# Information functions and expectation*

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**Abstract**

A DM (decision maker) with preferences over pairs consisting of a state of Nature and an action must choose an action in a situation of imperfect information, i.e., without knowing what state has attained. Coarser (finer) partitions of states correspond to more (less) imperfect information, and information functions assign to each partition precisely the maximum EU (expected utility) price the DM is willing to pay for it. Additive beliefs define information functions that are *additively separable*, i.e., their Möbius inversion lives only on the *modular elements* of the partition lattice. Thus, information functions may be classified according to the region of the partition lattice where their Möbius inversion does live; this yields *k-order solutions*, that are the analogue, in terms of partition functions, of *k-order additive* set functions. Information is evaluated very simply by those DMs whose information functions are 1-order solutions, while additive separability (and $1 < k$-order solutions) correspond to more (and more) complex evaluation methods. Eventually, maximal chains of partitions allow to define a novel discrete integral yielding the maximum EU price the DM is willing to pay for playing the 'whole game'. An appendix shows how to get the Choquet integral through the novel integration technique.

*Key words: subset and partition lattices, lattice function, Möbius inversion, non-additive integration.*

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## 1 Introduction

Consider a two-player game involving Nature and a DM (decision maker). Nature is assumed to play first by choosing some 'state of the world' from a finite set. The DM plays second by choosing some available action. In particular, the game displays imperfect information, so that the DM is assumed not to be able, by him/herself, to recognize what state has attained; yet, he/she may acquire

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information before playing. More specifically, information takes the form of a partition of the state set, so that the finer the partition purchased, the smaller the events (i.e., the less imperfect the information) available. Accordingly, information functions assign to each partition the maximum price the DM is willing to pay for it. Note that, in some sense, this generalizes Schmeidler (1989), where the DM chooses only one act, and 'an act maps states to outcomes', i.e., solely the coarsest partition gets considered. Conversely, Gilboa and Lehrer (1991a) introduce information functions for the general case where strategies assign actions to subsets of states, and outcomes consist of pairs of an action and a state, i.e., actions map states to outcomes. Nevertheless, while Schmeidler (1989) uses non-additive beliefs (i.e., a Choquet capacity), Gilboa and Lehrer (1991a) restrict attention to additive beliefs (i.e., probabilities). Most recently, Lehrer and Rosenberg (2004) consider the case where Nature is replaced with a second DM.

This paper and the companion Rossi (2004a) aim at developing, respectively, from Gilboa and Lehrer (1991a) and (1991b), who define G (global) games as real-valued, monotone and bottom-normalized partition functions (i.e., taking values on partitions of players). The solution problem associated with these games seems to deserve special attention: in order to copy with Shapley's (1953) solution of C (coalitional) games, which is a bottom-normalized and additive coalitional game, Rossi (2004a) defines solutions of G games as partition functions whose Möbius inversion (from below) lives only on the first level, i.e., the set of atoms, of the partition lattice. In fact, as known, any lattice function whose domain is a distributive lattice (such as any subset lattice) is additive and bottom-normalized iff (if and only if) its Möbius inversion lives only on atoms. Yet, the partition lattice is geometric indecomposable, and not distributive; also, additive partition functions, i.e., valuations of partition lattices, are constant (see Aigner (1979), ex. IV.4.12, p. 195, and pp. 60 and 69 on indecomposable geometric lattices; see also Gilboa and Lehrer (1991a) on full commutativity of information functions). Thus, valuations of partition lattices must be distinguished from those partition functions whose Möbius inversion lives only on the bottom and first levels, and for the latter do not have their own name (as far as the author knows), they are here named '1-order solutions' (of partition lattices). In particular, while solutions of G games are bottom-normalized 1-order solutions (see Rossi (2004a)), here bottom-normalization must be relaxed, as information functions need to be antitone (i.e., they must take their highest value on the finest partition; see Gilboa and Lehrer (1991a)). Accordingly, the Möbius inversion of information functions must be non-zero on the bottom element of the partition lattice. The atoms of the lattice of partitions of an $n$-cardinal set of states are all those partitions consisting of $n-2$ one-state blocks and 1 two-state block, thus there are $\binom{n}{2}$ atom-partitions.

Additive separability is introduced by Gilboa and Lehrer (1991a), who use it (together with further conditions) for characterizing information functions w.r.t. (with respect to) probabilities. In particular, a partition function is additively separated by a set function if the value taken by the former on any partition coincides with the sum of the values taken by the latter on the blocks of the partition itself. The recursive definition of the Möbius function of posets (see Rota (1964a)) allows to show (in section 3) that the Möbius inversion of additively separable partition functions lives only on the modular elements of the partition lattice (these are those partitions where the number of $k$-cardinal blocks, $k > 1$, is either 0 or 1; see Aigner (1979)). This result is not new (see Gilboa and Lehrer (1991b)); yet, when considered together with the novel definition of 1-order solutions it allows to note that all atoms of partition lattices are modular elements, and thus 1-order solutions turn out to be additively separable.

Grabisch (1997,2000) defines $k$-order additive set functions as lattice functions whose domain is a subset lattice, of course, and whose Möbius inversion is zero on all subsets whose cardinality is $> k$, and nonzero on at least one $k$-cardinal subset. Such an argument is used here (in section 4) for defining '$k$-order solutions' (of partition lattices) as partition functions whose Möbius inversion is zero on all partitions whose 'size' is $> k$, and non-zero on at least one partition of size $k$. The size of a partition is here defined as the number of atom-partitions the former dominates according to the coarsening order relation (in practice, the size is an integer number that gets defined through the class of the partition, which is a vector of integer numbers; see Rota (1964a)). In fact, while the rank and the cardinality of subsets coincide, the rank and the size of partitions are very different.

Those information functions that are both $k$-order solutions and additively separable are defined (in section 5) to be $k$-additively separable. In particular, these latter are shown to be additively separated by those $\widehat{k}$-order additive set functions, with $k = \binom{\widehat{k}}{2}$ and $1 \leq \widehat{k} \leq n$, introduced by Grabisch (1997,2000). Also, when regarded as information functions (in section 6), $k$-order solutions of partition lattices and additively separable partition functions are seen to correspond to different capabilities (of the DM) in terms of evaluation of information.

Following Rossi (2004a), the core of partition functions gets defined (in section 7) as a set of 1-order solutions of the partition lattice. As known, the core is used in decision theory for representing (i) supermodular (non-additive) beliefs, and (ii) Choquet integrals (w.r.t. supermodular beliefs; see, for example, Denneberg (2002)). Such an approach is translated here in terms of (i) supermodular monotone measures on partition lattices and (ii) chain integrals (w.r.t. supermodular monotone measures). In particular, one may ask what is the certainty equivalent, for the DM, of the whole game, i.e., played with both Nature and information sellers. In order to answer (in section 8), one needs to integrate partition functions w.r.t. partition measures. These latter are $[0, 1]$-normalized and monotone partition functions, and thus extend the concept of Choquet capacity to the partition lattice, while integration is achieved by means of maximal chains of partitions. In fact, the chain integral proposed here quantifies the expected worth of entering the game for a DM whose beliefs over what partitions will actually be available can be modeled through a partition measure.

An appendix shows that maximal chains lead not only to the chain integral (that seems most appropriate for partition lattices), but also to a general (non-additive) integration technique for integrating lattice functions w.r.t. lattice measures. Specializing such a technique to subset lattices yields an integral that coincides with the traditional Choquet integral whenever this latter may be computed, i.e., when the integrand is additive (and bottom-normalized).

## 2 Lattices and lattice functions

Any set $X$ endowed with an order, binary relation $\geqslant$ satisfying $x \geqslant x$ (reflexivity), $y \geqslant x, x \geqslant y \Rightarrow y = x$ (antisymmetry) and $z \geqslant y, y \geqslant x \Rightarrow z \geqslant x$ (transitivity) for all $x, y, z \in X$ is partially ordered, and thus named poset. If there exist $x_\perp, x^\top \in X$ such that $x_\perp \leqslant x, x^\top \geqslant x$ for all $x \in X$, then such elements are the bottom and top one, respectively. The antisymmetric part $>$ of $\geqslant$ is defined by $x > y \Rightarrow x \geqslant y, x \neq y$, as well as the covering relation $>^*$ is defined by $x >^* y \Rightarrow x \geqslant y$ and there is no $z \in X$ such that $x > z > y$. A $zy$-chain, $z, y \in X$, is any subset $K_{zy} = \left\{z = x_1, \ldots, x_{k_{zy}} = y\right\} \subset X$ such that $x_1 < \cdots < x_{k_{zy}}$, and its length is $k_{zy} - 1 =| K_{zy} | -1$. If, for any two fixed $z, y \in X, z \leqslant y$, all maximal $zy$-chains $K^*_{zy}$ (i.e., in terms of inclusion, that is $K^*_{zy} = \left\{z = x_1, \ldots, x_{k^*_{zy}} = y\right\} \subset X$ such that $x_{k^*_{zy}} >^* \cdots >^* x_1$) have the same length $k^*_{zy} - 1$, then $X$ is said to satisfy the JD (Jordan-Dedekind) condition, and this, in turn, allows to introduce the (integer-valued) rank function $r : X \rightarrow \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, defined recursively by $x >^* y \Rightarrow r(x) = r(y) + 1$ and $r(x_\perp) = 0$. Thus, $k^*_{zy} - 1 = r(y) - r(z)$, so that the rank $r(y)$ of lattice elements $y \in X$ measures their distance from the bottom element $x_\perp \in X$ as the length $k^*_{x_\perp y} - 1$ of any maximal $x_\perp y$-chain $K^*_{x_\perp y}$.

Defining both the join $\vee$ (i.e., the maximum or supremum) and the meet $\wedge$ (i.e., the minimum or infimum) operators (on $X \times X$) makes poset $X$ become a lattice. In particular, if $x_1 \vee \cdots \vee x_s \in X$ as well as $x_1 \wedge \cdots \wedge x_s \in X$ for all $\{x_1, \ldots, x_k\} = S \in 2^X \setminus \{\emptyset\}$, then $X$ is said a complete lattice (in which case the bottom and top elements do exist). A lattice $X$ is said atomic, or a point lattice, if every element $x \in X$ is a join of atoms, or points, i.e., elements covering the bottom one, that is $x = y_1 \vee \cdots \vee y_k$ such that $x \geqslant y_j >^* x_\perp, 1 \leq j \leq k$ .Thus, by means of the rank function, the set of atoms of $X$ is $\{y \in X : r(y) = 1\}$ (see Aigner (1979), p. 31). Any lattice function $h : X \rightarrow \mathbb{R}$ is defined to be:

top or bottom-normalized: if $h\left(x^\top\right) = 0$ or $h(x_\perp) = 0$,

monotone: if $x \geqslant y$ implies $h(x) \geq h(y)$ for all $x, y \in X$,

antitone: if $x \geqslant y$ implies $h(x) \leq h(y)$ for all $x, y \in X$,

additive or a valuation: if $h(x \vee y) = h(x) + h(y) - h(x \wedge y)$ for all $x, y \in X$,

$k$-monotone : if $h\left(\underset{1\leq i\leq k}{\vee} x_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|-1} h\left(\underset{i\in I}{\wedge} x_i\right)$ for every $k$-cardinal collection $x_1, \ldots, x_k \in X$,

$k$-antitone : if $h\left(\underset{1\leq i\leq k}{\wedge} x_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|-1} h\left(\underset{i\in I}{\vee} x_i\right)$ for every $k$-cardinal collection $x_1, \ldots, x_k \in X$, totally monotone: if it is monotone and $k$-monotone for all $k \in \{2, \ldots, | X |\}$,

totally antitone: if it is antitone and $k$-antitone for all $k \in \{2, \ldots, | X |\}$.

Strictly antitone lattice functions are obtained by replacing $\geqslant$ with $>$ and $\leq$ with $<$ in the above definition of antitone lattice functions. Top-normalization together with antitonicity (as well as bottom-normalization together with monotonicity) imply positivity, i.e., $h : X \rightarrow \mathbb{R}_+$. Recall that 2-monotone lattice functions are often said convex and/or supermodular, while replacing $\geq$ with $\leq$ in the definition of 2-monotonicity yields submodular lattice functions. As already mentioned, bottom-normalized and monotone lattice functions taking values on partitions of players are named G games in TU (transferable utility) cooperative game theory (see Gilboa and Lehrer (1991b)). In decision theory, antitone lattice functions taking values on partitions of states are named information functions (see Gilboa and Lehrer (1991a)). It may be anticipated that top-normalized information functions describe the situation where the DM associates zero worth to the coarsest partition. Eventually, it may be easily checked that any lattice function is 2-monotone iff it is 2-antitone, but $k$-monotone lattice functions need not be $k$-antitone, and *viceversa*, for $k > 2$.

The degree is the (integer-valued) lattice function $d : X \rightarrow \mathbb{N}_0$ defined by $d(x) =| \{y \in X : x >^* y\} |$ for all $x \in X$, i.e., the number of covered lattice elements. As known (see Gilboa and Lehrer (1991b), theorem 3.2, p. 133), any $h : X \rightarrow \mathbb{R}$ is totally monotone iff its **Möbius inversion** $\mu^h : X \rightarrow \mathbb{R}$ is positive, this latter being defined by

$$\mu^h(x) = h(x) - \sum_{\emptyset \neq I \subseteq \{1, \ldots, d(x)\}} (-1)^{|I|-1} h\left(\bigwedge_{i \in I} y_i\right) = \sum_{z \leqslant x} \mu_X(z, x) h(z),$$

where $\{y_1, \ldots, y_{d(x)}\} = \{y \in X : x >^* y\}$ denotes the set of lattice elements $y$ covered by $x$, while $\mu_X$ is the **Möbius function** of $X$, that takes values on $X \times X$ and is defined recursively by $\mu_X(z, x) = -\sum_{z \leqslant y < x} \mu_X(z, y)$ if $z < x$, 0 if $z \nleqslant x$ and 1 if $z = x$ (see Rota (1964a), propositions 1 and 2, p. 344).

