Finance 440 - Turbo Finance

Topic 6
Derivative Securities

Goals for Topic 6

• Understand the payoffs associated with derivative securities and the relationship between the payoffs of derivatives and underlying securities.

• Replicate the payoffs of an asset using options, stocks, and bonds.

• Implement hedge portfolio and risk-neutral pricing of securities.
I. Introduction to Derivatives

Derivative security and contingent claim are generic names given to a security whose payoff is determined by the price of some other security or portfolio of other securities.
A. Determining the value of a derivative security is “easier” than determining the value of a fundamental security:

1. The fact that a security is a derivative means that its payoff can be constructed using a portfolio of existing securities.
2. Given the prices of these securities, the prices of derivatives can be determined by appealing to the “no-arbitrage” property of equilibrium in financial markets.

II. Put and Call Options

A. A call option is a contract that gives its owner the right, but not the obligation, to:
   – purchase a pre-specified security (called the underlying security)
   – at a pre-specified price (called the strike price or exercise price)
   – on a future date (called the maturity date).
Example... If I own a May 1 call option on IBM stock with a strike price of $90, then come May 1, the person who sold me the option (the writer of the option) has to sell me a share of IBM stock for $90 if I notify him that I want to “exercise” the option. I am not required to exercise the option.

The option price (premium) was paid up front. The stock is purchased at the exercise price on May 1.

B. A put option is a contract that gives its owner the right, but not the obligation, to sell the underlying security at the strike price on the maturity date.
• Options that can be exercised only on the maturity date are called **European**.

• Options that can be exercised on or before the maturity date are called **American**.

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**C. Payoff and Position Diagrams**

Under what conditions would the owner of the call option exercise the option?

• If the price of IBM’s stock is less than $90 on May 1, the owner of the call would clearly not exercise it

• If the stock’s price exceeds $90 on May 1, the owner will exercise it

The payoff is the difference between the value of the stock (its market price) and $90.
C. Payoff and Position Diagrams

Payoff to a Call Option

Payoff at Maturity

Stock price at maturity

K (strike price)

C. Payoff and Position Diagrams

Payoff to a Put Option

Payoff at Maturity

Stock price at maturity

K (strike price)
D. Some Properties of Option Prices

1. The more volatile are the price changes of the underlying security, the more valuable the option.

2. For an American option, the longer is the option’s maturity, the greater its value.

This makes early exercise of options seldom optimal.

E. Put-Call Parity

- Call and put options are derivatives whose payoffs are based on the price of the underlying stock and a riskless bond.
- Of these four securities (a put, a call, a riskless bond, and a stock), one is redundant -- we can replicate its payoff using a portfolio of the other three.
- *The Put-Call Parity Relation:* if we know the prices any three, the value of the fourth must be equal to the price of the replicating portfolio.
Bond + Right to Buy

Payoff

K

K

Stock price

= Stock + Right to Sell

Payoff

K

K

Stock price
Payoff\{T\text{-}maturity call option with strike price } K \} + \\
Payoff\{T\text{-}maturity pure-discount bonds with face value } K \} \\
= Payoff\{One share of stock\} + \\
Payoff\{T\text{-}maturity put option with strike price } K \}

*The “no arbitrage” condition means that if the payoff at maturity is always equal for the two portfolios, then the price before maturity must always be identical for the two portfolios.*

III. Financial Engineering

A. Replicating Portfolios

*Using stocks, bonds and options, we can construct portfolios that produce almost any desired payoff pattern.*

*This helps us understand more complex securities and price untraded securities - especially important for hedging purposes.*
B. Corporate Debt and Levered Equity

The payoffs of a zero coupon corporate bond are structured so that….

if the assets are worth less than $F$ (face value), bondholders get the assets; otherwise, they receive $F$. 

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Payoff to a Zero Coupon Corporate Bond

- **Payoff**
- **$F$**
- **Asset value at Bond’s maturity**
We can construct this payoff function using a riskless bonds and a put option on the firm’s assets. It is the payoff from this portfolio:

(i) **long** $F$ pure-discount risk-free **bonds** and

(ii) short a put option on the firm’s assets with a strike price of $F$
Levered equity - equity in a firm with debt outstanding -

Payoff

\[ F \]  (face value)

Asset value at Bond’s maturity

IV. Pricing Contingent Claims

Options and other derivative securities are priced using “no arbitrage” conditions.

