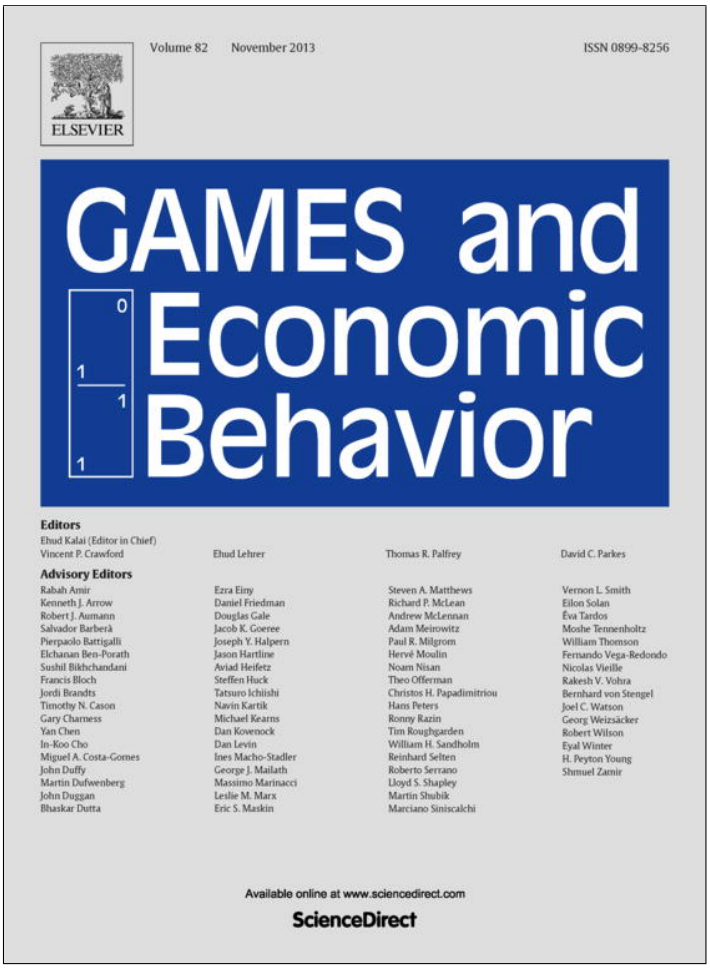
Provided for non-commercial research and education use.
Not for reproduction, distribution or commercial use.



Volume 82 November 2013 ISSN 0899-8256

ELSEVIER

GAMES and Economic Behavior

Editors
Ehud Kalai (Editor in Chief)
Vincent P. Crawford
Ehud Lehrer
Thomas R. Palfrey
David C. Parkes

Advisory Editors
Rabah Amir
Kenneth J. Arrow
Robert J. Aumann
Salvador Barberà
Pierpaolo Battigalli
Elchanan Ben-Porath
Sushil Bikhchandani
Francis Bloch
Jordi Brandts
Timothy N. Cason
Gary Charness
Yan Chen
In-Koo Cho
Miguel A. Costa-Gomes
John Duffy
Martin Dufwenberg
John Duggan
Bhaskar Dutta
Ezra Einy
Daniel Friedman
Douglas Gale
Jacob K. Goeree
Joseph Y. Halpern
Jason Hartline
Aviad Heifetz
Steffen Huck
Tatsuro Ichiishi
Navin Kartik
Michael Kearns
Dan Kovenock
Dan Levin
Ines Macho-Stadler
George J. Mailath
Massimo Marinacci
Leslie M. Marx
Eric S. Maskin
Steven A. Matthews
Richard P. McLean
Andrew McLennan
Adam Meirowitz
Paul R. Milgrom
Hervé Moulin
Noam Nisan
Theo Offerman
Christos H. Papadimitriou
Hans Peters
Ronny Razin
Tim Roughgarden
William H. Sandholm
Reinhard Selten
Roberto Serrano
Lloyd S. Shapley
Martin Shubik
Marciano Siniscalchi
Vernon L. Smith
Eilon Solan
Éva Tardos
Moshe Tennenholtz
William Thomson
Fernando Vega-Redondo
Nicolas Vieille
Rakesh V. Vohra
Bernhard von Stengel
Joel C. Watson
Georg Weizsäcker
Robert Wilson
Eyal Winter
H. Peyton Young
Shmuel Zamir

Available online at www.sciencedirect.com
ScienceDirect

This article appeared in a journal published by Elsevier. The attached copy is furnished to the author for internal non-commercial research and education use, including for instruction at the authors institution and sharing with colleagues.

Other uses, including reproduction and distribution, or selling or licensing copies, or posting to personal, institutional or third party websites are prohibited.

In most cases authors are permitted to post their version of the article (e.g. in Word or Tex form) to their personal website or institutional repository. Authors requiring further information regarding Elsevier's archiving and manuscript policies are encouraged to visit:

http://www.elsevier.com/authorsrights

Author's personal copy

Games and Economic Behavior 82 (2013) 468–489

Contents lists available at ScienceDirect

Games and Economic Behavior

www.elsevier.com/locate/geb

Minimum payments and induced effort in moral hazard problems ☆

Ohad Kadan [a], Jeroen M. Swinkels [b,*]

[a] Olin Business School, Washington University in St. Louis, United States
[b] Kellogg School of Management, Northwestern University, United States

ARTICLE INFO

Article history:
Received 14 September 2010
Available online 22 August 2013

JEL classification:
D86

Keywords:
Minimum payments
Moral hazard
Induced effort
Productivity
Incentives

ABSTRACT

Consider a moral hazard problem in which there is a constraint to pay the agent no less than some amount m. This paper studies the effect of changes in m on the effort that the principal chooses to induce from the agent. We present sufficient conditions on the informativeness of the signal observed by the principal and on the agent's utility under which when m increases, induced effort (and hence productivity) falls. We also study how the cost minimizing contract for any given effort level varies in m. We present an efficient algorithm for numerically calculating optimal contracts for given parameters and show that induced effort falls when m is increased in many cases even when our sufficient conditions fail.

© 2013 Elsevier Inc. All rights reserved.

1. Introduction

In many settings where incentives are used to induce effort, there is a lower bound on what can be paid. A retail worker has a minimum wage determined by law, but a commission structure is under the control of the retailer. A manager may already have a contractual right to a retention payment, but the board can choose the rest of his pay structure above that. A supplier may have limited liability, or courts may be unwilling to enforce large penalties on suppliers.

This paper looks at the effect of the level of such minimum payments on the contract that will be implemented. We consider a standard moral hazard problem using the first order approach (Holmström, 1979; Mirrlees, 1976, 1999), where we assume that pay is constrained to be at least some amount m. We ask whether in the face of an increase in m the principal chooses to adjust the contract of the agent in such a way that the agent chooses to exert more effort, or less. This is a problem of considerable economic import. When considering, for example, minimum wage legislation, a first order question is whether minimum wages do or do not get in the way of incentives and productivity.[1] Similarly, if a supplier has deeper pockets, so that the limited liability constraint is less binding, will contracts be adjusted so that the supplier faces stronger performance incentives than before, or weaker?

☆ We spent a fair bit of time thinking about some of the issues explored in this paper with Ian Jewitt. We learned a great deal from him. We also thank a seminar audience at UT Austin.

* Corresponding author.
E-mail address: j-swinkels@kellogg.northwestern.edu (J.M. Swinkels).

[1] Compensation contracts often include both a minimum payment and an incentive component. This is frequently the case in retail, sales, and hospitality jobs, among others. Our model does not, however, include the possibility that the agent is fired if his performance is low enough, a possibility that is relevant in some settings. All else equal, an increase in the minimum wage would increase incentives to work hard in such a setting.

0899-8256/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.geb.2013.08.004

Author's personal copy

It is not enough to note that, holding fixed the bonus structure, an increase in the minimum payment decreases effort (this follows because utility is concave, and so any given pay above the minimum has lower utility implications than before, and hence less of a motivating effect). Rather, one needs to trace what happens when the principal has optimally reacted to the change in the minimum payment.

When m increases, the principal is forced to pay more at low outcomes, and so, if payments are not increased at some higher outcomes, then effort will go down. But, because utility is concave, as pay levels rise, incremental increases in pay provide less motivation. Thus, there are significant forces in the direction that inducing effort is more expensive on the margin when m is increased. But, despite this intuition, and despite the obvious economic and theoretical importance of the question, results for this problem have proven to be quite elusive – the literature provides no results of any generality.[2] The key problem is that as the effort level changes, so does the statistical structure of the inference problem about the agent's effort.

To address this issue, we start with a discussion of the comparative statics of the optimal contract for the problem of minimizing costs for given effort level. This is interesting in its own right, as the agent may already be working as hard as possible (a corner solution), and so the only issue is to optimally implement this effort level. Moreover, this analysis provides a number of key constructs and results that we will need when we turn to the full analysis in which the choice of effort level is also on the table.

For low levels of m, *IR* will bind at the cost minimizing solution.[3] We show that for such m, a small increase in m results in a contract which is flat over a sufficiently long range that the agent is paid *less* over a range of outcomes. However, at high outcomes, incentives become more intense. Thus the agent is rewarded over a smaller range of outcomes, but pay is more responsive to output over that range. When m is high enough, *IR* does not bind. We show that a further increase in m then results in the agent being paid more at *all* outcomes.

Thus, for example, think about a retail chain that pays its employees the minimum wage plus a percentage of sales above some threshold, and assume that it wishes to hold the effort level of the employee constant in the face of an increased minimum wage. If *IR* is binding, so that retention is a key issue for the chain in question, then an increase in the minimum wage should be associated with an increase in the threshold above which commissions are paid, but also with an increase in the commission received by the employee once this threshold is reached. On the other hand, if retention is not a key issue, then an increase in the minimum wage should result in the agent being better compensated at all sales levels.

We then turn to our main focus: analysis of the setting in which the principal can adjust not only the contract for any given effort level, but the choice of the effort level as well. We present a number of complementary results all pointing in the direction that higher m induces lower effort.

Our approach for establishing those results is to find sufficient conditions under which the marginal cost of implementing any given effort level is increasing in m. Our first set of conditions is on the curvature of the utility function u. These assumptions impose varying levels of concavity/convexity on $\frac{1}{u'}$, the marginal cost to the principal of providing a util to the agent. Similar assumptions have a pedigree in the literature on the moral hazard problem (see for example Jewitt, 1988). Our second set of conditions is related to the statistical structure of the model. These conditions measure the extent to which an increase in effort mutes or intensifies the ability of the principal to use one signal versus another as an indication of effort.

Our results cover instances in which the *IR* constraint may or may not bind. We distinguish between two cases of a binding *IR* constraint. First, is a softly binding *IR* constraint. Here the *IR* constraint binds but it does so sufficiently mildly that the principal is not driven to make payments above m at outcomes which become less likely with higher effort. So, while part of the pay above m is indeed directed at retention, none of the pay above m has the effect of decreasing the incentive for effort. Second is the case of a severely binding *IR* constraint, where payments above m are made even at some outcomes that become more likely with low effort.

Our sufficient conditions for the marginal cost of effort to increase in the minimum payment are rather permissive when the *IR* constraint is slack and they are satisfied in many standard examples. When the *IR* constraint is softly binding we are able to provide sufficient conditions which are more demanding in terms of the utility function. We are not able to provide sufficient conditions for the case where the *IR* constraint is severely binding.

To examine the extent to which our sufficient conditions are also necessary, we develop an easy-to-implement and numerically efficient algorithm for calculating optimal contracts for any given statistical setting where the first order approach holds. We explore several examples that violate our sufficient conditions. In all cases (including when the *IR* constraint is severely binding) we find that the marginal cost of inducing effort goes up as m goes up, implying that optimal effort falls. Thus, it appears that our sufficient conditions are quite far from being necessary.

