 2, 3 \]

to get:

- \( r_0(1) = 10\% \),
- \( r_0(2) = 13\% \),
- \( r_0(3) = 15\% \).

Then use these interest rates to compute the PV of the stream of payments associated with the bond:

"New” Bond Price =.
The interest rates we just calculated, \( r_0(1) = 10\% \), \( r_0(2) = 13\% \), \( r_0(3) = 15\% \), ..., can be graphed to obtain a yield curve, generally of the form

![Yield Curve Diagram]

---

**A. Actual Pure Discount Bonds: Treasury Bills and STRIPS**

1. *Treasury Bills:*
   a) obligations of the U.S. federal government, with initial maturity of one year or less.

   b) the most common type of discount bond.

   c) quoted differently from most other securities.

2. *STRIPS*
   a) also Treasury obligations

   b) created by stripping payments from coupon bonds.
II. Constructing Spot Yield Curves for Coupon Bearing Bonds

• To construct a spot yield curve with coupon bonds, treat each bond as a portfolio of its component parts.

• In other words, decompose the total bond value today into the value of each coupon or principal payment today.

This approach is fundamentally different than using the yield to maturity

yield to maturity = internal rate of return

The yield to maturity answers the question, "what average return do I earn over the life of this investment?"

This is not a good tool for precisely measuring the value of individual cash flows.
For instance, say that the three year spot yield curve is: \( r(1) = 11\% \), \( r(2) = 12\% \), \( r(3) = 12.14\% \), and that you are considering a three year 10% coupon bond. The price of this bond is $95 per $100 face value, since

\[
95 = \frac{10}{1.11} + \frac{10}{(1.12)^2} + \frac{110}{(1.1214)^3}
\]

The yield to maturity of this bond is 12.08%, which solves

\[
95 = \frac{10}{1 + YTM} + \frac{10}{(1 + YTM)^2} + \frac{110}{(1 + YTM)^3}
\]

The yield to maturity tells you the average return over the life of this investment, but is not useful for pricing other similar bonds (e.g., a 3 year bond with a 7% coupon rate).

**Interpreting Yields with Default Risk**

- The yield on a bond represents the return if all payments are made.
- This may not be true due to default risk.
- The price of a bond with default risk is equal to the expected (not promised) payments, discounted at an appropriate risk-adjusted rate.
Example: UAL

- In Nov. 2002, bond with coupon of 10.67% (s.a.) maturing in May 2004 had a price of $43.40 per $100 face value.
- Implied YTM is 83.5%!
- More likely, market expected each payment with only 40% probability. Then implied yield = 4.73%, which follows from:

Finding the Spot Curve from Coupon Bond Price Information

- **Example.** You have the following information:
  - A one year 6% coupon bond (annual payments) sells for $100 per $100 face value.
  - A two year 10% coupon bond (annual payments) sells for $100 per $100 face value.
- Find the two year spot yield curve.
Step 1: Solve for the one year spot yield:

Step 2: Solve for the two year spot yield:

III. Implied Forward Rates

Consider two strategies:

a*: buy a bond that matures in two periods

b*: buy two bonds in succession each with a one-period horizon

The interest rate on the second one-period investment that equates the payoffs to the two strategies is the implied forward rate. It solves:

\[(1+r_{0}(2))^2 \cdot $1 = (1+f_{1}(1))(1+r_{0}(1)) \cdot $1\]
More generally, from the time-0 yield curve, we can compute implied one-period forward rates from the equation:

\[
(1+r_{0}(T))^{T} = (1+f_{T-1}(1))(1+r_{0}(T-1))^{T-1}.
\]

for any \(T\) we want.

A. The implied forward rates help explain the shape of the yield curve:

- If \(f_{T-1}(1) > r_{0}(T-1)\), then the yield curve slopes up between horizons \(T-1\) and \(T\);
- if \(f_{T-1}(1) < r_{0}(T-1)\), then the yield curve slopes down between horizons \(T-1\) and \(T\).

- Consequently, explaining differences in returns on investments having different horizons (i.e., slope in the yield curve) is equivalent to explaining differences in implied forward rates.
IV. Duration

A bond’s maturity does not take into account the arrival of coupons prior to maturity. “Duration” is a measure of time horizon that takes into account all cash flows. It is most often used as an indicator of the sensitivity of bond price to changes in interest rates.

If a bond has cash flows $C_t$ at times 1…N and current price, $P$, then its duration is

$$\text{Duration} = \frac{1 \times PV(C_1)}{P} + \frac{2 \times PV(C_2)}{P} + \ldots \frac{N \times PV(C_N)}{P}$$

- Duration is a value-weighted average of the time until cash flows arrive.
- Duration is also useful for summarizing the effects of interest rate changes on bond prices.
- It is routinely used in hedging strategies.
• **Example:** A bank finances a commercial real estate development with a 5-year loan, with annual payments of $2 million, a final payment of $15 million, and a yield of 12%.

\[
P = \frac{2}{1.12} + \frac{2}{(1.12)^2} + \frac{2}{(1.12)^3} + \frac{2}{(1.12)^4} + \frac{17}{(1.12)^5} \\
= 1.786 + 1.594 + 1.424 + 1.271 + 9.646 = 15.72
\]

\[
Duration = \frac{1\times 1.786 + 2\times 1.594 + 3\times 1.424 + 4\times 1.271 + 5\times 9.646}{15.72 + 15.72 + 15.72 + 15.72 + 15.72} \\
= (1\times.114) + (2\times.101) + (3\times.091) + (4\times.081) + (5\times.614) = 3.983
\]

---

**Graphical Interpretation of Duration and Convexity**

\[
\frac{\Delta P}{\Delta Y} = -D \cdot P
\]
V. Interpreting the Yield Curve

• Theories of the yield curve help to explain:
  – The shape of the yield curve at a point in time
  – How the yield curve moves over time
  – What one can infer about future economic conditions from the yield curve.