The **Zeta function** $\zeta : X \times X \rightarrow \{0, 1\}$ is defined by $\zeta(x, y) = 1$ if $x \leqslant y$ and 0 otherwise (see Aigner (1979), p. 140). For each $x \in X$ let $\zeta_x(y) := \zeta(x, y)$ for all $y \in X$. Thus, $\zeta_x : X \rightarrow \{0, 1\}$ is a lattice function for every $x \in X$. In particular, the dimension of the vector space of lattice functions $h$ on $X$ is $| X |$, and if the chosen basis is the 'unanimity' one, i.e., the set of lattice functions $\{\zeta_x : x \in X\}$, then the Möbius inversion $\mu^h$ of $h$ is 'the analog of the "fundamental theorem of calculus"' (see Rota (1964a), p. 341), in that $h(x) = \sum_{y \in X} \mu^h(y) \zeta_y(x)$ or, equivalently, $h(x) = \sum_{y \in X : y \leqslant x} \mu^h(y)$. In the sequel, such a representation $h = \sum_{y \in X} \mu^h(y) \zeta_y$ of lattice functions $h : X \rightarrow \mathbb{R}$ is referred to as the **Möbius representation** of $h$.

Any finite state set $\Omega = \{\omega_1, \ldots, \omega_n\}$ identifies two main atomic lattices: the subset lattice $(2^\Omega, \cap, \cup)$ (i.e., the set of subsets of $\Omega$, ordered by inclusion $\supseteq$), and the partition lattice $(\mathcal{P}^\Omega, \wedge, \vee)$ (i.e., the set of partitions of $\Omega$, ordered by coarsening $\geqslant$). In fact, set functions $v : 2^\Omega \rightarrow \mathbb{R}$ as well as partition functions

$f:\mathcal{P}^{\Omega}\rightarrow\mathbb{R}$ are real-valued lattice functions. The notations and definitions applying to the subset lattice are assumed to be known (and extensively used), while those applying to the partition lattice are briefly reported hereafter.

Let $\mathcal{P}^{\Omega}$ denote the set of collections $\{A_1,\ldots,A_m\}\subset 2^{\Omega}\backslash\{\emptyset\}$ such that $\underset{1\leq j\leq m}{\cup}A_j=\Omega$ and $A_i\cap A_j=\emptyset$ for $1\leq i\neq j\leq m$, endowed with the (coarsening) binary, order relation $\geqslant$ such that if $P,P'\in\mathcal{P}^{\Omega}$ are any two such collections (i.e., partitions), and $P=\{A_1,\ldots,A_m\}$ as well as $P'=\{A'_1,\ldots,A'_{m'}\}$, then $P\geqslant P'$ if for each $j'\in\{1,\ldots,m'\}$ there is some $j\in\{1,\ldots,m\}$ such that $A'_{j'}\subseteq A_j$. Also, $P>Q\Leftrightarrow P\geqslant Q,P\neq Q$ for all $P,Q\in\mathcal{P}^{\Omega}$ (as well as, of course, $A\supset B\Leftrightarrow A\supseteq B,A\neq B$ for all $A,B\in 2^{\Omega}$).

The meet and join operators for the partition lattice (where $P\wedge P'$ is the coarsest partition finer than both $P,P'\in\mathcal{P}^{\Omega}$, while $P\vee P'$ is the 'finest coarser than' partition) are defined by $P\wedge P'=\left\{A_j\cap A'_{j'}:A_j\cap A'_{j'}\neq\emptyset\right\}$ as well as $P\vee P'=\left\{\underset{1\leq j\leq m,1\leq j'\leq m'}{\cup}\left(A_j\cup A'_{j'}\right):A_j\cap A'_{j'}\neq\emptyset\right\}$. The bottom and top elements of $\mathcal{P}^{\Omega}$ clearly are, respectively, $P_{\perp}=\{\{\omega_1\},\ldots,\{\omega_n\}\}$ and $P^{\top}=\{\Omega\}$.

For all $A\in 2^{\Omega}\backslash\{\emptyset,\Omega\}$, let $\mathcal{P}^{A}$ denote the lattice of partitions of $A$ (noting that $\mathcal{P}^{A}\nsubseteq\mathcal{P}^{\Omega}$). For all $\{B_1,\ldots,B_m\}=P\in\mathcal{P}^{\Omega}$, let $P^A\in\mathcal{P}^A$ denote the partition of $A$ induced by $P$, i.e., the set of intersections $\emptyset\neq A\cap B_j,1\leq j\leq m$. This allows to note that if $\{B_1,\ldots,B_m\}=P>^{*}Q=\{A_1,\ldots,A_{m'}\}$, then $m'=m+1$ and there is exactly one $i\in\{1,\ldots,m\}$ such that $Q^{B_i}=\{A_i,A_{m'}\}$, with $A_{m'}=B_i\backslash A_i$, while $A_j=B_j$ for $j\neq i$. In words, any $Q$ covered by $P$ must equal this latter for all blocks $B_{j\neq i}$, while dividing some block $B_i,1\leq i\leq m$ in two (new) blocks, i.e., $\{A_i,B_i\backslash A_i\}=Q^{B_i}$ with $\emptyset\neq A_i\subset B_i$. In particular, each $P\in\mathcal{P}^{\Omega}$ covers $k_P=-\mid P\mid+\sum_{B\in P}2^{|B|-1}$ (distinct) such partitions $Q$ (see Aigner (1979), exercise I.4.6, p. 29). Concerning the subset lattice, clearly $A\supset^{*}B\Leftrightarrow\mid A\mid=\mid B\mid+1$. The rank functions are $r(P)=n-\mid P\mid$ for all $P\in\mathcal{P}^{\Omega}$ and $r(A)=\mid A\mid$ for all $A\in 2^{\Omega}$.

Finally, the Möbius inversion (from below) of set functions $v:2^{\Omega}\rightarrow\mathbb{R}$ is

$$\mu^{v}(A)=v(A)-\sum_{\emptyset\neq I\subseteq\{1,\ldots,|A|\}}(-1)^{|I|-1}v\left(\underset{i\in I}{\cap}B_i\right)=\sum_{B\subseteq A}(-1)^{|A\backslash B|}v(B),$$

with $A\supset^{*}B_1,\ldots,B_{|A|}$, while that of partition functions $f:\mathcal{P}^{\Omega}\rightarrow\mathbb{R}$ is

$$\mu^{f}(P)=f(P)-\sum_{\emptyset\neq I\subseteq\{1,\ldots,k_P\}}(-1)^{|I|-1}f\left(\underset{i\in I}{\wedge}Q_i\right)=\sum_{Q\leqslant P}\mu_{\mathcal{P}^{\Omega}}(Q,P)f(Q),$$

with $P>^{*}Q_1,\ldots,Q_{k_P}$. The Möbius function $\mu_{\mathcal{P}^{\Omega}}$ of the partition lattice is determined through the class of partitions; see Rota (1964a), p. 359.

Let $X$ be a lattice ordered by $\geqslant$ and with meet $\wedge$ and join $\vee$. The dual lattice $X_*$ is ordered by $\geqslant_*$ and has meet $\wedge_*$ and join $\vee_*$, where $x\geqslant_* y\Leftrightarrow y\geqslant x$ and $x\wedge_* y=x\vee y$ as well as $x\vee_* y=x\wedge y$ for all $X_*\ni x,y\in X$. The Möbius functions of any pair of dual lattices satisfy $\mu_{X_*}(y,x)=\mu_X(x,y)$ (see Rota (1964a), p. 345) and if $h$ is monotone and $k$-monotone on $X$, then it is antitone and $k$-antitone on $X_*$. In fact, information functions must be antitone on the partition lattice $\mathcal{P}^\Omega$ ordered by coarsening $\geqslant$. Accordingly, Gilboa and Lehrer (1991a) use the dual $\mathcal{P}_*^\Omega$ of $\mathcal{P}^\Omega$, i.e., ordered by refinement $\geqslant_*$. Conversely, the partition lattice used here is $\mathcal{P}^\Omega$, i.e., the same as for G games. The Möbius inversion of lattice functions $h : X \to \mathbb{R}$ can not only be done from below (yielding $\mu^h$), but also from above (yielding $\mu_*^h$), i.e., $h(x) = \sum_{y \geqslant x} \mu_*^h(y)$, where $\mu_*^h(y) = \sum_{z \geqslant y} \mu_X(y,z) h(z)$. Thus, $h$ is totally antitone on $X$ (i.e., totally monotone on $X_*$) iff $\mu_*^h \geq 0$.

# 3 Additively separable information functions

Following Gilboa and Lehrer (1991a), consider some DM with a bounded utility function $u : \Omega \times \mathcal{A} \to \mathbb{R}_+$, where $\mathcal{A}$ is some set of available actions. For each $a \in \mathcal{A}$ define $u_a(\omega) = u(\omega, a)$ for all $\omega \in \Omega$, and let beliefs take the form of probability $p : 2^\Omega \to [0,1]$, i.e., $p$ is additive and $p(\emptyset) = 1 - p(\Omega) = 0$. Partitions become valuable in that the EU (expected utility) gets maximized by choosing an optimal available action for each block that any given information (i.e., partition) allows to maximize over. Let $\mathcal{S}$ denote the set of all strategies $s : \Omega \to \mathcal{A}$ available to the DM, so that, given any partition $P \in \mathcal{P}^\Omega$, the set of $P$-admissible strategies is $\mathcal{S}_P = \{s \in \mathcal{S} : \omega_i, \omega_j \in B \in P \Rightarrow s(\omega_i) = s(\omega_j)\}$. In other terms, the DM cannot condition his/her choice of (optimal) actions on knowledge he/she has not. Note that $|\mathcal{S}| = |\mathcal{A}|^n$ and $|\mathcal{S}_P| = |\mathcal{A}|^{|P|}$. In particular, $|\mathcal{S}_P|$ is the number of distinct functions from $\Omega$ to $\mathcal{A}$ whose kernel is (weakly) coarser than $P$ (see Aigner (1979), p. 6, and Rota (1964b), p. 499).

In order to characterize information functions, Gilboa and Lehrer (1991a) consider partition functions $f$ associating to each partition $P$ the EU associated with an optimal $P$-admissible strategy. Thus, such an $f$ has form

$$f(P) = e^p_{(\mathcal{A},u)}(P) := \max_{s \in \mathcal{S}_P} \int_\Omega u_s dp \text{ for all } P \in \mathcal{P}^\Omega.$$

Note that $\int$ here denotes discrete integration, i.e., summation of finitely many terms, each of which is finite because $u$ is bounded, that is

$$e^p_{(\mathcal{A},u)}(P) = \max_{s \in \mathcal{S}_P} \sum_{i=1}^n u_{s(\omega_i)}(\omega_i) p(\omega_i).$$

Therefore, $e^p_{(\mathcal{A},u)}(P) = \sum_{B \in P} v(B)$, where set function $v : 2^\Omega \to \mathbb{R}$ is defined by $v(B) = \max_{a \in \mathcal{A}} \int_B u_a dp$ for all $B \in 2^\Omega$. In fact, if $\Omega$ is a continuum, then one needs to impose that $\max_{a \in \mathcal{A}} \int_B u_a dp$ exists for every $B$ in the algebra of events (see Gilboa and Lehrer (1991a), p. 446). Accordingly, consider the following

**Definition 1** *Partition function $f : \mathcal{P}^\Omega \to \mathbb{R}$ is additively separable if there is a set function $v : 2^\Omega \to \mathbb{R}$ such that $f(P) = \sum_{A \in P} v(A)$ for all $P \in \mathcal{P}^\Omega$.*

Gilboa and Lehrer (1991a) provide a characterization of additive separability in terms of non-intersecting partitions, where any two $P, Q \in \mathcal{P}^{\Omega}$ are non-intersecting if $P^A \leqslant Q^A, P^{A^c} \geqslant Q^{A^c}$ for some subset $A \in 2^N \backslash \{\emptyset, N\}$ (or, equivalently, if $P \cup Q = (P \wedge Q) \cup (P \vee Q)$). In fact, $f$ turns out to be additively separable iff it is additive over all pairs of non-intersecting partitions. Alternatively, one may determine the region of the partition lattice where the Möbius inversion of additively separable partition functions may take non-zero values. This may be done through the Möbius representation (see Gilboa and Lehrer (1991b), proof of proposition 4.4, p. 138), or, equivalently, through the recursive definition of the Möbius function, which is the approach adopted hereafter.

