Temptation and Revealed Preference*

Jawwad Noor

June 13, 2004

Abstract

Self-control problems have typically been understood to express themselves through a preference for commitment. For instance, an addict may deal with his addiction by making drugs unavailable to himself in the future, say, via rehabilitation. We suggest, however, that self-control problems may in fact cause the *absence* of a preference for commitment: an addict may be tempted to continue his addiction, and thus, may postpone rehabilitation indefinitely. How do self-control problems express themselves in such decision-makers?

To address this question, we provide revealed preference foundations for a Gul and Pesendorfer [7, 8] style model – the primitive of the model is a choice correspondence, rather than a preference over choice problems as in [7, 8]. The center-piece of our analysis is the derivation of a ranking that represents a decision-maker's normative preference, that is, his view of what choices he *should* make. The derivation is based on preference reversals.

*PRELIMINARY AND INCOMPLETE. Department of Economics, University of Rochester, Rochester, NY 14627. Email: jwdb@troi.cc.rochester.edu. I am greatly indebted to Larry Epstein for his guidance and encouragement. I would also like to thank Alvin Roth, William Thomson and participants at the Canadian Economic Theory Conference 2004 for comments. Any errors are my responsibility.

1 Introduction

Research in psychology has documented a behavioral regularity called preference reversals. In a typical experiment, subjects prefer a small immediate reward to a large delayed reward, but reverse their preference in favor of the latter when both rewards are delayed by a common number of periods. For instance, subjects may prefer \$20 now to \$30 in one month, but may prefer \$30 in two months to \$20 in one month.[1] The preference for the inferior, but immediate reward has been interpreted in terms of a temptation by opportunities of immediate gratification.

The idea that decision-makers experience temptation and have self-control problems suggests that they may make choices that are, in their own view, not the best choices. That is, there is a distinction to be made between two ways that an agent may rank alternatives: one that reflects the choices he makes, and one that reflects his view of which choices he *should* make (call it the normative ranking). This raises two questions: first, how can one identify an agent who has self-control problems, and secondly, can normative preference of such an agent be elicited from observed choices?

1.1 The Commitment Approach

In [7, 8], Gul and Pesendorfer (henceforth GP) provide an axiomatic model that addresses these questions. Their main hypothesis is that a preference for commitment identifies the existence of a self-control problem.[2] The following example illustrates this 'commitment approach'.

Let d denote the choice of taking an addictive drug, n the choice of not taking the drug, and let a menu be represented by any nonempty subset of $\{d, n\}$. Consider an addict who finds his addiction to be against his best-interest. He is contemplating whether or not to go for rehabilitation. If he declines rehabilitation, he faces the menu $\{d, n\}$ tomorrow; otherwise he faces $\{n\}$.[3] He is aware of his self-control problem, that is, he knows that if he faces $\{d, n\}$, he will find it hard to resist the temptation of d and may

[1] See Ainslie [1] for a survey of such experiments.

[2] Also see Strotz [18], Laibson [13] and O'Donoghue and Rabin [16].

[3] That is, if he declines rehabilitation, he has a choice of whether or not to take the drug, whereas if he goes for rehabilitation, he commits himself to not taking the drug. Disulfiram treatment for alcoholics is an example of a treatment procedure that removes

even succumb to it. Hence, he prefers to go for rehabilitation, exhibiting the preference

$$\{n\} \succ \{d, n\}.$$

That is, self-control problems lead to a preference for commitment. The above ranking also suggests that the addict normatively prefers n over d.

However, to the extent that the addict is tempted by drugs, *he may also be tempted to continue drug usage*. This may be understood in terms of temptation by menus: temptation by d leads to temptation by the menu $\{d, n\}$ which gives the option to indulge temptations later. In such a case, if d (and so, $\{d, n\}$) is tempting enough, the addict's self-control problem leads him to decline rehabilitation and to exhibit the preference

$$\{d, n\} \succsim \{n\}. \tag{1}$$

Therefore, when menus are tempting, self-control problems may cause the *absence* of a preference for commitment. If one adopts the commitment approach, the ranking (1) would imply that the addict is not struggling with a self-control problem, and that he finds consuming d is in his best interest!

It appears, then, that the commitment approach is appropriate only for those agents who are not tempted by menus. Therefore, in order to provide answers to the earlier questions, *we need a means of identifying an agent who is tempted by menus, and of eliciting his normative preference.* The relevance for welfare policy is clear. Should addicts who do not seek treatment (that is, exhibit the ranking (1)) be given less access to drugs? That depends on whether (1) is coming from an addict who finds addiction to be in his best interest (a 'happy addict'), or one who is unable to resist the temptation to continue drug usage. If a consumer is not observed to commit part of his savings to a retirement fund or IRA (that is, he is not exhibiting a demand for commitment), are we to conclude that he experiences no temptation to overspend and thus has no use for commitment mechanisms, or could it be that he is too tempted to overspend to want to tie his hands just yet? Without the answer, we cannot know whether or not the consumer saves 'too little' for retirement, and thus, whether or not government intervention is called for.

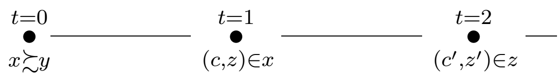
the option d from the menu; after taking disulfiram, ingesting even small quantities of alcohol leads to a severe reaction.

The purpose of this paper is to address these issues. We axiomatize two models, one of agents who are not tempted by menus, and one of those who are. By contrasting the axioms of the two models we identify the behavior that distinguishes the two types. The main step involves modeling the latter type.

1.2 A Model of Tempting Menus

Let Z denote the set of infinite horizon menus. A menu $x \in Z$ is a nonempty set of alternatives, where each alternative yields some present consumption $c \in C$ and a continuation menu $y \in Z$ to be faced in the next period.[4]

Take $\succsim$ to be a preference relation over Z, and consider the following time-line:



At time 0, the agent chooses a menu x. Then at time 1, he chooses an alternative (c, z) from x, which in turn yields immediate consumption c and a continuation menu z to be faced at time 2. At time 2 he chooses $(c', z') \in z$, and the process is repeated ad infinitum.

GP [8] adopt the preference $\succsim$ as a primitive and axiomatize Dynamic Self-Control (DSC) Preferences, which describe an agent who is tempted by immediate consumption, but not by menus. Future Temptation (FT) Preferences, axiomatized in [15], extend DSC preferences to allow for temptation by (continuation) menus in any period $t > 0$. However, temptation by menus in period 0 is excluded, and thus, period 0 is 'special' in that it is the period prior to the experience of temptation.

This is a problem for the FT model. Since the preference $\succsim$ is adopted as a primitive, the FT agent is characterized by restrictions on $\succsim$. That is, he is identified by restrictions on his behavior in the absence of temptation. But is this behavior observable? How do we tell if an agent is experiencing temptation in any given period? If, somehow, we know that he is experiencing temptation, is it possible to deduce how he would behave in the absence of

[4]The set Z may be viewed as homeomorphic to $\mathcal{K}(C \times Z)$, the set of nonempty subsets of $C \times Z$. This is simplified version of the set Z used in subsequent sections.

temptation? Without a means of identifying preference $\succsim$, it is not clear that the model's foundations are based on observables.

A solution to this problem is provided in this paper. We drop the period 0 preference $\succsim$ as a primitive, and instead, adopt a choice correspondence $\mathcal{C}$ defined over Z that summarizes choices in each period $t > 0$. That is, we characterize the FT model in terms of choices from menus instead of choices between menus. Specifically, we derive an FT preference $\succsim$ from $\mathcal{C}$ that, firstly, represents the ranking of menus in a *hypothetical* period 0 where no temptation is experienced, and secondly, generates $\mathcal{C}$ in a sense made precise in Section 2. This may be viewed as a non-standard revealed preference model. In standard revealed preference theory, choices over alternatives are generated by a preference over alternatives, while here choices over alternatives are, in a sense to be made precise, generated by a preference over sets of alternatives (menus).

It merits emphasis that choices captured by $\mathcal{C}$ are subject to temptation, whereas the preference $\succsim$ derived from $\mathcal{C}$ represents choices that would be made in the absence of temptation. The derivation of such a temptation-free preference is accomplished via preference reversals.

1.3 The Preference Reversal Approach

The opening example of a preference reversal suggests that the temptation of the earlier reward could not be resisted when subjects chose between \$20 now and \$30 in a month, but it became possible to resist when both rewards were delayed sufficiently. Hence, preference reversals lend themselves to the observation that *delayed temptations are easier to resist than immediate temptations.*

This observation suggests a way of deriving the hypothetical period 0 preference $\succsim$ from $\mathcal{C}$. For any menu x, let x^{+t} denote the menu in Z that gives fixed consumption c for t periods, and x in period $(t+1)$. Consider menus of the form $x^{+t} \cup y^{+t}$ that offer a choice between receiving x after t periods and receiving y after t periods. For t that is sufficiently large to induce a preference reversal, $\mathcal{C}(x^{+t} \cup y^{+t})$ reveals the agent's ranking of x and y after any possible temptation by the menus is resisted. This ranking, by construction, represents choice between menus in the absence of temptation, and thus uncovers $\succsim$.[5]

To connect with earlier discussion, we note that preference reversals provide a way to elicit normative preference from observed choices. The observed ranking of two alternatives a and b differs from the normative ranking only when the agent submits to temptations. Hence the ranking of the alternatives in the absence of temptation reflects the normative ranking. Since the post-preference reversal ranking of a and b is the ranking observed after temptation is resisted, it follows that this ranking uncovers normative preference. Indeed, the preference $\succsim$ derived above is the agent's normative preference over menus.

1.4 Outline of the Paper

Section 2 provides formal details of our model of tempting menus and elucidates the discussion in Section 1.2 regarding the problem associated with taking preference over menus as a primitive. Section 3 presents axioms and the representation result. Section 4 presents a behavioral definition of normative preference based on preference reversals, and clarifies how this definition is used to derive the preference $\succsim$ from $\mathcal{C}$. This section answers one of the two questions raised in this introduction: how can we elicit the normative preference of an agent who is tempted by menus.

Section 5 answers the other question: how can we distinguish between an agent who is tempted by menus, and one who is not. Still adopting $\mathcal{C}$ as a primitive, we axiomatize a model where menus are not tempting. By contrasting the axioms of this model and our model of tempting menus, we provide a test that distinguishes between the agents under consideration. The behavioral expression of a temptation by menus is a preference for early choice.

Section 6 summarizes the answers to the questions asked in the Introduction, and cites evidence in support of a preference for early choice.

Section 7 discusses in detail our definition of normative preference. It provides sufficient conditions for its existence, contrasts the definition with

[5]This is not a perfectly accurate description of $\succsim$. The actual construction involves taking the limit of a sequence of preferences relations.

that of GP, inquires into how normative preference can be approximated in practice and argues that in suitable settings, the definition captures the notion of an agent's ethics.

All proofs are collected in appendices.

2 The Model

For any compact metric space X, $\Delta(X)$ denotes the set of all probability measures on the Borel σ-algebra of X, endowed with the weak convergence topology; $\Delta(X)$ is compact and metrizable [17]. Let $\mathcal{K}(X)$ denote the set of all nonempty compact subsets of X. When endowed with the Hausdorff topology, $\mathcal{K}(X)$ is a compact metric space [4, p. 222].

The set C is a compact metric space that contains possible consumption levels. The set of choice problems is Z. Each choice problem $z \in Z$ is a compact set of lotteries, where each lottery is a measure over current consumption and a continuation menu. Thus Z can be identified with $\mathcal{K}(\Delta(C \times Z))$. See [8] for the formal definition of Z and the homeomorphism between Z and $\mathcal{K}(\Delta(C \times Z))$. In particular, Z is compact metric.

Future Temptation Preferences

In [15], we study a GP-style model that allows for tempting menus. The model adopts a binary relation $\succsim$ over Z as a primitive, and axiomatizes *Future Temptation (FT) preferences.* Say that $\succsim$ is an FT preference if it has a representation $W : Z \longrightarrow \mathbb{R}$ of the following form: there exist δ and γ, $0 < \gamma < \delta < 1$, and continuous functions $u, v : C \longrightarrow \mathbb{R}$, $U, V : \Delta(C \times Z) \longrightarrow \mathbb{R}$ and $\overline{V} : Z \longrightarrow \mathbb{R}$ such that for all $x \in Z$,

$$W(x) = \max_{\mu \in x} \{U(\mu) + V(\mu) - \max_{\eta \in x} V(\eta)\}, \tag{2}$$

where

$$\begin{aligned}
U(\mu) &= \int_{C \times Z} (u(c) + \delta W(y))\, d\mu(c, y), \\
V(\mu) &= \int_{C \times Z} \left(v(c) + \gamma \overline{V}(y)\right) d\mu(c, y), \\
\overline{V}(x) &= \max_{\eta \in x} V(\eta).
\end{aligned}$$

To understand the representation, begin by noting that $W(\{\mu\}) = U(\mu)$ for any singleton menu $\{\mu\}$. Thus U captures the agent's utility under commitment. Anticipating results in Section 4, we interpret U as a representation of the agent's normative preference, and thus refer to it as normative utility.[6] Interpreting V as temptation utility, the non-positive term $(V(\mu) - \max_{\eta \in z} V(\eta))$ can be understood as the cost of self-control, that is, the cost incurred when the most tempting item in z is not chosen. Hence, (2) states that the utility W of a menu x is the maximum value of normative utility net of self-control cost.

According to the functional form for U, normative utility depends on utility from current consumption, and the utility W from a continuation menu discounted by δ. Temptation utility V depends on current consumption and the temptation value $\overline{V}$ of continuation menus discounted by γ. The restriction $\gamma < \delta$ embodies the property that it is easier to resist a temptation when it is pushed into the future. As the form of $\overline{V}$ shows, continuation menus are as tempting as the most tempting item contained in them. The functional form with $\gamma = 0$ is a representation of GP's Dynamic Self-Control Preferences [8], which exhibit temptation by immediate consumption only.

Recall the time-line in the Introduction. The preference $\succsim$ is defined over menus, and thus describes how the agent would choose *between* menus in some period 0. But the representation is also suggestive of how he would choose *from* a menu in subsequent periods: The term '$\max_{\eta \in x} V(\eta)$' in (2) is a constant when x is given, and so, when maximizing over μ in x, he maximizes

$$U(\cdot) + V(\cdot).$$

Thus, the representation suggests that choice from menus is captured by the choice correspondence $\mathcal{C}(\cdot, \succsim)$ over Z defined by

$$\mathcal{C}(x, \succsim) = \arg\max_{\mu \in x} \{U(\mu) + V(\mu)\}. \tag{3}$$

That is, his choice from any given menu is a compromise between normative utility and temptation utility.

Though $\mathcal{C}(\cdot, \succsim)$ is defined in terms of one representation of $\succsim$, $\mathcal{C}(\cdot, \succsim)$ does

[6]GP adopt the more neutral term 'commitment utility'.

not depend on the particular representation.[7] That is, a given FT preference $\succsim$ generates a unique choice correspondence $\mathcal{C}(\cdot, \succsim)$.

A Question of Foundations

The preference $\succsim$ represents choice in the absence of temptation. To see this, observe that a period 0 choice between menus x and y is determined by solving

$$\max_{\{x,y\}} \{W(\cdot)\}. \tag{4}$$

A parallel choice problem arises in any period $t > 0$ if the agent chooses from the menu

$$\{(c, x), (c, y)\}.$$

Since no choice of current consumption is involved, the choice from this menu may be viewed as a period $t > 0$ choice between menus x and y. The choice between the menus is determined by $\mathcal{C}(\{(c, x), (c, y)\}, \succsim)$, that is, by solving

$$\max_{\{(c,x),(c,y)\}} \{u(\cdot) + \delta W(\cdot) + v(\cdot) + \gamma \overline{V}(\cdot)\}.$$

Ignoring current consumption, the problem becomes

$$\max_{\{x,y\}} \{W(\cdot) + \frac{\gamma}{\delta} \overline{V}(\cdot)\}. \tag{5}$$

Compare problems (4) and (5) and conclude that, in general, period 0 choice between x and y is different from the corresponding choice in any period $t > 0$. That is, period 0 is special. Since $\overline{V}$ captures temptation utility from menus, it is evident that period 0 choice between menus is not subject to temptation by menus.

The characterization of the FT model offered in [15] therefore involves restrictions on the ranking of menus in the absence of temptation – in order to verify that an agent has FT preferences, one needs to obtain this ranking. But can this be done? If there exists some period 0, how can it be identified,

[7] To see this, call (U, V) a representation of $\succsim$. If $\succsim$ exhibits a preference for commitment ($x \succ x \cup y$ for some x, y) then, by [7, Thm 4], (U', V') is another representation of $\succsim$ if and only if there exists $\alpha > 0$ and β_U, β_V such that $U' = \alpha U + \beta_U$ and $V' = \alpha V + \beta_V$. If $\succsim$ does not exhibit a preference for commitment, then without loss of generality, $V = 0$, and U is unique up to an affine transformation. In either case, any representation of $\succsim$ would give rise to the same $\mathcal{C}(\cdot, \succsim)$.

that is, how do we conclude that an agent is not experiencing temptation? Even if we could identify period 0, an agent who is tempted by menus will, in general, experience temptation in all periods. In such a scenario, can we deduce how he *would* behave in the absence of temptation? Indeed, it is not clear that the FT model's primitive preference $\succsim$ is observable.[8]

Alternative Foundations

We provide an alternative characterization of FT preferences that is in terms of restrictions on period $t > 0$ choices *from* menus, instead of period 0 choices *between* menus. Let these choices be summarized by a choice correspondence $\mathcal{C}(\cdot)$ over Z. For any choice correspondence $\mathcal{C}(\cdot)$ and any FT preference $\succsim$, say that $\succsim$ *generates* $\mathcal{C}(\cdot)$ if for all $x \in Z$,

$$\mathcal{C}(x) = \mathcal{C}(x, \succsim).$$

Our problem is to find restrictions on $\mathcal{C}(\cdot)$ that imply the existence of an FT preference $\succsim$ that generates $\mathcal{C}$. That is, under what conditions can we say that, in a hypothetical period 0 where no temptation is experienced, the agent's ranking of menus $\succsim$ is an FT preference? The problem is complicated by the fact that our data, $\mathcal{C}$, consists of choices that are subject to temptation, whereas we want to derive from $\mathcal{C}$ the agent's choices in the absence of temptation.

In the next section, we define a class of choice correspondences such that, for any $\mathcal{C}$ in this class, there exists a unique FT preference $\succsim$ that generates $\mathcal{C}$. Thus, any agent whose choice correspondence belongs to this class may be regarded as an FT agent. The converse also holds – any choice correspondence $\mathcal{C}(\cdot, \succsim)$ generated by an FT preference $\succsim$ belongs to this class.

3 Axioms and Representation Result

To ease notation, let Δ denote $\Delta(C \times Z)$. Generic elements of Z are x, y, z whereas generic elements of Δ are μ, η, ν. For $\alpha \in [0, 1]$, $\alpha\mu + (1-\alpha)\eta \in \Delta$ is the measure that assigns $\alpha\mu(A) + (1-\alpha)\eta(A)$ to each A in the Borel σ-algebra of $C \times Z$. Similarly, $\alpha x + (1-\alpha)y \equiv \{\alpha\mu + (1-\alpha)\eta : \mu \in x, \eta \in y\} \in Z$ is a

[8] GP note that DSC preferences do not suffer from this problem of non-observability. When only immediate consumption is tempting ($\gamma = 0$), problems (4) and (5) are identical, and hence, any period serves as period 0.

mixture of the choice problems x and y. Denote these mixtures more simply by $\mu\alpha\nu$ and $x\alpha y$ respectively.

The primitive of the model is a closed-valued choice correspondence $\mathcal{C} : Z \rightsquigarrow \Delta(C \times Z)$ such that for all $x \in Z$, $\mathcal{C}(x) \neq \phi$ and $\mathcal{C}(x) \subset x$. It represents the choices the agent would make out of menus at any point in time. We introduce some notation to aid exposition:

- Fix $\bar{c} \in C$ throughout. For any x, define $x^{+1} \equiv (\bar{c}, x)$ and inductively for $t > 1$, $x^{+t} = (\bar{c}, x^{+(t-1)})$. Then $x^{+t} \in \Delta$ is the alternative that yields menu x after $t > 0$ periods, and $\bar{c}$ in all periods between time 0 and t. We write $\{\mu\}^{+t}$ as μ^{+t} and identify μ^{+0} with μ.