*The general method is:*

- Find a portfolio of securities whose payoff replicates the option payoff
- Find the price of the replicating (or hedge) portfolio
- The option price must equal the price of the replicating portfolio
A. The Binomial Model

Within a sub-period, the price of a stock might go up or down …

$S_0 \leftarrow S_H \leftarrow S_L$

So the price of a derivative on that stock would also go up or down ...

$C_0 \leftarrow C_H \leftarrow C_L$

For example, if the derivative is a call option with exercise price $K$, then

$$C_H = \max\{0, S_H - K\}$$

and

$$C_L = \max\{0, S_L - K\}.$$
There are two approaches to finding \( C_0 \), the price of the derivative:

- Find the replicating (or hedge) portfolio and price that, or…
- Use risk-neutral pricing.

We’ll start with finding hedge portfolios….

B. The Hedge Portfolio Approach

Take a stock with current price $70.00, which will move up to $100 or drop to $50.

Suppose the contingent claim pays off $25.00 if the stock price moves up, and $0 if the stock price moves down.

The riskless rate is 4%:
What is the price of the contingent claim? --

Consider the returns from the portfolio

• purchase 1 share of stock
• short 50 pure discount bonds
  (that is, borrow 50/1+r)

Hedge
Portfolio

These payoffs are exactly twice that of the derivative, or equal to the payoffs to two derivatives.
But... if two portfolios have the same payoff for sure, then *their prices must also be the same*.

**Therefore, the value of two of the derivative securities is the same as the value of the portfolio.**

A share of stock costs $70.00; the proceeds from selling 50 pure discount bonds is 50 times the price of each PDB:

Sometimes easier than finding the hedging portfolio…

**Risk Neutral Pricing**

\[
S_0 \quad S_H \quad S_L
\]

Find the “risk neutral probabilities” -- these make the stock price at time 0, \( S_0 \), equal to the present value of the expected stock price at time 1 when discounting at the risk-free rate, \( r_f \).
Imagine probabilities of the high price ($\pi_H$) and the low price ($1-\pi_H$):

$$
\begin{align*}
&\pi_H & S_H \\
&S_0 & \downarrow \\
&1-\pi_H & S_L
\end{align*}
$$

Calculate the expected PV of future prices using the risk-free rate:

Solve for $\pi_H$:

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**A risk neutral pricing example:**

The current stock price is $70.00, which will move up to $100 or drop to $50. The riskless rate is 4%.

A contingent claim pays off $25.00 if the stock price moves up, and $0 if the stock price moves down.

*Solve for the risk-neutral probabilities:*

The value of the contingent claim is:
Implications of Pricing Results

- The Binomial Model becomes more exact as the time period becomes shorter
  
  * so, we need to consider multi-period cases
  
  (the Black-Scholes Option Pricing Formula is the extreme case - infinitely small steps)

- Option pricing results can be applied to real as well as financial assets: Real Options

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Options - Examples

1. Multi-Period Example

To use the binomial model more generally, we solve it working backwards.

**Example:** The current price of a stock is $70, which can either increase or decrease by 10% (per-period) over the next two periods. The riskless rate is 1% per-period.

**What is the value of a European call option with a $72 strike price?**

The evolution of the stock's price is:

70.00
  \[\rightarrow\] 77.00
  \[\rightarrow\] 84.70
  \[\rightarrow\] 69.30
  \[\rightarrow\] 69.30
  \[\rightarrow\] 63.00
  \[\rightarrow\] 56.70
which implies that the option’s terminal payoffs and values at each date are:

\[ C_0, C_{1H}, C_{1L} \]

The way to solve problems like this is to work backwards.

First, calculate the risk-neutral probabilities for each of the right-most trees.

\[ C_{1H} = \quad C_{1L} = \]

Notice that the probabilities for both branches were the same (.55). This will always be the case when the stock price is assumed to increase or decrease by a constant percentage, as in this case.