[2] Papers that study moral hazard problems with a minimum bound on payments (or limited liability) under different assumptions are Holmström (1979), Sappington (1983), Innes (1990), Park (1995), Kim (1997), Oyer (2000), Matthews (2001), Dewatripont et al. (2003), and Jewitt et al. (2008). These papers do not study the effect of a change in the minimum payment on the contract and on induced effort, which is the focus of this paper.

[3] Because of the minimum payment, *IR* can be non-binding, either simply because m already covers the agent's outside option, or because the principal has induced effort using payments above m for good performance, but cannot balance them with low payments for poor performance.

Author's personal copy

Our results here are of the "traditional" form, in that our assumptions are clearly on primitives and we build heavily on the validly of the first order approach (Mirrlees, 1976; Rogerson, 1985; Jewitt, 1988).[4] In Kadan and Swinkels (2013) we take a different approach. We develop a general expression for shadow value as one simultaneously tightens both the *IR* and minimum payment constraints that holds regardless of the validity of the first order approach. We then pursue applications of this expression to comparative statics. One of these applications is the effect of an increase in minimum wage on induced effort, which is the topic of this paper. However, since in Kadan and Swinkels (2013) we do not assume the validity of the first order approach, we cannot use the techniques developed here. Instead, we consider the behavior of the distribution of realized payments as the induced effort level changes. This distribution is a function of both the primitives and the contract, and so is not exogenous (and yet may be observable in data). We view the two approaches as useful complements.

Section 2 presents the model. Section 3 discusses how the contract changes in m for a fixed effort level. Section 4 discusses the effect of changes in the minimum wage on optimally induced effort. The algorithm and numerical examples are presented in Section 5. Section 6 concludes. Proofs can be found in Appendix A.

2. Model and preliminaries

To reduce clutter without sacrificing clarity, we make a number of notational conventions. If the variable or range of integration is obvious, we often suppress it. When we do include the argument of a function, we use round parentheses. For all other purposes we use squared brackets. So, $k(g+h)$ is the function k applied to the sum of g and h, while $k[g+h]$ is k times the sum of g and h. A variable as a subscript always indicates a derivative. So, $[g(y,z(y))]_y$ is $g_y(y,z(y)) + g_z(y,z(y))z_y(y)$.

An agent exerts effort $e \in [0,\bar{e}]$. A signal $x \in [0,1]$ is observed by the principal, where x is related to e via a distribution $F(x|e)$, with associated density $f(x|e)$ that has common support $[0,1]$ for all e, and is twice continuously differentiable in x and e. The principal uses x to construct a contract for the agent. Denote a generic such contract by $\pi(x)$. The agent has cost of effort $c(e)$ that is twice continuously differentiable, increasing and convex, outside option u_0, and utility of income $u(w)$ that is twice continuously differentiable, increasing, and strictly concave. We assume that the principal is constrained to pay at least a minimum $m \geqslant 0$ for any given signal x, where m is in the interior of the domain of u.

Denote $B(e)$ as the principal's gross expected revenue from effort level e. For example, if x is output, then $B(e) \equiv \int xf(x|e)\,dx$. The principal's problem is to maximize, by choice of $\pi(\cdot)$ and e,

$$B(e) - \int \pi(x) f(x|e)\,dx$$

subject to the individual rationality, incentive compatibility, and minimum payment constraints. As is typical in these settings, we de-couple the problem into first finding the cheapest way to induce e, and then solving for the optimal e. That is, we begin with the cost minimization (*CM*) problem, where for any given e and m the principal chooses a contract $\pi(\cdot)$ that solves,

$$\min_{\pi(\cdot)} \int \pi(x) f(x|e) \tag{CM}$$

s.t

$$\int u\big(\pi(x)\big) f(x|e) - c(e) \geqslant u_0 \tag{IR}$$

$$e \in \arg\max_{e'} \int u\big(\pi(x)\big) f\big(x\big|e'\big)\,dx - c\big(e'\big) \tag{IC}$$

$$\pi(x) \geqslant m \quad \forall x. \tag{MP}$$

As we often refer to the likelihood ratio $\frac{f_e}{f}$, it will be convenient to give this ratio its own notation $\boldsymbol{\ell}(x,e) \equiv \frac{f_e(x|e)}{f(x|e)}$. Assume that $\boldsymbol{\ell}(x,e)$ is uniformly bounded (this prevents near-forcing contracts as in Mirrlees, 1999). Also, assume that f satisfies the monotone likelihood ratio property (*MLRP*), so that $\ell(x,e)$ is increasing in x for each e. For any given e, denote $x^*(e)$ as the unique point at which $f_e(x^*(e)|e) = 0$, so that for each e, $f_e(x|e)$ and $\boldsymbol{\ell}(x,e)$ cross zero from below at $x^*(e)$. We also assume *CDFC* (convexity of the distribution function), i.e., $F_{ee} \geqslant 0$ so as to guarantee the validity of applying the first order approach to the agent's incentive problem (Mirrlees, 1976; Rogerson, 1985).[5] Then, the incentive constraint, *IC*, can be replaced by

$$\int u\big(\pi(x)\big) f_e(x|e)\,dx - c'(e) = 0. \tag{IC$'$}$$

[4] Thiele and Wambach (1999) study the effect of the agent's wealth on the well-being of the principal. Similarly to this paper, they make assumptions on the primitives of the model, but they also allow for settings where the first order approach fails.

[5] All of our examples satisfy both *CDFC* and *MLRP*. In a setting with a binding minimum wage, Jewitt's (1988) conditions fail because the contract has an upward kink at the output where the minimum wage ceases to bind.

Author's personal copy

Jewitt et al. (2008, henceforth JKS) show that under the conditions given, an optimal contract exists, is weakly increasing, and is unique.[6] Denote this contract $\pi(\cdot, e, m)$. Building on Holmström (1979) and Mirrlees (1976), JKS show that $\pi(\cdot, e, m)$ is implicitly defined by

$$\frac{1}{u'(\pi(x,e,m))} = \max\left(\frac{1}{u'(m)}, \lambda(e,m) + \mu(e,m)\ell(x,e)\right), \tag{1}$$

where the Lagrange multipliers $\lambda(e,m) \geqslant 0$ and $\mu(e,m) > 0$ are unique and continuous in e and m.

Write the expected cost to the principal of implementing e given m as

$$C(e,m) = \int \pi(x,e,m) f(x|e)\, dx. \tag{2}$$

JKS show that $C(e,m)$ is continuous and is differentiable on any region where the same set of constraints binds, with $C_e > 0$ and with $C_m > 0$ whenever m is being paid with positive probability.[7]

One can then write the principal's full optimization problem as simply

$$\max_e B(e) - C(e,m). \tag{P}$$

For given (e,m) let the point at which

$$\frac{1}{u'(m)} = \lambda(e,m) + \mu(e,m)\ell(x,e)$$

be denoted $\hat{x}(e,m)$. For $x < \hat{x}(e,m)$, the agent is paid m.[8]

By *MLRP*, when *IR* is not binding ($\lambda(e,m) = 0$) we have $\hat{x}(e,m) > x^*(e)$, so that pay is more than m only when incentives are enhanced by payments ($f_e(x|e) > 0$). When *IR* is binding, $\hat{x}(e,m)$ may be higher or lower than $x^*(e)$. We say that *IR* is *softly binding* when *IR* binds but $\hat{x}(e,m) \geqslant x^*(e)$. If $\hat{x}(e,m) < x^*(e)$ we say that *IR* is *severely binding*. In this case, *IR* forces the principal to pay above the minimum at pay levels that actually hurt incentives.

The following two useful lemmas are immediate.

Lemma 1. $\lambda(e,m) \int [u(\pi(x,e,m))]_e f(x|e)\, dx = 0$.

This is intuitive. Indeed, if $\lambda(e,m) > 0$, then the *IR* constraint is binding and holds as an identity. Therefore, the change in the agent's expected utility from a small change in effort is zero. Furthermore, since the agent is optimizing, this effect is equal to the direct effect only.

Lemma 2. *An expression for* $\int \pi_e(x,e,m) f(x|e)\, dx$ *is*

$$\mu(e,m)\left[c''(e) - \int u\big(\pi(x,e,m)\big) f_{ee}(x|e)\, dx\right] \geqslant 0.$$

The intuition is that μ is the shadow value of *IC*, while $c''(e) - \int u(\pi) f_{ee}$ is the amount by which *IC* is thrown out of balance as e increases. The shadow value λ of *IR* can be ignored in this calculation by Lemma 1.

3. Comparative statics in *m* for given effort level

In this section we study how the optimal contract varies in the minimum wage m for fixed e. For notational simplicity, we will often work with two levels of the minimum payment, m^L and m^H, where $m^L < m^H$, holding fixed e. Let $\pi^H(x) = \pi(x,e,m^H)$, let λ^H and μ^H be the associated multipliers, and let $\hat{x}^H = \hat{x}(e,m^H)$. Define π^L, λ^L, μ^L and $\hat{x}^L$ analogously. We assume throughout this section that m is binding at (e,m^L), so that $\hat{x}^L > 0$. We begin with two simple observations.

Lemma 3. $C(e,m^H) > C(e,m^L)$.

The weak inequality is obvious as any contract feasible under m^H is also feasible under m^L. The strict inequality follows because optimal contracts are unique and π^L pays m^L over an interval by assumption and so ceases to be feasible given m^H.

[6] If $\lim_{w\to\infty} u(w)$ is finite, then JKS (p. 64) also require that the problem is non-degenerate in the sense that both *IR* and *IC* can be met for some increasing, continuous, and piecewise continuously differentiable contract.

[7] Kinks in $C(e,m)$ can occur when *IR* and *MP* move from being binding to not binding.

[8] If m is never paid we set $\hat{x}(e,m) = 0$.

Author's personal copy

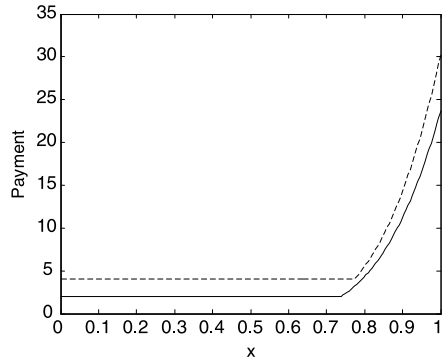


Fig. 1. The effect of m on the contract when *IR* does not bind.

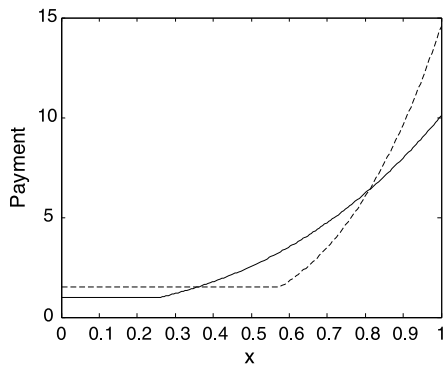


Fig. 2. The effect of m on the contract when *IR* binds.

Lemma 4. *Assume π^H single crosses π^L from above. Then,*

$$\int u\left(\pi^H(x)\right) f(x|e)\,dx > \int u\left(\pi^L(x)\right) f(x|e)\,dx.$$

Intuitively, π^H gives the agent lower risk than π^L (it is weakly flatter) and higher expected income by Lemma 3, and hence raises the agent's utility.

This in place, we can prove our first main result, which describes how contracts change in m when *IR* is not binding.