- The option that gives $x \cup y$ (resp. $x\alpha y$), after t periods is denoted $(x \cup y)^{+t}$ (resp. $(x\alpha y)^{+t}$).

- Let $\gtrapprox$ denote the revealed preference relation (defined on Δ) that is generated by choices from binary menus, that is,

$$\begin{aligned} \mu &\gtrapprox \eta \iff \mu \in \mathcal{C}(\{\mu, \eta\}) \\ \mu &\approx \eta \iff \{\mu, \eta\} = \mathcal{C}(\{\mu, \eta\}) \\ \mu &> \eta \iff \{\mu\} = \mathcal{C}(\{\mu, \eta\}). \end{aligned}$$

Consider the following axioms on $\mathcal{C}(\cdot)$. The quantifiers 'for all $\mu, \eta \in \Delta$, $x, y \in Z$, $c, c', c'' \in C$, and $\alpha \in [0, 1]$' should be understood.

Axiom 1 *(WARP)* *If $\mu, \eta \in x \cap y$, $\mu \in \mathcal{C}(x)$ and $\eta \in \mathcal{C}(y)$, then $\mu \in \mathcal{C}(y)$.*

This is the standard Weak Axiom of Revealed Preference. It imposes a minimal consistency requirement on choices.

Axiom 2 *(Continuity)* *$\mathcal{C}(\cdot)$ is upper hemicontinuous.*

Upper hemicontinuity of $\mathcal{C}$ suggests that $\gtrapprox$ is continuous.[9] Indeed, upper hemicontinuity results from the maximization of a continuous preference over compact sets.

[9]Formally, upper hemicontinuity implies that if $\{x_n\}$ is a sequence of menus converging to x, and $\mu_n \in \mathcal{C}(\{x_n\})$ for each n, then the sequence $\{\mu_n\}$ has a limit point in $\mathcal{C}(\{x\})$.

Axiom 3 *(Independence)* $\quad \mu > \eta \Longrightarrow \mu\alpha\nu > \eta\alpha\nu.$

This is the familiar Independence axiom.

Axiom 4 *(Separability)* $\quad$ *For all* $t \geq 0$*,*

$$(\frac{1}{2}(c,x) + \frac{1}{2}(c',x'))^{+t} \approx (\frac{1}{2}(c,x') + \frac{1}{2}(c',x))^{+t}.$$

Separability states that when comparing two lotteries (delayed by the same number of periods), the agent only cares about the marginal distributions on C and Z induced by the lotteries. That is, only marginals matter, and correlations between consumption and continuation menus do not affect the agent's choices.

Axiom 5 *(Indifference to Timing)* $\quad$ *For all* $t > 0$,

$$x^{+t}\alpha y^{+t} \approx (x\alpha y)^{+t}.$$

Under both rewards $x^{+t}\alpha y^{+t}$ and $(x\alpha y)^{+t}$, the agent faces x after t periods with probability α and y after t periods with probability $(1-\alpha)$. However, under $x^{+t}\alpha y^{+t}$, the uncertainty will be resolved today, whereas under $(x\alpha y)^{+t}$, the uncertainty will be resolved after t periods. That is, the two rewards differ only in the timing of resolution of uncertainty. Indifference between the rewards suggests indifference to the timing of resolution of uncertainty.

Axiom 6 *(Set-Betweenness)* $\quad$ *For all* $t > 0$*,*

$$x^{+t} \gtrapprox y^{+t} \Longrightarrow x^{+t} \gtrapprox (x \cup y)^{+t} \gtrapprox y^{+t}.$$

Set-Betweenness expresses the idea that the agent anticipates experiencing temptation in menus and may exert self-control when making choices from menus. To illustrate, fix some delay $t > 0$, and consider $\mu, \eta \in \Delta$ be such that $\{\mu\}^{+t} > \{\mu,\eta\}^{+t}$. The preference for commitment to μ reveals that η is tempting. When $\{\mu,\eta\}^{+t} \approx \{\eta\}^{+t}$ holds, the indifference suggests that the agent would choose the same item whether faced with $\{\mu,\eta\}$ or $\{\eta\}$. That is, choice from $\{\mu,\eta\}$ is η and so, the agent succumbs to temptation. On the other hand, the ranking $\{\mu,\eta\}^{+t} > \{\eta\}^{+t}$ suggests that μ is chosen from $\{\mu,\eta\}$ and so, the agent resists temptation, that is, he exerts self-control.

GP would interpret the ranking $\{\mu\}^{+t} \approx \{\mu, \eta\}^{+t} \gtrapprox \{\eta\}^{+t}$ as saying that no temptation is experienced in the menu $\{\mu, \eta\}$. However, this ranking permits a weaker interpretation: it is *also* consistent with an overwhelming temptation by μ. The reward μ is so tempting that he prefers $\{\mu, \eta\}^{+t}$ over $\{\eta\}^{+t}$, that is, he submits to the temptation of the *menu* $\{\mu, \eta\}$ that contains the tempting reward μ. The indifference between $\{\mu\}^{+t}$ and $\{\mu, \eta\}^{+t}$ is another expression of the overwhelming temptation by μ – he foresees that he will choose μ in either menu, and so he is indifferent between them.

Axiom 7 *(Sophistication)* *Suppose $\mu^{+t} > \eta^{+t}$ for $t > 0$. Then,*

$$\{\mu, \eta\}^{+t} \approx \{\eta\}^{+t} \Longleftrightarrow \eta \gtrapprox \mu.$$

As the name suggests, this axiom connects the agent's expectation of his future choices with his actual choices. The indifference between facing $\{\mu, \eta\}$ or $\{\eta\}$ after t periods suggests that the agent expects himself to choose η if, after t periods, he faces the menu $\{\mu, \eta\}$. The actual choice from $\{\mu, \eta\}$ after t periods is given by $\mathcal{C}(\cdot)$, since $\mathcal{C}(\cdot)$ is time-invariant. The axiom says that the agent's expected choice and actual choice coincide.

Axiom 8 *(Preference for Early Choice)* *For all $t > 0$,*

$$x^{+t} > (x \cup y)^{+t} \Longleftrightarrow \{(c, x)\}^{+t} > \{(c, x), (c, y)\}^{+t}.$$

Preference for Early Choice indicates that the presence of temptations in a menu make it tempting. The preference for commitment exhibited by the left-hand-side ranking implies that the menu y contains temptations. The right-hand-side ranking states that y is a tempting menu. Thus, the axiom states that y is tempting if and only if y contains tempting items.

The name of the axiom comes from the following observation about the right-hand-side ranking. Under $\{(c, x), (c, y)\}^{+t}$, the agent has the opportunity to decide after t periods whether to face x in the $(t + 1)^{th}$ period or y. Under $\{(c, x)\}^{+t}$, the agent has no choice but to facing x in the $(t + 1)^{th}$ period. Therefore right-hand-side ranking indicates that the agent would rather not leave himself with the flexibility offered by $\{(c, x), (c, y)\}^{+t}$. That is, he prefers to make an early decision about whether or not to face x in the $(t + 1)^{th}$ period. Hence the name.[10]

[10]Note that the axiom says more: such a preference arises if and only if y contains tempting items.

Axiom 9 *(Reversal)* *If* $\mu \lesssim \eta$ *and* $\mu^{+T} \gtrsim \eta^{+T}$ *with at least one ranking strict, then* $\mu^{+t} > \eta^{+t}$ *for all* $t > T$.

This axiom imposes the structure of preference reversals on $\mathcal{C}$. That is, if pushing a pair of rewards into the future changes its ranking, then the reversed ranking is maintained for all subsequent delays in the rewards. Following the evidence on preference reversals, the axiom allows no more than one reversal for any pair of rewards. However, the axiom can be weakened to allow for at most finite number of reversals for any pair of rewards without altering any results.

As a simple consequence of Reversal we obtain a function $\tau : \Delta \times \Delta \to \mathbb{R}$ such that for any pair of rewards μ, η, $\tau(\mu, \eta)$ is the number of periods that μ and η need to be delayed before a preference reversal is observed; if no reversal is observed, then $\tau(\mu, \eta) = 0$.[11]

Axiom 10 *(Commitment is Normative)* *Suppose* $x^{+t} > (x \cup y)^{+t}$. *Then*

$$\tau(x^{+t}, (x \cup y)^{+t}) = 0 \text{ and } \tau \text{ is continuous at } (x^{+t}, (x \cup y)^{+t}).$$

If, in addition, $(x \cup y)^{+t} > y^{+t}$, *then*

$$\tau((x \cup y)^{+t}, y^{+t}) = 0 \text{ and } \tau \text{ is continuous at } ((x \cup y)^{+t}, y^{+t}).$$

Before interpreting the axiom, note that the ranking $x^{+t} > (x \cup y)^{+t}$ not only reveals that y is a menu that contains temptations, but also that y is not an overwhelmingly tempting menu; if it were, we would have observed $(x \cup y)^{+t} \gtrsim x^{+t}$. Hence a t period delay makes the temptation by y is resistible. Consequently, and this is what the first part of the axiom essentially states, the ranking between x^{+t} and $(x \cup y)^{+t}$ is temptation-free. Temptation-freeness is expressed behaviorally in terms of the function τ, as explained below.

[11]Formally, for each μ, η there exists $\tau(\mu, \eta) \geq 0$ such that if $\tau(\mu, \eta) = 0$, then for all t,

$$\mu^{+\tau(\mu,\eta)} \gtrsim \eta^{+\tau(\mu,\eta)} \iff \mu^{+t} \gtrsim \eta^{+t},$$

and if $\tau(\mu, \eta) > 0$ and $\mu \gtrsim \eta$, then

$$\mu >_t \eta \text{ for all } t \geq \tau(\mu, \eta) \text{ and } \mu \lesssim_t \eta \text{ for all } t < \tau(\mu, \eta).$$

See Lemma 1, Appendix B.

As argued in Section 1.3, reversals occur only when the agent experiences overwhelming temptation. Since the ranking between x^{+t} and $(x \cup y)^{+t}$ is temptation-free, pushing both alternatives into the future should not change the ranking, that is, $\tau(x^{+t}, (x \cup y)^{+t}) = 0$.

Furthermore, the temptation-freeness of the ranking of x^{+t} and $(x \cup y)^{+t}$ suggests that the ranking of neighboring pairs of rewards should not be 'significantly' influenced by temptation. In particular, if a preference reversal is observed for neighboring rewards, the time of the reversal should be 'small' – the smaller a temptation, the smaller the delay required in order to resist it. This can be expressed thus: if a pair of rewards (μ', η') is close to (μ, η), then the time of the reversal $\tau(\mu', \eta')$ is close to $\tau(x^{+t}, (x \cup y)^{+t}) = 0$. This completes our explanation of the first part of the axiom.

The second part has the exact same justification once it is noted that if a t period delay makes the temptation by y resistible, then it should also make the temptation by $x \cup y$ resistible since $x \cup y$ is tempting only to the extent that it contains y. That is, $x^{+t} > (x \cup y)^{+t}$ implies that the ranking between $(x \cup y)^{+t}$ and y^{+t} is temptation-free.

Finally, $\mathcal{C}(\cdot)$ is said to be *nondegenerate* if there exist menus x and y such that $y^{+1} \gtrapprox x^{+1}$ and for some t,

$$x^{+t} > (x \cup y)^{+t} > y^{+t}.$$

Nondegeneracy implies that the agent experiences temptation and exerts self-control in some menu $x \cup y$. Since $y^{+1} \gtrapprox x^{+1}$ and $x^{+t} > y^{+t}$, nondegeneracy also implies that $\mathcal{C}(\cdot)$ exhibits a preference reversal for the pair x^{+1}, y^{+1}.

Before stating the representation theorem, recall from Section 2 that an FT preference $\succsim$ *generates* $\mathcal{C}(\cdot)$ if for all $x \in Z$,

$$\mathcal{C}(x) = \mathcal{C}(x, \succsim), \tag{6}$$

where $\mathcal{C}(\cdot, \succsim)$ is defined by (3). Also, say that $\succsim$ is *nondegenerate* if there exists x and y such that

$$x \succ x \cup y \succ y.$$

Theorem 1 *If the nondegenerate choice correspondence $\mathcal{C}(\cdot)$ satisfies Axioms 1-10, then there exists a unique nondegenerate FT preference $\succsim$ that generates $\mathcal{C}(\cdot)$. Conversely, if $\mathcal{C}(\cdot)$ is generated by an FT preference $\succsim$, then $\mathcal{C}(\cdot)$ satisfies Axioms 1-10.*

Thus, an agent whose choice correspondence satisfies Axioms 1-10 can be viewed as an FT agent. Conversely, the choices of an FT agent satisfy Axioms 1-10.

4 Normative Preference and $\succsim$

Turn to the construction of the FT preference $\succsim$ in Theorem 1. The construction is based on a behavioral definition of normative preference. We will also show that this definition gives us a way of justifying our interpretation in Section 2 of the function U as normative utility.

The choice correspondence $\mathcal{C}(\cdot)$ yields a set of preference relations $\{\gtrapprox_t\}_{t=0}^{\infty}$ defined over Δ, where for each $t \geq 0$ and $\mu, \eta \in \Delta$,

$$\mu \gtrapprox_t \eta \Longleftrightarrow \mu^{+t} \in \mathcal{C}(\{\mu^{+t}, \eta^{+t}\}).$$

That is, for any $\mu, \eta \in \Delta$, the preference $\gtrapprox_t$ ranks μ and η when both rewards are to be received t periods later. Thus, $\{\gtrapprox_t\}$ captures how the agent's current period preference over Δ changes as Δ is pushed into the future, so to speak.

By the Reversal axiom, $\mathcal{C}(\cdot)$ exhibits preference reversals. As observed earlier, preference reversals are a manifestation of the ability to resist delayed temptations. Thus, as t increases, the influence of temptation on choice diminishes and $\gtrapprox_t$ provides an increasingly better approximation of the agent's underlying *normative preference* $\succsim^*$ over Δ.[12] For this reason, it is intuitive that $\succsim^*$ be identified with the 'limit' of the sequence $\{\gtrapprox_t\}$.

To be formal, say that a binary relation B on Δ is nonempty if $\mu B \eta$ for some $\mu, \eta \in \Delta$. Adapting Kannai [10], identify any nonempty continuous binary relation on Δ with its graph, a nonempty compact subset of $\Delta \times \Delta$. Thus, the space of nonempty continuous preferences on Δ can be identified with $\mathcal{P} = \mathcal{K}(\Delta \times \Delta)$, the space of nonempty compact subsets of $\Delta \times \Delta$ endowed with the Hausdorff metric topology. See Appendix A for details about the topology.

[12] Note that temptation is what causes choice to deviate from normative preference, and hence, choice in the absence of temptation reveals normative preference.

Definition 1 *The normative preference $\succsim^*$ over Δ is the limit of the sequence $\{\gtrsim_t\}$ in $\mathcal{P}$.*

Normative preference plays a fundamental role in the construction of the temptation-free preference $\succsim$ over Z in Theorem 1. Since normative preference is essentially the temptation-free ranking of Δ, the preference $\succsim$ is simply the normative preference over Z defined by

$$x \succsim y \Longleftrightarrow x^{+1} \succsim^* y^{+1},$$

for all x and y. Theorem 1 establishes that under Axioms 1-10 imposed on $\mathcal{C}(\cdot)$, the normative preference $\succsim^*$, and hence the preference $\succsim$, is well-defined. Furthermore, the Theorem establishes that $\succsim$ is in fact the unique FT preference that generates $\mathcal{C}(\cdot)$.

We interpreted the function U in the representation of FT preference as normative utility (Section 2). Next, we provide justification for this interpretation.

For any FT preference $\succsim$, a normative preference $\succsim^*$ is said to be elicited from $\succsim$ if it is derived as above.[13] Call (U, V) a representation of the FT preference $\succsim$ if U and V have the functional form given by (2). Theorem 2 establishes that U is a representation of the normative preference $\succsim^*$ elicited from $\succsim$.

Theorem 2 *If (U, V) represents a nondegenerate FT preference $\succsim$ and the normative preference $\succsim^*$ is elicited from $\succsim$, then U is a representation of $\succsim^*$.*

5 No Temptation by Menus

Having modelled an agent who is tempted by menus, we now model one who is not tempted by menus.

As noted in Section 3, temptation by menus is captured by Preference for Early Choice. Observe that the axiom permits,

$$\{(c, x)\}^{+t} > \{(c, x), (c, y)\}^{+t},$$

[13] First, the choice correspondence $\mathcal{C}(\cdot, \succsim)$ is derived from $\succsim$ by defining it as in (3). Then, a set of preference relations $\{\gtrsim_t\}_{t=0}^{\infty}$ defined over Δ is obtained from $\mathcal{C}(\cdot, \succsim)$, where for each $t \geq 0$ and $\mu, \eta \in \Delta$, $\mu \gtrsim_t \eta \Longleftrightarrow \mu^{+t} \in \mathcal{C}(\{\mu^{+t}, \eta^{+t}\}, \succsim)$. Finally, $\succsim^*$ is defined as the limit of the sequence $\{\gtrsim_t\}$, as in Definition 1.

that is, committing now to receiving x in $t+1$ periods is preferred to leaving the decision open till the t^{th} period. The following axiom rules this out.

Axiom 11 *(Menus Do Not Tempt)* *For all* $t > 0$,

$$\{(c,x)\}^{+t} \gtrapprox \{(c,y)\}^{+t} \Longrightarrow \{(c,x)\}^{+t} \approx \{(c,x),(c,y)\}^{+t}.$$

That is, there is no benefit of making earlier decisions about menus which are to be faced in the future. This is to be expected from agents who are not tempted by menus. We explore the consequences for the representation of replacing Preference for Early Choice with Menus Do Not Tempt.

Let $\succsim^{CT}$ be a binary relation over Z that is represented by some function $W^{CT} : Z \longrightarrow R$. Say that $\succsim^{CT}$ is a *Current Temptation (CT) preference* if there exists $\delta \in (0,1)$, $\gamma \geq 0$ and continuous functions $u, v : C \longrightarrow R$ such that for all $x \in Z$,[14]

$$W^{CT}(x) = \max_{\mu \in x}\{U(\mu) + V(\mu) - \max_{\eta \in x} V(\eta)\},$$

where

$$\begin{aligned} U(\mu) &= \int_{C\times Z} \big(u(c) + \delta W^{CT}(y)\big)\, d\mu(c,y) \qquad (7) \\ V(\mu) &= \int_{C\times Z} \big(v(c) + \gamma W^{CT}(y)\big)\, d\mu(c,y). \end{aligned}$$

CT preferences differ from FT preferences in two ways. First, here γ may be zero, yielding GP's Dynamic Self-Control preferences where current consumption is the only source of temptation utility. Second, if $\gamma > 0$, then the temptation utility of a continuation menu is next-period utility from the menu, as opposed to the maximum temptation utility in the menu for FT preferences. Note that for CT preferences, if $\gamma > 0$, then commitment and temptation utility never disagree when it comes to choice between continuation menus, and hence, continuation menus do not tempt.

As before, say that $\succsim^{CT}$ *generates* $\mathcal{C}(\cdot)$ if

$$\mathcal{C}(x) = \arg\max_{\mu \in x}\{U(\mu) + V(\mu)\}.$$

Theorem 3 is the main result of this section.

[14]Need to be sure that W is well-defined for any given u, v, δ, γ. Need some fixed point theorem...Unique W not needed....

Theorem 3 *If the nondegenerate choice correspondence $\mathcal{C}(\cdot)$ satisfies Axioms 1-5,7-11, then there exists a nondegenerate CT preference $\succsim^{CT}$ that generates $\mathcal{C}(\cdot)$. Conversely, if $\mathcal{C}(\cdot)$ is generated by a CT preference $\succsim^{CT}$, then $\mathcal{C}(\cdot)$ satisfies Axioms 1-5,7-11.*

Theorem 3 tells us that the preferences studied by GP [8] (where $\gamma = 0$) and those studied by Krussel, Kurusçu and Smith [12] (where $\gamma > 0$) are models of agents who are not tempted by menus, and furthermore, these are the only models of this type.