**Theorem 2** *$f : \mathcal{P}^{\Omega} \to \mathbb{R}$ is additively separated by $v : 2^{\Omega} \to \mathbb{R}, v(\emptyset) = 0$ iff on each $P \in \mathcal{P}^{\Omega}$ its Möbius inversion $\mu^f$ takes value*

$$\mu^f(P) = \begin{cases} \sum_{\omega \in \Omega} v(\{\omega\}) = \sum_{\omega \in \Omega} \mu^v(\{\omega\}) \text{ if } P = P_{\perp}, \\ \mu^v(A) \text{ if } P = \{A\} \cup P_{\perp}^{A^c}, A \in 2^{\Omega}, 2 \leq |A| < n, \\ \mu^v(\Omega) \text{ if } P = P^{\top}, \\ 0 \text{ otherwise.} \end{cases}$$

**Proof.** Clearly, $\mu^f(P_{\perp}) = f(P_{\perp}) = \sum_{\omega \in \Omega} \mu^v(\{\omega\})$ by additive separability (and for $v(\emptyset) = 0$). Furthermore, if $P_{\perp} < P < P^{\top}$, then

$$\begin{aligned} \mu^f(P) &= \sum_{Q \leqslant P} \sum_{A \in Q} v(A) \mu_{\mathcal{P}^{\Omega}}(Q, P) = \sum_{A \in P} \sum_{\emptyset \neq B \subseteq A} v(B) \sum_{\substack{Q \leqslant P \\ Q \ni B}} \mu_{\mathcal{P}^{\Omega}}(Q, P) \\ &= \sum_{A \in P} \sum_{\emptyset \neq B \subseteq A} v(B) \sum_{Q^{B^c} \leqslant P^{B^c}} \mu_{\mathcal{P}^{\Omega}}\left(\{B\} \cup Q^{B^c}, P\right) \end{aligned}$$

$$= \sum_{A \in P} \sum_{\emptyset \neq B \subseteq A} v(B) \sum_{\substack{(Q^A, Q^{A^c}) \in \mathcal{P}^A \times \mathcal{P}^{A^c} \\ Q^A \ni B, Q^{A^c} \leqslant P^{A^c}}} \mu_{\mathcal{P}^A}\left(Q^A, \{A\}\right) \mu_{\mathcal{P}^{A^c}}\left(Q^{A^c}, P^{A^c}\right)$$

$$= \sum_{A \in P} \sum_{\emptyset \neq B \subseteq A} v(B) \sum_{\substack{Q^A \in \mathcal{P}^A \\ Q^A \ni B}} \left( \mu_{\mathcal{P}^A}\left(Q^A, \{A\}\right) \sum_{\substack{Q^{A^c} \in \mathcal{P}^{A^c} \\ Q^{A^c} \leqslant P^{A^c}}} \mu_{\mathcal{P}^{A^c}}\left(Q^{A^c}, P^{A^c}\right) \right),$$

where, as already mentioned, $\mathcal{P}^A$ denotes the lattice of partitions of any subset $\emptyset \neq A \in 2^{\Omega}$, and thus $\mu_{\mathcal{P}^A \times \mathcal{P}^{A^c}} = \mu_{\mathcal{P}^A} \mu_{\mathcal{P}^{A^c}}$ (see Rota (1964a), proposition 5, p. 345). In fact, $\sum_{Q^{A^c} \leqslant P^{A^c}} \mu_{\mathcal{P}^{A^c}}\left(Q^{A^c}, P^{A^c}\right) = 1$ if $P^{A^c} = P_{\perp}^{A^c}$, and 0 otherwise (by definition of the Möbius function). Thus, the above expression vanishes whenever $P^{A^c} > P_{\perp}^{A^c}$. It may also be observed that $\mathcal{P}^A \times \mathcal{P}^{A^c}$ is isomorphic to (i.e., coincides with) $\left\{P \in \mathcal{P}^{\Omega} : P \leqslant \{A, A^c\}\right\} = \mathcal{P}^{\Omega}_{\leqslant \{A, A^c\}}$, i.e., the subset of $\mathcal{P}^{\Omega}$ consisting of those partitions that do not intersect partition $\{A, A^c\} \in \mathcal{P}^{\Omega}$. In fact, $\mu_{\mathcal{P}^A \times \mathcal{P}^{A^c}}$ coincides with $\mu_{\mathcal{P}^{\Omega}_{\leqslant \{A, A^c\}}}$, i.e., the restriction of $\mu_{\mathcal{P}^\Omega}$ to (principal ideal) $\mathcal{P}^\Omega_{\leqslant\{A,A^c\}}$. On the other hand, if $P^{A^c} = P_\perp^{A^c}$, then firstly consider the case of atom-partitions, i.e., $\mid A \mid= 2$, so that

$$f\left(\{A\} \cup P_\perp^{A^c}\right) = v(A) + \sum_{\omega \in A^c} v(\{\omega\}) = \mu^f(P_\perp) + \mu^f\left(\{A\} \cup P_\perp^{A^c}\right)$$

clearly imply $\mu^f\left(\{A\} \cup P_\perp^{A^c}\right) = \mu^v(A)$. Now assume $\mu^f\left(\{B\} \cup P_\perp^{B^c}\right) = \mu^v(B)$ for all $\emptyset \neq B \subset \Omega$ such that $\mid B \mid< k$, and let $A \subset \Omega, \mid A \mid= k$. Then,

$$f\left(\{A\} \cup P_\perp^{A^c}\right) = v(A) + \sum_{\omega \in A^c} v(\{\omega\}) = \sum_{Q \leqslant \{A\} \cup P_\perp^{A^c}} \mu^f(Q)$$

and induction clearly imply $\mu^f\left(\{A\} \cup P_\perp^{A^c}\right) = \mu^v(A)$, as desired. Eventually, concerning the top element, noting that $f\left(P^\top\right) = \mu^f\left(P^\top\right) + \sum_{Q<P^\top} \mu^f(Q)$ completes the proof. In fact, it easily checked that any partition function with the above Möbius inversion is additively separable. ■

$\mathcal{P}^\Omega_{\text{mod}} := \left\{P_\perp, P^\top\right\} \cup \left\{\{A\} \cup P_\perp^{A^c} : A \in 2^\Omega, 2 \leq\mid A \mid\leq n-1\right\} \subset \mathcal{P}^\Omega$ constitutes the set of modular elements of $\mathcal{P}^\Omega$ (see Aigner (1979), exercise II.4.13, p. 71). Note that $\mid \mathcal{P}^\Omega_{\text{mod}} \mid= 2^n - 1 - (n-1) = 2^n - n$ as the void set $\emptyset \in 2^\Omega$ is disregarded, while the whole collection $\left\{\{\omega\} \in 2^\Omega : \omega \in \Omega\right\}$ of singletons is considered at once (i.e., it constitutes just one modular partition). Furthermore, valuations of partition lattices are, in fact, partition functions additively separated by additive set functions, and thus constant (see also Gilboa and Lehrer (1991a), corollary 5.2, p.457). On the other hand, if $f$ is additively separated by $v$, then it is also additively separated by all $v' : 2^\Omega \to \mathbb{R}$ equivalent to $v$ and such that $v(\Omega) = v'(\Omega)$ (see Gilboa and Lehrer (1991a), proposition 3.3, p. 452). Recall that $v$ and $v'$ are equivalent if $v'' = v - v'$ is an additive set function (see Shapley (1971), p. 12). Therefore, if a partition function is additively separable, then there may well be a continuum of set functions that additively separate it. In fact, any (bottom-normalized) $v : 2^\Omega \to \mathbb{R}$ belongs to a $2^n - 1$-dimensional vector space. Thus, if $v$ additively separates $f : \mathcal{P}^\Omega \to \mathbb{R}$, then $2^n - n \leq 2^n - 1$ yields that there are $n-1$ degrees of freedom.

**Lemma 3** *If $v : 2^\Omega \to \mathbb{R}$ additively separates $f : \mathcal{P}^\Omega \to \mathbb{R}$ and $v' : 2^\Omega \to \mathbb{R}$ satisfies (i) $\sum_{\omega\in\Omega} v'(\{\omega\}) = \sum_{\omega\in\Omega} v(\{\omega\})$, and (ii) $\mu^{v'}(A) = \mu^v(A)$ for all $A \in 2^\Omega, 2 \leq\mid A \mid$, then $v'$ additively separates $f$ as well.*

**Proof.** Consider the additively separable partition function $f'$ defined by $f'(P) = \sum_{B\in P} v'(B)$. Note that if $f$ is additively separated by $v$, then (i) and (ii) imply, respectively, $f'(P_\perp) = f(P_\perp)$ and $f'(P) = f(P)$ for all $P > P_\perp$. Therefore, $f = f'$, and thus $v'$ additively separates $f$. ■

Thus, given any set function $v$ that additively separates some partition function $f$, it is clear that, in order to construct a further $v'$ additively separating $f$, one can simply: (i) define $v'$ arbitrarily on singletons $\{\omega\} \in 2^\Omega$ (i.e., $\omega \in \Omega$), the only constraint being $\sum_{\omega\in\Omega} v'(\{\omega\}) = \sum_{\omega\in\Omega} v(\{\omega\})$, and (ii) define $v'$ on $A \in 2^\Omega, 2 \leq\mid A \mid$ according to $\mu^{v'}(A) = \mu^v(A)$, i.e., through recursion. Furthermore, this can be done in a continuum of different manners.

Any (bottom-normalized) set function $v$ defined by $v(B) = \max_{a \in \mathcal{A}} \int_B u_a dp$ for all $B \in 2^\Omega$ gets defined, in fact, by the triple $u, \mathcal{A}, p$, i.e., by preferences, available actions and beliefs, simultaneously. Let the action set $\mathcal{A}$ be fixed. Clearly, if one also fixes the utility function $u$, then finding some pair $p, p'$ of distinct additive beliefs satisfying $e^p_{(\mathcal{A},u)}(P) = e^{p'}_{(\mathcal{A},u)}(P)$ for all $P \in \mathcal{P}^\Omega$ might get hard. In fact, together with the 'initial' condition

$$\sum_{\omega \in \Omega} p(\omega) \max_{a \in \mathcal{A}} u_a(\omega) = \sum_{\omega \in \Omega} p'(\omega) \max_{a \in \mathcal{A}} u_a(\omega),$$

one must also check that for all $A \in 2^\Omega, |A| \geq 2$ the quantities

$$\max_{a \in \mathcal{A}} \sum_{\omega \in A} p(\omega) u_a(\omega) \text{ and } \max_{a \in \mathcal{A}} \sum_{\omega \in A} p'(\omega) u_a(\omega)$$

fulfill the above constraint on the Möbius inversion. This is not impossible, because changing (additive) beliefs also changes optimal actions (and thus expected utilities) associated with subsets $A \in 2^\Omega, |A| \geq 2$ when regarded as blocks of partitions. On the other hand, if both the utility function and beliefs are allowed to vary, then there may well be a continuum of distinct DMs (i.e., with different utility and beliefs) that share the same (additively separable) information function.

**Corollary 4** *The set of positive set functions $v: 2^\Omega \to \mathbb{R}_+$ that additively separate any positive and bottom-normalized partition function $f: \mathcal{P}^\Omega \to \mathbb{R}_+$ is either void or a singleton.*

**Proof.** Clearly, any positive and bottom-normalized partition function may well fail to be additively separable, in which case the set of set functions that additively separate is void. Otherwise, simply note that bottom-normalization of $f$ together with positivity of $v$ imply $v(\{\omega\}) = 0$ for all $\omega \in \Omega$. Therefore, $\mu^{v'}(A) = \mu^v(A)$ for all $A \in 2^\Omega, 2 \leq |A| \leq n$ can only be achieved for $v' = v$. ∎

**Remark 5** *If $f$ is additively separated by $v$, then*

$$\begin{aligned} f(P) &= \sum_{B \in P} v(B) = \sum_{Q \leqslant P} \mu^f(Q) = \sum_{B \in P} \sum_{\emptyset \neq A \subseteq B} \mu^v(A) \\ &= \sum_{Q \geqslant P} \mu^f_*(Q) = \sum_{B \in P} \sum_{A \supseteq B} \mu^v_*(A) = \sum_{B \in P} \sum_{A \supseteq B} \sum_{A' \supseteq A} (-1)^{|A' \backslash A|} v(A'), \end{aligned}$$

*where $\mu^v_*$ is the Möbius inversion from above of set function $v$. Furthermore,*

$$\mu^f_*(P) = \sum_{Q >^* P} \left( \sum_{B \in P \backslash Q} \sum_{A \supseteq B} \mu^v_*(A) - \sum_{B' \in Q \backslash P} \sum_{A' \supseteq B'} \mu^v_*(A') \right),$$

$$\sum_{Q\geqslant P}\mu_*^f(Q) = \sum_{Q\geqslant P}\sum_{Q'>^*Q}\left(\sum_{B\in Q\backslash Q'}\sum_{A\supseteq B}\mu_*^v(A)-\sum_{B'\in Q'\backslash Q}\sum_{A'\supseteq B'}\mu_*^v(A')\right)$$
$$= \sum_{B\in P}\sum_{A\supseteq B}\mu_*^v(A)=\sum_{B\in P}v(B).$$

**Remark 6** *Apart from additive separability, the Möbius inversions from below and from above of partition functions allow to note that, mathematically speaking, studying G games is equivalent to studying information functions. In fact, if $\widehat{f}$ is a bottom-normalized and monotone partition function (i.e., a global game), then $f$ defined by $f(P)=\widehat{f}(P^\top)-\widehat{f}(P)$ for all $P\in\mathcal{P}^\Omega$ is top-normalized and antitone (i.e., an information function). Furthermore, if $\widehat{v}:2^\Omega\to\mathbb{R}$ additively separates $\widehat{f}$, then $f$ is additively separated by any $v:2^\Omega\to\mathbb{R}$ defined by $\sum_{\omega\in\Omega}v(\{\omega\})=\widehat{v}(\Omega)$ (i.e., arbitrarily) on singletons, and by $\mu^v(A)=-\mu^{\widehat{v}}(A)$ (i.e., through recursion) on all $A\in 2^\Omega, \mid A\mid\geq 2$. In fact, monotonicity of $\widehat{f}$ clearly entails antitonicity of $f$, while the Möbius inversion of $f$ from below is $\mu^f(P)=\widehat{f}(P^\top)$ if $P=P_\perp$ and $-\mu^{\widehat{f}}(P)$ if $P>P_\perp$. Similarly, the Möbius inversion of $f$ from above is $\mu_*^f(P)=0$ if $P=P^\top$ and $-\mu_*^{\widehat{f}}(P)$ if $P<P^\top$.*

# 4 Lattices, valuations and solutions

Consider any lattice $X=x,y,z,\ldots$, with bottom element $x_\perp$ and top one $x^\top$, and with meet $\wedge$ and join $\vee$. Let $X^{(k)}=\{x\in X:r(x)=k\}$ denote the set of lattice elements whose rank is $k, 0\leq k\leq r(x^\top)$, or, equivalently, the $k$-th level of $X$; in particular, let $\{x_1^i:1\leq i\leq\mid X^{(1)}\mid\}=X^{(1)}$ denote the set of atoms. Any atomic lattice $X$ is said distributive if $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)$ and, similarly, $x\vee(y\wedge z)=(x\vee y)\wedge(x\vee z)$, for all $x,y,z\in X$; also, it is said geometric if for all $x,y\in X$ such that $x>^* y\neq x_\perp$ there is an atom $z\in X, z\nleqslant y$ such that $x=y\vee z$ (see Aigner (1979), proposition 2.28, p. 52). Subset lattices are distributive (and geometric), while partition lattices are geometric (only). A main distinction between distributive and (solely) geometric lattices is that while any element $x\in X_D$ of a distributive lattice $X_D$ admits a unique (irredundant) decomposition $x=x_1^1\vee\cdots\vee x_1^k$ as a join of atoms (see Aigner (1979), lemma 2.4, p. 33), any element $x\in X_G$ of a (solely) geometric lattice $X_G$ does not. Furthermore, a complement of any element $x\in X$ is any element $x'\in X$ such that $x\wedge x'=x_\perp$ as well as $x\vee x'=x^\top$. If $X=2^\Omega$, then every element $x\in X$ has exactly one complement. This is no longer true for geometric lattices, whose elements that differ from both the bottom and top ones admit several complements. For all $x,z\in X$, let $[x,z]$ denote the interval (i.e., the sublattice) $\{y\in X:x\leqslant y\leqslant z\}$. The center $Z(X)$ of $X$ is defined as the set of elements $x\in X$ such that $X$ is isomorphic to $[x_\perp,x]\times[x_\perp,x']$ for some (i.e., any) complement $x'$ of $x$. Note that $\emptyset\neq Z(X)\supseteq\{x_\perp,x^\top\}$ for all $X$ with bottom and top elements. In particular, if $X=2^\Omega$, then $Z(X)=X$. On the other hand, if $Z(X)=\{x_\perp,x^\top\}$, then $X$ is (defined to be) indecomposable, and $Z\left(\mathcal{P}^{\Omega}\right)=\left\{P_{\perp},P^{\top}\right\}$, i.e., partition lattices are geometric indecomposable. Eventually, if $Z(X)\supset\left\{x_{\perp},x^{\top}\right\}$, then $X$ is decomposable (see Aigner (1979), p. 60).