Proposition 1. *Assume IR is not binding given m^L. Then, $\mu^H > \mu^L$, $\lambda^H = \lambda^L = 0$, and π^H lies everywhere strictly above π^L.*

An implication is that once *IR* is non-binding, it remains so at any higher m. In particular, the agent has strictly higher utility under π^H than under π^L. Fig. 1 illustrates Proposition 1 for an example with a fixed effort level and for two levels of the minimum payment where *IR* is not binding.[9] As claimed, when m goes up, the contract goes up at all signal levels.

Now let us turn to the effect of change in m when *IR* is binding.

Proposition 2. *Assume IR is binding at both m^L and m^H. Then, π^H crosses π^L once from above (while π^H is still flat), and once from below (on the non-flat region of both contracts). In particular, $\mu^H > \mu^L$, and $\hat{x}^H > \hat{x}^L$. If IR is softly binding at m^H then $\lambda^H < \lambda^L$.*

Thus, where *IR* binds, an increase in m results in the agent being paid *less* over a range of outcomes. This is intuitive: as m increases, the principal has some accumulated slack in *IR* at the point where the contracts first cross. But, he is more challenged by *IC*, because the higher payments on the flat portion of the contract decrease incentives. He responds by providing no incentives over some range where he used to, but incentives that are more intense once they kick in. Fig. 2 illustrates this result for the same basic set-up as Fig. 1, but for choices of m and u_0 at which *IR* binds.[10]

From Propositions 1 and 2 we obtain the following corollary.

[9] We use CRRA utility, $u(w) = \frac{w^{1-\gamma}}{1-\gamma}$ with $\gamma = 0.3$, the FGM copula distribution, $f(x|e) = 1 + 0.5(1-2x)(1-2e)$, and $c(e) = e^2$. We set $e = 0.5$, $u_0 = 1$. We plot the cases $m = 2$ (solid line) and $m = 4$ (dotted line). We discuss the algorithm used to numerically solve for the contracts in Section 5.

[10] We use $e = 0.5$, $u_0 = 3$, and $m = 1$ and 1.5 for the solid and dotted contracts, respectively. In this example $x^*(e) = 0.5$, hence *IR* is severely binding for the solid contract and softly binding for the dotted contract.

Author's personal copy

Corollary 1. *$\mu(e,m)$ is strictly increasing in m for all m for which the minimum payment constraint is binding.*

This follows since if *IR* binds at m^L but not at m^H, then one can find an intermediate $\hat{m}$ that is the boundary between where *IR* does and does not bind. Proposition 1 gives $\mu^H > \mu(e,\hat{m})$, and Proposition 2 gives $\mu(e,\hat{m}) > \mu^L$.

We end this section with a result which we will need later.

Proposition 3. *For all e and m,*

$$C_m(e,m) = u'(m)\int\left[\frac{1}{u'(\pi(x,e,m))} - \lambda(e,m)\right] f(x|e)\,dx. \tag{3}$$

This is a corollary to Kadan and Swinkels (2013, Proposition 1), a more general result which does not rely on the validity of the first order approach (FOA), allows for multidimensional signals and effort, and requires no order structure on signals or differentiability of the distribution function. In the case discussed in this paper, one can derive (3) directly by integrating (1) and differentiating by m, followed with some algebra (we omit this exercise for brevity).

4. The effect of minimum pay on optimal implemented effort

We now turn to the main problem of how the optimal effort level varies with the minimum wage m. We present a set of results, each one of which has the conclusion that $C_{me}(e,m) > 0$ anywhere that *MP* binds. That is, the marginal cost of inducing effort goes up with the minimum wage. Since $B(e)$ is not affected by m, this implies that the principal's optimal choice of e falls when m rises. This follows from a standard monotone comparative statics argument, and does not rely on, for example, concavity of $B(e) - C(e,m)$.

For some intuition for $C_{me} > 0$, note first that if there are only two outcomes and if *MP* is binding, then the cost of inducing extra effort increases in m without any additional assumptions on either utility or the distribution function. Intuitively, inducing extra effort requires increasing the difference between the utility following a success versus a failure. Increasing this difference is more expensive when the utility of the agent following failure, which is determined by the minimum payment, is higher.

Another simple case is when $u(w) = \ln(w)$ and *IR* does not bind, and when $B_e > 0$. In this case, Proposition 3 becomes

$$C_m(e,m) = \frac{1}{m}\int \pi(x,e,m) f(x|e)\,dx = \frac{1}{m}C(m,e).$$

Hence,

$$C_{me}(e,m) = \frac{1}{m}C_e(m,e),$$

which is positive at any interior optimal effort level given $B_e > 0$. The first example involves a very simple information structure, while the second involves a very special utility function. A general result requires some combination of assumptions on information and on utility. We first discuss our assumptions and then present our main results.

4.1. Assumptions on the curvature of u

Most of our results require a restriction on the curvature of the utility function.[11] It is often convenient to state these restrictions as convexity/concavity requirements on $\frac{1}{u'}$, the marginal cost of providing a util to the agent. To interpret these conditions, we follow Thiele and Wambach (1999) who relate the curvature of the utility function to the relationship between the degree of absolute risk aversion $A = \frac{-u''}{u'}$ and the degree of absolute prudence $P = -\frac{u'''}{u''}$ (which (see Kimball, 1990) is a measure of the incentives of the agent to engage in precautionary savings).

The first restriction is extremely mild.

Condition U0. $\frac{1}{u'}$ is weakly log-concave, or equivalently, $\frac{P}{A} \geqslant 1$.

Condition **U0** is equivalent to weak DARA.[12] Thus, it is satisfied by many commonly used utility functions such as CARA utility, and CRRA utilities of the form $u(w) = \frac{w^{1-\gamma}}{1-\gamma}$, where $\gamma > 0$ is the coefficient of relative risk aversion. Next are two opposing conditions, each useful for different results.

Condition U1. $\frac{1}{u'}$ is weakly concave, or equivalently $\frac{P}{A} \geqslant 2$.

[11] There is no conflict between conditions on utility and the first order approach, since for example, CDFC is purely about F, and makes no requirements on u.

[12] This follows since $A' = A[A - P]$.

Author's personal copy

Condition U1′. $\frac{1}{u'}$ is weakly convex, or equivalently $\frac{P}{A} \leqslant 2$.

Both **U1** and **U1′** are plausible. For the case of CRRA utilities, **U1** (**U1′**) is satisfied when $\gamma \leqslant 1$ ($\gamma \geqslant 1$), with $\gamma = 1$ (log utility) the borderline case. Condition **U1′** is satisfied by CARA utility, but Condition **U1** is not.

The next condition is more restrictive.

Condition U2. $[\frac{1}{u'}]^2$ is weakly concave, or equivalently $\frac{P}{A} \geqslant 3$.

Condition **U2** is the opposite of that of Jewitt (1988). In the case of CRRA utilities it corresponds to risk aversion lower than $\frac{1}{2}$.

Finally, for one result, we will require that $\frac{P}{A}$ is decreasing. This is satisfied by all standard utility functions including the CRRA and CARA cases.

Condition U3. $\frac{P}{A}$ is weakly decreasing.

We now introduce two functions which become useful when applying these conditions. Let J be the function that translates $\frac{1}{u'}$ into u, so that J is implicitly defined by

$$u(p) = J\left(\frac{1}{u'(p)}\right),$$

and similarly, let W translate $\frac{1}{u'(p)}$ into p, so that

$$p = W\left(\frac{1}{u'(p)}\right).$$

Note that (1) implies that for $x > \hat{x}(e,m)$ we have $u(\pi(x,e,m)) = J(\lambda(e,m) + \mu(e,m)\ell(x,e))$ and $\pi(x,e,m) = W(\lambda(e,m) + \mu(e,m)\ell(x,e))$, so these two functions are clearly fundamental to moral hazard problems. The following remark is straightforward to verify.

Remark 1. Condition **U0** is equivalent to $W''(t)t/W'(t) \geqslant -1$, Condition **U1** (**U1′**) is equivalent to $W(\cdot)$ being weakly convex (concave) or to $J''(t)t/J'(t) \geqslant (\leqslant) -1$, and Condition **U2** is equivalent to $J(\cdot)$ being weakly convex.

A primary use of the conditions on utility is in ensuring that as m rises, contracts become steeper both in payment space and in utility space. The next lemma provides sufficient conditions for this to be the case when *IR* is not binding.

Lemma 5. *Fix an effort level e. Assume that IR is not binding and MP is binding for the cost minimizing contract $\pi(x,e,m)$. Then, for $x > \hat{x}(e,m)$*

1. *if Condition* **U0** *holds then $\pi_{xm} \geqslant 0$, and*
2. *if Condition* **U1** *holds then $[u(\pi)]_{xm} \geqslant 0$.*

When *IR* is binding this result is not available since contracts for different m levels cross (Proposition 2). However, the following useful lemma can be established, providing conditions under which the difference in slopes flips sign just once.

Lemma 6. *Assume that both IR and MP are binding and that* **U2** *and* **U3** *hold. Let $m^H > m^L$ and let π^H and π^L be the corresponding cost minimizing contracts. Then, $[u(\pi^L) - u(\pi^H)]_x$ has sign pattern $0/+/-$.*

That is, for low x, π^H and π^L have the same slope (while both are flat), then π^L is steeper than π^H, and finally π^H is steeper than π^L.

4.2. Assumptions on information

Our first two conditions make opposing assumptions on ℓ.

Condition I1. For $e^H > e^L$ and $x > \max(x^*(e^H), x^*(e^L))$,

$$\frac{\ell(x,e^H)}{\ell(x,e^L)}$$

is decreasing in x.

Author's personal copy

Condition I1′. For $e^H > e^L$ and $x > \max(x^*(e^H), x^*(e^L))$,

$$\frac{\ell(x, e^H)}{\ell(x, e^L)}$$

is increasing in x.

To interpret these conditions, note that $\ell(x, e)$ is a measure of how attractive it is for the principal to induce effort e by using rewards at x. In particular, adding rewards at x relaxes *IC* proportionally to $f_e(x|e)$ but increases the cost to the principal proportionally to $f(x|e)$. Thus, asking $\frac{\ell(x,e^H)}{\ell(x,e^L)}$ to be monotone (in either direction) in the outcome x is akin to asking that the relative difficulty of inducing effort e^H versus e^L using rewards at outcome x has a simple structure as the outcome changes. The conditions are implied by log-submodularity and supermodularity of ℓ, but weaker in that they apply only for $x > \max(x^*(e^H), x^*(e^L))$. Both conditions are plausible and consistent with the FOA as illustrated in the next example.

Example 1. Condition **I1** is satisfied by the FGM copula,

$$f(x|e) = 1 + 0.5(1-2x)(1-2e), \quad x, e \in [0, 1]$$

and by the LS1 distribution,

$$f(x|e) = 1 + \frac{1-2x}{e+1}, \quad x \in [0, 1],\ e \geqslant 0,$$

which is due to LiCalzi and Spaeter (2003). Condition **I1′** is satisfied by the LS2 distribution

$$f(x|e) = \exp\big(e(x-1)\big)(ex+1), \quad x \in [0, 1],\ e \geqslant 0,$$

which is again due to LiCalzi and Spaeter (2003). All of these distributions satisfy both MLRP and CDFC and so are also valid for the FOA.

Our next assumption imposes a concavity condition on F_e.

Condition I2. $[\frac{F_{ee}}{F_e}]_x \leqslant 0$ for all e and $x > x^*(e)$.