Next we provide the counterpart of Theorem 2 for CT preferences. Call (U, V) a representation of the CT preference $\succsim^{CT}$, where U and V have the above functional forms. The definition of 'normative preference $\succsim^*$ is elicited from $\succsim^{CT}$' is analogous to the counterpart in the previous section.

Theorem 4 *If (U, V) represents a nondegenerate CT preference $\succsim^{CT}$ and the normative preference $\succsim^*$ is elicited from $\succsim^{CT}$, then U is a representation of $\succsim^*$.*

6 The Addict Revisited

In the Introduction we asked two questions: first, how can we distinguish an addict who finds taking drugs to be in his best interest (a happy addict) from one who thinks otherwise but cannot resist the temptation to take drugs (an unhappy addict), and second, how can the normative preference of the latter be elicited. We answer these questions by summarizing and discussing results from preceding sections.

6.1 Behavioral Distinction

One way to try distinguishing the two is through their respective demands for commitment mechanisms. But as argued in the Introduction, in the presence of a temptation by menus, the demand for commitment cannot distinguish a happy and unhappy addict. To recall the argument, consider the following table. Say that an unhappy addict is of type CT if he is not tempted by menus, and of type FT if he is.

	happy addict	unhappy CT addict	unhappy FT addict
commits	no	yes	maybe

A happy addict has no use for commitment mechanisms, whereas an unhappy addict does. However, while a CT addict demands commitment, an FT addict may be tempted not to commit, and so may not demand commitment. Hence, an addict who does not commit may either be a happy addict, or an FT addict. Conclude that a happy and unhappy addict cannot be distinguished by observing their demand for commitment.

The problem is solved if we can distinguish between a CT and FT addict. As noted in the previous section, an agent who is tempted by menus prefers to make early choices in order to deal with temptation, that is, he exhibits the following ranking for some x, y and t:

$$\{(c, x)\}^{+t} > \{(c, x), (c, y)\}^{+t}.$$

The absence of such behavior implies the absence of temptation by menus. Hence, a preference for early choice separates an FT addict from a CT addict. The following table summarizes the discussion.

	happy addict	unhappy CT addict	unhappy FT addict
commits	no	yes	maybe
prefers early choice	no	no	yes

Having given behavioral meaning to the term 'temptation by menus', we can ask if there is any evidence of such temptation. That is, is there any evidence of a preference for early choice? The results of Benartzi and Thaler [3] provide support. Benartzi and Thaler introduce a saving-enhancement plan, called the 'Save More Tomorrow' (SMT) plan. Subjects in a firm are given the opportunity to commit in advance to allocating a portion of their future salary increases towards retirement savings. Observe that opting for this plan implies a preference for early choice since subjects would rather not leave the allocation decision for when the salary increase actually takes place.

The authors implemented the plan in several firms, and found a significant demand for it. In one implementation, 27% (216 of 816) of the firm' employees opted to use the SMT plan. Of those employees who actually attended a seminar on the plan, 46% (196 of 426) opted to use it. In another implementation, the authors report that 78% of subjects (162 of 207) who were offered the plan opted to use it. Of those who participated in the plan, 80% remained in the plan through four pay raises. Furthermore, average saving rates for the participants increased almost four-fold (from 3.5% to 13.6%) over the course of 40 months.

6.2 Eliciting Normative Preference

Now we come to the question of how the normative preference of unhappy addicts may be elicited. Since FT preferences model an FT addict and CT preferences model a CT addict, Theorems 2 and 4 tell us that normative preference $\succsim^*$ of each type can be elicited by taking the limit of the ranking of alternatives as the delay with which the alternatives are to be received goes to infinity (Definition 1). See Section 7.5 for more on how to elicit $\succsim^*$.

An alternative means of obtaining information about $\succsim^*$ is through a preference for commitment or preference for early choice. For instance, consider an addict's choice of whether or not to enter rehabilitation. Entering rehabilitation leads the addict to face the menu $\{n\}$ in the next period and declining it leads to $\{n, d\}$, where n denotes 'no drugs' and d denotes 'drugs'. A CT addict exhibits a preference for entering rehabilitation immediately:

$$\{n\}^{+1} > \{n, d\}^{+1},$$

revealing $n \succ^* d$. That is, a preference for rehabilitation reveals that he finds n better for his welfare. An FT addict may not prefer rehabilitation immediately due to a temptation to postpone rehabilitation. However, he prefers to commit in advance to entering rehabilitation:

$$\{\{n\}^{+1}\}^{+t} > \{\{n\}^{+1}, \{n, d\}^{+1}\}^{+t},$$

for some large t. This commitment to entering rehabilitation in the future reveals that he finds n preferable to d in terms of best-interest, that is $n \succ^* d$.

7 Discussion: Normative Preference

This section presents and discusses an abstract definition of normative preference that is independent of the setting of the models in this paper. We provide sufficient conditions for the existence of normative preference, contrast our definition with that of GP, inquire how normative preference can be approximated in practice and finally, look at another interpretation of the definition.

7.1 Definition

First define normative preference for a more general setting. Let Δ be *any* compact metric space that represents some set of rewards with generic elements μ, η, ν, and take as given a set of preference relations $\{\gtrapprox_t\}_{t=0}^{\infty}$ on Δ which captures current period preference over Δ as Δ is delayed. Normative preference $\succsim^*$ over Δ is defined as in Definition 1, repeated below:[15]

Definition 1 *The normative preference $\succsim^*$ over Δ is the limit of the sequence $\{\gtrapprox_t\}$ in $\mathcal{P}$.*

As before, the definition rests on the observation that delayed temptations are easier to resist than immediate ones. Roughly speaking, the ranking of alternatives that are delayed by an infinite number of periods reveals the ranking in the absence of temptation. Normative preference is identified with this temptation-free ranking, since temptation is what drives a wedge between choice and normative preference.

7.2 Existence

In order to explore sufficient conditions for the existence of $\succsim^*$, consider the following axioms on $\{\gtrapprox_t\}$.

Axiom A1 *(Order*)* *$\gtrapprox_t$ is complete and transitive, for all t.*

[15] The set $\mathcal{P}$ is defined as $\mathcal{K}(\Delta \times \Delta)$, the space of nonempty compact subsets of $\Delta \times \Delta$ endowed with the Hausdorff metric topology.

Axiom A2 *(Continuity*)* *The sets* $\{\eta : \mu \gtrapprox_t \eta\}$ *and* $\{\eta : \eta \gtrapprox_t \mu\}$ *are closed, for all* t*.*

Axiom A3 *(Reversal*)* *If* $\mu \lessapprox_0 \eta$ *and* $\mu \gtrapprox_T \eta$ *with at least one ranking strict, then* $\mu >_t \eta$ *for all* $t > T$*.*

These arguably mild axioms are sufficient for the existence of a normative preference $\succsim^*$.

Theorem 5 *If* $\{\gtrapprox_t\}$ *satisfies A1-3, then* $\succsim^*$ *is well-defined, complete, transitive and continuous.*

7.3 GP's Definition

In order to define normative preference over Δ at some time (say period 1), GP's definition requires an ex-ante (say period 0) preference relation $\succsim$ defined over $\mathcal{K}(\Delta)$, the set of nonempty compact subsets of Δ. Their definition of normative preference depends on the period 0 preference over alternatives available for consumption in period 1. Formally, define commitment preference as follows.

Definition 2 *The commitment preference over* Δ *is the restriction of* $\succsim$ *to* $\Delta \subset \mathcal{K}(\Delta)$*.*

That is, commitment preference over alternatives in Δ is defined as the agent's preferences under commitment. GP claim that commitment preference defines normative preference – a preference to go to restaurant $\{v\}$ that serves only vegetarian meals v rather than restaurant $\{b\}$ that serves only burgers b supposedly reflects a dieter's period 0 view about what he *should* have for dinner in period 1.

However, there are at least two objections to identifying normative preference with commitment preference. Firstly, as before, a temptation to have a burger can induce a dieter to be tempted to go to a restaurant that serves only burgers, thereby creating a gap between commitment preference and normative preference (see the next subsection). Indeed, a necessary condition for the two to coincide is that the agent not be tempted by menus.

A second objection arises from the fact that commitment preference is a period 0 ranking of alternatives in Δ, and by itself, this ranking does not communicate anything about the agent's normative ranking of alternatives in Δ in period 1.[16] The agent's notion of his best interest in period 1 may be inconsistent with his period 0 perspective. That is, he may have *dynamically inconsistent normative preferences*. It follows that commitment preference is not an appropriate definition of welfare in such a case. Indeed, the idea of 'welfare' is not even well-defined for this agent.

Our definition is immune to these objections. Firstly, whether an agent is tempted by menus or not does not affect the intuition that goes into our definition. Secondly, in order to derive the normative preference of an agent at time t, we only need information about his preferences over delayed alternatives *at time t*. Hence, any dynamic model that contains the data necessary to derive normative preference at every t can also be checked for dynamic consistency of normative preference. The models axiomatized in this paper have dynamically consistent normative preference – normative preference at every $t > 0$ is identical and stationary, and Theorems 2 and 4 assure us that the period 0 ex-ante preference $\succsim$ is constructed such that there is no dynamic inconsistency between normative preference in period 0 and any other period.

7.4 Commitment Preference vs Normative Preference

We demonstrate that when menus are tempting, observed ($t > 0$) commitment preference does not coincide with normative preference. Consider an agent with a nondegenerate choice correspondence $\mathcal{C}$ that satisfies Axioms 1-10. The observed ranking of any pair of menus x, y is captured by

$$\mathcal{C}(\{(c, x), (c, y)\}).$$

That is, we observe a ranking $\succeq$ defined by

$$x \succeq y \Longleftrightarrow (c, x) \in \mathcal{C}(\{(c, x), (c, y)\}).$$

From the representation of $\mathcal{C}$ it can be deduced that $\succeq$ is represented by the function

[16] The objection holds for any ex-ante definition of welfare. For instance, see O'Donoghue and Rabin [16].

$$\phi(x) = \max_{\mu\in x}(U(\mu) + V(\mu)) - (1 - \frac{\gamma}{\delta}) \max_{\eta\in x} V(\eta).$$

However, $\succeq$ satisfies the axioms of GP [7].[17] Therefore, by GP [7, Theorem 1], there exist functions $\widehat{\phi} : Z \longrightarrow \mathbb{R}$ and $\widehat{U}, \widehat{V} : \Delta \longrightarrow \mathbb{R}$ such that

$$\widehat{\phi}(x) = \max_{\mu\in x} \widehat{U}(\mu) + \widehat{V}(\mu) - \max_{\eta\in x} \widehat{V}(\eta),$$

where $\widehat{\phi}$ represents $\succeq$. Assuming that $\widehat{U}$ is not an affine transformation of $\widehat{V}$, the functions $\widehat{U}, \widehat{V}$ are related to U, V by

$$\begin{aligned} \widehat{U}(\mu) &= U(\mu) + \frac{\gamma}{\delta} V(\mu), \\ \widehat{V}(\mu) &= (1 - \frac{\gamma}{\delta}) V(\mu). \end{aligned}$$

The function $\widehat{U}$ is a representation of commitment preference in GP's model: observe that $\widehat{\phi}(\{\mu\}) = \widehat{U}(\mu)$ for all μ, and apply Definition 2. By Theorem 2, the function U is a representation of normative preference (Definition 1). Conclude that $\widehat{U}$ does not represent normative preference since some temptations are incorrectly attributed to normative preference. This establishes that commitment preference does not coincide with normative preference when menus are tempting. When menus are not tempting ($\gamma = 0$), the two coincide.

7.5 Approximation

Our definition of normative preference $\succsim^*$ (Section 7.1) relies on a large amount of information $\{\gtrsim_t\}_{t=0}^{\infty}$. Eliciting the normative preference over two alternatives apparently requires an infinite number of observations. This section shows that if each $\gtrsim_t$ satisfies the vNM axioms, $\succsim^*$ may be conveniently approximated by the preference $\gtrsim_\tau$ that summarizes post-preference reversal rankings. Formally, define $\gtrsim_\tau$ by

$$\mu \gtrsim_\tau \eta \Longleftrightarrow \mu \gtrsim_{\tau(\mu,\eta)} \eta,$$

for all $\mu, \eta \in \Delta$.

[17] This verified in Appendix C.

First we show that $\succsim^*$ and $\gtrapprox_\tau$ may differ in general. Consider the following characterization of $\succsim^*$. The set Ω of points in $\Delta \times \Delta$ on which τ is upper semicontinuous is given by

$$\Omega = \{(\mu, \eta) \in \Delta \times \Delta : (\mu_n, \eta_n) \to (\mu, \eta) \Longrightarrow \limsup_{n \to \infty} \tau(\mu_n, \eta_n) \leq \tau(\mu, \eta)\}.$$

Lemma 4 (Appendix B) establishes that

$$\mu \succ^* \eta \Longleftrightarrow [\mu >_\tau \eta \text{ and } (\mu, \eta) \in \Omega]. \tag{8}$$

Hence, $\mu \succ^* \eta$ if and only if, firstly, the post-preference reversal ranking is given by $\mu >_{\tau(\mu,\eta)} \eta$ and secondly, τ is upper semicontinuous at (μ, η). Note that if $\mu \approx_\tau \eta$, then $\mu \sim^* \eta$. It follows that the set $\Lambda \subset \Delta \times \Delta$ on which $\succsim^*$ and $\gtrapprox_\tau$ differ is

$$\Lambda = \{(\mu, \eta) : \mu >_\tau \eta \text{ and } (\mu, \eta) \notin \Omega\}.$$

By (8), $\mu \sim^* \eta$ for every $(\mu, \eta) \in \Lambda$.

To understand why $\succsim^*$ and $\gtrapprox_\tau$ may differ, consider $(\mu, \eta) \in \Lambda$ such that $\mu > \eta$, $\tau(\mu, \eta) = 0$ and $(\mu, \eta) \notin \Omega$. That is, we have post-reversal non-indifference $\mu >_\tau \eta$ and normative indifference $\mu \sim^* \eta$. This gap between the two preferences is filled by *temptation*. If there is normative indifference and temptation non-indifference, then we will observe the agent to be non-indifferent in his choice between μ and η regardless of how far we push the rewards into the future.

Hence, $\succsim^*$ and $\gtrapprox_\tau$ differ when there is normative indifference and temptation non-indifference. When such a situation arises, we may say that there is a preference reversal at infinity. Intuitively, normative preference is the ranking between rewards that are delayed by an infinite number of periods. Hence $\mu >_t \eta$ for all t and $\mu \sim^* \eta$ implies that there is a switch in the preference to indifference *at infinity*. Another reason for the name 'reversal at infinity' is that for any $(\mu, \eta) \in \Lambda$, there is a sequence $\{(\mu_n, \eta_n)\}$ such that $(\mu_n, \eta_n) \to (\mu, \eta)$ and $\tau(\mu_n, \eta_n) \to \infty$ (see the proof of Lemma 27 in Appendix D.3). The fact that $\tau(\mu_n, \eta_n)$ goes to infinity suggests that for rewards (μ_n, η_n) that are close to (μ, η), temptation preference becomes increasingly stronger relative to normative preference, and that at (μ, η), temptation overwhelms to the extent that a reversal is observed only at infinity.

Given that it is easier to elicit $\gtrapprox_\tau$ than $\succsim^*$, it is natural to ask whether $\gtrapprox_\tau$ may serve as an approximation for $\succsim^*$. Theorem 6 tells us that under the following axioms on $\{\gtrapprox_t\}_{t=0}^\infty$, the preference $\gtrapprox_\tau$ serves as an approximation. In what follows, Δ is assumed to be a mixture space.

Axiom A1 *(Order*)* $\gtrapprox_t$ *is complete and transitive, for all* t.

Axiom A2 *(Continuity*)* *The sets* $\{\eta : \mu \gtrapprox_t \eta\}$ *and* $\{\eta : \eta \gtrapprox_t \mu\}$ *are closed, for all* t.

Axiom A3 *(Reversal*)* *If* $\mu \lessapprox_0 \eta$ *and* $\mu \gtrapprox_T \eta$ *with at least one ranking strict, then* $\mu >_t \eta$ *for all* $t > T$.

Axiom A4 *(Independence*)* $\mu >_t \eta \Longrightarrow \mu\alpha\nu >_t \eta\alpha\nu$, *for all* t.

Axiom A5 *(Non-triviality*)* $\mu >_0 \eta$ *and* $\mu <_T \eta$ *for some* μ,η *and* T.

Theorem 6 is the main result of this section.

Theorem 6 *If* $\{\gtrapprox_t\}$ *satisfies Axioms A1-5, then* Λ *is no-where dense.*

That is, the set Λ of pairs of rewards for which $\succsim^*$ and $\gtrapprox_\tau$ differ is 'sparse'. Hence, $\gtrapprox_\tau$ approximates $\succsim^*$.

Conclude with the observation that Λ is empty if τ is bounded: as observed earlier, $(\mu, \eta) \in \Lambda$ implies that there is a sequence $\{(\mu_n, \eta_n)\}$ such that $(\mu_n, \eta_n) \to (\mu, \eta)$ and $\tau(\mu_n, \eta_n) \to \infty$. If τ is bounded then such a sequence cannot exist. Hence in such a case, Λ is empty, and so the normative preference $\succsim^*$ coincides with $\gtrapprox_\tau$.

7.6 Individual Ethics

A number of game-theoretic models (see for instance Fedderson and Sandroni [5], Fehr and Schmidt [6]) have hypothesized that agents are motivated not just by material concerns, but also psychological concerns that include ethics. Karni and Safra [11] present a choice-theoretic model of an agent who has a sense of justice, but whose actual choices reflect a compromise between this sense and his self-interest. In their model, the 'more-just-than' ranking (call it the ethical ranking) is one of the primitives. Therefore, the authors leave open the question of whether the ethical ranking can be given choice-theoretic foundations.

Let Δ denote, say, a set of income tax regimes that redistribute income in favor of the poor to varying degrees. We suggest that it is possible for a rich person to believe in more redistribution, but nevertheless be tempted to vote for less redistribution in order to avoid paying higher taxes. If delayed temptations are easier to resist than immediate ones, then we can expect him to prefer 'more redistribution in the distant future' to 'less redistribution in the distant future'. Thus, his preferences over Δ when redistribution is to take place in the distant future would give us information about his ethical ranking over Δ.

That is, we suggest that in suitable settings, normative preference may be interpreted as the ethical ranking. Choice between alternatives in the distant future is less affected by myopic self-interest (temptations) and so leads to impartial choices. Thus, Definition 1 provides foundations for an individual sense of ethics. We saw in Section 7.2 that under Axioms A1-3, the only restriction on the ethical ranking is that it satisfies order and continuity (Theorem 5). That is, under Axioms A1-3, the set of possible ethical rankings an agent may have is not unduly restricted.

We conclude by acknowledging that GP's model can also be interpreted as one that identifies an agent's ethics. However, such an interpretation has the undesirable implication that all agents succeed (through the use of commitment mechanisms) in behaving ethically: a criminal with a sense of ethics would turn himself in to the police before committing a crime. This is a consequence of the assumption that choice between menus is not subject to temptation. By contrast, in our model with tempting menus, an agent may be tempted to postpone commitment indefinitely, and therefore persistently behave in a way that he himself believes he should not.

8 Conclusion

Choice has traditionally been viewed as a guide for welfare policy. However, the existence of self-control problems can drive a wedge between choice and welfare; an agent may make choices that are, in his own view, not in his best interest. This raises the question of whether one can identify an agent who has self-control problems, and whether one can elicit from an agent's choice behavior his view of what is best for his welfare.

This paper started with the question of what is the behavioral manifestation of self-control problems. The temptation literature starting from Strotz [18] isolates a preference for commitment as the expression of self-control problems. We noted that, when agents are tempted by menus, self-control problems may be responsible for the absence of a preference for commitment. This paper established that the behavioral manifestation of a temptation by menus is a preference for early choice, evidence for which was also cited. This completed our inquiry into the observable implications of self-control problems. Next, we studied a behavioral definition of normative preference based on preference reversals. This definition provided an answer to the question of how normative preference can be elicited from choice behavior.