By definition (of the covering relation), the set $\mathcal{P}^{\Omega(1)}$ of atoms of the partition lattice consists of those partitions having $n-1$ blocks, i.e., $n-2$ one-state blocks and 1 two-state block (see Aigner (1979), p. 52). Thus, the second level $2^{\Omega(2)}$ of the lattice $2^{\Omega}$ of subsets of $\Omega$ is isomorphic to the first level $\mathcal{P}^{\Omega(1)}$ of the lattice $\mathcal{P}^{\Omega}$ of partitions of $\Omega$. Furthermore, any $P\in\mathcal{P}^{\Omega(k)},k\geq 2$ may (and does) admit, in general, several irredundant decompositions as a join of atoms. Yet, there is a unique (definitely redundant) such a decomposition that involves a maximum number of atoms.

The Möbius inversion of valuations of distributive lattices is well known to live only on the bottom and first levels; this is shown hereafter for the sake of completeness.

**Lemma 7** *If lattice $X$ is distributive and $h:X\rightarrow\mathbb{R}$ is additive, then $\mu^{h}(x)=0$ for all $x\in X$ such that $r(x)>1$.*

**Proof.** (By induction on $r(x)$.) Firstly note that $\mu^{h}(x_{\perp})=h(x_{\perp})$. Secondly, if $r(x_1)=1$, then $\mu^{h}(x_1)=h(x_1)-h(x_{\perp})$, and thus nonzero for generic additive lattice functions $h$. Conversely, $r(x_2)=2$, additivity and uniqueness of the decomposition $x_2=x_1^1\vee x_1^2$, with $r\left(x_1^1\right)=r\left(x_1^2\right)=1$, imply

$$\mu^{h}(x_2)=h(x_2)-h\left(x_1^1\right)-h\left(x_1^2\right)=h\left(x_1^1\right)+h\left(x_1^2\right)-h\left(x_1^1\right)-h\left(x_1^2\right)=0.$$

Now assume $\mu^{h}(x)=0$ for $1<r(x)\leq k$ and let $r(x_{k+1})=k+1$. Accordingly, the Möbius representation of $h$ and the unique irredundant decomposition $x_{k+1}=x_1^1\vee\cdots\vee x_1^{k+1}$ yield $\mu^{h}(x_{k+1})=h(x_{k+1})-\sum_{y<x}\mu^{h}(y)=$

$$=\sum_{\emptyset\neq I\subseteq\{1,\ldots,k+1\}}(-1)^{|I|-1}h\left(\underset{i\in I}{\wedge}x_1^i\right)-\sum_{i=1}^{k+1}h\left(x_1^i\right)=(1-1)^1\sum_{i=1}^{k+1}h\left(x_1^i\right)=0,$$

where the third equality obtains by means of the general sieve formula (see Aigner (1979), formula 4.62, p. 189) together with $\underset{i\in I}{\wedge}x_1^i=x_{\perp}$ for all subsets $I\subseteq\{1,\ldots,k+1\}$ such that $\mid I\mid\geq 2$ (see also Aigner (1979), theorem 4.63, pp. 190-1). ■

Thus, if $X$ is distributive, then the dimension of the vector space of valuations of $X$ is $\mid X^{(1)}\mid+1$, i.e., the number of atoms of $X$ plus one, and if bottom-normalization applies, then such a dimension clearly is $\mid X^{(1)}\mid$ (in fact, the general sieve formula constitutes a generalization of the exclusion-inclusion principle; see Aigner (1979)). Therefore, if attention is restricted to set functions (or, more generally, to lattice functions whose domain is a distributive lattice), then valuations may be defined as additive set (or lattice) functions or, equivalently, as set (or lattice) functions whose Möbius inversion lives only on the bottom and first levels. But if the lattice under concern is geometric indecomposable, then a clear distinction has to be made. In fact, valuations

$\upsilon : \mathcal{P}^{\Omega} \to \mathbb{R}$ of partition lattices are constant, i.e., there is a $V \in \mathbb{R}$ such that $\upsilon(P) = V$ for all $P \in \mathcal{P}^{\Omega}$ (see Aigner (1979), exercise IV.4.12, p. 195, noting that partition lattices are semimodular, but not modular, so that the (ii) part of the exercise applies). Thus, borrowing some terminology from game theory allows the following

**Definition 8** *1-order solutions $\phi : X \to \mathbb{R}$ of any lattice $X$ satisfy $\mu^{\phi}(x) = 0$ for all $x \in X$ such that $r(x) > 1$.*

For every pair of states $\{\omega_i, \omega_j\} \in 2^{\Omega(2)}$, let $[ij] = \{\omega_i, \omega_j\} \cup P_{\perp}^{\{\omega_i,\omega_j\}^c}$ denote the atom of $\mathcal{P}^{\Omega}$ whose unique two-state block is $\{\omega_i, \omega_j\}$ (and $A^c = \Omega \backslash A$ for all $A \in 2^{\Omega}$ denotes complementation). Note that such pairs are unordered, thus, $[ij] = [ji]$; furthermore, $\mathcal{P}^{\Omega(1)} = \left\{[ij] : \{\omega_i, \omega_j\} \in 2^{\Omega(2)}\right\}$ or, by means of an alternative notation, $\mathcal{P}^{\Omega(1)} = \left\{[ij]_t : t = 1, \ldots, \binom{n}{2}\right\}$. Clearly, if $f : \mathcal{P}^{\Omega} \to \mathbb{R}$ and $[ij]_1, \ldots, [ij]_k \in \mathcal{P}^{\Omega(1)}$, then $f([ij]_1 \vee \cdots \vee [ij]_k) \in \mathbb{R}$ is well defined. Yet, most likely $f\left(\underset{1 \leq t \leq k : \tilde{t} \neq t}{\vee} [ij]_t\right) = f\left(\underset{1 \leq t \leq k}{\vee} [ij]_t\right)$ for several $\tilde{t} \in \{1, \ldots, k\}$. This is due to linear dependence (see Whitney (1935)). In particular, the partition lattice $\mathcal{P}^{\Omega}$ is not isomorphic to $2^{\mathcal{P}^{\Omega(1)}}$, i.e., the set of collections of its atoms, which is, in turn, isomorphic to the probabilistic space of the random graph on vertex set $\Omega$, i.e., $2^{2^{\Omega(2)}}$ (see Spencer (2001), p. 13). In fact, $\mathcal{P}^{\Omega}$ is isomorphic (solely) to the set of graphs (on vertex set $\Omega$) each of whose components is a complete subgraph. More precisely, $2^{\mathcal{P}^{\Omega(1)}} \sim 2^{2^{\Omega(2)}}$, where $\sim$ denotes isomorphism and $\mid 2^{2^{\Omega(2)}} \mid = 2^{\binom{n}{2}}$, while the number $\mathcal{B}_n$ of partitions of an $n$-set is $\mathcal{B}_n = \sum_{k=0}^{n-1} \binom{n-1}{k} \mathcal{B}_k$, i.e., to be obtained through recursion by setting $\mathcal{B}_0 = 1$ (see Aigner (1979) on Bell numbers); thus, if $n \geq 3$, then $\mathcal{B}_n < 2^{\binom{n}{2}}$. This leads to regard partition functions (taking values on $\mathcal{P}^{\Omega}$) as peculiar set functions (taking values on a subset $\mathcal{M} \subset 2^{2^{\Omega(2)}}$). Formally, for any edge set $E \subseteq 2^{\Omega(2)}$, with $\{E_1 \cup \cdots \cup E_k\} = E \in 2^{2^{\Omega(2)}}$ and $E_1, \ldots, E_k$ denoting the edge sets of the components of the graph on $\Omega$ defined by $E$, let $\left\{\overline{E}_1 \cup \cdots \cup \overline{E}_k\right\} = \overline{E} \supseteq E$ denote the minimal superset (i.e., the transitive closure) consisting of the unique collection of (minimal) edge sets of complete subgraphs that includes $E$ (see also Aigner (1979), remark 2.31, p. 54, and section VII.3 on the use of such a closure operator $E \to \overline{E}$ in graph theory). Let $\mathcal{M} \subset 2^{2^{\Omega(2)}}$ denote the combinatorial geometry (or matroid) defined by $\mathcal{M} = \left\{E \in 2^{2^{\Omega(2)}} : E = \overline{E}\right\}$, so that $\mathcal{P}^{\Omega}$ is easily seen to be isomorphic to $\mathcal{M}$, in that any $\{B_1, \ldots, B_m\} = P \in \mathcal{P}^{\Omega}$ corresponds (bijectively) to a unique $E = \overline{E} \in \mathcal{M}$ such that $\mid E \mid = \sum_{i=1}^{m} \binom{|B_i|}{2}$. Hence, partition functions $f : \mathcal{P}^{\Omega} \to \mathbb{R}$ are, in fact, set functions $f : \mathcal{M} \to \mathbb{R}$.

**Lemma 9** *If $\phi : \mathcal{P}^{\Omega} \to \mathbb{R}$ is a 1-order solution, then for all $P \in \mathcal{P}^{\Omega}$*

$$\phi(P) = \phi(P_{\perp}) + \sum_{[ij] \in \mathcal{P}^{\Omega(1)} : P \geqslant [ij]} \mu^{\phi}([ij]) .$$

**Proof.** The Möbius representation of $\phi$ (i.e., $\phi(P) = \sum_{Q \leqslant P} \mu^{\phi}(Q)$ for all $P \in \mathcal{P}^{\Omega}$) together with the above definition of solution straightforwardly yield the desired conclusion. ■

Grabisch (1997,2000) defines set functions $v : 2^{\Omega} \to \mathbb{R}$ to be $k$-order additive (or $k$-additive) if $\mu^{v}(A) = 0$ for all $A \in 2^{\Omega}$ such that $| A |> k$, and there is at least one $k$-cardinal $A$ such that $\mu^{v}(A) \neq 0$. In order to define something similar in terms of partition functions, one needs to associate an integer number to each partition; in particular, such a number should translate the cardinality of subsets into some feature of partitions. One possibility could be to use the rank, but the size, defined hereafter, seems more appropriate.

The class of any partition $P \in \mathcal{P}^{\Omega}$ is defined (see Rota (1964a), p. 359) as the collection $c^P = \left\{c_1^P, c_2^P, \ldots, c_n^P\right\}$ of (possibly zero) integers such that $c_k^P$ is the number of blocks of $P$ consisting of $k$ states. Thus, if, for example, $c_n^P = 1$, then $P = P^{\top}$ and $c_k^P = 0$ for $1 \leq k < n$. Note that the class of partitions is much more informative than their rank (see section 2), for the latter is an integer number, while the former is an $n$-dimensional vector of integer numbers. In fact, $\sum_{1 \leq k \leq n} c_k^P k = n$ and $\sum_{1 \leq k \leq n} c_k^P = n - r(P) =| P |$ for all $P$, so that there are many more distinct classes than distinct ranks of partitions. Describing partitions as 'closed graphs' (i.e., graphs $E$ that coincide with their closure $\overline{E}$) and borrowing some terminology from graph theory lead to the following

**Definition 10** *The size $s : \mathcal{P}^{\Omega} \to \mathbb{N}_0$ of partitions is defined by*

$$s^P = \sum_{1 \leq k \leq n} c_k^P \binom{k}{2} =\left| \left\{[ij] \in \mathcal{P}^{\Omega(1)} : P \geqslant [ij]\right\} \right|$$

*for all $P \in \mathcal{P}^{\Omega}$ (recall that $\binom{1}{2} = 0$).*

The size describes partitions in terms of a single integer number, like the rank. Also, there are several distinct classes for each size. Note that the size of any partition is precisely the number of atoms whose join constitutes the largest decomposition as a join of atoms of such a partition.