Equivalently, $-F_e$ is log-concave on $(x^*(e), 1]$. Note also that

$$\frac{F_{ee}}{F_e} = \frac{\int_x^1 \frac{f_{ee}}{f_e} f_e}{\int_x^1 f_e} \tag{4}$$

and so is a conditional expectation of $\frac{f_{ee}}{f_e}$. Thus, Condition **I2** states that on average, $\frac{f_{ee}}{f_e}$ is lower when x is higher. To further interpret this condition note that the ratio $\frac{f_e(x^H|e)}{f_e(x^L|e)}$ is a measure of by how much an increase in the signal from x^L to x^H is associated with extra effort. But,

$$\frac{d}{de} \ln \frac{f_e(x^H|e)}{f_e(x^L|e)} = \frac{f_{ee}(x^H|e)}{f_e(x^H|e)} - \frac{f_{ee}(x^L|e)}{f_e(x^L|e)}.$$

Hence, Condition **I2** asserts that on average, $\frac{f_e(x^H|e)}{f_e(x^L|e)}$ deteriorates in e.

Condition **I2** is satisfied by the FGM copula and the LS1 distributions defined above, but fails for the LS2 distribution. To see the relation between **I1** and **I2** note that both are implied by $\frac{f_{ee}}{f_e}$ being weakly decreasing in x (or equivalently $[\ln f_e]_{ex} \leqslant 0$) for $x > x^*(e)$.[13]

For our final condition on information denote

$$\tau(x, e) \equiv \frac{-F_e(x|e)}{1 - F(x|e)} \left[1 - \frac{f(x|e) F_{ee}(x|e)}{f_e(x|e) F_e(x|e)}\right], \tag{5}$$

for all e and $x \in [0, 1)$, and $\tau(1, e) \equiv \lim_{x \uparrow 1} \tau(x, e)$.

[13] For **I2** this follows directly from (4). For **I1** this follows from *MLRP* and the fact that **I1** is implied by log-submodularity of $\ell(x, e)$, noting that

$$[\ln \ell(x, e)]_{ex} = \left[\frac{f_{ee}}{f_e} - \frac{f_e}{f}\right]_x.$$

Author's personal copy

Condition I3. $[\tau(x,e)]_x \geqslant 0$ for all e and $x > x^*(e)$.

To interpret Condition **I3** note that $\frac{-F_e}{1-F}$ is positive and increasing by MLRP, while under CDFC, the bracketed term in (5) is also positive. So, it is sufficient for **I3** that

$$\frac{f}{f_e}\frac{F_{ee}}{-F_e}$$

is increasing. Using (4) we see that Condition **I3** is satisfied when $\frac{f_e(x^H|e)}{f_e(x^L|e)}$ deteriorates fast enough. An example of a distribution that satisfies **I3** is the FGM copula discussed above.

We now present two sets of results. In the first we primarily rely on crossing properties of optimal contracts associated with different effort levels when m is kept fixed. In the second set of results we primarily rely on crossing properties of optimal contracts associated with different m when e is kept fixed.

4.3. A result using crossing properties of contracts when e changes

In this section we sign C_{me} by considering optimal contracts for different effort levels and imposing sufficient conditions implying that these contracts either don't cross or single cross. Our result in this section applies when *IR* is not binding. The economic intuition behind this result has the flavor of stochastic dominance rules. The idea is to apply Proposition 3, which when *IR* is not binding becomes

$$C_m(e,m) = u'(m)\int \frac{1}{u'(\pi(x,e,m))} f(x|e)\,dx. \tag{6}$$

We first present sufficient conditions in terms of the informational structure of the model implying that the contract becomes either more or less "risky" when e increases, as reflected in the crossing properties of $\pi(x,e,m)$ for different effort levels. We then align these conditions with assumptions on the curvature of the utility function, implying that $\frac{1}{u'}$ be either convex or concave. Jointly, these conditions imply that the RHS of (6) increases in e. Thus, we obtain that $C_{me} > 0$.

Proposition 4. *Assume that IR does not bind. Assume further that either*

(i) **I1** *and* **U1** *hold or*
(ii) **I1′** *and* **U1′** *hold. Then* $C_{me}(e,m) > 0$.

The first key step in the proof is showing that under Conditions **I1** or **I1′** contracts for different e can only cross in simple ways. For any given m and $e^H > e^L$, denote $\pi^H(x) \equiv \pi(x,e^H,m)$ and $\pi^L(x) \equiv \pi(x,e^L,m)$.

Lemma 7. *Assume IR is not binding at* e^H *and* e^L. *Then*

(i) *if* **I1** *holds then, either* $\pi^H(x) \geqslant \pi^L(x)$ *everywhere or* π^H *single crosses* π^L *from above, and*
(ii) *if* **I1′** *holds then either* $\pi^H(x) \geqslant \pi^L(x)$ *everywhere or* π^H *single crosses* π^L *from below.*

Thus, a high effort contract can cross a low effort contract either from above or from below. When **I1** holds, the high effort contract is "flatter" than the low effort contract, but incentive pay starts at lower levels of the signal. By contrast, when **I1′** holds, the high effort contract is "steeper" than the low effort contract, and incentive pay starts at higher levels of the signal. Intuitively, recall that $\frac{\ell(x,e^H)}{\ell(x,e^L)}$ is a measure of the relative attractiveness for the principal to induce effort by using rewards at x. Thus, **I1** implies that lower realizations of x are relatively attractive for inducing effort given e^H, whereas higher realizations of x are relatively attractive for inducing effort given e^L. Correspondingly, assuming the contracts cross, π^H specifies relatively high payments at low outcomes and low payments at high outcomes compared to π^L. When we instead impose Condition **I1′**, the crossing structure is reversed. Both cases are plausible as highlighted in Example 1.

When a contract single crosses another contract from below, it can be intuitively thought of as being "riskier" in the sense that the distribution of payments it induces tends to be more spread out.[14] Setting the case when the contracts do not cross aside, the next lemma considers the cases where either (i) π^H is "less risky" than π^L (crossing from above) and $\frac{1}{u'}$ is concave; and (ii) π^H is "riskier" than π^L (crossing from below) and $\frac{1}{u'}$ is convex. In both cases the expected value of $\frac{1}{u(\pi^H)}$ given effort level e^H is higher than the expected value of $\frac{1}{u'(\pi^L)}$ given effort level e^L.

[14] More formally, note that when $\pi^H \geqslant \pi^L$ everywhere the payment distribution under e^H dominates the payment distribution under e^L in the sense of first order stochastic dominance. Similarly, having π^H single cross π^L from above (below) creates a force in the direction of the payment distribution given π^H being less (more) spread out than that given π^L in a manner similar to second order stochastic dominance (the distribution of payments is also affected by the fact that $f(\cdot|e^H)$ differs from $f(\cdot|e^L)$). Kadan and Swinkels (2013) study how stochastic dominance conditions on the (endogenous) payment distribution relate to the effect of a change in minimum pay or outside option on the induced effort in a setting where the first order approach is dispensed with.

Author's personal copy

Lemma 8. *Fix m and $e^H > e^L$. Assume IR is not binding at e^H and e^L. Assume further that either*

(i) **I1** *and* **U1** *hold or*
(ii) **I1′** *and* **U1′** *hold.*

Then,

$$\int \frac{1}{u'(\pi^H(x))} f\left(x|e^H\right) dx > \int \frac{1}{u'(\pi^L(x))} f\left(x|e^L\right) dx. \tag{7}$$

The final ingredient in the proof comes from Proposition 3, which in the case that *IR* is not binding ($\lambda = 0$), says that (7) is equivalent to

$$C_m\left(e^H, m\right) > C_m\left(e^L, m\right).$$

As this is true for all $e^H > e^L$ where *IR* does not bind, we have proven Proposition 4.

4.4. Results using crossing properties of contracts when m changes

In the previous section we used assumptions on information implying crossing properties of contracts with the same m but different e. The approach in this section is to exploit the results in Sections 3 and 4.1, in which we used assumptions on the utility function to develop crossing and other properties of contracts with different m when effort is kept fixed. This approach yields results when the *IR* constraint is either non-binding or softly binding. We begin by noting that

$$C_e = \int \pi_e f + \int \pi f_e. \tag{8}$$

The first term on the RHS is the cost of changing the contract in such a way as to elicit higher effort. The second term is the cost of the agent actually taking the deal. Both terms are positive, the first by Lemma 2 and the second by *FOSD*. To sign C_{me} we differentiate (8) by m and, either sign each resulting term separately or the entire expression.

This approach yields three sets of sufficient conditions, all leading to the conclusion that $C_{me} > 0$. When we sign each term in (8) separately, we obtain two results – one for the case where *IR* is not binding and one for the case where *IR* is softly binding. When the *IR* constraint is not binding we are able to sign the entire expression in (8). Compared to the first two results, this approach yields conditions imposing very mild requirements on utility, but quite strong restrictions on information.

To understand the technical issues that arise in these results, consider the term $\int \pi_e f$ in (8). By Lemma 2,

$$\begin{aligned}\left[\int \pi_e f\right]_m &= \left[\mu\left[c''(e) - \int u(\pi) f_{ee}\right]\right]_m \\ &= \underbrace{\mu_m\left[c''(e) - \int u(\pi) f_{ee}\right]}_{A} + \underbrace{\mu\left[c''(e) - \int u(\pi) f_{ee}\right]_m}_{B}.\end{aligned}$$

Since $\mu_m > 0$ (Proposition 1), and using Lemma 2, we have that Term A is non-negative. The difficulty is in showing that Term B is positive. Note that this term is equal to $-\int [u(\pi)]_m f_{ee}$. It is tempting to rewrite this expression using integration by parts as $\int [u(\pi)]_{xm} F_{ee}$. Using that $[u(\pi)]_{xm} = 0$ over $[0, \hat{x}]$ one would like to conclude that this integral equals $\int_{\hat{x}}^1 [u(\pi)]_{xm} F_{ee}$. Then, one would hope to use the conditions on utility and the results in Lemmas 5 and 6 combined with *CDFC* to conclude that this integral is positive.

This simple and intuitive approach is, however, misguided. Indeed, it ignores the fact that $\hat{x}$ depends on m, implying that the contract's slope changes discontinuously at $\hat{x}$ as m changes. Moreover, it is often the case that $\hat{x}_m > 0$, implying that the contract's slope can jump down at $\hat{x}$ as m increases, introducing a force working against the positivity of the integral. To account for this we first integrate by parts and only then differentiate by m obtaining that term B equals

$$\left[\int_{\hat{x}}^1 \left[u(\pi)\right]_x F_{ee}\right]_m = \left[-\hat{x}_m \left[u(\pi)\right]_x F_{ee}\right](\hat{x}, e, m) + \int_{\hat{x}}^1 \left[u(\pi)\right]_{xm} F_{ee}.$$

Here it is clear that the first term (reflecting the movement of $\hat{x}$) works against the positivity of this expression whenever $\hat{x}_m > 0$. Our results then use the conditions on preferences to guarantee that the second term is positive, and the conditions on information to guarantee that the first term (even if negative) is dominated by the second term. A similar approach is used when analyzing the second term in (8) or when analyzing the sum of the terms.

The first two results use a unified approach to establish conditions under which each term in (8) increases in m. The first result applies when *IR* is not binding and the second, under stricter sufficient conditions, when *IR* is softly binding.

Author's personal copy

Proposition 5. *Assume that IR does not bind,* **U1**, *and* **I2**. *Then* $C_{me}(e,m) > 0$.

Proposition 6. *Assume that IR binds softly,* **U2**, **U3** *and* **I2**. *Then* $C_{me}(e,m) > 0$.

Below we sketch the main ideas behind the proofs of both results.

Showing that $\int \pi_e f$ increases in m. As noted above, it enough to show that $[\int u(\pi) f_{ee}]_m \leqslant 0$, or, integrating by parts, $[\int_{\hat{x}} [u(\pi)]_x (-F_{ee})]_m \leqslant 0$.