A Appendix: Topology on $\mathcal{P}$

Let $(\Delta \times \Delta, d)$ be a compact metric space and denote the space of nonempty compact subsets of $\Delta \times \Delta$ by $\mathcal{P}$. For any $A, B \in \mathcal{P}$, let $d(a, B) = \inf_{b \in B} d(a, b)$ and $d(b, A) = \inf_{a \in A} d(b, a)$. The Hausdorff metric h_d induced by d is defined by

$$h_d(A, B) = \max\{\sup d(a, B), \sup d(b, A)\},$$

for all $A, B \in \mathcal{P}$. An ε-ball centered at A is defined by

$$B(A, \varepsilon) = \{B : h_d(A, B) < \varepsilon\}.$$

The Hausdorff metric topology on $\mathcal{P}$ is the topology for which the collection of balls $\{B(A, \varepsilon)\}_{A \in \mathcal{P}, \varepsilon \in (0, \infty)}$ is a base.

View the set $\mathcal{P}$ as the space of nonempty and continuous binary relations on Δ by identifying any such binary relation B on Δ with $\Gamma(B)$, the graph of B:

$$\Gamma(B) = \{(\mu, \eta) \in \Delta \times \Delta : \mu B \eta\}.$$

The relation B is continuous if and only if $\Gamma(B)$ is closed (see Ali-Border..closed graph theorem). Since Δ (and hence $\Delta \times \Delta$) is compact metric, it follows that if B is continuous, then $\Gamma(B) \subset \Delta \times \Delta$ is compact. By [2, Thm 3.71(3)], compactness of $\Delta \times \Delta$ implies that $\mathcal{P}$ is compact. Also, under compactness of $\Delta \times \Delta$, $\Gamma(B)$ is the Hausdorff metric limit of a sequence $\{\Gamma(B_n)\} \subset \mathcal{P}$ if and only if $\Gamma(B)$ is the 'closed limit' of $\{\Gamma(B_n)\}$ [2, Thm 3.79]. To define the closed limit of a sequence $\{\Gamma(B_n)\}$, first define the topological limit superior $Ls\Gamma(B_n)$ and topological limit inferior $Li\Gamma(B_n)$ of the sequence:

$$Ls\Gamma(B_n) = \{a \in \Delta \times \Delta : \text{for every neighborhood } V \text{ of } a,$$

$$V \cap \Gamma(B_n) \neq \phi \text{ for infinitely many } n\}$$

$$Li\Gamma(B_n) = \{a \in \Delta \times \Delta : \text{for every neighborhood } V \text{ of } a,$$

$$V \cap \Gamma(B_n) \neq \phi \text{ for all but a finite number of } n\}.$$

The sequence $\{\Gamma(B_n)\}$ converges to a closed limit $\Gamma(B)$ if $\Gamma(B) = Ls\Gamma(B_n) = Li\Gamma(B_n)$.

B Appendix: Normative Preference and Proof of Theorem 5

Define normative preference $\succsim^*$ as in Section 7.1. Consider the following axioms on $\{\gtrapprox_t\}$. Axiom A4 requires that Δ be a mixture space.

Axiom A1 *(Order*)* *$\gtrapprox_t$ is complete and transitive, for all t.*

Axiom A2 *(Continuity*)* *The sets $\{\eta : \mu \gtrapprox_t \eta\}$ and $\{\eta : \eta \gtrapprox_t \mu\}$ are closed, for all t.*

Axiom A3 *(Reversal*)* *If $\mu \lessapprox_0 \eta$ and $\mu \gtrapprox_T \eta$ with at least one ranking strict, then $\mu >_t \eta$ for all $t > T$.*

Axiom A4 *(Independence*)* *$\mu >_t \eta \Longrightarrow \mu\alpha\nu >_t \eta\alpha\nu$, for all t.*

In this Appendix, we prove two theorems, the first of which is stated as Theorem 5 in Section 7.2.

Theorem A *If $\{\gtrapprox_t\}$ satisfies A1-3, then $\succsim^*$ is well-defined, complete, transitive and continuous.*

Theorem B *If Δ is a mixture space and $\{\gtrapprox_t\}$ also satisfies A4, then $\succsim^*$ also satisfies independence axiom.*

B.1 Proof of Theorem A

This is proved in a series of lemmas.

Lemma 1 defines a function $\tau : \Delta \times \Delta \to \mathbb{R}$ which captures the time at which a reversal takes place in $\{\mu, \eta\}$. Note that since time is discrete, $\tau(\Delta \times \Delta) \subset \mathbb{N} \cup \{0\}$.

Lemma 1 *For each μ, η there exists $\tau(\mu, \eta) \geq 0$ such that if $\tau(\mu, \eta) = 0$, then,*

$$\mu \gtrapprox_{\tau(\mu,\eta)} \eta \Longleftrightarrow \mu \gtrapprox_t \eta, \quad \textit{for all } t \geq 0,$$

and if $\tau(\mu, \eta) > 0$ and $\mu \gtrapprox_0 \eta$, then

$$\begin{aligned} \mu &< {}_t\eta \textit{ for all } t \geq \tau(\mu, \eta), \\ \mu &\gtrapprox {}_t\eta \textit{ for all } t < \tau(\mu, \eta). \end{aligned}$$

Proof. Define $\tau : \Delta \times \Delta \to \mathbb{R}$ in the following way. Take any μ and η, and without loss of generality suppose that $\mu \gtrapprox_0 \eta$. If for all t,

$$\mu \gtrapprox_0 \eta \Longleftrightarrow \mu \gtrapprox_t \eta,$$

then define $\tau(\mu, \eta) = 0$. If there exists T such that $\mu <_T \eta$, then define

$$\tau(\mu, \eta) = \min\{t : \mu <_t \eta\}.$$

Given Reversal, this $\tau(\mu, \eta)$ has the desired properties. Finally, if there is no T such that $\mu <_T \eta$, then we must be in one of the two earlier cases. For instance, if $\mu >_0 \eta$ and $\mu \approx_T \eta$ for some T, then by Reversal, $\mu <_t \eta$ for all $t > T$, and hence we are in the second case. ■

Define $\succsim^*$ over Δ by

$$\mu \succsim^* \eta \Longleftrightarrow \text{there exists a sequence } \{(\mu_n, \eta_n)\} \text{ that converges to } (\mu, \eta) \text{ and } \mu_n \gtrapprox_{\tau(\mu_n,\eta_n)} \eta_n \text{ for all } n.$$

Lemma 2 $\mu \gtrapprox_{\tau(\mu,\eta)} \eta \Longrightarrow \mu \succsim^* \eta$.

Proof. This is implied directly by the definition of $\succsim^*$. ■

Lemma 3 $\succsim^*$ *satisfies order and continuity.*

Proof. To establish completeness, suppose $\eta \not\succsim^* \mu$. By Lemma 2, $\eta \not\gtrapprox_{\tau(\mu,\eta)} \mu$, and by completeness of $\gtrapprox_{\tau(\mu,\eta)}$, $\mu \gtrapprox_{\tau(\mu,\eta)} \eta$, and so, again by Lemma 2, $\mu \succsim^* \eta$. To establish transitivity, suppose $\mu \succsim^* \eta \succsim^* \nu$. Then by definition of $\succsim^*$, there exist sequences $\{(\mu_n, \eta_n)\}, \{(\eta_n, \nu_n)\}$ such that $(\mu_n, \eta_n) \to (\mu, \eta)$ and $(\eta_n, \nu_n) \to (\eta, \nu)$, and for each n, $\mu_n \gtrapprox_{\tau(\mu_n,\eta_n)} \eta_n$ and $\eta_n \gtrapprox_{\tau(\eta_n,\nu_n)} \nu_n$. By Lemma 1, $\mu_n \gtrapprox_t \eta_n \gtrapprox_t \nu_n$ for all $t \geq T = \max\{\tau(\mu_n, \eta_n), \tau(\eta_n, \nu_n)\}$. By transitivity of $\gtrapprox_t$, $\mu_n \gtrapprox_t \nu_n$ for all $t \geq T$, implying that for each n, $\mu_n \gtrapprox_{\tau(\mu_n,\nu_n)} \nu_n$. It follows that $\{(\mu_n, \nu_n)\}$ is a sequence that converges to (μ, ν) and $\mu_n \gtrapprox_{\tau(\mu_n,\nu_n)} \nu_n$ for all n. By definition of $\succsim^*$, $\mu \succsim^* v$, thus establishing transitivity of $\succsim^*$.

To establish continuity, we show that $\{\eta : \eta \succsim^* \mu\}$ is closed; the other case holds by an analogous argument. Take a sequence $\{\nu_n\}$ such that $\nu_n \succsim^* \mu$ for all n and $\nu_n \to \nu$. Also, take a sequence $\{V_i\}$ where each $V_i \subset \Delta \times \Delta$ is a ball of radius 2^{-i} that contains (ν, μ). Because $\nu_n \to \nu$, for every i there exists n such that $(\nu_n, \mu) \in V_i$. Furthermore, $\nu_n \succsim^* \mu$ and the definition of $\succsim^*$ imply the existence a sequence $\{(\nu'_m, \mu'_m)\}$ such that $(\nu'_m, \mu'_m) \to (\nu_n, \mu)$ and $\nu'_m \gtrapprox_{\tau(\nu'_m,\mu'_m)} \mu'_m$ for all m. Since V_i is also a neighborhood of (ν_n, μ), there exists m such that $(\nu'_m, \mu'_m) \in V_i$. Define

$$(\nu'_{m,i}, \mu'_{m,i}) \equiv (\nu'_m, \mu'_m),$$

and note that $\nu'_{m,i} \gtrapprox_{\tau(\nu'_{m,i},\mu'_{m,i})} \mu'_{m,i}$. Furthermore, by construction, $(\nu'_{m,i}, \mu'_{m,i}) \to (\nu, \mu)$ as $i \to \infty$, and so we are done. ■

The set Ω of points in $\Delta \times \Delta$ on which τ is upper semicontinuous will be important. Formally,

$$\Omega = \{(\mu,\eta) \in \Delta \times \Delta : (\mu_n,\eta_n) \to (\mu,\eta) \Longrightarrow \limsup_{n\to\infty} \tau(\mu_n,\eta_n) \leq \tau(\mu,\eta)\}.$$

Lemma 4 is a useful characterization of $\succsim^*$.

Lemma 4

$$\mu \succ^* \eta \Longleftrightarrow [\mu >_{\tau(\mu,\eta)} \eta \textit{ and } (\mu,\eta) \in \Omega].$$

Proof. $\Longleftarrow$: Take μ and η such that $\mu >_{\tau(\mu,\eta)} \eta$ and $(\mu,\eta) \in \Omega$. Lemma 1 implies $\mu >_{\tau(\mu,\eta)+1} \eta$. Since $\gtrapprox_{\tau(\mu,\eta)+1}$ is continuous, for every sequence $\{(\mu_n,\eta_n)\}$ that converges to (μ,η), there exists N such that

$$\mu_n >_{\tau(\mu,\eta)+1} \eta_n, \text{ for all } n \geq N.$$

By hypothesis, $\limsup_{n\to\infty} \tau(\mu_n,\eta_n) \leq \tau(\mu,\eta)$. Therefore, there exists N' such that

$$\tau(\mu,\eta) + 1 > \tau(\mu_n,\eta_n), \text{ for all } n \geq N'.$$

It follows by definition of $\tau(\mu_n,\eta_n)$ that

$$\mu_n >_{\tau(\mu_n,\eta_n)} \eta_n, \quad \text{for all } n \geq \max\{N,N'\}.$$

This establishes that for any sequence $\{(\mu_n,\eta_n)\}$ that converges to (μ,η), there exists M such that $\mu_n >_{\tau(\mu_n,\eta_n)} \eta_n$ for all $n \geq M$. In particular, there is no sequence $\{(\mu_n,\eta_n)\}$ that converges to (μ,η) such that $\eta_n \gtrapprox_{\tau(\mu_n,\eta_n)} \mu_n$ for all n. Thus $\eta \not\succsim^* \mu$, as desired.

$\Longrightarrow$: Take μ,η such that $\mu \succ^* \eta$. Lemma 2 yields

$$\mu >_{\tau(\mu,\eta)} \eta \tag{9}$$

thus establishing the first assertion in the implication. To establish the second assertion, take any sequence $\{(\mu_n,\eta_n)\}$ that converges to (μ,η). Since $\mu \succ^* \eta$ and since $\succsim^*$ is continuous (Lemma 3), there exists N such that

$$\mu_n \succ^* \eta_n, \quad \text{for all } n \geq N.$$

By Lemma 2,

$$\mu_n >_{\tau(\mu_n,\eta_n)} \eta_n, \quad \text{for all } n \geq N. \tag{10}$$

Without loss of generality, let $N = 1$. Suppose by way of contradiction that

$$\limsup_{n\to\infty} \tau(\mu_n, \eta_n) > \tau(\mu, \eta).$$

Then, there exists a subsequence $\{(\mu_{n(m)}, \eta_{n(m)})\} \subset \{(\mu_n, \eta_n)\}$ where for all m,

$$\tau(\mu_{n_m}, \eta_{n_m}) > \tau(\mu, \eta). \tag{11}$$

By construction, $\mu_{n_m} >_{\tau(\mu_{n_m},\eta_{n_m})} \eta_{n_m}$ for all m. Thus, by Lemma 1 and (11),

$$\eta_{n_m} \gtrapprox_{\tau(\mu,\eta)} \mu_{n_m}, \quad \text{for all } m.$$

However, since $\gtrapprox_{\tau(\mu,\eta)}$ is continuous and $(\mu_{n_m}, \eta_{n_m}) \to (\mu, \eta)$, we have $\eta \gtrapprox_{\tau(\mu,\eta)} \mu$, contradicting (9). ■

We now prove that $\succsim^*$ is a normative preference. Since each $\gtrapprox_t$ is a continuous weak order, $\{\Gamma(\gtrapprox_t)\}$ is a sequence in $\mathcal{P}(\Delta)$. Define:

$$\Gamma(\succsim_\tau) = \{(\mu, \eta) \in \Delta \times \Delta : \exists T \text{ such that } (\mu, \eta) \in \Gamma(\gtrapprox_t) \text{ for all } t \geq T\},$$

and note that $\Gamma(\succsim^*) = \overline{\Gamma(\succsim_\tau)}$.

Lemma 5 $\Gamma(\succsim^*) = \lim_{t\to\infty} \Gamma(\gtrapprox_t)$.

Proof. To establish the existence of a closed limit, it suffices to show that $Ls\Gamma(\gtrapprox_t) \subset Li\Gamma(\gtrapprox_t)$, since $Li\Gamma(\gtrapprox_t) \subset Ls\Gamma(\gtrapprox_t)$ always holds.

Step 1: $Ls\Gamma(\gtrapprox_t) \subset \Gamma(\succsim^*)$.

Let $(\mu, \eta) \in Ls\Gamma(\gtrapprox_t)$. By Lemma 1, there exists $T^* < \infty$ such that either $(\mu, \eta) \in \Gamma(\gtrapprox_t)$ for all $t \geq T^*$ or $(\mu, \eta) \notin \Gamma(\gtrapprox_t)$ for all $t \geq T^*$. If $(\mu, \eta) \in \Gamma(\gtrapprox_t)$ for all $t \geq T^*$, then $(\mu, \eta) \in \Gamma(\succsim_\tau) \subset \Gamma(\succsim^*)$ and we are done. Therefore, suppose $(\mu, \eta) \notin \Gamma(\gtrapprox_t)$ for all $t \geq T^*$. By Lemma 4, it suffices to show $(\mu, \eta) \notin \Omega$. So suppose by way of contradiction that $(\mu, \eta) \in \Omega$.

By definition (for instance, see [4, pg 110]), $(\mu, \eta) \in Ls\Gamma(\gtrapprox_t)$ implies that there is a subsequence $\{\Gamma(\gtrapprox_{t(n)})\}$ and a sequence $\{(\mu_n, \eta_n)\}$ that converges to (μ, η) such that $(\mu_n, \eta_n) \in \Gamma(\gtrapprox_{t(n)})$ for each n. By assumption, $(\mu, \eta) \in \Omega$, and so,

$$\limsup_{n\to\infty} \tau(\mu_n, \eta_n) \leq T^*.$$

It follows that there exists M such that $\tau(\mu_n, \eta_n) < T^* + 1$ for all $n \geq M$. Without loss of generality, assume $M = 1$. Also, since $\Gamma(\gtrapprox_{T^*+1})$ is closed and $(\mu,\eta) \notin \Gamma(\gtrapprox_{T^*+1})$, there exists N such that $(\mu_n,\eta_n) \notin \Gamma(\gtrapprox_{T^*+1})$ for all $n \geq N$. By Lemma 1, these observation, namely $\tau(\mu_n,\eta_n) < T^*+1$ and $(\mu_n,\eta_n) \notin \Gamma(\gtrapprox_{T^*+1})$ for all $n \geq N$, imply that for all $n \geq N$ and $t \geq T^*+1$,

$$(\mu_n,\eta_n) \notin \Gamma(\gtrapprox_t). \tag{12}$$

However, by construction of the sequence $\{(\mu_n,\eta_n)\}$, we have $(\mu_n,\eta_n) \in \Gamma(\gtrapprox_{t(n)})$ for all n. Let N' be such that $t(N') \geq T^*+1$. It follows that for all $n \geq \max\{N,N'\}$,

$$(\mu_n,\eta_n) \in \Gamma(\gtrapprox_{t(n)}),$$

contradicting (12).

Step 2: $\Gamma(\succsim^*) \subset Li\Gamma(\gtrapprox_t)$.

First show that $\Gamma(\succsim_\tau) \subset Li\Gamma(\gtrapprox_t)$. Observe that if $(\mu,\eta) \in \Gamma(\succsim_\tau)$, then there exists $T < \infty$ such that $(\mu,\eta) \in \Gamma(\gtrapprox_t)$ for all $t \geq T$. Hence for every neighborhood V of (μ,η),

$$V \cap \Gamma(\gtrapprox_t) \neq \phi \text{ for all but a finite number of } t.$$

It follows that $(\mu,\eta) \in Li\Gamma(\gtrapprox_t)$, thus establishing that $\Gamma(\succsim_\tau) \subset Li\Gamma(\gtrapprox_t)$, as desired. To complete the proof of Step 2, note that since $Li\Gamma(\gtrapprox_t)$ is closed (see Ali-Border..), it follows that $\overline{\Gamma(\succsim_\tau)} \subset Li\Gamma(\gtrapprox_t)$. But $\Gamma(\succsim^*) = \overline{\Gamma(\succsim_\tau)}$. The assertion follows.

By Steps 1 and 2, $Ls\Gamma(\gtrapprox_t) \subset \Gamma(\succsim^*) \subset Li\Gamma(\gtrapprox_t)$. Hence,

$$Li\Gamma(\gtrapprox_t) = Ls\Gamma(\gtrapprox_t) = \Gamma(\succsim^*).$$

This completes the proof. ■

B.2 Proof of Theorem B

This is proved in 5 steps.

Step 1:

$$\mu >_t \eta \Longleftrightarrow \mu\alpha\nu >_t \eta\alpha\nu, \text{ for all } t.$$

Axioms A1, A2 and A4 together imply this stronger version of Independence*.

Step 2:

$$\begin{aligned} \tau(\mu,\eta) &= \tau(\mu\alpha\nu,\eta\alpha\nu) \\ \text{and } \mu &> {}_{\tau(\mu,\eta)}\eta \Longleftrightarrow \mu\alpha\nu >_{\tau(\mu\alpha\nu,\eta\alpha\nu)} \eta\alpha\nu. \end{aligned}$$

This follows from Step 1.

Step 3:

$$(\mu,\eta)\notin\Omega \Longrightarrow (\mu\alpha\nu,\eta\alpha\nu)\notin\Omega.$$

If $\{(\mu_n,\eta_n)\}$ is a sequence that converges to (μ,η) and

$$\limsup_{n\to\infty}\tau(\mu_n,\eta_n) > \tau(\mu,\eta),$$

then $\{(\mu_n\alpha\nu,\eta_n\alpha\nu)\}$ is a sequence that converges to $(\mu\alpha\nu,\eta\alpha\nu)$ and, by the first assertion in Step 2,

$$\limsup_{n\to\infty}\tau(\mu_n\alpha\nu,\eta_n\alpha\nu) > \tau(\mu\alpha\nu,\eta\alpha\nu).$$

Thus, $(\mu\alpha\nu,\eta\alpha\nu)\notin\Omega$.