Therefore \( \pi_{H} \) is also .55 at time 0.

Applying this to the left tree gives us

which is the current price of the option.
1. The Black Scholes Model

The binomial tree will better reflect the distribution of the stock price if time is divided into many short intervals.

The limiting value of this procedure results in the famous Black-Scholes Option Pricing Model.

The inputs into the Black-Scholes model include:

1) the stock’s expected return
2) the standard deviation of the stock price per unit time
3) the risk-free rate
4) the strike price of the call option
5) the time to maturity of the call option.

1. Real Options

Option pricing can be applied to real assets as well as to financial assets. We have already seen the flexibility in timing can be handled with the NPV rule, now we see that the "value of flexibility" can be priced….

Example: Option to Wait... You have a project whose value depends on whether demand will be strong or weak next year.

The risk-free rate is 10%.

You estimate the NPV of the project to be $30 million based on the following possibilities for the time 1 value of the project’s future cash flows:

$100 million
NPV(invest now)=$30
-$20 million

This analysis did not include the fact that you have flexibility in timing.
If you wait until next year to see whether demand is high or not you can:

(a) avoid the loss entirely if demand is low by not undertaking the project, or
(b) realize future cash flows whose time 1 value is $75 million if demand is high.

What is the value of the option to wait? Should you wait? How much would the project be worth if you could sell it to someone now?

First, note that the option’s payoffs are

Option Value = ?

We can compute the state prices from the project because it is the underlying asset:

So the value of the option to wait is

The fact that the option is valuable means that you should not kill it; so wait.

If you were to sell the project to someone, what is the lowest price you would sell for?

Alternatively, how would you value the project taking into account that you will wait?
V. Summary

- Key Concepts
- Definitions
- Notation

Key Concepts

THE PUT-CALL PARITY RELATION SAYS THAT IF WE KNOW THE PRICES OF ANY THREE (OF THE STOCK, BOND, AND OPTIONS), THE VALUE OF THE FOURTH MUST BE EQUAL TO THE PRICE OF THE REPLICATING PORTFOLIO.

THE “NO ARBITRAGE” CONDITION MEANS THAT IF THE PAYOFF AT MATURITY IS ALWAYS EQUAL FOR THE TWO PORTFOLIOS, THEN THE PRICE BEFORE MATURITY MUST ALWAYS BE IDENTICAL FOR THE TWO PORTFOLIOS.

USING STOCKS, BONDS AND OPTIONS, WE CAN CONSTRUCT PORTFOLIOS THAT PRODUCE ALMOST ANY DESIRED PAYOFF PATTERN.
Definitions

DERIVATIVE SECURITY AND CONTINGENT CLAIM ARE GENERIC NAMES GIVEN TO A SECURITY WHOSE PAYOFF IS DETERMINED BY THE PRICE OF SOME OTHER SECURITY OR PORTFOLIO OF OTHER SECURITIES.

CALL AND PUT OPTIONS CONFER RIGHTS, BUT NOT OBLIGATIONS, TO TRADE IN THE UNDERLYING SECURITY AT THE STRIKE PRICE ON THE MATURITY DATE.

OPTIONS THAT CAN BE EXERCISED ONLY ON THE MATURITY DATE ARE CALLED EUROPEAN.

OPTIONS THAT CAN BE EXERCISED ON OR BEFORE THE MATURITY DATE ARE CALLED AMERICAN.

Notation

PUT-CALL PARITY:

\[
\text{PAYOFF}\{\text{T-MATURITY CALL OPTION WITH STRIKE PRICE } K\} + \text{PAYOFF}\{\text{T-MATURITY PURE-DISCOUNT BONDS WITH FACE VALUE } K\} = \text{PAYOFF}\{\text{ONE SHARE OF STOCK}\} + \text{PAYOFF}\{\text{T-MATURITY PUT OPTION WITH STRIKE PRICE } K\}
\]
Next Time:

- Interest rates are market prices!
- Variation in interest rates
  - over time
  - across maturities