When *IR* is not binding, we obtain from Lemma 5 under Condition **U1** that $[u(\pi)]_{xm} \geqslant 0$ for $x > \hat{x}$. Using this, and the fact that under **I2**, $\frac{-F_{ee}}{F_e}$ is increasing, we are able to show that $[\int_{\hat{x}} [u(\pi)]_x (-F_{ee})]_m \geqslant [\int_{\hat{x}} [u(\pi)]_x (-F_e)]_m$, accounting for the jump in pay at $\hat{x}$ as m changes. But, the last expression (again by an integration by parts) is just the change in incentives brought about as m changes, and since *IC* is satisfied for all m, is thus 0.

When *IR* is softly binding, we rely on Lemma 6 to obtain that

$$\left[u\left(\pi^L(x)\right) - u\left(\pi^H(x)\right)\right]_x$$

has sign pattern $0/+/-$ whenever $m^H > m^L$. An integration by parts of $\int u(\pi) f_{ee}$, and application of Condition **I2** completes this step of the proof.

Showing that $\int \pi f_e$ increases in m. When *IR* is not binding this follows fairly directly from integrating this term by parts and using that $\pi_{xm} \geqslant 0$ (Lemma 5). When *IR* is softly binding this is again more subtle as we cannot conclude that $\pi_{xm} \geqslant 0$. The key is to show that the difference between $u(\pi^H)$ and

$$u(\hat{\pi}) \equiv u\left(\pi^L\right) + u\left(m^H\right) - u\left(m^L\right)$$

has sign pattern $0/-/+$. This sign pattern, the concavity of $[\frac{1}{u'}]^2$, and the fact that

$$\int u\left(\pi^H\right) f_e = \int \left[u(\hat{\pi})\right] f_e = \int u\left(\pi^L\right) f_e$$

together imply that

$$\int \pi^H f_e \geqslant \int \hat{\pi} f_e > \int \pi^L f_e,$$

completing the proof.

Our third result in this section is permissive on utility but quite demanding on information.

Proposition 7. *Assume that MP binds, that IR does not bind, and* **U0** *and* **I3**. *Then* $C_{me}(e,m) > 0$.

The proof uses the fact that for $x \in [\hat{x}, 1]$, $\mu u'(\pi) = \frac{f}{f_e}$ and Lemma 2 to combine the two terms of (8). We then differentiate the resulting expression by m and use Condition **I3** and Lemma 5 (part 1) to account for the discontinuous jump in pay at $\hat{x}$, and show that $C_{me} \geqslant \tau(\hat{x}, e)[C_m - 1]$. The final step of the proof comes from Proposition 3, which in the case that *IR* is not binding easily implies that $C_m - 1 > 0$, establishing the result.

5. Numerical exploration

So far we have presented several sets of sufficient conditions implying that an increase in the minimum feasible payment lowers the level of induced effort. To supplement these results, we now present an algorithm that allows the efficient numerical calculation of optimal contracts in constrained moral hazard problems, and use it to present some examples covering cases where our sufficient conditions are not met.[15] Despite some effort, we have not found an example where an increase in m results in higher induced effort. Our conclusion is that the sufficient conditions we have provided are rather far from being necessary.

[15] Matlab code implementing the algorithm is available for download at http://apps.olin.wustl.edu/faculty/kadan/index/Research.html.

Author's personal copy

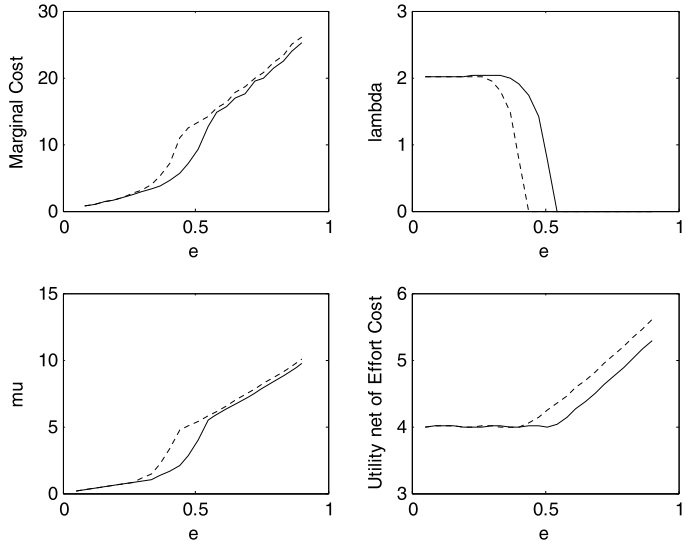


Fig. 3. Graphs for Example 2.

5.1. An algorithm for solving the constrained moral hazard problem

The algorithm is based on the existence part of the proof of Proposition 1 in JKS. The first key point for the proof is that, fixing e, if one can find λ and μ such that the contract $\pi(x, \lambda, \mu)$ defined by

$$\frac{1}{u'(\pi(x, \lambda, \mu))} = \max\left(\frac{1}{u'(m)}, \lambda + \mu \frac{f_e(x|e)}{f(x|e)}\right),$$

satisfies *IR* and *IC* with appropriate complementary slackness, then the contract is in fact the (unique) optimal solution to the relaxed cost minimization problem. So, the search for an optimal contract is reduced to the search for an appropriate λ and μ. The existence proof in JKS is based on the idea that an appropriate λ and μ are guaranteed to exist by two intermediate value theorem arguments that use continuity and monotonicity properties of the problem.

Specifically, for any given μ, consider the class of contracts defined by varying λ. If at $\lambda = 0$, *IR* is already satisfied, define $\lambda(\mu) = 0$. Otherwise, JKS show that as λ becomes arbitrarily large, *IR* is eventually slack, and so, by the intermediate value theorem, there is some $\lambda(\mu)$ such that *IR* is satisfied exactly at $(\lambda(\mu), \mu)$. JKS then argue that when $\mu = 0$, $\pi(\cdot, \lambda(\mu), \mu)$ leaves *IC* unsatisfied, while when μ is large enough, $\pi(\cdot, \lambda(\mu), \mu)$ leaves *IC* slack. The intermediate value theorem then tells us that there is a μ such that $\pi(\cdot, \lambda(\mu), \mu)$ satisfies *IC* with equality. But then, the resulting $\lambda(\mu)$ and μ in fact satisfy the required conditions.

The algorithm computationally implements the steps of this argument. For any given μ, finding $\lambda(\mu)$ within a given tolerance can be accomplished in an amount of time that is logarithmic in the size of $\lambda(\mu)$ and in the inverse of the tolerance (keep doubling an initial guess to find a λ where *IR* is slack, and then successively divide the interval to get to the given tolerance). Finding μ to meet *IC* numerically consists of the same procedure as for λ, noting that for each guess at μ, one needs to enter a subroutine to numerically find $\lambda(\mu)$.

We use the algorithm to first calculate $C(e, m)$ on a fine grid of e values, and then approximate $C_e(e, m)$ by looking at first differences. We then plot $C_e(e, m)$ for different values of m to check whether C_e rises or falls with m. We also plot $\mu(e, m)$ and $\lambda(e, m)$.

5.2. Numerical examples

Example 2. Consider the following specification:

Distribution	FGM copula, $f(x\|e) = 1 + 0.5(1 - 2x)(1 - 2e)$
Utility and cost of effort	CRRA with $\gamma = 0.5$, $c(e) = e^2$
Outside utility	$u_0 = 4$
Minimum pay	$m^L = 2$ (solid line), $m^H = 2.5$ (dotted line)

This example satisfies Conditions **I1**, **I2**, **I3**, **U0**, **U1**, **U2**, and **U3**. Our results guarantee that when the *IR* constraint is non-binding or softly binding, effort is decreasing when the minimum payment goes up. Our sufficient conditions do not cover the case of a severely binding *IR* constraint. The top left graph in Fig. 3 plots the marginal cost $C_e(e, m)$ for two levels of m. It can be seen that C_e is higher for m^H than for m^L, consistent with $C_{me} > 0$. The top right figure shows that for low levels of e, the two contracts coincide, implying that *MP* does not bind, while for high levels of e we have $\lambda = 0$, implying

Author's personal copy

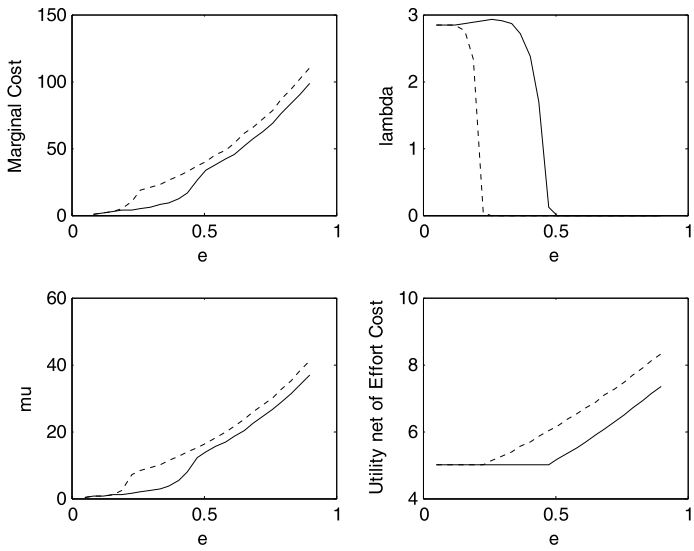


Fig. 4. Graphs for Example 3.

that *IR* does not bind. At intermediate levels of e, *IR* first binds severely and then softly. Note also that λ is increasing for a while (when *IR* binds severely) and then decreasing, while μ is always increasing. The bottom right graph plots the agent's expected utility net of effort costs. In this example, the agent is happy to be asked to work harder: the increase in his utility from the contract outweighs the disutility from the increase in effort.

Example 3. Consider the following specification:

Distribution	LS2, $f(x\|e) = \exp(e(x-1))(ex+1)$
Utility and cost of effort	CRRA with $\gamma = 0.6$, $c(e) = e^2$
Outside utility	$u_0 = 5$
Minimum pay	$m^L = 2$ (solid line), $m^H = 4$ (dotted line)

This example satisfies Conditions **I1′**, **U0, U1**, and **U3**. None of our results covers this combination of properties. The results are in Fig. 4. As before $C_{me} \geqslant 0$ whether *IR* is binding or not. λ is increasing for a while and then decreasing, and μ is increasing. Once again, the agent is happy to be asked to work harder.

Example 4. Consider a case with CRRA utility and $\gamma > 1$ using the following assumptions:

Distribution	FGM copula, $f(x\|e) = 1 + 0.5(1-2x)(1-2e)$
Utility and cost of effort	CRRA: $u(w) = \frac{w^{1-\gamma}}{1-\gamma}$ with $\gamma = 1.2$.[16] $c(e) = e^2$
Outside utility	$u_0 = 5$
Minimum pay	$m^L = 2$ (solid line), $m^H = 2.5$ (dotted line)

This example satisfies Conditions **I1**, **I2**, **I3**, **U0**, **U1′**, and **U3**. Our results guarantee that when *IR* is non-binding an increase in m results in lower effort level. However, our sufficient conditions do not cover the case of a binding *IR* constraint. The results presented in Fig. 5 show that even when the *IR* constraint binds (softly or severely) the marginal cost goes up as m is increased. Note that when $\gamma > 1$ the utility function is bounded, and it becomes very flat at high outcomes. This, along with the minimum payment make it very hard for the principal to provide the agent with additional utility, and causes the marginal cost of providing effort to explode.