Step 4:

$$\mu\sim^*\eta \Longrightarrow \mu\alpha\nu\sim^*\eta\alpha\nu.$$

Suppose $\mu\sim^*\eta$. Then
$\mu\sim^*\eta$
$\Longrightarrow \mu\approx_{\tau(\mu,\eta)}\eta$ or $(\mu,\eta)\notin\Omega$ by Lemma 4
$\Longrightarrow \mu\alpha\nu\approx_{\tau(\mu\alpha\nu,\eta\alpha\nu)}\eta\alpha\nu$ or $(\mu\alpha\nu,\eta\alpha\nu)\notin\Omega$ by Steps 2 and 3
$\Longrightarrow \mu\alpha\nu\sim^*\eta\alpha\nu$, as desired.

Step 5:

$$\mu\succ^*\eta \Longrightarrow \mu\alpha\nu\succ^*\eta\alpha\nu.$$

By Herstein and Milnor [9], under order and continuity of $\succsim^*$, Step 4 implies the result.

C Appendix: Proof of Theorem 1 (Necessity)

Define the function $\phi:\Delta(C\times Z)\to\mathbb{R}$ by $\phi(\mu)=U(\mu)+V(\mu)$ for all $\mu\in\Delta$ and consider the correspondence $\mathcal{C}:Z\rightsquigarrow\Delta(C\times Z)$ defined by

$$\mathcal{C}(x)=\arg\max_{\mu\in x}\phi(\mu).$$

Clearly, $\mathcal{C}(x)\subset x$. Since ϕ is continuous, the Maximum Theorem [2, Thm 16.31] yields that $\mathcal{C}$ is nonempty, compact-valued (in particular closed-valued) and upper hemicontinuous. Hence ϕ generates a closed-valued choice correspondence $\mathcal{C}$ that satisfies Axiom 2. That $\mathcal{C}$ satisfies Axiom 1 can be checked easily. We need to show that $\mathcal{C}$ satisfies Axioms 3-10. Proofs for Independence, Separability and Indifference to Timing are omitted. Let $\gtrapprox$ be the the binary relation that is represented by ϕ. For $t \geq 1$, define $\gtrapprox_t$ over Z by

$$x \gtrapprox_t y \Longleftrightarrow x^{+t} \gtrapprox y^{+t}.$$

Lemma 6 *$\mathcal{C}$ satisfies Set-Betweenness.*

Proof. Note that

$$\begin{aligned} \phi(x^{+t}) &= \delta^t W(x) + \gamma^t \overline{V}(y) + \text{constant} \\ &= \delta^t (\max_{\mu \in x} U(\mu) + V(\mu) - \max_{\eta \in x} V(\eta)) + \gamma^t \overline{V}(x) + \text{constant} \\ &= \delta^t (\max_{\mu \in x} U(\mu) + V(\mu) - (1 - \frac{\gamma^t}{\delta^t}) \max_{\eta \in x} V(\eta)) + \text{constant}. \end{aligned}$$

Hence, $\gtrapprox_t$ is represented by

$$\phi'(x) = \max_{\mu \in x} U(\mu) + V(\mu) - (1 - \frac{\gamma^t}{\delta^t}) \max_{\eta \in x} V(\eta).$$

Defining $V'(\mu) = (1 - \frac{\gamma^t}{\delta^t})V(\mu)$ and $U'(\mu) = U(\mu) + \frac{\gamma^t}{\delta^t}V(\mu)$, we have

$$\phi'(x) = \max_{\mu \in x} U'(\mu) + V'(\mu) - \max_{\eta \in x} V'(\eta),$$

and so ϕ' represents a Self-Control preference [7]. Therefore, $\mathcal{C}$ satisfies Set-Betweenness. ■

Lemma 7 *$\mathcal{C}$ satisfies Sophistication.*

Proof. In the proof of the previous lemma, we showed that $\gtrapprox_t$ is represented by

$$\phi'(x) = \max_{\mu \in x} U(\mu) + V(\mu) - (1 - \frac{\gamma^t}{\delta^t}) \max_{\eta \in x} V(\eta),$$

which can be re-written as

$$\phi'(x) = \max_{\mu \in x} U'(\mu) + V'(\mu) - \max_{\eta \in x} V'(\eta),$$

with $U' + V' = U + V$. Let $\mu >_t \eta$. Then, by [15, Lemma 1(b)],

$$\{\mu, \eta\} >_t \{\eta\} \Longleftrightarrow U(\mu) + V(\mu) > U(\eta) + V(\eta).$$

But $\mathcal{C}(x) = \arg\max_{\mu \in x}(U(\mu) + V(\mu))$,and so, $U(\mu) + V(\mu) > U(\eta) + V(\eta)$ is equivalent to $\{\mu\} = \mathcal{C}(\{\mu, \eta\})$, which in turn is equivalent to $\mu > \eta$. ■

Lemma 8 *$\mathcal{C}$ satisfies Preference for Early Choice.*

Proof. Note that $\gtrsim_t$ is represented by

$$W(\cdot) + \frac{\gamma^t}{\delta^t}\overline{V}(\cdot).$$

Therefore, it suffices to show that

$$\begin{aligned} W(x) + \frac{\gamma^t}{\delta^t}\overline{V}(x) \quad &> \quad W(x \cup y) + \frac{\gamma^t}{\delta^t}\overline{V}(x \cup y) \\ &\Longrightarrow \quad W(\{(c,x)\}) + \frac{\gamma^t}{\delta^t}\overline{V}(\{(c,x)\}) > W(\{(c,x),(c,y)\}) + \frac{\gamma^t}{\delta^t}\overline{V}(\{(c,x),(c,y)\}) \end{aligned}$$

Therefore, suppose $W(x) + \frac{\gamma^t}{\delta^t}\overline{V}(x) > W(x \cup y) + \frac{\gamma^t}{\delta^t}\overline{V}(x \cup y)$. First some preliminary results.

Step 1: $U(c,x) + \frac{\gamma^t}{\delta^t}V(c,x) > U(c, x \cup y) + \frac{\gamma^t}{\delta^t}V(c, x \cup y)$

By hypothesis, $W(x) + \frac{\gamma^t}{\delta^t}\overline{V}(x) > W(x \cup y) + \frac{\gamma^t}{\delta^t}\overline{V}(x \cup y)$. Then, $\overline{V}(x \cup y) \geq \overline{V}(x)$ implies

$$W(x) > W(x \cup y)\ , \tag{13}$$

which in turn implies

$$\overline{V}(x \cup y) = \overline{V}(y) > \overline{V}(x).$$

Given that $\frac{\gamma}{\delta} < 1$, it follows from the hypothesis that

$$W(x) + \frac{\gamma^{t+1}}{\delta^{t+1}}\overline{V}(x) > W(x \cup y) + \frac{\gamma^{t+1}}{\delta^{t+1}}\overline{V}(x \cup y). \tag{14}$$

But adding $u(c) + \frac{\gamma^t}{\delta^t}v(c)$ to both sides of (14) yields

$$U(c,x) + \frac{\gamma^t}{\delta^t}V(c,x) > U(c, x \cup y) + \frac{\gamma^t}{\delta^t}V(c, x \cup y).$$

Step 2: $U(c,x) + \frac{\gamma^t}{\delta^t} V(c,x) > U(c,y) + \frac{\gamma^t}{\delta^t} V(c, x \cup y)$

The representation implies that if $W(x) > W(x \cup y)$, then $W(x) > W(y)$. Thus, $U(c,x) > U(c, x \cup y)$ implies $U(c,x) > U(c,y)$. Applying this to the result in Step 1 yields

$$U(c,x) + \frac{\gamma^t}{\delta^t} V(c,x) > U(c,y) + \frac{\gamma^t}{\delta^t} V(c, x \cup y).$$

Step 3: $\overline{V}(\{(c,x),(c,y)\}) = V(c, x \cup y) = V(c,y) > V(c,x)$

By (13), $\overline{V}(y) > \overline{V}(x)$, which implies

$$V(c,y) > V(c,x), \tag{15}$$

and furthermore,

$$\overline{V}(\{(c,x),(c,y)\}) = V(c,y) = V(c, x \cup y). \tag{16}$$

Step 4: The result.

Consider two possibilities:

a) $W(c,x) > W(\{(c,x),(c,y)\}) = W(c,y)$

Then,

$$W(\{(c,x),(c,y)\}) = U(c,y). \tag{17}$$

Now,

$W(\{(c,x)\}) + \frac{\gamma^t}{\delta^t} \overline{V}(\{(c,x)\})$

$= U(c,x) + \frac{\gamma^t}{\delta^t} V(c,x)$

$> U(c,y) + \frac{\gamma^t}{\delta^t} V(c, x \cup y)$ by Step 2

$= W(\{(c,x),(c,y)\}) + \frac{\gamma^t}{\delta^t} \overline{V}(\{(c,x),(c,y)\})$, by (17) and (16)

as desired.

b) $W(\{(c,x)\}) > W(\{(c,x),(c,y)\}) > W(\{(c,y)\})$.

Then,

$$W(\{(c,x),(c,y)\}) = U(c,x) + (V(c,x) - V(c,y)).$$

Hence,

$W(\{(c,x)\}) + \frac{\gamma^t}{\delta^t} \overline{V}(\{(c,x)\}) - \frac{\gamma^t}{\delta^t} \overline{V}(\{(c,x),(c,y)\})$

$= U(c,x) + \frac{\gamma^t}{\delta^t} (V(c,x) - V(c,y))$ by (16)

$> U(c,x) + (V(c,x) - V(c,y))$ since $\frac{\gamma^t}{\delta^t} < 1$ and (15).

$= W(\{(c,x),(c,y)\})$.

That is, $W(\{(c,x)\})+\frac{\gamma^t}{\delta^t}\overline{V}(\{(c,x)\}) > W(\{(c,x),(c,y)\})+\frac{\gamma^t}{\delta^t}\overline{V}(\{(c,x),(c,y)\})$, as desired ■

The proof of Lemma 9 establishes that $\mathcal{C}$ satisfies Reversal.

Lemma 9 *Suppose* $U(\mu) \geq U(\eta)$*. Then,*

$$\begin{aligned}
\frac{U(\mu)-U(\eta)}{V(\eta)-V(\mu)} &\in (0,1] \Longrightarrow \tau(\mu,\eta) = \min\{k \in \mathbb{N} : k > \frac{\ln \frac{U(\mu)-U(\eta)}{V(\eta)-V(\mu)}}{\ln \frac{\gamma}{\delta}}\} > 0 \\
\frac{U(\mu)-U(\eta)}{V(\eta)-V(\mu)} &\notin (0,1] \text{ and } V(\eta) \neq V(\mu) \Longrightarrow \tau(\mu,\eta) = 0 \\
V(\eta) &= V(\mu) \Longrightarrow \tau(\mu,\eta) = 0.
\end{aligned}$$

Proof. By the representation, for $t \geq 0$,

$$\mu^{+t} \gtrapprox \eta^{+t} \Longleftrightarrow U(\mu) + \frac{\gamma^t}{\delta^t}V(\mu) \geq U(\eta) + \frac{\gamma^t}{\delta^t}V(\eta).$$

Suppose $\frac{U(\mu)-U(\eta)}{V(\eta)-V(\mu)} \in (0,1)$. Since $U(\mu) \geq U(\eta)$, we have $U(\mu) > U(\eta)$ and $V(\mu) < V(\eta)$. Furthermore, $\frac{U(\mu)-U(\eta)}{V(\eta)-V(\mu)} \leq 1$ implies $U(\mu) + V(\mu) \leq U(\eta) + V(\eta)$. Since $\frac{\gamma}{\delta} < 1$ and $U(\mu) > U(\eta)$, there exists $\tau(\mu,\eta)$ such that

$$\begin{aligned}
\forall t &< \tau(\mu,\eta),\ U(\mu) + \frac{\gamma^t}{\delta^t}V(\mu) \leq U(\eta) + \frac{\gamma^t}{\delta^t}V(\eta) \\
\forall t &\geq \tau(\mu,\eta),\ U(\mu) + \frac{\gamma^t}{\delta^t}V(\mu) > U(\eta) + \frac{\gamma^t}{\delta^t}V(\eta). \qquad (18)
\end{aligned}$$

To find $\tau(\mu,\eta)$, first find the t^* that solves

$$U(\mu) + \frac{\gamma^{t^*}}{\delta^{t^*}}V(\mu) = U(\eta) + \frac{\gamma^{t^*}}{\delta^{t^*}}V(\eta).$$

The solution is $t^* = \frac{\ln \frac{U(\mu)-U(\eta)}{V(\eta)-V(\mu)}}{\ln \frac{\gamma}{\delta}}$. Then $\tau(\mu,\eta)$ is the smallest integer greater than $\frac{\ln \frac{U(\mu)-U(\eta)}{V(\eta)-V(\mu)}}{\ln \frac{\gamma}{\delta}}$, that is,

$$\tau(\mu,\eta) = \min\{k \in \mathbb{N} : k > \frac{\ln \frac{U(\mu)-U(\eta)}{V(\eta)-V(\mu)}}{\ln \frac{\gamma}{\delta}}\}.$$

If $V(\eta) = V(\mu)$ or if $V(\eta) \neq V(\mu)$ and $\frac{U(\mu)-U(\eta)}{V(\eta)-V(\mu)} = 0$, then it is straightforward to establish that

$$\mu \gtrapprox \eta \Longleftrightarrow \mu^{+t} \gtrapprox \eta^{+t},$$

and so $\tau(\mu, \eta) = 0$. Suppose $V(\eta) \neq V(\mu)$ and $\frac{U(\mu)-U(\eta)}{V(\eta)-V(\mu)} < 0$. Then, $V(\mu) > V(\eta)$. By hypothesis, $U(\mu) \geq U(\eta)$, and so $\frac{\gamma}{\delta} < 1$ implies $\tau(\mu, \eta) = 0$. Finally, suppose $V(\eta) \neq V(\mu)$ and $\frac{U(\mu)-U(\eta)}{V(\eta)-V(\mu)} > 1$. Then $V(\mu) < V(\eta)$ and $U(\mu) + V(\mu) > U(\eta) + V(\eta)$. Again it follows that $\tau(\mu, \eta) = 0$. Hence, $\tau(\mu, \eta) = 0$ if $\frac{U(\mu)-U(\eta)}{V(\eta)-V(\mu)} \notin (0, 1)$ and $V(\eta) \neq V(\mu)$ or if $V(\eta) = V(\mu)$. ■

Lemma 10 *If $U(\mu) > U(\eta)$ then $\mu >_t \eta$ for all $t \geq \tau(\mu, \eta)$. In particular, $U(\mu) > U(\eta)$ and $\mu > \eta$ imply $\tau(\mu, \eta) = 0$.*

Proof. The first assertion is a corollary of Lemma 9. The second assertion follows from the first. ■

Define $\Omega = \{\{\mu, \eta\}$: for any sequence $\{(\mu_n, \eta_n)\}$ that converges to (μ, η),

$$\limsup_{n\to\infty} \tau(\mu_n, \eta_n) \leq \tau(\mu, \eta)\}.$$

Lemma 11 *If $U(\mu) \neq U(\eta)$ then $(\mu, \eta) \in \Omega$.*

Proof. If $U(\mu) > U(\eta)$ and $V(\mu) > V(\eta)$, then for any sequence $\{(\mu_n, \eta_n)\}$ that converges to (μ, η), there exists N such that for all $n \geq N$,

$$U(\mu_n) > U(\eta_n) \text{ and } V(\mu_n) > V(\eta_n).$$

Hence $\frac{U(\mu_n)-U(\eta_n)}{V(\eta_n)-V(\mu_n)} \to \frac{U(\mu)-U(\eta)}{V(\eta)-V(\mu)}$. Note that $U(\mu) > U(\eta)$ implies $\frac{U(\mu)-U(\eta)}{V(\eta)-V(\mu)} \neq 0$. Now consider the possibilities:

(a) $V(\mu) \neq V(\eta)$ and U, V agree on $\{\mu, \eta\}$.

Then $\frac{U(\mu)-U(\eta)}{V(\eta)-V(\mu)} \notin (0, 1]$. Since $\frac{U(\mu_n)-U(\eta_n)}{V(\eta_n)-V(\mu_n)} \to \frac{U(\mu)-U(\eta)}{V(\eta)-V(\mu)}$, there exists N such that $\frac{U(\mu_n)-U(\eta_n)}{V(\eta_n)-V(\mu_n)} \notin (0, 1]$ for all $n \geq N$, and hence by Lemma 9, $\tau(\mu_n, \eta_n) = 0$ for all $n \geq N$. Hence $\limsup_{n\to\infty} \tau(\mu_n, \eta_n) = 0 \leq \tau(\mu, \eta)$, implying $(\mu, \eta) \in \Omega$.

(b) $V(\mu) \neq V(\eta)$ and U, V disagree on $\{\mu, \eta\}$.

Consider three possibilities.

(i) If $\frac{U(\mu)-U(\eta)}{V(\eta)-V(\mu)} \notin (0,1]$ then argue as in (a).

(ii) If $\frac{U(\mu)-U(\eta)}{V(\eta)-V(\mu)} \in (0,1)$, then for any sequence $\{(\mu_n, \eta_n)\}$ that converges to (μ, η), there exists N such that for all $n \geq N$, $\frac{U(\mu_n)-U(\eta_n)}{V(\eta_n)-V(\mu_n)} \in (0,1)$. Wlog let $N = 1$. By Lemma 9, $\tau(\mu_n, \eta_n) = \min\{k \in \mathbb{N} : k > \frac{\ln \frac{U(\mu_n)-U(\eta_n)}{V(\eta_n)-V(\mu_n)}}{\ln \frac{\gamma}{\delta}}\}$ for each n. Since $\frac{U(\mu_n)-U(\eta_n)}{V(\eta_n)-V(\mu_n)} \to \frac{U(\mu)-U(\eta)}{V(\eta)-V(\mu)}$, we have $\lim_{n\to\infty} \tau(\mu_n, \eta_n) = \tau(\mu, \eta)$, which establishes the result.

(iii) If $\frac{U(\mu)-U(\eta)}{V(\eta)-V(\mu)} = 1$, then take any sequence $\{(\mu_n, \eta_n)\}$ that converges to (μ, η). If there exists N such that $\forall n \geq N$, $\frac{U(\mu_n)-U(\eta_n)}{V(\eta_n)-V(\mu_n)} \geq 1$, then note that $\tau(\mu_n, \eta_n) \leq \tau(\mu, \eta)$ for all $n \geq N$ and so $(\mu, \eta) \in \Omega$. If no such N exists, then construct a subsequence $\{(\mu_{n(m)}, \eta_{n(m)})\}$ by deleting all (μ_n, η_n) in $\{(\mu_n, \eta_n)\}$ such that $\frac{U(\mu_n)-U(\eta_n)}{V(\eta_n)-V(\mu_n)} \notin (0,1)$. The subsequence $\{(\mu_{n(m)}, \eta_{n(m)})\}$ converges to (μ, η) and for all m, $\frac{U(\mu_n)-U(\eta_n)}{V(\eta_n)-V(\mu_n)} \in (0,1)$. Note that $\tau(\mu_n, \eta_n) = 0$ for all these discarded (μ_n, η_n). If we show that $\limsup_{n\to\infty} \tau(\mu_{n(m)}, \eta_{n(m)}) \leq \tau(\mu, \eta)$, then that establishes $\limsup_{n\to\infty} \tau(\mu_n, \eta_n) \leq \tau(\mu, \eta)$. But the former assertion is proved in (ii).

(c) $V(\mu) = V(\eta)$.