**Definition 11** *$k$-order solutions of partition lattice $\mathcal{P}^{\Omega}$ are partition functions $\phi^k : \mathcal{P}^{\Omega} \to \mathbb{R}$ satisfying $s^P > k \Rightarrow \mu^{\phi^k}(P) = 0$ for all $P \in \mathcal{P}^{\Omega}$, and $\mu^{\phi^k}(P) \neq 0$ for at least one $P \in \mathcal{P}^{\Omega}$ such that $s^P = k$.*

Thus, valuations of partition lattices are 0-order solutions (of such lattices), i.e., partition functions whose Möbius inversion (from below) lives only on the bottom element. On the other hand, solutions of G games may be defined as bottom-normalized 1-order solutions of the lattice of partitions of some $n$-cardinal player set (see Rossi (2004a)). Furthermore, $k$-order solutions of partition lattices, with $k \geq 0$, get defined by specifying a region where the Möbius inversion is zero-valued. In view of section 3, additively separable partition functions may also be defined in the same way, i.e., as partition functions whose Möbius inversion lives only on the modular elements. In particular,

**Lemma 12** *1-order solutions of $\mathcal{P}^{\Omega}$ are additively separable.*

**Proof.** In view of lemma 9 above, if $\phi : \mathcal{P}^{\Omega} \rightarrow \mathbb{R}$ is a 1-order solution, then

$$\phi(P) = \phi(P_{\perp}) + \sum_{A \in P} \sum_{[ij] \leqslant \{A\} \cup P_{\perp}^{A^c}} \mu^{\phi}([ij])$$

for all $P \in \mathcal{P}^{\Omega}$, so that $\phi$ is additively separated by any set function $v_{\phi} : 2^{\Omega} \rightarrow \mathbb{R}$ defined (arbitrarily) on singletons so to satisfy $\sum_{\omega \in \Omega} v_{\phi}(\{\omega\}) = \phi(P_{\perp})$, and (through recursion) by $\mu^{v_{\phi}}(A) = \sum_{[ij] \leqslant \{A\} \cup P_{\perp}^{A^c}} \mu^{\phi}([ij])$ on all $A \in 2^{\Omega}$ such that $2 \leq | A | \leq n$. ∎

# 5 $k$-additively separable information functions

Consider those partition functions that are both additively separable and $k$-order solutions. As already observed, $\mathcal{P}^{\Omega}$ is isomorphic to a matroid $\mathcal{M} \subset 2^{2^{\Omega(2)}}$. In particular, only those edge sets $E$ that coincide with their closure, i.e., $E = \overline{E}$, have a size $| E |$ that corresponds to the size $s^P$ of some partition $P \in \mathcal{P}^{\Omega}$. Accordingly, $k$-order solutions are only defined for $k = \binom{\widehat{k}_1}{2} + \binom{\widehat{k}_2}{2} + \cdots + \binom{\widehat{k}_m}{2}$, where $\widehat{k}_1 + \widehat{k}_2 + \cdots + \widehat{k}_m$ constitutes a number-partition of $n$ into $m$ parts (see Aigner (1979), recursion 3.36, p. 97).

Let $\mathcal{I}_{k\text{-s}}$ denote the set of $k$-order solutions of $\mathcal{P}^{\Omega}$. In particular, $\mathcal{I}_{k\text{-s}}$ may be regarded as the set of information functions (i.e., antitone and possibly top-normalized partition functions) that take the form of $k$-order solutions. Accordingly, $\mathcal{I}_{k\text{-s}} := \emptyset$ if there is no $P \in \mathcal{P}^{\Omega}$ such that $k = s^P$. Clearly, if $\phi^k, \phi'^k \in \mathcal{I}_{k\text{-s}}$ are any two such functions and $\alpha \in \mathbb{R}_{++}$, then both $\phi^k + \phi'^k$ and $\alpha\phi^k$ are in $\mathcal{I}_{k\text{-s}}$. Thus, $\mathcal{I}_{k\text{-s}} \subset \mathbb{R}^{\mathcal{B}_n - 1}$ is a cone. In particular, for each $k$ as above (i.e., such that $k = s^P$ for some $P \in \mathcal{P}^{\Omega}$), its dimension is given by $| \left\{ Q \in \mathcal{P}^{\Omega} : s^Q \leq k \right\} |$. Now consider the set $\mathcal{I}_{\text{as}} \subset \mathbb{R}^{\mathcal{B}_n - 1}$ of additively separable partition (i.e., information) functions. $\mathcal{I}_{\text{as}}$ is also easily recognized to be a cone, and its dimension is $2^n - n$ (see section 3).

For all $0 \leq k \leq \binom{n}{2}$, define the set $\mathcal{I}_{k\text{-as}}$ of $k$-additively separable partition functions as the intersection $\mathcal{I}_{k\text{-as}} = \mathcal{I}_{k\text{-s}} \cap \mathcal{I}_{\text{as}}$. Thus, $\mathcal{I}_{k\text{-s}} = \emptyset \Rightarrow \mathcal{I}_{k\text{-as}} = \emptyset$.

**Theorem 13** *$f \in \mathcal{I}_{k\text{-}as}, 0 \leq k \leq \binom{n}{2}$ iff there is a $\widehat{k} \in \{1, \ldots, n\}$ such that (i) $k = \binom{\widehat{k}}{2}$, and (ii) $f$ is additively separated by a $\widehat{k}$-order additive set function $v$.*

**Proof.** Clearly, if $f$ is additively separated by a $\widehat{k}$-order additive set function $v$, then, by definition, $f$ is a $\binom{\widehat{k}}{2}$-order solution of $\mathcal{P}^{\Omega}$, i.e., $f \in \mathcal{I}_{\binom{\widehat{k}}{2}\text{-as}} \neq \emptyset$. To see the converse, firstly note that, as already mentioned, any partition function that is additively separated by an additive set function is, in fact, a constant function, i.e., it associates to each and every partition the same value, so that $\mathcal{I}_{0\text{-as}}$ (i.e., $\mathcal{I}_{k\text{-as}}$ for $k = n\binom{1}{2} = 0$) may be disregarded. In fact, in terms of decision making, it may be seen as corresponding to the situation where $| \mathcal{A} | = 1$, i.e., there is just one available action, so that information really does not matter. Secondly, as already observed, any 1-order solution is additively separable (in that every atom-partition is modular). Thus, $\mathcal{I}_{1\text{-as}} = \mathcal{I}_{1\text{-s}}$. Thirdly, consider that 2-order solutions $\phi^2 \in \mathcal{I}_{2\text{-s}}$ may have nonzero Möbius inversion only on (i) the bottom element $P_\perp$, (ii) all atom-partitions $[ij] \in \mathcal{P}^{\Omega(1)}$ and (iii) all partitions $P$ of the form $P = [ij]_t \vee [ij]_{t'}$ such that $[ij]_t, [ij]_{t'} \in \mathcal{P}^{\Omega(1)}$. These latter partitions (i.e., of type (iii)) are not modular elements, and there must be at least one of them where the Möbius inversion is nonzero. Therefore, 2-order solutions are not additively separated, that is, $\mathcal{I}_{2\text{-s}} \cap \mathcal{I}_{\text{as}} = \mathcal{I}_{2\text{-as}} = \emptyset$. Now consider 3-order solutions $\phi^3 \in \mathcal{I}_{3\text{-s}}$, noting that (i) partitions of the form $Q = \{A\} \cup P_\perp^{Ac}$ with $| A |= 3$ have size $s^Q = 3$ and are modular elements, but (ii) there are partitions $P$ of the form $P = [ij]_t \vee [ij]_{t'} \vee [ij]_{t''}$ such that $[ij]_t, [ij]_{t'}, [ij]_{t''} \in \mathcal{P}^{\Omega(1)}$ and $P \notin \mathcal{P}^{\Omega}_{\text{mod}}$, i.e., $P$ has size $s^P = 3$ but is not modular. Hence, there exist 3-order solutions that are additively separable: if $\phi^3 \in \mathcal{I}_{3\text{-as}} = \mathcal{I}_{3\text{-s}} \cap \mathcal{I}_{\text{as}} \neq \emptyset$, then $\phi^3$ is additively separated by some 3-order additive set function $v$, i.e., such that $\mu^v(A) = 0$ for all $A : 4 \leq| A |$ and $\mu^v(A) \neq 0$ for at least one 3-cardinal subset $A$ (see Grabisch (1997,2000)). The same argument leads to easily check that $\mathcal{I}_{4\text{-s}} \cap \mathcal{I}_{\text{as}} = \mathcal{I}_{5\text{-s}} \cap \mathcal{I}_{\text{as}} = \emptyset$, while $\mathcal{I}_{6\text{-s}} \cap \mathcal{I}_{\text{as}} = \mathcal{I}_{6\text{-as}} \neq \emptyset$ consists of those 6-order solutions that are additively separated by 4-order additive set functions, as $6 = \binom{4}{2}$. More generally, repeating the argument for all $k$-order solutions, with $0 \leq k \leq \binom{n}{2}$ (noting that $s^{P^\top} = \binom{n}{2}$), leads to observe that $\mathcal{I}_{k\text{-as}} \neq \emptyset$ iff $k = \binom{\widehat{k}}{2}$, with $1 \leq \widehat{k} \leq n$. ■

In particular, $\binom{n}{2}$-additively separable partition functions are generically additively separable, while 0-additively separable partition functions are valuations of partition lattices.

# 6 Alternative evaluations of information

The analysis conducted so far aims at showing that information (or, more generally, partition) functions may be classified according to the region where their Möbius inversion does live. Thus, it now comes natural to consider the implications, in terms of evaluation of information, of (i) 1-order solutions, (ii) additively separable information functions, and (iii) $k$-order solutions, $k > 1$. In particular, recall that, given any lattice $X$ and any lattice function $h : X \to \mathbb{R}$, the value taken by the Möbius inversion $\mu^h : X \to \mathbb{R}$ of $h$ on any lattice element $x \in X$, i.e., $\mu^h(x)$, is, in fact, the net worth (defined by $h$) added by $x$ w.r.t. all $y \in X, y < x$. In terms of information functions (that are antitone), this means that the finest (i.e., bottom) partition adds the highest, strictly positive net worth, while elements of upper levels shall add, in general, negative net worth (i.e., a loss). Thus, if the (positive) net worth of the finest partition equals the (negative) sum of the net worth added by all partitions $P > P_\perp$, then the information function is top-normalized.

As already mentioned, the DM is characterized by the triple $u, \mathcal{A}, p$, i.e., by preferences, available actions and beliefs. In fact, it may be easily checked that for any choice of beliefs (i.e., whether additive or non-additive) there exist choices of preferences $u$ that result in strictly antitone information functions (see section 2) iff each state has its own optimal action, requiring $\mid \mathcal{A} \mid \geq \mid \Omega \mid = n$. Put it differently,

**Lemma 14** *If an information function is strictly antitone, then* $\mid \mathcal{A} \mid \geq \mid \Omega \mid = n$.

**Proof.** If $f = e^p_{(\mathcal{A},u)}$ is strictly antitone, then $f(P_\perp) - f([ij]) > 0$ for all atom-partitions $[ij] \in \mathcal{P}^{N(1)}$. But this can only occur if each state $\omega_i \in \Omega$ has its own, distinct optimal action in $\mathcal{A}$. ■

Firstly note that this result applies to the case of non-additive beliefs as well. More precisely, if $p : 2^\Omega \rightarrow [0,1]$ is monotone and $p(\emptyset) = 0 = 1 - p(\Omega)$, then $e^p_{(\mathcal{A},u)}$ becomes a CEU (Choquet expected utility) as proposed by Schmeidler (1989). In fact, Lefort (2003) is gratefully acknowledged for highlighting that the problem of characterizing information functions w.r.t. non-additive beliefs seems to be open (see also Lehrer (2003), section 7.3). Secondly observe that the implication is not double-sided, i.e., $\mid \mathcal{A} \mid \geq n$ does not imply that each state has its own, distinct optimal action. Furthermore, even when there exist a distinct optimal action for each state, the case $\mid \mathcal{A} \mid > n$ deserves special attention, as it may formalize the situation where there are actions that are optimal for subsets $B \in 2^\Omega, \mid B \mid \geq 2$ but not for singletons $\{\omega\} \in 2^\Omega, \omega \in \Omega$.

Assume (i) $\mid \mathcal{A} \mid = n$ and (ii) each state has its own optimal action. Concerning evaluation of information, firstly consider information functions that take the form of 1-order solutions, i.e., antitone $\phi : \mathcal{P}^\Omega \rightarrow \mathbb{R}$ such that for all $P \in \mathcal{P}^\Omega$, if $\mid P \mid < n - 1$, then $\mu^\phi(P) = 0$. This is the easiest way to start, as

$$\phi(P) = \phi(P_\perp) + \sum_{[ij] \in \mathcal{P}^{\Omega(1)} : P \geqslant [ij]} \mu^\phi([ij])$$

for all $P \in \mathcal{P}^\Omega$ (see above). Clearly, if both (i) and (ii) hold, then $\mu^\phi([ij]) < 0$ for all $[ij] \in \mathcal{P}^{\Omega(1)}$. Thus, this is the case where each atom-partition $[ij]$ has an associated net loss of expected utility (i.e., $\mu^\phi([ij])$). Furthermore, the gross loss associated to each (non-atomic) partition $P \in \mathcal{P}^\Omega$ is the sum of the net losses associated to all atom-partitions it dominates. In terms of 'interaction among atomic losses' (see Grabisch and Roubens (1999)), this means that being forced to choose a unique action for both states $\omega_i, \omega_j \in \Omega$ always adds the same loss of utility, i.e., for all partitions $P \geqslant [ij]$, and thus independently from what action is actually chosen! In particular, to see how simple such an evaluation of information is, compare $[ij]$ with some coarser partition $P > [ij]$ such that $\{\omega_i, \omega_j\} \subset B \in P$. In addition, assume block $B$ also contains (at least) two further states $\omega_{i'}, \omega_{j'}$ that may both lead to the best or else to the worst conceivable outcome. More precisely, let $a^\omega, a_\omega \in \mathcal{A}$ denote, respectively, the optimal and the worst action for each state $\omega \in \Omega$. In fact, (i) and (ii) above imply that $\omega \rightarrow a^\omega$ is a bijection; in order to avoid additional notation, assume $\omega \rightarrow a_\omega$ is a bijection as well, i.e., $\max_{a \in \mathcal{A}} u_a(\omega) = u_{a^\omega}(\omega)$ and $\min_{a \in \mathcal{A}} u_a(\omega) = u_{a_\omega}(\omega)$ for every $\omega \in \Omega$. Next let $u_{a^{\omega_{i'}}}(\omega_{i'}) = u_{a^{\omega_{j'}}}(\omega_{j'})$ be the highest utility level the

DM may attain and $u_{a^{\omega_{i'}}}(\omega_{j'}) = u_{a_{\omega_{j'}}}(\omega_{j'}) = u_{a^{\omega_{j'}}}(\omega_{i'}) = u_{a_{\omega_{i'}}}(\omega_{i'})$ be the lowest utility level the DM may attain. In words, the best conceivable outcome, for the given DM, is that state $\omega_{i'}$ (or $\omega_{j'}$) attains and action $a^{\omega_{i'}}$ (or $a^{\omega_{j'}}$) is chosen. But if state $\omega_{i'}$ (or $\omega_{j'}$) attains and action $a^{\omega_{j'}} = a_{\omega_{i'}}$ (or $a^{\omega_{i'}} = a_{\omega_{j'}}$) is chosen, then the worst conceivable outcome realizes. Clearly, if the DM has some partition $P \geqslant [i'j']$ as the available information, then most of his/her effort (in terms of decision making) is put in dealing precisely with states $\omega_{i'}, \omega_{j'}$. In fact, if $[ij] \in \mathcal{P}^{\Omega(1)}$ is such that $u_{a^{\omega_{i'}}}(\omega_i) = u_{a_{\omega_{j'}}}(\omega_i) = u_{a^{\omega_i}}(\omega_i) - \varepsilon$ and $u_{a^{\omega_{j'}}}(\omega_j) = u_{a_{\omega_{i'}}}(\omega_j) = u_{a^{\omega_j}}(\omega_j) - \varepsilon$, where $\varepsilon$ is a strictly positive, arbitrarily small real number, then states $\omega_i, \omega_j$ allow to 'hedge', in some sense, whenever they are in a same block with states $\omega_{i'}, \omega_{j'}$. Yet, if the information function is a 1-order solution, then whenever $\omega_i, \omega_j$ are in a same block they produce the same loss of utility. Thus, 1-order solutions are information functions of rather simple DMs.