• Three traditional theories:
  – expectations hypothesis, liquidity preference, market segmentation

Market Segmentation Theory

• Some investors/borrowers prefer long maturities (e.g., life insurers & pension funds)
• Others like short maturities (e.g., banks)
• The forces of supply and demand operate somewhat separately in the long and short-term markets
• Some weak evidence for this, but not dominant theory of the term structure.
Expectations Hypothesis

- The “Expectations Hypothesis” says that investors choose a portfolio of securities with the highest expected holding period return (all else equal).

- Therefore, due to the forces of supply and demand, the expected rate of return on any security over a given holding period is the same, regardless of maturity (all else equal).

Example: Consider an investor with a two year investment horizon. The expected return is the same from:

- Buying a one-year security and rolling it over into another one-year security at the end of the first year;

- Buying a two-year security;

- Buying a five-year security and selling it after two years.
a) The most important implication of this theory is that the forward rates implied by the term structure are equal to the market's expectation of future spot rates over the same period.

It follows that long-term yields are geometric averages of current and expected short-term yields.

The unbiased expectations theory relates current forward interest rates with expected future spot rates with the simple equation: $f_t(n) = E_0(r_t(n))$

---

**Liquidity Preference**

Investors are not indifferent between bonds of different maturities, but require a premium for investing in longer term debt.

The premium required is called a "liquidity premium."

The empirical evidence suggests that both the expectations hypothesis and liquidity preference help to explain the term structure.
One frequently suggested explanation for the liquidity premium is the much higher sensitivity of long-term bond prices to interest rate changes than short-term bond prices.
Example: Consider two discount bonds. Each has a $1000 face value, but one has two years to maturity and the other has 15 years to maturity. Suppose the current yield curve is flat at 8%.

\[
\text{Price of Bond 1} = \frac{1000}{(1.08)^2} = 857.34 \\
\text{Price of Bond 2} = \frac{1000}{(1.08)^{15}} = 315.24
\]

Now, suppose that the interest rates rise 1% to 9%.

\[
\text{Price of Bond 1} = \frac{1000}{(1.09)^2} = 841.68 \\
\text{% Change} = \frac{(841.68 - 857.34)}{857.34} = -1.83%
\]

\[
\text{Price of Bond 2} = \frac{1000}{(1.09)^{15}} = 274.54 \\
\text{% Change} = \frac{(274.54 - 315.24)}{315.24} = -12.91%
\]

Thus, long-term bond prices tend to be much more sensitive to interest rate changes.

V. Summary

- Key Concepts
- Definitions
- Notation
Key Concepts

THE EQUILIBRIUM PRICE EQUATES THE SUPPLY AND DEMAND FOR BONDS AT EACH MATURITY.

IF WE KNOW THE PRICES OF PURE DISCOUNT BONDS WITH MATURITIES 1, 2, 3,... THEN WE ALSO KNOW WHAT THE PRICE OF A RISKLESS BOND WITH A COMPLICATED STREAM OF FUTURE PAYMENTS HAS TO BE.

TO CONSTRUCT A SPOT YIELD CURVE WITH COUPON BONDS, TREAT EACH BOND AS A PORTFOLIO OF ITS COMPONENT PARTS.

Definitions

THE INTEREST RATE ON A SECOND ONE-PERIOD INVESTMENT THAT MAKES THE PAYOFF EQUAL TO A SINGLE TWO-PERIOD INVESTMENT IS CALLED THE FORWARD RATE.

TRADITIONAL EXPECTATIONS: DUE TO SUPPLY AND DEMAND, EXPECTED RATE OF RETURN ON ANY SECURITY OVER A GIVEN HOLDING PERIOD IS THE SAME, REGARDLESS OF MATURITY.

LIQUIDITY PREFERENCE: INVESTORS ARE NOT INDIFFERENT BETWEEN BONDS OF DIFFERENT MATURITIES, BUT REQUIRE A PREMIUM FOR INVESTING IN LONGER TERM DEBT.
THE T-BILL DISCOUNT RATE, “d”. FOR A T-BILL WITH A FACE VALUE OF $1 MILLION: PRICE = $1M - $1M \times d \times (# \text{ days}/360)

FROM THE TIME-0 YIELD CURVE, COMPUTE IMPLIED FORWARD RATES FROM THE EQUATION: $(1+r_0(T))^T = (1+f_{T-1}(1))(1+r_0(T-1))^{T-1}$

\[
\text{Duration} = \frac{1 \times PV(C_1)}{P} + \frac{2 \times PV(C_2)}{P} + \ldots + \frac{N \times PV(C_N)}{P}
\]

Next Time:

- Raising Capital in Perfect Markets