Take any sequence $\{(\mu_n, \eta_n)\}$ that converges to (μ, η). By continuity of U, V,

$$\begin{aligned} \lim_{n\to\infty} U(\mu_n) - U(\eta_n) &> 0 \\ \lim_{n\to\infty} V(\eta_n) - V(\mu_n) &= 0. \end{aligned}$$

Thus, there exists N such that for each $n \geq N$,

$$U(\mu_n) - U(\eta_n) > V(\eta_n) - V(\mu_n),$$

that is, for all $n \geq N$, $[\frac{U(\mu_n)-U(\eta_n)}{V(\eta_n)-V(\mu_n)} \notin (0,1)$ or $V(\eta_n) = V(\mu_n)]$. It follows from Lemma 9 that $\tau(\mu_n, \eta_n) = 0$ for all $n \geq N$. Hence, $\lim_{n\to\infty} \tau(\mu_n, \eta_n) = 0$. ■

The last lemma verifies that $\mathcal{C}$ satisfies Commitment is Normative. Note that $\tau(\mu, \eta) = 0$ and $(\mu, \eta) \in \Omega$ imply that that τ is continuous at (μ, η).

Lemma 12 *(a)* $x >_t x \cup y \Longrightarrow \tau(x^{+t}, (x \cup y)^{+t}) = 0$ *and* $(x^{+t}, (x \cup y)^{+t}) \in \Omega$.

(b) $x >_t x \cup y >_t y \Longrightarrow \tau((x \cup y)^{+t}, y^{+t}) = 0$ *and* $((x \cup y)^{+t}, y^{+t}) \in \Omega$.

Proof. (a) Let $x >_t x \cup y$, that is,

$$W(x) + \frac{\gamma^t}{\delta^t}\overline{V}(x) > W(x \cup y) + \frac{\gamma^t}{\delta^t}\overline{V}(x \cup y).$$

From $x \subset x \cup y$ it follows that $\overline{V}(x \cup y) \geq \overline{V}(x)$. Hence the displayed inequality implies $W(x) > W(x \cup y)$, which in turn implies $U(x^{+t}) > U((x \cup y)^{+t})$. By Lemmas 10 and 11, $\tau(x^{+t}, (x \cup y)^{+t}) = 0$ and $(x^{+t}, (x \cup y)^{+t}) \in \Omega$.

(b) Note that by [15, Lemma 1(a)], $W(x) > W(x \cup y)$ implies

$$\overline{V}(x \cup y) = \overline{V}(y) > \overline{V}(x).$$

Hence, $x \cup y >_t y$ implies

$$W(x \cup y) + \frac{\gamma^t}{\delta^t}\overline{V}(x \cup y) > W(y) + \frac{\gamma^t}{\delta^t}\overline{V}(y).$$

It follows that $W(x \cup y) > W(y)$, which implies $U((x \cup y)^{+t}) > U(y^{+t})$. By Lemmas 10 and 11, $\tau((x \cup y)^{+t}, y^{+t}) = 0$ and $((x \cup y)^{+t}, y^{+t}) \in \Omega$. ■

D Appendix: Proof of Theorem 1 (Sufficiency)

The proof is divided into three sections. The first studies $\gtrapprox$ and the second studies the normative preference $\succsim^*$ derived from $\gtrapprox$. The third defines a candidate preference $\succsim$ in terms of $\succsim^*$ and verifies that $\succsim$ is indeed an FT preference that generates $\mathcal{C}$.

D.1 Properties of $\gtrapprox$

Define the choice correspondence $\mathcal{C}^*(\cdot, \gtrapprox)$ by

$$\mathcal{C}^*(x, \gtrapprox) \equiv \{\mu \in x : \mu \gtrapprox \eta \text{ for all } \eta \in x\}.$$

Say that $\gtrapprox$ rationalizes $\mathcal{C}(\cdot)$ if $\mathcal{C}(x) = \mathcal{C}^*(x, \gtrapprox)$ for all x.

Lemma 13 $\gtrapprox$ *is the unique preference relation that rationalizes* $\mathcal{C}(\cdot)$ *and satisfies the vNM axioms.*

Proof. Step 1: $\succsim$ is continuous.

We want to show that

$$\{\eta : \eta \succsim \mu\} \text{ and } \{\eta : \mu \succsim \eta\} \text{ are closed.}$$

Take $\{\eta_n\}$ such that $\eta_n \succsim \mu$ for all n and $\eta_n \to \eta$. Consider the sequence of menus $\{\{\eta_n, \mu\}_n\}$. Since Z is endowed with the Hausdorff metric, $\eta_n \to \eta$ implies $\{\eta_n, \mu\} \to \{\eta, \mu\}$. Note that for all n, $\eta_n \in \mathcal{C}(\{\eta_n, \mu\})$. By Continuity (upper hemicontinuity of $\mathcal{C}$) and by [2, Thm 16.20], $\eta \in \mathcal{C}(\{\eta, \mu\})$, that is, $\eta \succsim \mu$, as desired. This establishes that $\{\eta : \eta \succsim \mu\}$ is closed. To see that $\{\eta : \mu \succsim \eta\}$ is closed, take $\{\eta_n\}$ such that $\mu \succsim \eta_n$ for all n and $\eta_n \to \eta$, and consider the sequence of menus $\{\{\eta_n, \mu\}_n\}$. Since $\{\eta_n, \mu\} \to \{\eta, \mu\}$ and $\mu \in \mathcal{C}(\{\eta_n, \mu\})$ for all n, we have $\mu \in \mathcal{C}(\{\eta, \mu\})$, that is, $\mu \succsim \eta$ as desired. Hence continuity of $\succsim$ is established.

For the next steps, say that a binary relation is a weak order if it is complete and transitive and define the revealed preference relation $\succsim'$ (with domain Δ) by

$$\mu \succsim' \eta \Longleftrightarrow \exists x \text{ such that } \mu, \eta \in x \text{ and } \mu \in \mathcal{C}(x).$$

Step 2: $\succsim' = \succsim$.

Suppose $\mu \succsim \eta$. Therefore, by definition, there exists x (which is $\{\mu, \eta\}$) such that $\mu, \eta \in x$ and $\mu \in \mathcal{C}(x)$, and so $\mu \succsim' \eta$. Hence, $\mu \succsim \eta \Longrightarrow \mu \succsim' \eta$. Conversely, if $\mu \succsim' \eta$, then $\exists x$ such that $\mu, \eta \in x$ and $\mu \in \mathcal{C}(x)$. Nonemptiness of $\mathcal{C}$ and WARP imply $\mu \in \mathcal{C}(\{\mu, \eta\})$. Hence $\mu \succsim' \eta \Longrightarrow \mu \succsim \eta$.

Step 3: $\mathcal{C}^*(\cdot, \succsim)$ is nonempty.

The domain Z consists of compact menus, $\succsim$ is continuous (Step 1) and thus $\mathcal{C}^*(\cdot, \succsim) \neq \phi$ by [2, Thm 2.41].

Step 4: $\succsim$ is the unique weak order that rationalizes $\mathcal{C}(\cdot)$.

The result follows from Steps 2 and 3, and [14, Prop 1.D.2].

Step 5: $\succsim$ satsifies the vNM axioms.

This follows from Steps 1 and 5, and by Independence. ∎

For $t > 0$, define $\succsim_t$ over Z by

$$x \succsim_t y \Longleftrightarrow x^{+t} \succsim y^{+t}.$$

We establish that each $\succsim_t$ satisfies the following properties.

B1 (Order*) $\succsim_t$ *is complete and transitive.*
B2 (Continuity*) *The sets* $\{y : x \succsim_t y\}$ *and* $\{y : y \succsim_t x\}$ *are closed.*
B3 (Independence*) $x >_t y \Longrightarrow \alpha x + (1-\alpha)z >_t \alpha y + (1-\alpha)z$.
B4 (Set-Betweenness*) $x \succsim_t x \cup y \succsim_t y$.
B5 (Separability*) *If* $\mu^1 = \pi^1, \mu^2 = \pi^2, \eta^1 = \nu^1$ *and* $\eta^2 = \nu^2$*, then,*

$$\{\mu, \eta\} \approx_t \{\pi, \nu\}.$$

Let $\Delta_s \subset \Delta$ be the set of lotteries on $C \times Z$ with finite support and $\Delta_s(Z)$ the set of lotteries on Z with finite support. Let δ_z denote the lottery degenerate at menu z. Define $\varphi : \Delta_s(Z) \longrightarrow Z$ by

$$\varphi(\sum p(x)\delta_x) = \sum p(x)x.$$

B6 (Indifference to Timing*) *For all* $\mu, \eta, \pi, \nu \in \Delta_s$*, if* $\mu^1 = \pi^1, \eta^1 = \nu^1, \varphi(\mu^2) = \varphi(\pi^2)$ *and* $\varphi(\eta^2) = \varphi(\nu^2)$*, then,*

$$\{\mu, \eta\} \approx_t \{\pi, \nu\}.$$

Start by showing that $\succsim_t$ satisfies Order*, Continuity*, Independence* and Set-Betweenness*, that is, $\succsim_t$ is a Self-Control preference [7].

Lemma 14 $\succsim_t$ *satisfies Order*, Continuity*, Independence* and Set-Betweenness*.*

Proof. It is clear that by Lemma 13, $\succsim_t$ satisfies Order* and Continuity*. By Set-Betweenness, $\succsim_t$ satisfies Set-Betweenness*. To see that $\succsim_t$ satisfies Independence*, observe that by Independence, for any x, y, z,

$$x^{+t} > y^{+t} \Longrightarrow \alpha x^{+t} + (1-\alpha)z^{+t} > \alpha y^{+t} + (1-\alpha)z^{+t},$$

and that by Indifference to Timing,

$$\begin{aligned} \alpha x^{+t} + (1-\alpha)z^{+t} &\approx (\alpha x + (1-\alpha)z)^{+t} \\ \alpha y^{+t} + (1-\alpha)z^{+t} &\approx (\alpha y + (1-\alpha)z)^{+t}. \end{aligned}$$

Therefore, by Order* and definition of $\succsim_t$,

$$x >_t y \Longrightarrow \alpha x + (1-\alpha)z >_t \alpha y + (1-\alpha)z,$$

that is, $\succsim_t$ satisfies Independence*. ■

By Lemma 14 and [7, Theorem 1], each $\gtrsim_t$ is represented by $W_t : Z \to \mathbb{R}$ such that

$$W_t(x) = \max_{\mu \in x}(U_t(\mu) + V_t(\mu)) - \max_{\eta \in x} V_t(\eta), \tag{19}$$

where $U_t, V_t : \Delta \to \mathbb{R}$ are linear and continuous.

Lemma 15 $\mu \gtrsim \eta \iff U_t(\mu) + V_t(\mu) \geq U_t(\eta) + V_t(\eta)$.

Proof. By [15, Lemma 1(b)], if $\eta >_t \mu$, then

$$\{\eta, \mu\} \approx_t \{\mu\} \iff U_t(\mu) + V_t(\mu) \geq U_t(\eta) + V_t(\eta).$$

Therefore, Sophistication implies that for any η, μ, and all $t \geq 1$,

$$\eta >_t \mu \Longrightarrow [\mu \gtrsim \eta \iff U_t(\mu) + V_t(\mu) \geq U_t(\eta) + V_t(\eta)]. \tag{20}$$

We need to show that the conclusion holds even when $\eta \not>_t \mu$. First make some observations.

By nondegeneracy, there exists ρ, ν such that[18]

$$\nu \gtrsim \rho \text{ and } \rho^{+1} > \nu^{+1}.$$

Furthermore, by Reversal,

$$\rho >_t \nu, \text{ for all } t \geq 1. \tag{21}$$

Note that by (20) and (21), for all $t \geq 1$,

$$\rho^{+1} > \nu^{+1} \iff U_t(\rho^{+1}) + V_t(\rho^{+1}) > U_t(\nu^{+1}) + V_t(\nu^{+1}). \tag{22}$$

Now prove the lemma. We want to show

$$\mu \gtrsim_t \eta \Longrightarrow [\mu \gtrsim \eta \iff U_t(\mu) + V_t(\mu) \geq U_t(\eta) + V_t(\eta)].$$

Suppose $\mu \gtrsim_t \eta$, $\mu \gtrsim \eta$ and by way of contradiction, $U_t(\mu) + V_t(\mu) < U_t(\eta) + V_t(\eta)$. For all $\alpha \in (0,1)$, [19]

$$\mu\alpha\rho^{+1} >_t \eta\alpha\nu^{+1} \text{ and } \mu\alpha\rho^{+1} > \eta\alpha\nu^{+1},$$

[18] To be precise, $\mu = (c,x)^{+(\tau(x,y)-1)}$ and $\eta = (c,y)^{+(\tau(x,y)-1)}$, where x, y are as in the definition of nondegeneracy.

[19] This relies on earlier results that $\gtrsim_t$ and $\gtrsim$ satisfy Independence* and Independence, respectively. Note that by (21), $\rho^{+1} >_t \nu^{+1}$.

and so by (20), for all $\alpha \in (0,1)$,

$$U_t(\mu\alpha\rho^{+1}) + V_t(\mu\alpha\rho^{+1}) > U_t(\eta\alpha\nu^{+1}) + V_t(\eta\alpha\nu^{+1}).$$

As $\alpha \to 1$, continuity of $U_t + V_t$ yields

$$U_t(\mu) + V_t(\mu) \geq U_t(\eta) + V_t(\eta),$$

a contradiction.

Next suppose $\mu \gtrapprox_t \eta$, $U_t(\mu) + V_t(\mu) \geq U_t(\eta) + V_t(\eta)$ and by way of contradiction, $\mu < \eta$. For all $\alpha \in (0,1)$,

$$\mu\alpha\rho^{+1} >_t \eta\alpha\nu^{+1} \text{ and } U_t(\mu\alpha\rho^{+1}) + V_t(\mu\alpha\rho^{+1}) > U_t(\eta\alpha\nu^{+1}) + V_t(\eta\alpha\nu^{+1}),$$

and so by (20), for all $\alpha \in (0,1)$,

$$\mu\alpha\rho^{+1} > \eta\alpha\nu^{+1}.$$

As $\alpha \to 1$, continuity of $\gtrapprox$ yields

$$\mu \gtrapprox \eta,$$

a contradiction.

This establishes that for all $\mu, \eta \in \Delta$ and all $t \geq 1$,

$$\mu \gtrapprox \eta \Longleftrightarrow U_t(\mu) + V_t(\mu) \geq U_t(\eta) + V_t(\eta).$$

as desired. ■

Lemma 16 $\gtrapprox_t$ *satisfies Separability*.*

Proof. Step 1: Show that

$$U_t(\mu) = \int_{C\times Z} \left(u_t(c) + \widehat{W}_t(y)\right) d\mu(c,y).$$

Take μ, η such that

$$\begin{aligned} \mu &= \frac{1}{2}(c,x) + \frac{1}{2}(\overline{c},\overline{x}) \\ \eta &= \frac{1}{2}(c,\overline{x}) + \frac{1}{2}(\overline{c},x). \end{aligned}$$

By Separability,

$$\{\frac{1}{2}(c,x)+\frac{1}{2}(\overline{c},\overline{x})\} \approx_t \{\frac{1}{2}(c,\overline{x})+\frac{1}{2}(\overline{c},x)\}.$$

It follows that $\gtrsim_t$ satisfies GP's version of Separability, and so, by [8, Lemma 9(1)], U is additively separable, thus establishing Step 1.

Step 2: Show that

$$\begin{aligned} U_t(\mu)+V_t(\mu) &= \int_{C\times Z}\left(u_t(c)+\widehat{W}_t(y)+v_t(c)+\widehat{V}_t(y)\right)d\mu(c,y) \\ \text{and } V_t(\mu) &= \int_{C\times Z}\left(v_t(c)+\widehat{V}_t(y)\right)d\mu(c,y). \end{aligned}$$

Since V is linear and continuous, there exists a continuous function $\overline{v}_t : C\times Z \longrightarrow \mathbb{R}$ such that for all $\mu\in\Delta$,

$$V_t(\mu)=\int \overline{v}_t(c,x)d\mu.$$

By Separability, $\frac{1}{2}(c,x)+\frac{1}{2}(\overline{c},\overline{x}) \approx \frac{1}{2}(c,\overline{x})+\frac{1}{2}(\overline{c},x)$. Hence, by Lemma (15),

$$\begin{aligned} & U_t(\frac{1}{2}(c,x)+\frac{1}{2}(\overline{c},\overline{x}))+V_t(\frac{1}{2}(c,x)+\frac{1}{2}(\overline{c},\overline{x})) \\ = \;& U_t(\frac{1}{2}(c,\overline{x})+\frac{1}{2}(\overline{c},x))+V_t(\frac{1}{2}(c,\overline{x})+\frac{1}{2}(\overline{c},x)). \end{aligned}$$

By Step 1, $U_t(\frac{1}{2}(c,x)+\frac{1}{2}(\overline{c},\overline{x}))=U_t(\frac{1}{2}(c,\overline{x})+\frac{1}{2}(\overline{c},x))$. Therefore,

$U_t(\frac{1}{2}(c,x)+\frac{1}{2}(\overline{c},\overline{x}))+V_t(\frac{1}{2}(c,x)+\frac{1}{2}(\overline{c},\overline{x})) = U_t(\frac{1}{2}(c,\overline{x})+\frac{1}{2}(\overline{c},x))+V_t(\frac{1}{2}(c,\overline{x})+\frac{1}{2}(\overline{c},x))$

$\Longrightarrow V_t(\frac{1}{2}(c,x)+\frac{1}{2}(\overline{c},\overline{x}))=V_t(\frac{1}{2}(c,\overline{x})+\frac{1}{2}(\overline{c},x))$

$\Longrightarrow V_t(c,x)+V_t(\overline{c},\overline{x}))=V_t(c,\overline{x})+V_t(\overline{c},x))$

$\Longrightarrow \overline{v}_t(c,x)+\overline{v}_t(\overline{c},\overline{x})=\overline{v}_t(c,\overline{x})+\overline{v}_t(\overline{c},x)$

$\Longrightarrow \overline{v}_t(c,x)=\overline{v}_t(c,\overline{x})-\overline{v}_t(\overline{c},\overline{x})+\overline{v}_t(\overline{c},x)$.

Define $v_t(c)\equiv\overline{v}_t(c,\overline{x})-\overline{v}_t(\overline{c},\overline{x})$ and $\widehat{V}_t(x)\equiv\overline{v}_t(\overline{c},x)$. We can then write,

$$V_t(\mu)=\int_{C\times Z}\left(v_t(c)+\widehat{V}_t(y)\right)d\mu(c,y).$$

It also follows that,

$$U_t(\mu)+V_t(\mu)=\int_{C\times Z}\left(u_t(c)+\widehat{W}_t(y)+v_t(c)+\widehat{V}_t(y)\right)d\mu(c,y).$$

Step 3: Show that if $\mu^1 = \pi^1, \mu^2 = \pi^2, \eta^1 = \nu^1$ and $\eta^2 = \nu^2$, then

$$\{\mu, \eta\} \approx_t \{\pi, \nu\}.$$

Since $\mu^1 = \pi^1, \mu^2 = \pi^2, \eta^1 = \nu^1$ and $\eta^2 = \nu^2$, by Step 2 we have

$$\begin{aligned} U_t(\mu) + V_t(\mu) &= U_t(\pi) + V_t(\pi) \\ V_t(\mu) &= V_t(\pi) \\ U_t(\eta) + V_t(\eta) &= U_t(\nu) + V_t(\nu) \\ V_t(\eta) &= V_t(\nu). \end{aligned}$$

Observe that

$$\max_{\{\mu,\eta\}}(U_t + V_t) = \max_{\{\pi,\nu\}}(U_t + V_t) \text{ and } \max_{\{\mu,\eta\}} V_t = \max_{\{\pi,\nu\}} V_t.$$

Hence by (19), $\{\mu, \eta\} \approx_t \{\pi, \nu\}$. ■

Lemma 17 $\gtrapprox_t$ *satisfies Indifference to Timing**.