Now consider information functions $f$ that are additively separated by some set function $v$. In terms of evaluation of information, such functions imply, roughly speaking, that there is no interaction among non-atomic losses, i.e., losses associated with different $m$-cardinal blocks, $m \geq 2$. More precisely, they model the situation where each conceivable block $B \in 2^{\Omega}, \mid B \mid \geq 2$ of any partition $P \ni B$ has an associated gross loss, i.e., $\sum_{A \subseteq B : |A| \geq 2} \mu^v(A)$, which it adds to all partitions $P, Q \ni B$, i.e., independently from $P^{B^c}$ and $Q^{B^c}$. Assumptions (i) and (ii) above do not imply $\mu^v(B) \leq 0$ for all $B \in 2^{\Omega}, \mid B \mid \geq 2$; yet, if $v$ additively separates an information function (that is antitone), then $\mu^v(B) \leq 0$ for most $B \in 2^{\Omega}, \mid B \mid \geq 2$ is quite likely. Consider any such $B$ and partitions $\{B\} \cup P_{\perp}^{B^c}$, $\{B^c\} \cup P_{\perp}^{B}$ and their join $\{B\} \cup P_{\perp}^{B^c} \vee \{B^c\} \cup P_{\perp}^{B} = \{B, B^c\}$. Now, is there any reason why the gross EU loss associated with $\{B, B^c\}$ should be higher than the sum of the gross losses associated with $\{B\} \cup P_{\perp}^{B^c}$ and $\{B^c\} \cup P_{\perp}^{B}$? The answer is yes, and the reason is that strategies have to be chosen *ex ante*, i.e., before knowing which block of the available partition contains the 'true' state of Nature. In particular, firstly consider the *ex post* case, i.e., the DM firstly knows which block contains the state that attained, and secondly declares his/her choice of an optimal action for each block (i.e., an optimal $P$-admissible strategy, where $P$ is the information the DM managed to buy). It is clear that in such a case DMs choose a (possibly unique) optimal action for each (conceivable) block independently from what the available partition is, i.e., an action that is optimal for block $B$ of partition $\{B, B^c\}$ remains optimal in partition $\{B\} \cup P_{\perp}^{B^c}$ (as well as in any $\{B\} \cup P^{B^c}$ with $P_{\perp}^{B^c} < P^{B^c} < B^c$). Nevertheless, the evaluation (i.e., the pricing) of information must occur *ex ante*, i.e., before knowing what block contains the 'true' state, by definition. In fact, the *ex post* case just described requires information to take the form of a subset, rather than a partition, of states. If the optimal action for each block must be (i.e., gets) defined in advance, then some DM might reason as follows. If his/her beliefs are $p$ and the information to be evaluated is $\{B\} \cup P_{\perp}^{B^c}$, then with probability $p(B^c)$ the state that will attain would be perfectly 'recognized', so that no risk would be taken on the '$B^c$-side' of the state set. In this case, the attitude towards risk could induce, in the pricing process, to take some higher level of risk (i.e., to be more optimistic) on the '$B$-side' of the state set, where some risk would have to be taken anyway, given that the available information shall be $\{B\} \cup P_{\perp}^{B^c}$. On the other hand, if such a DM evaluates information $\{B, B^c\}$, then he/she has to bear much risk on the whole state set $\Omega = B \cup B^c$, and thus more pessimistic strategies would prevail. In other terms, the optimal action for $B$ varies across the different partitions $B$ is a block of. Interestingly, this feature may be checked to arise naturally when $p$ is allowed to be non-additive. Thus, those DMs whose information function is additively separable are definitely more skilled (in evaluating information) than those whose information function is a 1-order solution. Yet, the former are not so skilled, in that they do not consider different optimal actions for each conceivable block (see Gilboa and Lehrer (1991a), observation 3.4, p. 452). Clearly, additively separable information functions include, as special cases, $k$-additively separable information functions for all $k = \binom{\widehat{k}}{2}$ and $1 \leq \widehat{k} \leq n$.

Thirdly (and lastly) consider those information functions that are $k$-order solutions $\phi^k$, with $k = s^P$ for some $P \in \mathcal{P}^\Omega$. In particular, firstly let $k = \binom{n-1}{2}$, where $\binom{n-1}{2}$ is the size of any partition consisting of one 1-cardinal block and one $n-1$-cardinal block. By definition, $\mu^{\phi^k}\left(P^\top\right) = \phi^k\left(P^\top\right) - \sum_{P<P^\top} \mu^{\phi^k}(P) = 0$, so that the net loss of expected utility added by the coarsest partition $P^\top$ is zero, i.e., $\phi^k\left(P^\top\right) = \sum_{P<P^\top} \mu^{\phi^k}(P)$. In general, an information function that is a $k$-order solution for some $k = s^P, P \in \mathcal{P}^\Omega$ describes the situation where partitions $P > P_\perp$ (may) produce a non-zero net loss of expected utility only if they are fine enough (or, equivalently, not too coarse), i.e., if their size is $\leq k$. Conversely, those partitions $P$ whose size is $> k$ clearly have an associated gross loss of expected utility, i.e., $\phi^k(P) = \sum_{Q \leqslant P} \mu^{\phi^k}(Q)$, but the net loss of expected utility they produce is zero, i.e., $\phi^k(P) = \sum_{Q<P} \mu^{\phi^k}(Q)$. Thus, in view of the above analysis of additively separable information functions, $k$-order solutions (when regarded as information functions) describe the situation where each conceivable block $B \in 2^\Omega, \mid B \mid \geq 2$ of any partition $P \ni B$ has an associated loss of expected utility that varies across different partitions $P \ni B$; nevertheless, such a loss is the same for all partitions $P \ni B$ such that $s^P > k$.

The argument just used for interpreting $k$-order solutions in terms of information functions may be inverted. In fact, the Möbius inversion from above $\mu_*^f$ of any information function $f$ defines the (signed) net gain of expected utility that every partition $P$ produces w.r.t. all coarser partitions $Q \geqslant P$. Formally (from sections 2 and 3), for all $f : \mathcal{P}^\Omega \to \mathbb{R}$ and all $P \in \mathcal{P}^\Omega$,

$$f(P) = \sum_{Q \geqslant P} \mu_*^f(Q) = \sum_{Q \geqslant P} \sum_{Q' \geqslant Q} \mu_{\mathcal{P}^\Omega}(Q, Q') f(Q').$$

Recall that if $x^\top$ is the top-element of some lattice $X$, then the co-atoms of $X$ (i.e., the atoms of $X_*$) are those $x \in X$ such that $x^\top >^* x$. Thus, the co-atoms of $\mathcal{P}^\Omega$ are those partitions consisting of two blocks. Accordingly,

**Definition 15** *Co-solutions of $\mathcal{P}^\Omega$ are partition functions $\phi^c : \mathcal{P}^\Omega \to \mathbb{R}$ such* *that: (i) if $n$ is even, then $s^P < 2\binom{\frac{n}{2}}{2} \Rightarrow \mu_*^{\phi^c}(P) = 0$ and $\mu_*^{\phi^c}(P) \neq 0$ for at least one $P$ with $s^P \geq 2\binom{\frac{n}{2}}{2}$; (ii) if $n$ is odd, then $s^P < \binom{\frac{n-1}{2}}{2} + \binom{\frac{n+1}{2}}{2} \Rightarrow \mu_*^{\phi^c}(P) = 0$ and $\mu_*^{\phi^c}(P) \neq 0$ for at least one $P$ with $\binom{n}{2} > s^P \geq \binom{\frac{n-1}{2}}{2} + \binom{\frac{n+1}{2}}{2}$.*

In words, the Möbius inversion from above $\mu_*^{\phi^c}$ of co-solutions $\phi^c$ of $\mathcal{P}^\Omega$ lives only on the top element and on the set of co-atoms of $\mathcal{P}^\Omega$.

**Definition 16** *If $k = s^P$ for some $P \in \mathcal{P}^\Omega$, then $k$-order co-solutions of $\mathcal{P}^\Omega$ are partition functions $\phi^{k\text{-}c} : \mathcal{P}^\Omega \to \mathbb{R}$ such that $s^Q < k \Rightarrow \mu_*^{\phi^c}(Q) = 0$ and $\mu_*^{\phi^c}(Q) \neq 0$ for at least one $Q$ with $s^Q = k$.*

Thus, a DM whose information function is a $k$-order co-solution (of $\mathcal{P}^\Omega$) has a net gain of expected utility only for those partitions $P < P^\top$ that are coarse enough (or not too fine). Form another viewpoint, $k$-order co-solutions (when regarded as information functions) describe the situation where each conceivable block $B \in 2^\Omega, \mid B \mid < n$ of any partition $P \ni B$ has an associated gain of expected utility that varies across different partitions $P \ni B$; nevertheless, such a gain is the same for all partitions $P \ni B$ such that $s^P < k$.

# 7 Solutions, cores and representation

1-order solutions of partition lattices $\mathcal{P}^\Omega$ allow to define the core $\mathcal{C}(f)$ of partition functions $f$ as a set of 1-order solutions $\phi$ above $f$ and coinciding with this latter on both the bottom and top elements, i.e.,

**Definition 17** *The core $\mathcal{C}(f)$ of any partition function $f : \mathcal{P}^\Omega \to \mathbb{R}$ is $\mathcal{C}(f) =$*

$$\left\{\phi : \phi \text{ 1-order solution of } \mathcal{P}^\Omega, \phi \geq f, \phi(P_\perp) = f(P_\perp), \phi\left(P^\top\right) = f\left(P^\top\right)\right\},$$

*where $\phi \geq f$ reads $\phi(P) \geq f(P)$ for all $P \in \mathcal{P}^\Omega$.*

Concerning the differences between the core of set functions and the core of partition functions, firstly note, following Shapley (1971), that $\mathcal{C}(f)$ is a (possibly empty) compact convex polyhedron of dimension, at most, $\binom{n}{2} - 1$. In fact, once $f$ is fixed, $\mathcal{C}(f)$ is a set of partition functions $\phi$ whose Möbius inversion lives only on the bottom and first levels of $\mathcal{P}^\Omega$, and there are $\binom{n}{2}$ atom-partitions, while $\phi(P_\perp) = f(P_\perp)$ and $\phi\left(P^\top\right) = f\left(P^\top\right)$ by definition. If the concern is on G games, then bottom-normalization becomes a natural assumption (see Gilboa and Lehrer (1991b)), so that Rossi (2004a) omits $\phi(P_\perp) = f(P_\perp) = 0$ when defining the core of G games.

Both in decision theory and in cooperative game theory an important role is played by bottom-normalized and monotone set functions $v : 2^\Omega \to \mathbb{R}$. In particular, in decision theory $v$ is usually assumed to be a Choquet capacity (or non-additive belief), i.e., monotone and satisfying $v(\emptyset) = 1 - v(\Omega) = 0$. Basically, according to some DM's beliefs, $v(A)$ is the 'ambiguous probability' that the true state will be some $\omega \in A \in 2^\Omega$. Furthermore, many representations (of capacities and/or Choquet integrals) used in decision theory rely upon the assumption that $v$ is supermodular (see, for example, Dennebrg (2002)). Accordingly, consider some monotone partition measure $\eta : P^{\Omega} \rightarrow [0,1]$ satisfying $\eta\left(P^{\top}\right) = 1$. Note that Gilboa and Lehrer (1991a) (as well as this paper so far) implicitly assume that the DM is already 'into' the game. On the other hand, if the DM is also allowed to choose whether to enter the game or not, then such a choice is clearly based on his/her beliefs over what partitions will actually be available (and affordable). In fact, $\eta$ is intended to model precisely such beliefs. In particular, for all $P \in \mathcal{P}^{\Omega}$, let $\eta(P)$ denote the ambiguous probability, as perceived by the DM, that the partition he/she will actually use for choosing the optimal strategy (i.e., the optimal action actually played, eventually) is some $Q \leqslant P$. Note that $\eta(P_{\perp}) \geq 0$, as there is no reason why the finest partition should (automatically) be excluded. In the remainder of this paper attention is focused on (i) $[0,1]$-normalized, monotone and 2-monotone (i.e., supermodular) partition measures $\eta$ (that model beliefs over what partitions will actually be available), and (ii) antitone and, possibly, 2-antitone partition functions (that may well be regarded as a type of information functions of DMs whose beliefs $p$ over what state will attain need not be additive). Technically, 2-monotonicity and 2-antitonicity are needed for non-emptiness of, respectively, the core of monotone partition functions and the anticore of antitone partition functions.