[16] The actual utility function used was

$$u(w) = \frac{w^{1-1.2}}{1-1.2} + 9.$$

We added a constant to the utility function to make sure that the expected utility net of effort costs in this example is positive on the relevant range. Equivalently, when $\gamma > 1$ one can take the outside option u_0 to be negative.

Author's personal copy

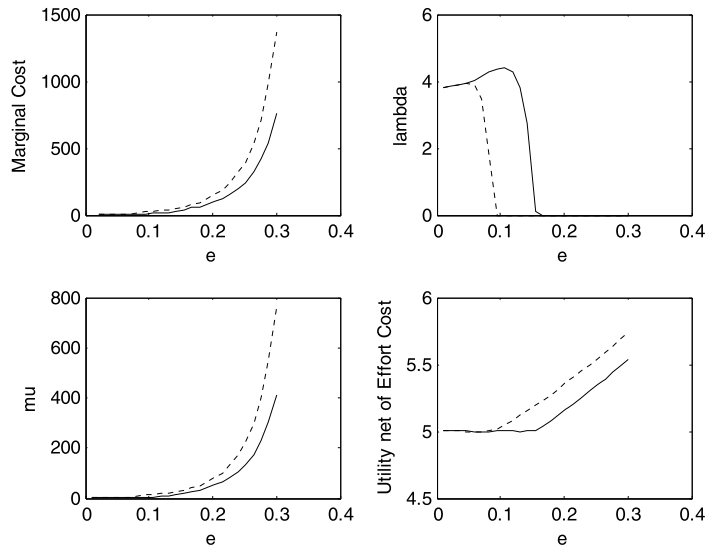


Fig. 5. Graphs for Example 4.

6. Conclusion

In this paper, we examined the question of whether increasing the minimum allowable payment in a moral hazard model causes the principal to increase or decrease the effort induced from the agent. Under a variety of conditions, we get the same result: As the minimum payment goes up, the cost of inducing effort on the margin goes up as well, and so the principal optimally settles for lower productivity.

We supplement our theoretical results by developing and then using an algorithm for efficiently generating optimal contracts in constrained moral hazard problems. These examples are supportive of the prediction that effort is more costly to induce when minimum payments are higher.

The fact that our results all point in the direction that a higher minimum payment implies a lower induced effort level suggests that in settings where this model is appropriate, there is both a set of testable implications and some policy implications. In particular, our results suggest that increases in minimum wage will result in lower individual productivity in sectors where incentive pay is a material component of compensation. But, as explored in Kadan and Swinkels (2010), there are interesting caveats to this prediction in settings where for example a firm can adjust the number of workers it has, and not just the effort induced from a single worker, and in settings where the benefit to the firm of inducing effort is itself determined as part of a broader market equilibrium.

We view this paper as a step in the direction of exploring comparative statics results in principal agent models. As suggested by our numerical results, the sufficient conditions we provide are likely too strong. Additionally, there is a clear gap in our results since we did not find sufficient conditions for the case where the *IR* constraint is severely binding. On the other hand, it seems unlikely to us that there is no combination of parameter values in existence under which an increase in minimum payment results in an increase in induced effort. However, an example of this sort (which would be highly desirable) simply evades us. We leave these to future research.

Appendix A. Proofs

Proof of Lemma 1. If $\lambda(e,m)=0$ we are done. Assume then that $\lambda(e,m)>0$. In this case, *IR* is binding on a neighborhood of effort levels. Thus, we can differentiate *IR* to get

$$c'(e)=\left[\int u(\pi)f\right]_e=\int\left[u(\pi)\right]_e f+\int u(\pi)f_e$$
$$=\int\left[u(\pi)\right]_e f+c'(e),$$

using *IC*. So $\int[u(\pi)]_e f=0$, as required. □

Proof of Lemma 2. Note that

$$\int \pi_e f=\int \frac{1}{u'(\pi)}u'(\pi)\pi_e f$$

Author's personal copy

$$= \int \left[\lambda(e,m) + \mu(e,m) \frac{f_e}{f} \right] [u(\pi)]_e f$$
$$= \underbrace{\lambda(e,m) \int [u(\pi)]_e f}_{A} + \underbrace{\mu(e,m) \int [u(\pi)]_e f_e}_{B}.$$

The second equality follows using (1) over $[\hat{x}(e,m), 1]$, and since over $[0, \hat{x})$, $[u(\pi)]_e = u'(\pi)\pi_e = 0$. By Lemma 1, $A = 0$. To evaluate B, note that from *IC*

$$\int u(\pi) f_e - c'(e) = 0$$

holds as an identity, and so

$$\int [u(\pi)]_e f_e + \int u(\pi) f_{ee} - c''(e) = 0,$$

from which

$$\int [u(\pi)]_e f_e = c''(e) - \int u(\pi) f_{ee}.$$

This is weakly positive from the second order necessary condition for the agent to be optimizing. □

Proof of Lemma 4. Let y be the point such that $\pi^H(x) > \pi^L(x)$ for all $x < y$, and $\pi^H(x) \leqslant \pi^L(x)$ for all $x \geqslant y$. Then,

$$\int [u(\pi^H) - u(\pi^L)] f \geqslant \int u'(\pi^H(y)) [\pi^H - \pi^L] f$$
$$= u'(\pi^H(y)) \int [\pi^H f - \pi^L f] > 0,$$

where the first inequality follows by the definition of y and by concavity of u, and the last inequality follows by Lemma 3. □

Proof of Proposition 1. If $\mu^H \leqslant \mu^L$, then by Lemma 4, $\lambda^H = 0$. By assumption, $\lambda^L = 0$ as well. So, since $\mu^H \leqslant \mu^L$, $\hat{x}^H$ occurs at or after the point $y > x^*(e)$ at which $\pi^L = m^H$, and everywhere to the right of y, $\pi^L \geqslant \pi^H$. But then,

$$\int [u(\pi^H) - u(\pi^L)] f_e = \int_0^y [u(\pi^H) - u(\pi^L)] f_e + \int_y^1 [u(\pi^H) - u(\pi^L)] f_e$$
$$\leqslant \int_0^y [u(m^H) - u(\pi^L)] f_e$$
$$\leqslant \int_0^y [u(m^H) - u(m^L)] f_e \underset{s}{=} \int_0^y f_e < 0,$$

where $\underset{s}{=}$ means 'has the same sign as.' The first inequality follows because the second term is weakly negative, since $f_e(x) > 0$ for $x > y > x^*(e)$, and since $u(\pi(x^H)) - u(\pi(x^L)) \leqslant 0$. It also makes the substitution $\pi^H(x) = m^H$ for $x \leqslant y$. The second inequality follows since $\pi^L = m^L$ where $f_e < 0$ and $\pi^L \geqslant m^L$ where $f_e > 0$. But, the last inequality contradicts that *IC* binds for both π^H and π^L. □

Proof of Proposition 2. Since π^H and π^L are continuous and both satisfy *IR* with equality, they must cross. Since $m^H > m^L$, π^H first crosses π^L from above. If this is the only crossing (or if the only other "crossing" is at 1), then by Lemma 4 the agent strictly prefers π^H to π^L, a contradiction. Since the contracts are of the form given by (1), the only remaining possibility is that $\mu^H > \mu^L$, and that π^H also crosses π^L once from below.

Consider now the case where *IR* is softly binding for m^H, and let y be the crossing point of the contracts when they are both strictly increasing. We have that $y > x^*(e)$, and hence $\ell(y|e) > 0$. Moreover,

$$\lambda^H + \mu^H \ell(y|e) = \lambda^L + \mu^L \ell(y|e).$$

Since $\mu^H > \mu^L$ and $\ell(y|e) > 0$ it must be that $\lambda^H < \lambda^L$. □

Author's personal copy

Proof of Lemma 5. Since *IR* is slack, by (1) for $x > \hat{x}(e,m)$, $\pi = W(\mu\ell)$. It follows that

$$\pi_{xm} = \left[\frac{W''(\mu\ell)\mu\ell}{W'(\mu\ell)} + 1\right]\mu_m \ell_x W'(\mu\ell) \geqslant 0,$$

where the bracketed term is non-negative by **U0** and Remark 1, $\mu_m > 0$ (from Proposition 2), and $\boldsymbol{\ell}_x > 0$ (by MLRP). The second part of the lemma follows similarly noting that for $x > \hat{x}(e,m)$ $u(\pi) = J(\mu\ell)$, and applying **U1** and Remark 1. □

Proof of Lemma 6. Note that $[u(\pi^L(x)) - u(\pi^H(x))]_x$ is 0 on $[0, \hat{x}^L]$ and is positive (by Proposition 2) on $[\hat{x}^L, \hat{x}^H]$. Let z be the point at which π^H and π^L cross while having positive slope (such a point exists by Proposition 2). Beyond z we have $\pi^H(x) > \pi^L(x)$, and hence $\lambda^H + \mu^H\ell(x) > \lambda^L + \mu^L\ell(x)$. Thus, beyond z

$$\begin{aligned}\left[u\left(\pi^L(x)\right) - u\left(\pi^H(x)\right)\right]_x &= \left[J\left(\lambda^L + \mu^L\ell\right) - J\left(\lambda^H + \mu^H\ell\right)\right]_x \\ &= \left[J'\left(\lambda^L + \mu^L\ell\right)\mu^L - J'\left(\lambda^H + \mu^H\ell\right)\mu^H\right]\ell_x < 0,\end{aligned}$$

where the inequality follows from $\mu^L < \mu^H$ (Proposition 2), the convexity of $J(\cdot)$ (Condition **U2** and Remark 1), and since $\ell_x > 0$ (*MLRP*).

It is thus sufficient to show that on $[\hat{x}^H, z]$, the sign pattern never crosses from negative to positive. We can write

$$\left[u\left(\pi^L(x)\right) - u\left(\pi^H(x)\right)\right]_x = K(x)\ell_x$$

where

$$K(x) = J'\left(\lambda^L + \mu^L\ell\right)\mu^L - J'\left(\lambda^H + \mu^H\ell\right)\mu^H.$$

By *MLRP*,

$$\left[u\left(\pi^L(x)\right) - u\left(\pi^H(x)\right)\right]_x \underset{s}{=} K(x),$$

and so it is enough to show that $K(x)$ cannot go from negative to positive, or that when $K(x) = 0$, $K'(x) \leqslant 0$.

But,

$$K'(x) = \left[J''\left(\lambda^L + \mu^L\ell\right)\left[\mu^L\right]^2 - J''\left(\lambda^H + \mu^H\ell\right)\left[\mu^H\right]^2\right]\ell_x,$$

and where $K(x) = 0$,

$$\mu^H = \frac{J'(\lambda^L + \mu^L\ell)}{J'(\lambda^H + \mu^H\ell)}\mu^L,$$

and so

$$\begin{aligned}K'(x) &\underset{s}{=} J''\left(\lambda^L + \mu^L\ell\right)\left[\mu^L\right]^2 \\ &\quad - J''\left(\lambda^H + \mu^H\ell\right)\left[\frac{J'(\lambda^L + \mu^L\ell)}{J'(\lambda^H + \mu^H\ell)}\right]^2\left[\mu^L\right]^2 \\ &\underset{s}{=} \frac{J''}{[J']^2}\left(\lambda^L + \mu^L\ell\right) - \frac{J''}{[J']^2}\left(\lambda^H + \mu^H\ell\right).\end{aligned}$$

Since $(\lambda^L + \mu^L\ell) > (\lambda^H + \mu^H\ell)$ on $(\hat{x}^H, z)$ this expression is negative if $\frac{J''}{[J']^2}$ is decreasing.