Proof. The proof follows steps that are similar to those in Lemma 16: for the $\widehat{W_t}$ and $\widehat{V_t}$ in the proof of Lemma 16, use Indifference to Timing to show that $\widehat{W_t}$ is linear, then show that $\widehat{V_t}$ is linear, and then use (19) to establish the result. ■

D.2 Normative Preference $\succsim^*$

For each $t > 0$, denote the restriction of $\gtrapprox_t$ to $\Delta \subset Z$ by $\gtrapprox_{t|_\Delta}$, and define $\gtrapprox_{0|_\Delta} = \gtrapprox$. Then $\{\gtrapprox_{t|_\Delta}\}_{t=0}^{\infty}$ is a set of preference relations defined over Δ. Since $\mathcal{C}(\cdot)$ satisfies WARP, Continuity and Reversal, $\{\gtrapprox_{t|_\Delta}\}$ satisfies the conditions in Theorem A, Appendix B. Thus, there is a well-defined normative preference $\succsim^*$ defined over Δ and a well-defined function $\tau : \Delta \times \Delta \to \mathbb{R}$ as in Lemma 1. Let Ω be the subset of $\Delta \times \Delta$ on which τ is upper semicontinuous, that is,

$$\Omega = \{(\mu, \eta) \in \Delta \times \Delta : (\mu_n, \eta_n) \to (\mu, \eta) \Longrightarrow \limsup_{n\to\infty} \tau(\mu_n, \eta_n) \leq \tau(\mu, \eta)\}.$$

Lemma 18 states some results from Appendix B.

Lemma 18 *(a)* $\mu \succ^* \eta \Longleftrightarrow [\mu >_{\tau(\mu,\eta)} \eta$ *and* $(\mu,\eta) \in \Omega]$

(b) $\mu \succsim^* \eta \Longleftrightarrow$ *there exists a sequence* $\{(\mu_n, \eta_n)\}$ *that converges to* (μ, η) *and* $\mu_n \gtrapprox_{\tau(\mu_n,\eta_n)} \eta_n$ *for all* n.

(c) $\mu \gtrapprox_{\tau(\mu,\eta)} \eta \Longrightarrow \mu \succsim^* \eta$.

Lemma 19 $\succsim^*$ *satisfies order, continuity and independence*

Proof. Theorem A establishes that $\succsim^*$ is complete, transitive and continuous. To prove independence, it suffices by Theorem B (Appendix B) to show that each $\gtrapprox_{t|_\Delta}$ satisfies independence. Recall that $\gtrapprox_{t|_\Delta}$ is the restriction of $\gtrapprox_t$ to Δ, and apply Lemma 14. ■

In the remainder of the subsection we establish a stationarity property of $\succsim^*$.

Lemma 20 $(c,\mu) \succsim^* (c,\eta) \Longleftrightarrow (c',\mu) \succsim^* (c',\eta)$

Proof. Step 1: $(c,x)\frac{1}{2}(c',x') \sim^* (c,x')\frac{1}{2}(c',x)$.

By Separability, for all t,

$$(\frac{1}{2}(c,x) + \frac{1}{2}(c',x'))^{+t} \approx (\frac{1}{2}(c,x') + \frac{1}{2}(c',x))^{+t}.$$

Hence, $\tau((c,x)\frac{1}{2}(c',x'),(c,x')\frac{1}{2}(c',x)) = 0$ and $(c,x)\frac{1}{2}(c',x') \approx (c,x')\frac{1}{2}(c',x)$. Apply Lemma 18(c) to obtain $(c,x)\frac{1}{2}(c',x') \sim^* (c,x')\frac{1}{2}(c',x)$.

Step 2: The result.

Suppose by way of contradiction that $(c,\mu) \succsim^* (c,\eta)$ and $(c',\mu) \prec^* (c',\eta)$. Since $\succsim^*$ satisfies the vNM axioms,

$$(c,\mu)\frac{1}{2}(c',\eta) \succ^* (c,\eta)\frac{1}{2}(c',\mu). \tag{23}$$

But by Step 1,

$$(c,\eta)\frac{1}{2}(c',\mu) \sim^* (c,\mu)\frac{1}{2}(c',\eta) \text{ and } (c,\mu)\frac{1}{2}(c',\eta) \sim^* \ (c,\eta)\frac{1}{2}(c',\mu),$$

and so $(c,\eta)\frac{1}{2}(c',\mu) \succ^* (c,\mu)\frac{1}{2}(c',\eta)$, contradicting (23). ■

Lemma 21 $\mu^{+1}\alpha\eta^{+1} \sim^* (\mu\alpha\eta)^{+1}$

Proof. By Indifference to Timing

$$\mu^{+1}\alpha\eta^{+1} \approx (\mu\alpha\eta)^{+1}$$

and by Lemma 17, for all $t \geq 1$,

$$\{\mu^{+1}\alpha\eta^{+1}\}^{+t} \approx \{(\mu\alpha\eta)^{+1}\}^{+t}$$

Hence, $\tau(\mu^{+1}\alpha\eta^{+1}, (\mu\alpha\eta)^{+1}) = 0$ and $\mu^{+1}\alpha\eta^{+1} \approx (\mu\alpha\eta)^{+1}$. Apply Lemma 18(c) to obtain $\mu^{+1}\alpha\eta^{+1} \sim^* (\mu\alpha\eta)^{+1}$. ∎

Lemma 22 $\mu \succsim^* \eta \Longrightarrow (c, \mu) \succsim^* (c, \eta)$.

Proof. Step 1: $\mu \gtrapprox_{\tau(\mu,\eta)} \eta \Longleftrightarrow \mu^{+1} \gtrapprox_{\tau(\mu^{+1},\eta^{+1})} \eta^{+1}$.

First show that if $\tau(\mu, \eta) > 0$, then

$$\tau(\mu^{+1}, \eta^{+1}) = \tau(\mu, \eta) - 1. \tag{24}$$

For this purpose, suppose without loss of generality that $\mu \lessapprox \eta$. If $\tau(\mu, \eta) > 0$, then by Lemma 1, $\mu^{+t} > \eta^{+t}$ for all $t \geq \tau(\mu, \eta)$ and $\mu^{+t} \lessapprox \eta^{+t}$ for all $t < \tau(\mu, \eta)$. It follows that

$$\begin{aligned} \mu^{+t+1} &> \eta^{+t+1}, \text{ for all } t \geq \tau(\mu, \eta) - 1. \\ \text{and } \mu^{+t+1} &\lessapprox \eta^{+t+1}, \text{ for all } t < \tau(\mu, \eta) - 1. \end{aligned}$$

The assertion follows. Now prove the result. It follows by definition of τ if $\tau(x, y) = 0$ (in which case $\tau(x^{+1}, y^{+1}) = 0$ as well). When $\tau(\mu, \eta) > 0$, then note that $\mu^{+\tau(\mu,\eta)} = (\bar{c}, \mu)^{+(\tau(\mu,\eta)-1)}$ and $\eta^{+\tau(\mu,\eta)} = (\bar{c}, \eta)^{+(\tau(\mu,\eta)-1)}$. The result follows from (24).

Step 2: $\mu \succsim^* \eta \Longrightarrow \mu^{+1} \succsim^* \eta^{+1}$.

If $\mu \succsim^* \eta$, then by Lemma 18(b), there exists a sequence $\{(\mu_n, \eta_n)\}$ such that $(\mu_n, \eta_n) \to (\mu, \eta)$ and $\mu_n \gtrapprox_{\tau(\mu_n,\eta_n)} \eta_n$ for all n. It follows by Step 1 that $\{(\mu_n^{+1}, \eta_n^{+1})\}$ is a sequence such that $(\mu_n^{+1}, \eta_n^{+1}) \to (\mu^{+1}, \eta^{+1})$ and $\mu_n^{+1} \gtrapprox_{\tau(\mu_n^{+1},\eta_n^{+1})} \eta_n^{+1}$ for all n. But then by Lemma 18(b), $\mu^{+1} \succsim^* \eta^{+1}$. The result follows by Lemma 20. ∎

Lemma 23

$$\mu \succsim^* \eta \Longleftrightarrow (c, \mu) \succsim^* (c, \eta).$$

Proof. By Lemma 20, it suffices to show

$$\mu \succsim^* \eta \iff (\bar{c}, \mu) \succsim^* (\bar{c}, \eta).$$

Define a binary relation $\succsim^{**}$ over Δ by

$$\mu \succsim^{**} \eta \iff (\bar{c}, \mu) \succsim^* (\bar{c}, \eta).$$

Make three observations. Firstly, by Lemma 22, $\mu \succsim^* \eta \Longrightarrow \mu \succsim^{**} \eta$. Hence,

$$\mu \sim^* \eta \Longrightarrow \mu \sim^{**} \eta. \tag{25}$$

Secondly, establish that $\succsim^{**}$ satisfies the vNM axioms: Since $\succsim^*$ is a continuous weak order, so is $\succsim^{**}$. To show

$$\mu \succ^{**} \eta \Longrightarrow \alpha\mu + (1-\alpha)\nu \succ^{**} \alpha\eta + (1-\alpha)\nu,$$

note that

$\mu \succ^{**} \eta$
$\Longrightarrow (\bar{c}, \mu)\} \succ^* (\bar{c}, \eta)$
$\Longrightarrow \alpha(\bar{c}, \mu) + (1-\alpha)(\bar{c}, \nu) \succ^* \alpha(\bar{c}, \eta) + (1-\alpha)(\bar{c}, \nu)$
$\Longrightarrow (\bar{c}, \alpha\mu + (1-\alpha)\nu) \succ^* (\bar{c}, \alpha\eta + (1-\alpha)\nu)$ by Lemma 21
$\Longrightarrow \alpha\mu + (1-\alpha)\nu \succ^{**} \alpha\eta + (1-\alpha)\nu$. Hence, $\succsim^{**}$ satisfies the vNM axioms.

Lastly, show that $\succsim^{**}$ (and hence, $\succsim^*$) is non-trivial, in the sense that there exist $\rho, \nu \in \Delta$ such that $\rho \succ^{**} \nu$. By Nondegeneracy of $\mathcal{C}$, there exists x, y such that $x^{+t} > (x \cup y)^{+t}$. By Reversal, we can assume that $t > 1$. By Commitment is Normative, $\tau(x^{+t}, (x \cup y)^{+t}) = 0$ and τ is continuous at $(x^{+t}, (x \cup y)^{+t})$, and so $(x^{+t}, (x \cup y)^{+t}) \in \Omega$. Hence, by Lemma 18(a), $x^{+t} \succ^* (x \cup y)^{+t}$. It follows that $x^{+(t-1)} \succ^{**} (x \cup y)^{+(t-1)}$, that is, $\succsim^{**}$ is non-trivial.

Now prove the lemma. By continuity of $\succsim^*$ and compactness of Δ, there exist $\succsim^*$-best and $\succsim^*$-worst elements $\overline{\mu}, \underline{\mu} \in \Delta$, respectively. By Lemma 22, for all $\mu \in \Delta$,

$$\overline{\mu} \succsim^* \mu \succsim^* \underline{\mu} \Longrightarrow \overline{\mu} \succsim^{**} \mu \succsim^{**} \underline{\mu},$$

and so $\overline{\mu}, \underline{\mu}$ are also the $\succsim^{**}$ $-$best and $\succsim^{**}$ $-$worst elements, respectively. As noted, $\succsim^*$ and $\succsim^{**}$ have EU representations, say, W^* and W^{**}, respectively. By non-triviality of $\succsim^*$ and $\succsim^{**}$, we can normalize W^* and W^{**} so that

$W^*(\overline{\mu}) = 1$ and $W^*(\underline{\mu}) = 0$, and similarly for W^{**}. Note that, for any $\alpha \in [0,1]$,

$$W^*(\overline{\mu}\alpha\underline{\mu}) = \alpha = W^{**}(\overline{\mu}\alpha\underline{\mu}).$$

For any $\mu \in \Delta$ there exists $\alpha_\mu \in [0,1]$ such that $\mu \sim \overline{\mu}\alpha_\mu\underline{\mu}$. By (25),

$$\mu \sim^* \overline{\mu}\alpha_\mu\underline{\mu} \Longrightarrow \mu \sim^{**} \overline{\mu}\alpha_\mu\underline{\mu},$$

that is,

$$W^*(\mu) = W^*(\overline{\mu}\alpha_\mu\underline{\mu}) \Longrightarrow W^{**}(\mu) = W^{**}(\overline{\mu}\alpha_\mu\underline{\mu}).$$

But then

$$W^*(\mu) = W^*(\overline{\mu}\alpha_\mu\underline{\mu}) = \alpha_\mu = W^{**}(\overline{\mu}\alpha_\mu\underline{\mu}) = W^{**}(\mu).$$

Thus, W^* and W^{**} represent the same preferences and

$$\mu \succsim^* \mu' \Longleftrightarrow \mu \succsim^{**} \mu'.$$

The assertion follows. ■

D.3 FT Preference $\succsim$

The candidate for the FT preference $\succsim$ over Z that generates $\mathcal{C}(\cdot)$ is defined by

$$x \succsim y \Longleftrightarrow (c,x) \succsim^* (c,y),$$

for some $c \in C$. By Lemma 20, the preference $\succsim$ is invariant to the choice of c. We verify that $\succsim$ is an FT preference by checking that it satisfies the conditions in [15, Theorems 1 and 3].

Lemma 24 *For any x, y and $t \geq 1$,*
(a) $x >_t x \cup y \Longrightarrow x \succ x \cup y$.
(b) $x >_t x \cup y >_t y \Longrightarrow x \succ x \cup y \succ y$.
(c) $x >_t x \cup y \approx_t y \Longrightarrow x \succ x \cup y \sim y$.

Proof. Suppose $x >_t x \cup y$, or equivalently, $x^{+t} > (x \cup y)^{+t}$. By Commitment is Normative, $\tau(x^{+t}, (x \cup y)^{+t}) = 0$ and τ is continuous at $(x^{+t}, (x \cup y)^{+t})$, and so $(x^{+t}, (x \cup y)^{+t}) \in \Omega$. Hence, by Lemma 18(a), $x^{+t} \succ^* (x \cup y)^{+t}$, and repeated application of Lemma 23 establishes $x^{+1} \succ^* (x \cup y)^{+1}$, which in turn establishes (a). A similar argument establishes (b).

Turn to (c). In what follows we prove that $x \cup y \approx_{t'} y$ for all $t' \geq t$, since then $(x \cup y)^{+1} \approx_{\tau((x\cup y)^{+1}, y^{+1})} y^{+1}$, and so by Lemma 18(c), $(x \cup y)^{+1} \sim^* y^{+1}$, that is, $x \cup y \sim y$.

As before, $x >_t x \cup y$ implies $x >_{t'} x \cup y$ for all $t' \geq t$. It follows by Set-Betweenness that $x >_{t'} y$ for all $t' \geq t$. By [15, Lemma 1(b)], for all $t' \geq t$,

$$x \cup y \approx_{t'} y \Longleftrightarrow \max_{\mu \in y}\{U_{t'} + V_{t'}\} \geq \max_{\eta \in x}\{U_{t'} + V_{t'}\}. \tag{26}$$

where $U_{t'}$ and $V_{t'}$ are as in (19). By Lemma 15, for all $t' \geq t$,

$$\max_{\mu \in y}\{U_{t'} + V_{t'}\} \geq \max_{\eta \in x}\{U_{t'} + V_{t'}\} \Longleftrightarrow \max_{\mu \in y}\{U_t + V_t\} \geq \max_{\eta \in x}\{U_t + V_t\}. \tag{27}$$

By hypothesis, $x \cup y \approx_t y$. Thus, by (26) and (27), $x \cup y \approx_{t'} y$ for all $t' \geq t$, as desired. ■

In the remainder of the proof we verify that $\succsim$ is a nondegenerate FT preference and that it generates $\mathcal{C}(\cdot)$. In A.1 we stated some axioms for the preference $\gtrapprox_t$. The same names will be used for axioms that impose similar restrictions on $\succsim$.

Lemma 25 *$\succsim$ satisfies Order*, Continuity*. Moreover, $\succsim$ satisfies Independence*: for all $\alpha \in (0,1)$,*

$$\{\mu\} \succ \{\eta\} \Longrightarrow \{\alpha\mu + (1-\alpha)\nu\} \succ \{\alpha\eta + (1-\alpha)\nu\}.$$

Proof. Follows from Lemma 19 and 23. ■

Lemma 26 *$\succsim$ satisfies Stationarity*:*

$$z \succsim z' \Longleftrightarrow \{(c,z)\} \succsim \{(c,z')\}.$$

Proof. This follows from Lemmas 23 and 20, and by definition of $\succsim$. ■

Lemma 27 *$\succsim$ satisfies Set-Betweenness*:*

$$x \succsim y \Longrightarrow x \succsim x \cup y \succsim y.$$

Proof. Begin by establishing that if $\mu >_{\tau(\mu,\eta)} \eta$ then

$$(\mu^{+1}, \eta^{+1}) \in \Omega \Longrightarrow (\mu, \eta) \in \Omega. \tag{28}$$

Suppose $(\mu, \eta) \notin \Omega$, so that there exists a sequence $\{(\mu_n, \eta_n)\}$ that converges to (μ, η) and $\limsup_{n\to\infty} \tau(\mu_n, \eta_n) > \tau(\mu, \eta)$. Without loss of generality, $\tau(\mu_n, \eta_n) > \tau(\mu, \eta)$ for all n. Suppose by way of contradiction that $\limsup_{n\to\infty} \tau(\mu_n, \eta_n) = T < \infty$. Thus, there exists N such that for all $n \geq N$, $T + 1 > \tau(\mu_n, \eta_n)$. Also, for large enough n, $\mu_n >_{\tau(\mu,\eta)} \eta_n$, and since

$$T + 1 > \tau(\mu_n, \eta_n) > \tau(\mu, \eta),$$

it follows that for all large enough n, $\eta_n \gtrapprox_{T+1} \mu_n$. By continuity of $\gtrapprox_{T+1}$, $\eta \gtrapprox_{T+1} \mu$. But since $T + 1 > \tau(\mu, \eta)$, this contradicts the hypothesis that $\mu >_{\tau(\mu,\eta)} \eta$. Therefore, $\limsup_{n\to\infty} \tau(\mu_n, \eta_n) = \infty$.

To complete the argument, observe that $\{(\mu_n^{+1}, \eta_n^{+1})\}$ is a sequence that converges to (μ^{+1}, η^{+1}), and by (24) in Lemma 22, $\limsup_{n\to\infty} \tau(\mu_n^{+1}, \eta_n^{+1}) = \infty$.[20] It follows that $(\mu^{+1}, \eta^{+1}) \notin \Omega$, thus proving (28).

To prove Set-Betweenness*, we need to show

$$x^{+1} \succsim^* y^{+1} \Longrightarrow x^{+1} \succsim^* (x \cup y)^{+1} \succsim^* y^{+1}.$$

Denote $\tau(x^{+1}, y^{+1})$ by τ and consider two cases.

Case (a): $x^{+1} \succ^* y^{+1}$

Then by Lemma 18(a), $x^{+1} >_\tau y^{+1}$. By Set-Betweenness, $x^{+1} \gtrapprox_t (x \cup y)^{+1} \gtrapprox_t y^{+1}$ for all $t \geq \tau$. Hence

$$x^{+1} \gtrapprox_{\tau(x^{+1},(x\cup y)^{+1})} (x \cup y)^{+1} \gtrapprox_{\tau((x\cup y)^{+1},y^{+1})} y^{+1},$$

and by Lemma 18(c), $x^{+1} \succsim^* (x \cup y)^{+1} \succsim^* y^{+1}$.

Case (b): $x^{+1} \backsim^* y^{+1}$

By Lemma 23, $x^{+\tau} \backsim^* y^{+\tau}$. Note that $\tau(x^{+\tau}, y^{+\tau}) = 0$. By Lemma 18(a), either $x^{+\tau} \approx y^{+\tau}$ or $[x^{+\tau} \not\approx y^{+\tau}$ and $(x^{+\tau}, y^{+\tau}) \notin \Omega]$ holds.

(i) Suppose first that $x^{+\tau} \approx y^{+\tau}$. Then by Set-Betweenness, $x^{+\tau} \approx_t (x \cup y)^{+\tau} \approx_t y^{+\tau}$ for all t. Thus, $\tau(x^{+\tau}, (x \cup y)^{+\tau}) = \tau((x \cup y)^{+\tau}, y^{+\tau}) = 0$

[20]Note that we had assumed $\tau(\mu_n, \eta_n) > \tau(\mu, \eta)$ for all n, and so $\tau(\mu_n, \eta_n) > 0$ for all n, as required by (24).

and $x^{+\tau} \approx (x \cup y)^{+\tau} \approx y^{+\tau}$. By Lemma 18(c), $x^{+\tau} \backsim^* (x \cup y)^{+\tau} \backsim^* y^{+\tau}$. By repeated application of Lemma 23, $x^{+1} \backsim^* (x \cup y)^{+1} \backsim^* y^{+1}$, as desired.