Any $n$-cardinal collection $\{P_0, P_1, \ldots, P_{n-1}\} = K^* \subset \mathcal{P}^{\Omega}$ of partitions satisfying $P_k >^* P_{k-1}$ for all $1 \leq k \leq n-1$ is a maximal chain. Let $\mathcal{K}^*$ denote the whole set of maximal chains in $\mathcal{P}^{\Omega}$, and recall that if $f$ is 2-monotone, then $f(P \vee Q) + f(P \wedge Q) \geq f(P) + f(Q)$ for all $P, Q \in \mathcal{P}^{\Omega}$ (see section 2). The following result clearly parallels Shapley (1971), theorem 4, p. 21, and also appears in Rossi (2004a).

**Theorem 18** *If $f$ is monotone and 2-monotone, then $\mathcal{C}(f) \neq \emptyset$.*

**Proof.** Fix a maximal chain $\{P_0, P_1, \ldots, P_{n-1}\} = K^* \in \mathcal{K}^*$ of partitions, and for $1 \leq k \leq n-1$ let $\mathcal{P}^{\Omega(1)}_{K^*,k} = \left\{[ij] \in \mathcal{P}^{\Omega(1)} : P_{k-1} \ngeqslant [ij] \leqslant P_k\right\}$ denote the set of atom-partitions $[ij] \nleqslant P_{k-1}$ such that $P_{k-1} \vee [ij] = P_k$. Define 1-order solution $\phi^{f,K^*} : \mathcal{P}^{\Omega} \rightarrow \mathbb{R}$ of generic monotone and 2-monotone $f$ by

$$\phi^{f,K^*}([ij]) = \frac{f(P_k) - f(P_{k-1})}{s^{P_k} - s^{P_{k-1}}}$$

for all $[ij] \in \mathcal{P}^{\Omega(1)}_{K^*,k}, 1 \leq k \leq n-1$, noting that the denominator equals $\mid \mathcal{P}^{\Omega(1)}_{K^*,k} \mid$. Now let $Q \in \mathcal{P}^{\Omega} \backslash \{P_{\perp}\}$ be any partition, noting that if $Q = P_k \in K^*$ for some $k$, then $\phi^{f,K^*}(Q) = f(Q)$ by construction. Otherwise, let $P_{k_Q} \in K^*$ satisfy $Q > P_{k_Q - 1}$ and $Q \ngeqslant P_{k_Q}$, with $1 \leq k_Q < n-1$. Thus, for $P_{k_Q} >^* P_{k_Q-1}$, it must be $Q \wedge P_{k_Q} = P_{k_Q-1}$. Also, $f(Q) + f\left(P_{k_Q}\right) \leq f\left(P_{k_Q-1}\right) + f\left(Q \vee P_{k_Q}\right)$ as $f$ is 2-monotone. Hence,

$$\phi^{f,K^*}(Q) - f(Q) \geq \sum_{[ij] \in \mathcal{P}^{N(1)}_{K^*,k_Q}} \phi^{f,K^*}([ij]) + \phi^{f,K^*}(Q) - f\left(Q \vee P_{k_Q}\right).$$

Note that $Q \wedge [ij] = P_\perp$ for all $[ij] \in \mathcal{P}^{\Omega(1)}_{K^*,k_Q}$. Repeating the argument for all $P_k \in K^*$, with $k_Q < k \leq k^Q \leq n-1$, such that $P_{k^Q-1} \not\geqslant Q < P_{k^Q}$ yields

$$\phi^{f,K^*}(Q) - f(Q) \geq \phi^{f,K^*}(P_{k^Q}) - f(P_{k^Q}) = 0.$$

In particular, it must be $\mathcal{P}^{\Omega(1)}_{K^*,k^Q} \cap \left\{[ij] \in \mathcal{P}^{\Omega(1)} : [ij] \leqslant Q\right\} \neq \emptyset$. Therefore, $\phi^{f,K^*} \in \mathcal{C}(f)$ for all $K^* \in \mathcal{K}^*$, as desired. ■

Thus, following Shapley (1971), it can be noted that if $f$ is monotone and 2-monotone, then $\left\{\phi^{f,K^*} : K^* \in \mathcal{K}^*\right\}$ is the set of extreme points of $\mathcal{C}(f)$ (see also Rossi (2004a)). In fact, concerning representation, consider the following

**Lemma 19** *If $f$ is monotone and 2-monotone, then $f(P) = \min_{\phi \in \mathcal{C}(f)} \phi(P)$ for all $P \in \mathcal{P}^\Omega$.*

**Proof.** Simply note that for all $P \in \mathcal{P}^\Omega$, if $f$ is monotone and 2-monotone, then, from above, $\min_{\phi \in \mathcal{C}(f)} \phi(P) = \phi^{f,K^*}(P) = f(P)$ for all $K^* \in \mathcal{K}^*$ such that $K^* \ni P$. ■

This may be compared, for example, with Denneberg (2002), proposition 3.1. On the other hand, in order to represent antitone (and possibly 2-antitone) partition functions (i.e., information functions) consider the following

**Definition 20** *The anticore of any partition function $f$ is $\mathcal{AC}(f) =$*

$$\left\{\phi : \phi \text{ 1-order solution of } \mathcal{P}^\Omega, \phi \leq f, \phi(P_\perp) = f(P_\perp), \phi\left(P^\top\right) = f\left(P^\top\right)\right\},$$

*where $\phi \leq f$ reads $\phi(P) \leq f(P)$ for all $P \in \mathcal{P}^\Omega$.*

As already mentioned, 2-antitonicity is sufficient for non-emptiness of the anticore, as shown hereafter.

**Theorem 21** *If $f$ is antitone and 2-antitone, then $\mathcal{AC}(f) \neq \emptyset$.*

**Proof.** Define 1-order solution $\phi_{f,K^*} : \mathcal{P}^\Omega \to \mathbb{R}$ of generic antitone and 2-antitone $f$ by $\phi_{f,K^*}([ij]) = f(P_\perp) + \mu^{\phi_{f,K^*}}([ij])$, with

$$\mu^{\phi_{f,K^*}}([ij]) = \frac{f(P_k) - f(P_{k-1})}{s^{P_k} - s^{P_{k-1}}}$$

for all $P_{k-1} \not\geqslant [ij] \leqslant P_k$ and $1 \leq k \leq n-1$. For $f$ is antitone, $\mu^{\phi_{f,K^*}}([ij]) \leq 0$ for all $[ij] \in \mathcal{P}^{\Omega(1)}$; also, $\phi_{f,K^*}(P) = f(P_\perp) + \sum_{[ij] \leqslant P} \mu^{\phi_{f,K^*}}([ij])$ for all $P \in \mathcal{P}^\Omega$. Now let $Q \in \mathcal{P}^\Omega \backslash \{P_\perp\}$ be any partition, noting that if $Q = P_k \in K^*$ for some $k$, then $\phi_{f,K^*}(Q) = f(Q)$ by construction. Otherwise, let $P_{k_Q} \in K^*$ satisfy $Q > P_{k_Q-1}$ and $Q \not\geqslant P_{k_Q}$, with $1 \leq k_Q < n-1$. Thus, for $P_{k_Q} >^* P_{k_Q-1}$, it must be $Q \wedge P_{k_Q} = P_{k_Q-1}$. Also, $f(Q) + f\left(P_{k_Q}\right) \geq f\left(P_{k_Q-1}\right) + f\left(Q \vee P_{k_Q}\right)$ as $f$ is 2-antitone. Thus,

$$f\left(Q \vee P_{k_Q}\right) - \sum_{P_{k_Q-1} \not\geqslant [ij] \leqslant P_{k_Q}} \mu^{\phi_{f,K^*}}([ij]) - \phi_{f,K^*}(Q) \leq f(Q) - \phi_{f,K^*}(Q).$$

Repeating the argument for all $P_k \in K^*$, with $k_Q < k \leq k^Q \leq n-1$, such that $P_{k^Q} > Q, P_{k^Q-1} \ngtr Q$ yields

$$f(Q) - \phi_{f,K^*}(Q) \geq f(P_{k^Q}) - \phi_{f,K^*}(P_{k^Q}) = 0.$$

so that $\phi_{f,K^*} \in \mathcal{AC}(f)$, as desired. ■

Thus, if $f$ is antitone and 2-antitone, then $\{\phi_{f,K_*} : K^* \in \mathcal{K}^*\}$ is the set of extreme points of $\mathcal{AC}(f)$; furthermore, $f(P) = \max_{\phi \in \mathcal{AC}(f)} \phi(P)$.

The remainder of this paper is mainly concerned with the certainty equivalent (or, equivalently, the expected worth), for the given DM, of the whole game, i.e., played with both Nature and information sellers. Such a certainty equivalent is obtained, of course, as an expectation, i.e., as the *chain integral* (defined hereafter) of some information function $f$ w.r.t. some partition (monotone) measure $\eta$ as above.

# 8 Chain integral and expectation

Conceptually, the idea of top-normalized information functions sounds quite disturbing. In fact, on the one hand, the coarsest partition corresponds to no information at all, and no DM who is already 'into' the game would pay anything for buying it. On the other hand, the expected utility of playing an optimal $P^{\top}$-admissible strategy may well be strictly positive, so that some DM could pay for entering the game endowed with no information at all (i.e., with the coarsest partition $P^{\top}$). In fact, Gilboa and Lehrer's (1991a) discussion on full commutativity of information functions seems to rely upon the idea that the DM is in a game where he/she may well buy several distinct partitions and use, eventually, the meet of such a collection of acquired partitions. In fact, in such a case, it is very reasonable that a DM buys the coarsest partition for entering the game, in that he/she is assumed to buy finer partitions once into the game. Yet, in such a case each time the DM buys a new partition he/she evaluates it as the meet of all the partitions previously purchased and such a partition itself. Conversely, assume the DM buys only twice: firstly a 'ticket' for entering the game, and secondly a unique partition. In this case, if the DM has information function $f : \mathcal{P}^{\Omega} \to \mathbb{R}$ and pays $\theta \in \mathbb{R}_+$ for entering the game, then for each partition $P$ he/she will be willing to pay $\max\{0, f(P) - \theta\}$ (once into the game). The aim now turns at determining the maximum price $\theta$ that any DM is willing to pay for entering the game.

Assume the DM is characterized by $\mathcal{A}, u, p$ and beliefs $\eta : \mathcal{P}^{\Omega} \to [0,1]$ as above, i.e., $\eta$ is monotone and 1-normalized on the top-element. As already mentioned, $\eta(P)$ is the ambiguous probability that, once into the game and given the market of information, the unique partition that will be purchased is any $Q \leqslant P$. Let $f : \mathcal{P}^{\Omega} \to \mathbb{R}$ be the (antitone) information function, i.e., $f = e^p_{\mathcal{A},u}$, and consider any maximal chain $K^*_f = \left\{P^f_0, P^f_1, \ldots, P^f_{n-1}\right\}$ such that $f\left(P^f_k\right) = \min_{P^f_{k+1} >^* P} f(P)$ for all $n-2 \geq k \geq 0$. Note that there surely exists at least one maximal chain satisfying such a condition. Accordingly, the expected worth of entering the game is $\theta = \theta(\mathcal{A}, u, p, \eta) =$

$$\int_{\mathcal{P}^\Omega}^{Chain} f d\eta = f\left(P^\top\right) + \left[f\left(P^f_{n-2}\right) - f\left(P^\top\right)\right]\eta\left(P^f_{n-2}\right) + \cdots +$$

$$+\left[f\left(P^f_1\right) - f\left(P^f_2\right)\right]\eta\left(P^f_1\right) + \left[f\left(P_\perp\right) - f\left(P^f_1\right)\right]\eta\left(P_\perp\right) =$$

$$= \sum_{k=0}^{n-1}\left[f\left(P^f_{n-1-k}\right) - f\left(P^f_{n-k}\right)\right]\eta\left(P^f_{n-1-k}\right), \text{ with } f\left(P^f_n\right) := 0.$$

Concerning representation,

**Lemma 22** *If $\eta$ is 2-monotone, then*

$$\int_{\mathcal{P}^\Omega}^{Chain} f d\eta = \min_{\phi^\eta \in \mathcal{C}(\eta)} \int_{\mathcal{P}^\Omega}^{Chain} f d\phi^\eta.$$

**Proof.** Fix the integrand $f$, and let $K^*_f = \left\{P^f_0, P^f_1, \ldots, P^f_{n-1}\right\}$ as above, i.e., $f\left(P^f_k\right) = \min_{P^f_{k+1} >^* P} f(P)$ for all $n-2 \geq k \geq 0$. Now, as already observed, if $\eta$ is monotone (by definition) and 2-monotone (by assumption), then the set of extreme points of $\mathcal{C}(\eta)$ are those 1-order solutions $\phi^{\eta,K^*} : \mathcal{P}^\Omega \to \mathbb{R}$ defined by

$$\phi^{\eta,K^*}([ij]) = \frac{\eta(P_k) - \eta(P_{k-1})}{s^{P_k} - s^{P_{k-1}}}$$

for all $[ij] \in \mathcal{P}^{\Omega(1)}_{K^*,k} = \left\{[ij] \in \mathcal{P}^{\Omega(1)} : P_{k-1} \ngeqslant [ij] \leqslant P_k\right\}, 1 \leq k \leq n-1$, where $K^* = \{P_0, P_1, \ldots, P_{n-1}\}$ is any maximal chain of partitions (see above). Therefore, the number of extreme points of $\mathcal{C}(\eta)$ equals (at most) the number of distinct maximal chains. If the maximal chain is $K^*_f = \left\{P^f_0, P^f_1, \ldots, P^f_{n-1}\right\}$, then $\phi^{\eta,K^*_f}$ coincides with $\eta$ on $K^*_f$, i.e., $\phi^{\eta,K^*_f}\left(P^f_k\right) = \eta\left(P^f_k\right)$ for all $P^f_k \in K^*_f$. On the other hand, by definition (of the core) $\phi^\eta\left(P^f_k\right) \geq \eta\left(P^f_k\right)$ for all $P^f_k \in K^*_f$ and all $\phi^\eta \in \mathcal{C}(\eta)$. ■

This is the analogue of a well-known representation of the Choquet integral w.r.t. supermodular capacities (see Denneberg (2002), proposition 3.1).