By the definition of $J(\cdot)$,

$$J'\left(\frac{1}{u'(w)}\right) = \frac{[u'(w)]^3}{-u''(w)}.$$

So,

$$J''\left(\frac{1}{u'(w)}\right)\frac{-u''(w)}{(u'(w))^2} = \left[\frac{(u'(w))^3}{-u''(w)}\right]'$$

or

$$J''\left(\frac{1}{u'(w)}\right) = \left[\frac{(u'(w))^3}{-u''(w)}\right]'\frac{(u'(w))^2}{-u''(w)}.$$

Thus,

Author's personal copy

$$\frac{J''}{(J')^2}\left(\frac{1}{u'(w)}\right) = \frac{[\frac{(u'(w))^3}{-u''(w)}]'}{\frac{(u'(w))^3}{-u''(w)}}\frac{\frac{(u'(w))^2}{-u''(w)}}{\frac{(u'(w))^3}{-u''(w)}} = \left[3\frac{u''(w)}{u'(w)} - \frac{u'''(w)}{u''(w)}\right]\frac{1}{u'(w)}$$
$$= \frac{u''(w)}{u'(w)}\left[3 - \frac{u'''(w)u'(w)}{(u''(w))^2}\right]\frac{1}{u'(w)}$$
$$= \frac{-u''(w)}{(u'(w))^3}\left[\frac{u'''(w)u'(w)}{(u''(w))^2} - 3\right]u'(w).$$

But, each term is positive (the middle one by Condition **U2**), and so it would be enough that each term decreased. The first does so by Condition **U2**, the last by concavity of u, and the middle one by **U3**. □

Proof of Lemma 7. It cannot be that π^L is everywhere above π^H. Indeed, in that case we would have $\hat{x}^H > \hat{x}^L > x^*(e^L)$, since *IR* is not binding. Thus, π^L differs from π^H only where $f_e(x|e^L) > 0$, implying that

$$\int u(\pi^L) f_e(\cdot|e^L) > \int u(\pi^H) f_e(\cdot|e^L).$$

Thus, π^H provides inadequate incentives to implement e^L and so by *CDFC* is a *fortiori* inadequate to implement e^H.

Suppose the two contracts cross at y, so that $\mu^L \ell(y, e^L) = \mu^H \ell(y, e^H)$, or equivalently,

$$\frac{\ell(y, e^H)}{\ell(y, e^L)} = \frac{\mu^L}{\mu^H}.$$

If **I1** holds then for $x > (<) y$,

$$\frac{\ell(x, e^H)}{\ell(x, e^L)} < (>) \frac{\mu^L}{\mu^H}.$$

Thus, if there is a crossing point, then π^H crosses π^L from above, and the crossing is unique. Analogously, when **I1′** holds, if there is a crossing point, then π^H crosses π^L from below, and the crossing is unique. □

Proof of Lemma 8. Consider first the case where **I1′** and **U1′** hold. By Lemma 7, either π^H crosses π^L once from below, or the two contracts don't cross at all. We examine each case separately.

(i) The two contracts cross at some point y. Let p be the common value of π^H and π^L at y.

For $x < y$, $p \geqslant \pi^L(x) \geqslant \pi^H(x)$. Using that $\frac{1}{u'}$ is increasing and convex (by **U1′**) we have

$$\frac{1}{u'(\pi^L(x))} - \frac{1}{u'(\pi^H(x))} \leqslant \left(\pi^L(x) - \pi^H(x)\right)\left[\frac{1}{u'(p)}\right]'$$
$$= \left(\pi^L(x) - \pi^H(x)\right)\frac{-u''(p)}{u'(p)^2}.$$

Similarly, for $x \geqslant y$, $p \leqslant \pi^L(x) \leqslant \pi^H(x)$, implying that

$$\frac{1}{u'(\pi^H(x))} - \frac{1}{u'(\pi^L(x))} \geqslant \left(\pi^H(x) - \pi^L(x)\right)\left[\frac{1}{u'(p)}\right]'$$
$$= \left(\pi^H(x) - \pi^L(x)\right)\frac{-u''(p)}{u'(p)^2}.$$

Combining the two cases we have

$$\int\left[\frac{1}{u'(\pi^H(x))} - \frac{1}{u'(\pi^L(x))}\right] f\left(x|e^L\right) \geqslant \frac{-u''(p)}{u'(p)^2}\int\left[\pi^H(x) - \pi^L(x)\right] f\left(x|e^L\right)$$
$$\geqslant 0,$$

where the last inequality follows since $\int \pi_e f \geqslant 0$ from Lemma 2.

By *FOSD* we obtain

$$\int \frac{1}{u'(\pi^H(x))} f\left(x|e^H\right) > \int \frac{1}{u'(\pi^L(x))} f\left(x|e^L\right),$$

as required.

(ii) The two contracts do not cross. Then, π^H lies everywhere (weakly) above π^L. Consequently,

Author's personal copy

$$\int \left[\frac{1}{u'(\pi^H(x))} - \frac{1}{u'(\pi^L(x))}\right] f\left(x|e^L\right) \geqslant 0,$$

and the result follows as before by *FOSD*.

The proof for the case where **I1** and **U1** hold is analogous with the appropriate sign changes. □

Proof of Proposition 5. We will show that each term in (8) increases in m.

The first term. By Lemma 2,

$$\begin{aligned}\left[\int \pi_e f\right]_m &= \left[\mu\left[c''(e) - \int u(\pi) f_{ee}\right]\right]_m \\ &= \underbrace{\mu_m\left[c''(e) - \int u(\pi) f_{ee}\right]}_{A} + \underbrace{\mu\left[c''(e) - \int u(\pi) f_{ee}\right]_m}_{B}.\end{aligned}$$

Since $\mu_m > 0$ by Proposition 1, and using Lemma 2, $A \geqslant 0$.

Integrating by parts,

$$\begin{aligned}-\int u(\pi) f_{ee} &= -\underbrace{u\big(\pi(x,e,m)\big) F_{ee}(x|e)|_0^1}_{=0} + \int_0^1 \left[u(\pi)\right]_x F_{ee} \\ &= \int_{\hat{x}}^1 \left[u(\pi)\right]_x F_{ee}\end{aligned}$$

since $\pi_x(x,e,m) = 0$ for $x < \hat{x}$.[17] Thus, B is given by[18]

$$\begin{aligned}&\left[\int_{\hat{x}}^1 \left[u(\pi)\right]_x F_{ee}\right]_m \\ &= \left[-\hat{x}_m\left[u(\pi)\right]_x F_{ee}\right](\hat{x},e,m) + \int_{\hat{x}}^1 \left[u(\pi)\right]_{xm} F_{ee} \\ &= \left[-\hat{x}_m\left[u(\pi)\right]_x\left[-\frac{F_{ee}}{F_e}\right][-F_e]\right](\hat{x},e,m) + \int_{\hat{x}}^1 \left[u(\pi)\right]_{xm}\left[-\frac{F_{ee}}{F_e}\right][-F_e].\end{aligned}$$

From Lemma 5, $[u(\pi)]_{xm} \geqslant 0$. By *MLRP*, $-F_e$ is positive as well. Thus, since by **I2,** $-\frac{F_{ee}}{F_e}$ is non-decreasing in x for $x \geqslant \hat{x} > x^*(e)$, if we replace $-\frac{F_{ee}(x|e)}{F_e(x|e)}$ by $-\frac{F_{ee}(\hat{x}|e)}{F_e(\hat{x}|e)}$ in the second term, the integral becomes smaller. Thus,

$$\begin{aligned}B &\geqslant -\frac{F_{ee}(\hat{x}|e)}{F_e(\hat{x}|e)}\left[\left[-\hat{x}_m\left[u(\pi)\right]_x[-F_e]\right](\hat{x},e,m) + \int_{\hat{x}}^1 \left[u(\pi)\right]_{xm}[-F_e]\right] \\ &= -\frac{F_{ee}(\hat{x}|e)}{F_e(\hat{x}|e)}\left[\int_{\hat{x}}^1 \left[u(\pi)\right]_x[-F_e]\right]_m \\ &= -\frac{F_{ee}(\hat{x}|e)}{F_e(\hat{x}|e)}\left[\int_0^1 \left[u(\pi)\right]_x[-F_e]\right]_m \quad \big(\text{again, since } \pi_x(x,e) = 0 \text{ for } x < \hat{x}\big)\end{aligned}$$

[17] The reader will note that we laboriously drop from range of integration $[0,1]$ to $[\hat{x},1]$, and then a few steps later, reverse the process. The step is needed because $[u(\pi)]_x$ changes discontinuously in m near $\hat{x}$.

[18] We take $[u(\pi)]_x(\hat{x},e,m)$ to be equal to $\lim_{x\downarrow\hat{x}}[u(\pi)]_x(\hat{x},e,m)$.

Author's personal copy

$$= -\frac{F_{ee}(\hat{x}|e)}{F_e(\hat{x}|e)}\left[\int u(\pi) f_e\right]_m \quad \text{(integrating by parts)}$$
$$= 0 \quad \text{(since } IC \text{ holds for all } m\text{)}.$$

The second term. Now, let's consider $[\int \pi f_e]_m$. Since *IC* binds for all m we have (integrating the *IC* constraint by parts) that

$$\left[\int_{\hat{x}}^{1} u'(\pi)\pi_x[-F_e]\right]_m = 0.$$

Since $u'(\pi(x)) = \frac{1}{\mu}\frac{f(x|e)}{f_e(x|e)}$ on $[\hat{x}, 1]$ this is equivalent to

$$\left[\frac{1}{\mu}\int_{\hat{x}}^{1} \frac{f}{f_e}\pi_x[-F_e]\right]_m = 0.$$

Equivalently,

$$\left[\frac{1}{\mu}\right]_m \int_{\hat{x}}^{1} \frac{f}{f_e}\pi_x[-F_e] + \frac{1}{\mu}\left[\int_{\hat{x}}^{1} \frac{f}{f_e}\pi_x[-F_e]\right]_m = 0.$$

The first term is strictly negative since $[\frac{1}{\mu}]_m < 0$ (Proposition 1) and since $\hat{x} > x^*(e)$ as *IR* is not binding, and so $\frac{f}{f_e} > 0$ everywhere on the domain of integration. Thus,

$$0 < \left[\int_{\hat{x}}^{1} \frac{f}{f_e}\pi_x[-F_e]\right]_m$$
$$= \left[-\hat{x}_m \frac{f}{f_e}\pi_x[-F_e]\right](\hat{x}, e, m) + \int_{\hat{x}}^{1} \frac{f}{f_e}\pi_{xm}[-F_e]. \tag{9}$$

Since $\frac{1}{u'}$ is positive and weakly concave, it is also weakly log-concave. Thus, by Lemma 5, $\pi_{xm} \geqslant 0$ and so $\pi_{xm}[-F_e] \geqslant 0$. Since $\frac{f}{f_e}$ is decreasing in x (by *MLRP*) we can replace it in the second term in (9) by its maximal value $\frac{f(\hat{x}|e)}{f_e(\hat{x}|e)} > 0$ and maintain the inequality:

$$0 < \frac{f(\hat{x}|e)}{f_e(\hat{x}|e)}\left[-\hat{x}_m(e, m)\pi_x(\hat{x}, e, m)\big[-F_e(\hat{x}|e)\big] + \int_{\hat{x}}^{1} \pi_{xm}[-F_e]\right]$$
$$\underset{s}{=} \left[\int_{\hat{x}}^{1} \pi_x(-F_e)\right]_m$$
$$= \left[\int_{0}^{1} \pi_x(-F_e)\right]_m \quad \big(\text{as } \pi_x = 0 \text{ on } [0, \hat{x}]\big)$$
$$= \left[\int \pi f_e\right]_m. \qquad \square$$

Proof of Proposition 6. Let $m^H > m^L$ and let π^H and π^L be the corresponding cost minimizing contracts. We will show that each term in (8) is larger at m^H than at m^L.