According to the approach adopted here the DM decides twice: firstly whether to enter the game or not, and secondly what partition to buy (once into the game). In particular, $f(P)$ is the expected worth of being into the game and endowed with information $P$. Furthermore, once into the game after paying $\theta$, the DM will buy some partition $P$ with price $\leq f(P) - \theta > 0$. In fact, beliefs $\eta$ concern precisely what partitions will be affordable. Thus, for all $1 \leq k \leq n-1$, the quantity $f\left(P^f_{k-1}\right) - f\left(P^f_k\right)$ is expected with probability $\eta\left(P^f_{k-1}\right)$. From this viewpoint, the chain integral is obtained by adapting the

Choquet (non-additive) integration technique to the partition lattice. In fact, conversely, chain integration is a novel, general non-additive integration technique that may be adapted to subset lattices so to obtain the Choquet integral whenever this latter may be computed. (This is detailed in the appendix.)

# 9 Concluding remarks

In general, a main novelty of this paper is the study of information functions in terms of their Möbius inversion. Such an approach incorporates additively separable information functions (see Gilboa and Lehrer (1991a)) within a larger, more comprehensive picture. In fact, the size of partitions allows to consider distinct regions of the partition lattice where the Möbius inversion of partition functions is allowed to live; this leads to obtain the analogue of $k$-order additive set functions (see Grabisch (1997,2000)) in terms of partition functions, i.e., $k$-order solutions (and co-solutions) of partition lattices. In addition, 1-order solutions of partition lattices allow to define the core (and anticore) of partition functions in the usual manner (i.e., following Shapley (1971)), so that the traditional representation results applying to set functions (in terms of their core) can be derived for partition functions as well. Eventually, but perhaps most importantly, studying information functions in terms of their Möbius inversion enables to get a precise understanding of the different conceivable evaluations of information.

Secondly, this paper proposes a novel integral, i.e., the chain integral, that furnishes the expectation of a random variable taking values on the lattice of partitions of discrete sets. This is achieved by means of maximal chains of partitions. Clearly, expectation must be taken w.r.t. some beliefs; accordingly, one needs to introduce partition measures, that are the analogue of Choquet capacities in terms of partition lattices. In fact, as already mentioned, maximal chains allow to introduce a novel technique for (non-additively) integrating lattice functions w.r.t. lattice measures. Such a technique may be specialized for the case of subset lattices (see below).

Here partition measures formalize beliefs over what partitions will be affordable once into the game. Thus, the whole approach leading to chain integration requires to conceive the case where information was auctioned. In fact, partition measures may be derived from a mechanism according to which the available information gets sold, i.e., bought by those DMs who previously decided to enter the whole game. Designing a mechanism for auctioning information seems not easy. On the one hand, one may assume that there are several sellers and several buyers, so that, at any time before Nature plays (i.e., before the market for information closes), each seller is interested in selling any partition coarser than the finest partition he/she has, as well as any buyer is interested in buying any partition leading to a refinement of the meet of all partitions already bought. Such a situation seems to correspond to a market, rather than an auction; it fits Gilboa and Lehrer's (1991a) discussion on full commutativity of information functions, and might be approached in terms of two-sided matching (see Roth and Sotomayor (1990), chapters 8 and 9). On the other hand, one may assume that there exists a unique mechanism designer (see Milgrom (2004), p. 55) whose production of information is costly, and who faces many buyers. Clearly, once produced any information (i.e., partition), the mechanism designer can sell such an information to many buyers without additional costs, like in the provision of public goods.

The (indecomposable geometric) lattice of partitions is far more complex than the (distributive) lattice of subsets. In fact, while the latter is central in decision theory, whether under uncertainty or w.r.t. multiple criteria, the former is not. More precisely, as known, using partitions for modeling information has a long tradition; yet, the lattice of partitions as a whole seems to be much less used, especially in MCDM (multicriteria decision making). The same applies to cooperative game theory, where C games are far better known and extensively used than G games. Accordingly, this paper and the companion Rossi (2004a) aim at developing, respectively, from Gilboa and Lehrer (1991a) and (1991b), and thus at showing that the partition lattice may be a useful modeling tool in, respectively, decision theory and cooperative game theory. In particular, the partition lattice seems a useful tool for modeling interesting real-world situations of international interaction (see, Gilboa and Lehrer (1991b), Myerson (1977), Rossi (2003,2004b,c)). In order to fully exploit partition functions for modeling such situations, two important analytical tools are missing, i.e., the interaction indices and the multilinear extension (see Grabisch and Roubens (1999) and Owen (1988)), that shall be object of future research.

Concerning G games, it can be shown that if (i) solutions of G games are defined as bottom-normalized 1-order solutions of the lattice of partitions of players, and (ii) the Shapley (1953) axioms are suitably adapted to such a solution concept, then there exists a unique Shapley value, i.e., a $\binom{n}{2}$-dimensional real-valued vector of payoffs for pairs of players. Also, such a Shapley value gets defined, as usual, as a weighted average of marginal contributions of atoms (i.e., atom-partitions) to those lattice elements (i.e., partitions) they are not dominated by (according to the coarsening order relation). Yet, the weights used for such an averaging may be determined through recursion only, and seem related to the Stirling numbers of the second kind $\mathcal{S}_{n,n-k}, 0 \leq k \leq n-1$, that recursively define the cardinality of each $k$-th level of the partition lattice (see Aigner (1979)). The needed recursion is provided by Rossi (2004a), main theorem, section 4, where the implications in terms of maximal chains of partitions are highlighted. Eventually, one may define invariant solutions of G games as the analogue of invariant valuations of distributive lattices (see Klain and Rota (1997), p. 22), i.e., as solutions of G games that are invariant under the group of permutations of players; note that such permutations act on pairs of players (i.e., atom-partitions) as well. Accordingly, the dimension of the vector space of such invariant solutions of G games may be shown to equal the number of distinct classes of partitions of an $n$-set (see Rossi (2004a)).

Eventually, as already mentioned, the behavior of information functions w.r.t. to non-additive beliefs over the power set of the state set (i.e., according to the CEU model of Schmeidler (1989)) is unknown (see Lehrer (2003), section

7.3). This definitely deserves separate investigation.

# 10 Appendix: chains and integration

Let $\mathcal{V}_\emptyset^M \subset \mathbb{R}_+^{2^n-1}$ denote the set (i.e., the vector space) of all monotone set functions $v : 2^\Omega \rightarrow \mathbb{R}_+$ such that $v(\emptyset) = 0$. Also let $\mathcal{V}_\emptyset^A \subset \mathbb{R}_+^n$ denote the set (i.e., the vector space) of all additive $v \in \mathcal{V}_\emptyset^M$. Thus, clearly, $\mathcal{V}_\emptyset^A \subset \mathcal{V}_\emptyset^M$. Eventually, let $\gamma$ be a Choquet capacity on $2^\Omega$, i.e., $\gamma \in \mathcal{V}_\emptyset^M$ and $\gamma(\Omega) = 1$. Then, the Choquet integral of any $v \in \mathcal{V}_\emptyset^A$ w.r.t. $\gamma$ over $\Omega$ is defined by

$$\int_\Omega^{Choquet} v d\gamma = \sum_{1 \leq k \leq n} \left[ v\left(\left\{\omega_{(k)}\right\}\right) - v\left(\left\{\omega_{(k-1)}\right\}\right)\right] \gamma\left(\left\{\omega_{(m)} : k \leq m \leq n\right\}\right),$$

where $(\bullet) : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ satisfies $v\left(\left\{\omega_{(1)}\right\}\right) \leq \cdots \leq v\left(\left\{\omega_{(n)}\right\}\right)$, while $v\left(\left\{\omega_{(0)}\right\}\right) := v(\emptyset)$. Accordingly, $\int_\Omega^{Choquet} (\bullet)\, d\gamma : \mathcal{V}_\emptyset^A \rightarrow \mathbb{R}_+$ may well be regarded as an aggregation function (see Marichal (2000)), as it associates a (positive) real number to any $n$-tuple $v \in \mathcal{V}_\emptyset^A$ of (positive) real numbers. The aim now turns at obtaining an extension $\int_\Omega^{Choquet-EXT} (\bullet)\, d\gamma : \mathcal{V}_\emptyset^M \rightarrow \mathbb{R}_+$ of the Choquet integral to the case where the integrand is bottom-normalized, monotone, but may fail to be additive. This shall be achieved by means of maximal chains of subsets. Clearly, when regarded as an aggregation function, such an extended Choquet integral is seen to associate a (positive) real number to any $2^n - 1$-tuple $v \in \mathcal{V}_\emptyset^M \supset \mathcal{V}_\emptyset^A$ of (positive) real numbers. Eventually, recall that, by definition (of extension), for all $v \in \mathcal{V}_\emptyset^A$ it must be

$$\int_\Omega^{Choquet-EXT} v d\gamma = \int_\Omega^{Choquet} v d\gamma.$$

For all $v \in \mathcal{V}_\emptyset^M$, let $K_v^* = \{A_0^v, A_1^v, \ldots, A_n^v\} \subset 2^\Omega$ be a maximal chain of subsets (i.e., $A_k^v \supset^* A_{k-1}^v, 1 \leq k \leq n$) such that $v(A_k^v) = \min_{A \supset^* A_{k-1}^v} v(A)$ for all $1 \leq k \leq n$. Here again, there surely exists at least one such a $K_v^*$. Define

$$\begin{aligned} \int_\Omega^{Choquet-EXT} v d\gamma &= \sum_{1 \leq k \leq n} \left[ g\left(v\left(A_k^v\right)\right) - g\left(v\left(A_{k-1}^v\right)\right)\right] \gamma\left(\Omega \backslash A_{k-1}^v\right), \\ \text{with } g\left(v\left(A_k^v\right)\right) &= v\left(A_k^v\right) - v\left(A_{k-1}^v\right), 1 \leq k \leq n \text{ and } v\left(A_{-1}^v\right) := 0. \end{aligned}$$

**Lemma 23** *If $v \in \mathcal{V}_\emptyset^A$, then $\int_\Omega^{Choquet-EXT} v d\gamma = \int_\Omega^{Choquet} v d\gamma$.*

**Proof.** Let $v \in \mathcal{V}_\emptyset^A$. Any permutation $(\bullet) : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ identifies a (unique) maximal chain of subsets. If $v\left(\left\{\omega_{(1)}\right\}\right) \leq \cdots \leq v\left(\left\{\omega_{(n)}\right\}\right)$, then $K_v^* = \{A_0^v, A_1^v, \ldots, A_n^v\}$ defined by $A_k^v = \bigcup_{1 \leq m \leq k} \left\{\omega_{(m)}\right\}$ for all $1 \leq k \leq n$ satisfies $v(A_k^v) = \min_{A \supset^* A_{k-1}^v} v(A)$ for all $1 \leq k \leq n$. Eventually, if $v \in \mathcal{V}_\emptyset^A$, then

$g\left(v\left(A_k^v\right)\right)=v\left(\left\{\omega_{(k)}\right\}\right)$ for all $1\leq k\leq n$. At this point, the desired conclusion obtains simply by substitution. ■

$\int_\Omega^{Choquet-EXT} vd\gamma$ is defined for all $v\in\mathcal{V}_\emptyset^M$, and coincides with $\int_\Omega^{Choquet} vd\gamma$ on $\mathcal{V}_\emptyset^A$; thus the former is an extension of the latter from $\mathcal{V}_\emptyset^A$ to the whole $\mathcal{V}_\emptyset^M$.

**Lemma 24** *If $\gamma$ is supermodular, then for all $v\in\mathcal{V}_\emptyset^M$*

$$\int_\Omega^{Choquet-EXT} vd\gamma=\min_{p^\gamma\in\mathcal{C}(\gamma)}\int_\Omega^{Choquet-EXT} vdp^\gamma.$$

**Proof.** Let $v\in\mathcal{V}_\emptyset^M$ and consider any maximal chain $K_v^*=\{A_0^v,A_1^v,\ldots,A_n^v\}$ satisfying $v\left(A_k^v\right)=\min_{A\supset^* A_{k-1}^v} v\left(A\right)$ for all $1\leq k\leq n$. Here again, if $\gamma$ is monotone (by definition) and 2-monotone (by assumption), then the extreme points $p^{\gamma,K^*}$ of its core

$$\mathcal{C}\left(\gamma\right)=\left\{p^\gamma:p^\gamma\in\mathcal{V}_\emptyset^A,p^\gamma\left(B\right)\geq\gamma\left(B\right)\text{ for all }B\in 2^\Omega,p^\gamma\left(\Omega\right)=1\right\}$$

get defined through maximal chains $K^*=\{A_0,A_1,\ldots,A_n\}$ of subsets by

$$p^{\gamma,K^*}\left(\{\omega_i\}\right)=\left[\gamma\left(A_k\right)-\gamma\left(A_{k-1}\right)\right]:A_k\backslash A_{k-1}=\{\omega_i\}$$

for all $1\leq i\leq n$ (see Shapley (1971)). Accordingly, if the maximal chain is $K_v^*$, then $p^{\gamma,K_v^*}$ coincides with $\gamma$ on $K_v^*$, i.e., $p^{\gamma,K_v^*}\left(A_k^v\right)=\gamma\left(A_k^v\right)$ for all $A_k^v\in K_v^*$, while $p^\gamma\left(A_k^v\right)\geq\gamma\left(A_k^v\right)$ for all $A_k^v\in K_v^*$ and all $p^\gamma\in\mathcal{C}\left(\gamma\right)$. ■

The extended Choquet integral together with the chain integral of section 8 seem to suggest that maximal chains allow for a general technique for integrating monotone lattice (or, more generally) poset functions w.r.t. monotone measures. As known, there is no universal agreement on how to update ambiguous beliefs or, equivalently, on how to condition non-additive integrals (see Gilboa and Schmeidler (1993), Denneberg (2002) and Lehrer (2003)). Such an issue also applies, of course, to the above chain and Choquet-EXT (non-additive) integrals.

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