The first term. Much as in the proof of Proposition 5, from Lemma 2, and since $\mu_m > 0$ it is enough to show that

$$\int u(\pi^H) f_{ee} \leqslant \int u(\pi^L) f_{ee},$$

Author's personal copy

where

$$\int u(\pi^L) f_{ee} - \int u(\pi^H) f_{ee} = \int \left[u(\pi^L) - u(\pi^H)\right]_x \left[\frac{-F_{ee}}{F_e}\right] F_e.$$

From Lemma 6 $[u(\pi^L(x)) - u(\pi^H(x))]_x$ has sign pattern $0/+/-$. Let y be the turning point from $+$ to $-$. Then, for all x,

$$\left[u(\pi^L(x)) - u(\pi^H(x))\right]_x \frac{-F_{ee}(x|e)}{F_e(x|e)} F_e(x|e) \geqslant \left[u(\pi^L(x)) - u(\pi^H(x))\right]_x \frac{-F_{ee}(y|e)}{F_e(y|e)} F_e(x|e),$$

where the inequality follows from Condition **I2** (remembering that $F_e < 0$). Hence,

$$\int \left[u(\pi^L) - u(\pi^H)\right]_x \frac{-F_{ee}}{F_e} F_e \geqslant \frac{-F_{ee}(y)}{F_e(y)} \int \left[u(\pi^L) - u(\pi^H)\right]_x F_e = 0,$$

where the equality follows since both π^L and π^H satisfy *IC*.

The second term. Let $\hat{\pi}$ be defined by

$$u(\hat{\pi}(x)) = u(\pi^L) + u(m^H) - u(m^L),$$

and note that $\int u(\hat{\pi}) f_e = \int u(\pi^L) f_e$. Let $\hat{\Delta}(x) = u(\pi^H(x)) - u(\hat{\pi}(x))$. We first argue that $\hat{\Delta}$ has sign pattern $0/-/+$. Since $\hat{x}^H > \hat{x}^L$ by Proposition 2, $\hat{\Delta} = 0$ on $[0, \hat{x}(e, m^L))$, and $\hat{\Delta} < 0$ on some interval beginning at $\hat{x}(e, m^L)$. But, since $f_e(x|e) > 0$ for all $x \geqslant \hat{x}^L$ (*IR* is softly binding) it cannot be that $\hat{\Delta} < 0$ for all $x > \hat{x}(e, m^L)$ since then

$$\int \left[u(\hat{\pi}) - u(\pi^H)\right] f_e > 0,$$

contradicting that both contracts satisfy *IC*. Thus $\hat{\Delta}$ crosses 0 from below at some $y > \hat{x}(e, m^H)$.

To see that y is unique, note that since $m^H > m^L$, we have $\pi^H(y) > \pi^L(y)$. It follows from **U2** and Proposition 2 that π^H is steeper than π^L in utility space at y.[19] However, by construction, the slope of $\hat{\pi}$ in utility space is equal to the slope of π^L in utility space. Hence, $u(\pi^H(x))$ crosses $u(\hat{\pi}(x))$ at y from below, and there cannot be another crossing.

Let $p = \hat{\pi}(y) = \pi^H(y)$. Then, since u is concave,

$$u'(p)\left[\pi^H(x) - \hat{\pi}(x)\right] \geqslant u(\pi^H(x)) - u(\hat{\pi}(x)),$$

for all x. So,

$$u'(p) \int \left[\pi^H - \hat{\pi}\right] f_e \geqslant \int \left[u(\pi^H) - u(\hat{\pi})\right] f_e = 0$$

(noting that $\pi^H - \hat{\pi} = 0$ where f_e is negative) and thus

$$\int \left[\pi^H - \hat{\pi}\right] f_e \geqslant 0. \tag{10}$$

Now, note that $\hat{\pi} - \pi^L$ is positive and is increasing since u is concave. Thus,

$$\int \left[\hat{\pi} - \pi^L\right] f_e > 0 \tag{11}$$

by *MLRP*.

Thus, from (11) and (10),

$$\int \pi^H f_e \geqslant \int \hat{\pi} f_e > \int \pi^L f_e,$$

as required. □

[19] The argument for this is as in the beginning of the proof of Lemma 6 where

$$\left[u(\pi^L(y)) - u(\pi^H(y))\right]_x = \left[J(\lambda^L + \mu^L \ell(y, e)) - J(\lambda^H + \mu^H \ell(y, e))\right]_x < 0.$$

Author's personal copy

Proof of Proposition 7. By Lemma 2,

$$\begin{aligned}\int \pi_e f &= \mu c''(e) - \mu \int u(\pi) f_{ee} \\ &= \mu c''(e) + \int \mu u'(\pi)\pi_x F_{ee} \quad \text{(using integration by parts)} \\ &= \mu c''(e) + \int \pi_x \frac{f F_{ee}}{f_e},\end{aligned}$$

where the last equality follows since $\pi_x = 0$ over $[0, \hat{x}]$, and for $x \in [\hat{x}, 1]$, $\mu u'(\pi) = \frac{f}{f_e}$.

Plugging into (8) we obtain

$$\begin{aligned}C_e &= \mu c''(e) + \int \pi_x \frac{f F_{ee}}{f_e} + \int \pi f_e \\ &= \mu c''(e) + \int \pi_x \frac{f F_{ee}}{f_e} - \int \pi_x F_e \quad \text{(using integration by parts)} \\ &= \mu c''(e) + \int \left(\frac{f F_{ee}}{f_e} - F_e\right)\pi_x.\end{aligned}$$

Multiplying and dividing by $1 - F$ and using the definition of $\tau = \tau(x, e)$ gives

$$\begin{aligned}C_e &= \mu c''(e) + \int \left[\frac{f F_{ee}}{f_e(1-F)} - \frac{F_e}{1-F}\right][1-F]\pi_x \\ &= \mu c''(e) + \int \tau[1-F]\pi_x \\ &= \mu c''(e) + \int_{\hat{x}}^{1} \tau[1-F]\pi_x,\end{aligned}$$

where the last equality follows since $\pi_x = 0$ for $x < \hat{x}$.

Differentiating by m gives

$$\begin{aligned}C_{em} &= \mu_m c''(e) - \hat{x}_m \tau(\hat{x}, e)\big[1 - F(\hat{x}|e)\big]\pi_x(\hat{x}, e) + \int_{\hat{x}}^{1} \tau[1-F]\pi_{xm} \\ &\geqslant -\hat{x}_m \tau(\hat{x}, e)\big[1 - F(\hat{x}|e)\big]\pi_x(\hat{x}, e) + \int_{\hat{x}}^{1} \tau(x, e)\big[1 - F(x|e)\big]\pi_{xm}(x, e) \\ &\geqslant -\hat{x}_m \tau(\hat{x}, e)\big[1 - F(\hat{x}|e)\big]\pi_x(\hat{x}, e) + \tau(\hat{x}, e)\int_{\hat{x}}^{1} [1-F]\pi_{xm},\end{aligned}$$

where the first inequality follows since $\mu_m > 0$ (Corollary 1) and $c'' \geqslant 0$, and the second inequality follows since $\pi_{xm} \geqslant 0$ (Lemma 5), $1 - F \geqslant 0$, and Condition **I3**. Hence,

$$\begin{aligned}C_{em} &\geqslant \tau(\hat{x}, e)\left[-\hat{x}_m\big[1 - F(\hat{x}|e)\big]\pi_x(\hat{x}, e) + \int_{\hat{x}}^{1} \big[1 - F(x|e)\big]\pi_{xm}(x, e)\right] \\ &= \tau(\hat{x}, e)\frac{\partial}{\partial m}\int_{\hat{x}}^{1} \pi_x(x, e)\big(1 - F(x|e)\big) \\ &= \tau(\hat{x}, e)\frac{\partial}{\partial m}\left[\pi(x, e)\big(1 - F(x|e)\big)\Big|_{\hat{x}}^{1} + \int_{\hat{x}}^{1} \pi(x, e) f(x|e)\, dx\right] \quad \text{(using integration by parts)}\end{aligned}$$

Author's personal copy

$$= \tau(\hat{x}, e) \frac{\partial}{\partial m} \left[-m\big(1 - F(\hat{x}|e)\big) + \int_{\hat{x}}^{1} \pi(x, e) f(x|e)\, dx \right] \quad \big(\text{using that } \pi(\hat{x}, e) = m\big)$$

$$= \tau(\hat{x}, e) \frac{\partial}{\partial m} \left[-m + \int_{0}^{1} \pi(x, e) f(x|e)\, dx \right] \quad \big(\text{using that } \pi(x) = m \text{ for } x \leqslant \hat{x}\big)$$

$$= \tau(\hat{x}, e)[C_m - 1].$$

But, from Proposition 3, when *IR* is not binding

$$C_m - 1 = \int \frac{u'(m)}{u'(\pi(x, e))} f(x|e)\, dx - 1 > 0,$$

where the inequality follows since $u'(m) \geqslant u'(\pi(x, e))$ for all x and strictly so for $x > \hat{x}$. Since $\tau(\hat{x}, e) > 0$ (by CDFC) we conclude that $C_{em} > 0$ as required. □

References

Dewatripont, M., Matthews, S.A., Legros, P., 2003. Moral hazard and capital structure dynamics. J. Europ. Econ. Assoc. 1, 890–930.
Holmström, B., 1979. Moral hazard and observability. Bell J. Econ. 10, 74–91.
Innes, R.D., 1990. Limited liabilities and incentive contracting with ex ante action choices. J. Econ. Theory 52, 45–67.
Jewitt, I., 1988. Justifying the first order approach to principal-agent problems. Econometrica 56, 1177–1190.
Jewitt, I., Kadan, O., Swinkels, J., 2008. Moral hazard with bounded payments. J. Econ. Theory 143, 59–82.
Kadan, O., Swinkels, J., 2010. Minimum payments, incentives, and markets. Available at http://www.kellogg.northwestern.edu/Faculty/Directory/Swinkels_Jeroen.aspx.
Kadan, O., Swinkels, J., 2013. On the moral hazard problem without the first order approach. Available at http://www.kellogg.northwestern.edu/Faculty/Directory/Swinkels_Jeroen.aspx. J. Econ. Theory, forthcoming.
Kim, S.K., 1997. Limited liability and bonus contracts. J. Econ. Manage. Strategy 6, 899–913.
Kimball, M.S., 1990. Precautionary saving in the small and in the large. Econometrica 58, 53–73.
LiCalzi, M., Spaeter, S., 2003. Distributions for the first-order approach to principal-agent problems. Econ. Theory 21, 167–173.
Matthews, S.A., 2001. Renegotiating moral hazard contracts under limited liability and monotonicity. J. Econ. Theory 97, 1–29.
Mirrlees, J., 1976. The optimal structure of incentives and authority within an organization. Bell J. Econ. 7, 105–131.
Mirrlees, J., 1999. The theory of moral hazard and unobservable behavior: Part I. Rev. Econ. Stud. 66, 3–21.
Oyer, P., 2000. A theory of sales quotas with limited liability and rent sharing. J. Lab. Econ. 18, 405–426.
Park, E., 1995. Incentive contracting under limited liability. J. Econ. Manage. Strategy 4, 477–490.
Rogerson, W.P., 1985. The first-order approach to principal-agent problems. Econometrica 53, 1357–1367.
Sappington, D., 1983. Limited liability contracts between principal and agent. J. Econ. Theory 29, 1–21.
Thiele, Henrik, Wambach, Achim, 1999. Wealth effects in the principal agent model. J. Econ. Theory 89, 247–260.