(ii) Now suppose $[x^{+\tau} > y^{+\tau}$ and $(x^{+\tau}, y^{+\tau}) \notin \Omega]$. Since, $\tau(x^{+\tau}, y^{+\tau}) = 0$, we have $x^{+\tau} >_t y^{+\tau}$ for all t. By Set-Betweenness, $x^{+\tau} \gtrapprox_t (x \cup y)^{+\tau}$ for all t. If $x^{+\tau} >_t (x \cup y)^{+\tau}$ for some t, then Commitment is Normative implies that τ is continuous at $(x^{+\tau+t}, y^{+\tau+t})$, which implies $(x^{+\tau+t}, y^{+\tau+t}) \in \Omega$. By repeated application of (28), $(x^{+\tau}, y^{+\tau}) \in \Omega$, a contradiction. Therefore, $x^{+\tau} \approx_t (x \cup y)^{+\tau}$ for all t , and so, by an argument similar to the one in (i), Lemma 18(c) and repeated application of Lemma 23 implies $x^{+1} \backsim^* (x \cup y)^{+1}$.

By hypothesis $x^{+1} \backsim^* y^{+1}$ and we have seen that $x^{+1} \backsim^* (x \cup y)^{+1}$. Hence by transitivity of $\succsim^*$, $(x \cup y)^{+1} \backsim^* y^{+1}$. Put together, $x^{+1} \backsim^* (x \cup y)^{+1} \backsim^* y^{+1}$, as desired. ■

Lemma 28 $\succsim$ *satisfies Separability*: if* $\mu^1 = \pi^1, \mu^2 = \pi^2, \eta^1 = \nu^1$ *and* $\eta^2 = \nu^2$*, then,*

$$\{\mu, \eta\} \backsim \{\pi, \nu\}.$$

Proof. By Lemma 16, $\{\mu, \eta\} \approx_t \{\pi, \nu\}$ for all $t \geq 1$. Thus, $\tau(\{\mu, \eta\}^{+1}, \{\pi, \nu\}^{+1}) = 0$ and

$$\{\mu, \eta\}^{+1} \approx \{\pi, \nu\}^{+1}.$$

From Lemma 18(c), we see that $\{\mu, \eta\}^{+1} \backsim^* \{\pi, \nu\}^{+1}$, and so $\{\mu, \eta\} \backsim \{\pi, \nu\}$. ■

Lemma 29 $\succsim$ *satisfies Indifference to Timing*: for any* $\mu, \eta, \pi, \nu \in \Delta_s$*, if* $\mu^1 = \pi^1, \eta^1 = \nu^1, \varphi(\mu^2) = \varphi(\pi^2)$ *and* $\varphi(\eta^2) = \varphi(\nu^2)$*, then,*

$$\{\mu, \eta\} \backsim \{\pi, \nu\}.$$

Proof. Observe that by Lemma 17, $\{\mu, \eta\} \approx_t \{\pi, \nu\}$ for all $t \geq 1$, and then argue as in Lemma 28. ■

Lemma 30 $\succsim$ *satisfies Temptation Stationarity*:*

$$x \succ x \cup y \Longleftrightarrow \{(c, x)\} \succ \{(c, x), (c, y)\}.$$

Proof. By Preference for Early Choice, $\tau(x^{+1}, (x \cup y)^{+1}) = \tau(\{(c, x)\}^{+1}, \{(c, x), (c, y)\}^{+1})$. Denote these by τ. Observe that

$x \succ x \cup y$

$\iff x^{+1} \succ^* (x \cup y)^{+1}$
$\iff x^{+\tau} > (x \cup y)^{+\tau}$, by Lemma 18(a)
and $(x^{+\tau}, (x \cup y)^{+\tau}) \in \Omega$ by Commitment is Normative
$\iff \{(c, x)\}^{+\tau} > \{(c, x), (c, y)\}^{+\tau}$ by Preference for Early Choice,
and $(\{(c, x)\}^{+\tau}, \{(c, x), (c, y)\}^{+\tau}) \in \Omega$ by Commitment is Normative
$\iff \{(c, x)\}^{+1} \succ^* \{(c, x), (c, y)\}^{+1}$ by Lemma 18(a)
$\iff \{(c, x)\} \succ \{(c, x), (c, y)\}$, as desired. ■

Lemma 31 $\succsim$ *satisfies Preference Reversal*: there exist* μ *and* η *such that* $\{\mu\} \succ \{\mu, \eta\}$ *and* $T > 0$ *such that*

$$\begin{aligned} \{\mu^{+t}, \eta^{+t}\} &\sim \{\eta^{+t}\} \text{ for all } t < T, \\ \{\mu^{+t}, \eta^{+t}\} &\succ \{\eta^{+t}\} \text{ for all } t \geq T. \end{aligned}$$

Proof. By nondegeneracy of $\mathcal{C}(\cdot)$, there exists μ, η such that $\tau(\mu, \eta) > 0$ and

$$\begin{aligned} \mu^{+t} &\leq \eta^{+t} \text{ for all } t < \tau(\mu, \eta), \\ \mu^{+t} &> \eta^{+t} \text{ for all } t \geq \tau(\mu, \eta). \end{aligned}$$

Consider any $t' \geq \tau(\mu, \eta)$ and $t < \tau(\mu, \eta)$. We know $\{\mu^{+t}\}^{+t'} > \{\eta^{+t}\}^{+t'}$ and $\mu^{+t} \leq \eta^{+t}$, and by Sophistication this implies $\{\mu^{+t}, \eta^{+t}\}^{+t'} \approx \{\eta^{+t}\}^{+t'}$. Hence, for all $t' \geq \tau(\mu, \eta)$ and $t < \tau(\mu, \eta)$,

$$\{\mu^{+t}\}^{+t'} > \{\mu^{+t}, \eta^{+t}\}^{+t'} \approx \{\eta^{+t}\}^{+t'}, \tag{29}$$

and so, by Lemma 24(c), for all $t < \tau(\mu, \eta)$,

$$\{\mu^{+t}\} \succ \{\mu^{+t}, \eta^{+t}\} \sim \{\eta^{+t}\}.$$

Note that this implies, in particular, $\{\mu\} \succ \{\mu, \eta\}$ and $\{\mu^{+t}, \eta^{+t}\} \sim \{\eta^{+t}\}$ for all $t < \tau(\mu, \eta)$. It remains to show that $\{\mu^{+t}, \eta^{+t}\} \succ \{\eta^{+t}\}$ for all $t \geq \tau(\mu, \eta)$.

By (29), for all $t' \geq \tau(\mu, \eta)$ and $t < \tau(\mu, \eta)$,

$$\{\mu^{+t}\}^{+t'} > \{\mu^{+t}, \eta^{+t}\}^{+t'}.$$

Repeated application of Preference for Early Choice yields that for any $t' \geq \tau(\mu, \eta)$ and $t \geq \tau(\mu, \eta)$,

$$\{\mu^{+t}\}^{+t'} > \{\mu^{+t}, \eta^{+t}\}^{+t'}.$$

Moreover, for any $t, t' \geq \tau(\mu, \eta)$, we know $\{\mu^{+t}\}^{+t'} > \{\eta^{+t}\}^{+t'}$ and $\mu^{+t} > \eta^{+t}$, and by Sophistication, this implies $\{\mu^{+t}, \eta^{+t}\}^{+t'} > \{\eta^{+t}\}^{+t'}$. Put together, for any $t, t' \geq \tau(\mu, \eta)$,

$$\{\mu^{+t}\}^{+t'} > \{\mu^{+t}, \eta^{+t}\}^{+t'} > \{\eta^{+t}\}^{+t'}.$$

By Lemma 24(b),

$$\{\mu^{+t}, \eta^{+t}\} \succ \{\eta^{+t}\} \text{ for all } t \geq \tau(\mu, \eta),$$

as desired. ■

Lemma 32 $\succsim$ *is nondegenerate.*

Proof. By nondegeneracy of $\mathcal{C}(\cdot)$ and Lemma 24(b). ■

The above Lemmas establish that $\succsim$ is a nondegenerate FT preference and thus, by [15, Thm 1 and 3], $\succsim$ has the representation,

$$\begin{aligned} W(z) &= \max_{\mu \in z} \int_{C \times Z} \big(u(c) + \delta W(x) + v(c) + \gamma \overline{V}(x)\big)\, d\mu(c, x) \\ &\quad - \max_{\eta \in z} \int_{C \times Z} \big(v(c) + \gamma \overline{V}(y)\big)\, d\eta(c, y), \\ \text{where } \overline{V}(z) &= \max_{\mu \in z} \int_{C \times Z} \big(v(c) + \gamma \overline{V}(x)\big)\, d\mu(c, x) \text{ and } 0 < \gamma < \delta < 1. \end{aligned}$$

It remains to show that $\succsim$ generates $\mathcal{C}(\cdot)$. Let

$$U(\mu) = \int_{C \times Z} (u(c) + \delta W(x))\, d\mu(c, x) \text{ and } V(\mu) = \int_{C \times Z} \big(v(c) + \gamma \overline{V}(y)\big)\, d\mu(c, y).$$

Lemma 33 $\succsim$ *generates* $\mathcal{C}(\cdot)$.

Proof. By nondegeneracy, U is not an affine transformation of V (see proof of [15, Theorem 2]). Lemmas 24(a) and 24(b) establish that for all t, $\succsim$ has *more preference for commitment* than $\gtrapprox_t$ and $\succsim$ has *more self-control* than $\gtrapprox_t$; see GP for definitions of these terms. By [7, Theorem 8],

$$\begin{aligned} U_t &= \alpha U + (1 - \alpha) V \\ V_t &= \alpha' U + (1 - \alpha') V, \end{aligned}$$

for $\alpha, \alpha' \in [0,1]$, which implies

$$U_t + V_t = (\alpha + \alpha')U + (2 - \alpha - \alpha')V. \tag{30}$$

Furthermore, by [7, Theorem 9],

$$U_t + V_t = \beta(U + V) + (1 - \beta)V = \beta U + V, \tag{31}$$

for $\beta \in [0,1]$. Since U is not an affine transformation of V, (30) and (31) imply

$$U_t + V_t = U + V.$$

Hence, by Lemma 15,

$$\mu \gtrapprox \eta \Longleftrightarrow U(\mu) + V(\mu) \geq U(\eta) + V(\eta),$$

as desired. ■

E Appendix: Proof of Theorem 1 (Uniqueness)

Lemma 34 *If $U(\mu) = U(\eta)$ and $\mu > \eta$, then $(\mu, \eta) \notin \Omega$.*

Proof. By the definition of $\gtrapprox$, $U(\mu) = U(\eta)$ and $\mu < \eta$ imply $V(\mu) < V(\eta)$. By nondegeneracy, there exists ν, ρ such that $U(\nu) > U(\rho)$ and $V(\nu) < V(\rho)$. Consider the sequence $\{(\mu\alpha_n\nu, \rho\alpha_n\eta)\}$ that converges to (μ, η). Since U, V are linear, for each n, $U(\mu\alpha_n\nu) > U(\rho\alpha_n\eta)$ and $V(\mu\alpha_n\nu) < V(\rho\alpha_n\eta)$ and by Lemma 10,

$$\mu\alpha_n\nu >_t \rho\alpha_n\eta \text{ for all } t \geq \tau(\mu\alpha_n\nu, \rho\alpha_n\eta). \tag{32}$$

The hypothesis (that is, $U(\mu) = U(\eta)$ and $\mu < \eta$) implies that $\forall t$,

$$\mu <_t \eta. \tag{33}$$

Suppose by way of contradiction that $\limsup_{n\to\infty} \tau(\mu\alpha_n\nu, \rho\alpha_n\eta) = T \leq \tau(\mu, \eta)$. Then there exists N such that

$$\tau(\mu\alpha_n\nu, \rho\alpha_n\eta) < T + 1, \quad \text{for all } n \geq N. \tag{34}$$

However, $\mu <_{T+1} \eta$ by (32) and so there exists N' such that

$$\mu\alpha_n\nu <_{T+1} \rho\alpha_n\eta, \quad \text{for all } n \geq N'.$$

But by (32) this implies $\tau(\mu\alpha_n\nu, \rho\alpha_n\eta) > T + 1$ for all $n \geq \max\{N, N'\}$, a contradicting (34). ■

Lemma 35 *If $x \succ y$, then there exists T such that $(c, x) >_t (c, y)$ for all $t \geq T$.*

Proof. The hypothesis implies $U(c, x) > U(c, y)$, and the result follows from Lemma 10. $\blacksquare$

Lemma 36 *If $x \sim y$, then $\tau(x^{+1}, y^{+1}) = 0$.*

Proof. Since, for any t,

$$(c, x) \gtrapprox_t (c, y) \Longleftrightarrow W(x) + \frac{\gamma^t}{\delta^t}\overline{V}(x) \geq W(y) + \frac{\gamma^t}{\delta^t}\overline{V}(y),$$

the hypothesis $x \sim y$ implies

$$(c, x) \gtrapprox_t (c, y) \Longleftrightarrow \overline{V}(x) \geq \overline{V}(y).$$

It follows that for all t, t', $(c, x) \gtrapprox_t (c, y) \Longleftrightarrow (c, x) \gtrapprox_{t'} (c, y)$, that is, $\tau(x^{+1}, y^{+1}) = 0$. $\blacksquare$

Lemma 37 *$\mathcal{C}(\cdot)$ is generated by a unique FT preference $\succsim$.*

Proof. Suppose, by way of contradiction, that $\succsim$ and $\succsim'$ are two different FT preferences that generate $\mathcal{C}(\cdot)$. Then, there exist x and y such that $x \succ y$ and $y \succsim' x$. Let (U, V) and (U', V') be representations of $\succsim$ and $\succsim'$, respectively. Consider three possibilities:

(a) $x \succ y$ and $y \succ' x$.
By Lemma 35, $\succsim$ and $\succsim'$ do not generate the same choice correspondence, a contradiction.

(b) $x \succ y$, $\tau(x^{+1}, y^{+1}) > 0$, and $y \sim' x$.
By Lemma 36, $y \sim' x$ implies $\tau(x^{+1}, y^{+1}) = 0$, and so $\succsim$ and $\succsim'$ do not generate the same choice correspondence, a contradiction.

(c) $x \succ y$, $\tau(x^{+1}, y^{+1}) = 0$, and $y \sim' x$.
By the representation, $x \succ y$ implies $\{(c, x)\} \succ \{(c, y)\}$, and so, $U(c, x) > U(c, y)$. By Lemma 11, $(x^{+1}, y^{+1}) \in \Omega$. Furthermore, Lemma 35, $x \succ y$ and $\tau(x^{+1}, y^{+1}) = 0$ imply $(c, x) > (c, y)$. However, $y \sim' x$ and $(c, x) > (c, y)$ imply $U'(c, x) = U'(c, y)$ and $V'(c, x) > V'(c, y)$, and so, by Lemma 34, $(x^{+1}, y^{+1}) \notin \Omega$, a contradiction. $\blacksquare$

F Appendix: Proof of Theorem 2

First prove the theorem for a representation (U, V) of a nondegenerate FT preference $\succsim$ for which $V \geq 0$. Let $\gtrapprox$ be the preference relation that is represented by $\varphi : \Delta(C \times Z) \to \mathbb{R}$ where for all $\mu \in \Delta$,

$$\varphi(\mu) = U(\mu) + V(\mu).$$

For each $t > 0$, define $\gtrapprox_t$ on Δ by

$$\mu \gtrapprox_t \eta \Longleftrightarrow \mu^{+t} \gtrapprox \eta^{+t}.$$

It is straightforward to establish that $\gtrapprox_t$ is represented by $\varphi_t : \Delta(C \times Z) \to \mathbb{R}$ where for all $\mu \in \Delta$,

$$\varphi_t(\mu) = U(\mu) + \left(\frac{\gamma}{\delta}\right)^t V(\mu).$$

Lemma 38 *The sequence $\{\varphi_t\}$ uniformly converges to U.*

Proof. The sequence $\{\varphi_t\}$ is a sequence of continuous real functions defined on a compact space Δ. Since $V \geq 0$ and $\frac{\gamma}{\delta} < 1$, the sequence is monotone decreasing and φ_t converges pointwise to the continuous function U. Therefore, by Dini's Theorem [2, Theorem 2.62], the convergence is uniform. ■

Since $\succsim$ is nondegenerate, there is x, y such that $x \succ y$. By the representation, $U(c, x) > U(c, y)$. Thus, there exists $\rho, \nu \in \Delta$ such that $U(\rho) > U(\nu)$. By linearity of U,

$$U(\mu) \geq U(\eta) \Longrightarrow U(\mu\alpha\rho) > U(\eta\alpha\nu), \text{ for all } \alpha \in (0,1)\,. \tag{35}$$

This observation will be used in the next Lemma. Let $\succsim_U$ be the preference relation represented by U. As in the proof of Theorem 2, identify any binary relation B on Δ with its graph $\Gamma(B) \subset \Delta \times \Delta$.

Lemma 39 $\Gamma(\succsim_U) = \lim_{t\to\infty} \Gamma(\gtrapprox_t)$.

Proof. First establish $Ls\Gamma(\gtrapprox_t) \subset \Gamma(\succsim_U)$. If $(\mu, \eta) \in Ls\Gamma(\gtrapprox_t)$ then there is a subsequence $\{\Gamma(\gtrapprox_{t(n)})\}$ and a sequence $\{(\mu_n, \eta_n)\}$ that converges to (μ, η) such that $(\mu_n, \eta_n) \in \Gamma(\gtrapprox_{t(n)})$ for each n. Therefore, for each n,

$$\varphi_{t(n)}(\mu_n) \geq \varphi_{t(n)}(\eta_n).$$

Since $\varphi_{t(n)}$ converges to U uniformly, it follows that $U(\mu) \geq U(\eta)$. Hence $(\mu, \eta) \in \Gamma(\succsim_U)$, as desired.

Next establish $\Gamma(\succsim_U) \subset Li\Gamma(\gtrapprox_t)$. Let $(\mu, \eta) \in \Gamma(\succsim_U)$ and take any neighborhood V of (μ, η). By (35), there exists $\alpha \in (0, 1]$ s.t $(\mu\alpha\rho, \eta\alpha\nu) \in V$ and $U(\mu\alpha\rho) > U(\eta\alpha\nu)$. By Lemma 38, there exists $T < \infty$ such that $\varphi_t(\mu\alpha\rho) > \varphi_t(\eta\alpha\nu)$ for all $t \geq T$, that is, $(\mu\alpha\rho, \eta\alpha\nu) \in \Gamma(\gtrapprox_t)$ for all $t \geq T$. Hence,

$$V \cap \Gamma(\gtrapprox_t) \neq \phi \text{ for all but a finite number of } t,$$

that is, $(\mu, \eta) \in Li\Gamma(\gtrapprox_t)$.

This completes the proof. ■

By Lemma 39 and Theorem 2, $\succsim_U = \succsim^*$, that is, U is a representation of normative preference $\succsim^*$, as desired.

To complete the proof, let (U, V) be any representation of $\succsim$. Given nondegeneracy of $\succsim$, [7, Theorem 4] implies that for any α such that $V + \alpha \geq 0$, $(U, V + \alpha)$ is also a representation of $\succsim$. Hence, it follows from the preceding that U is a representation of normative preference $\succsim^*$.

G Appendix: Proof of Theorem 3

$\Longleftarrow$: First address the question of whether a representation exists for any continuous u, v, and discount factors $\delta \in (0, 1)$, $\gamma \geq 0$. Let $C_b(Z)$ denote the space of bounded continuous real functions on Z endowed with the sup norm topology. Consider the mapping from $C_b(Z)$ to itself defined by

$$\begin{aligned} W'_{n+1}(z) &= \max_{\mu \in z} \int_{C \times Z} (u(c) + \delta W'_n(x) + v(c) + \gamma W'_n(x))\, d\mu(c, x) \\ &\quad - \max_{\eta \in z} \int_{C \times Z} (v(c) + \gamma W'_n(y))\, d\eta(c, y). \end{aligned}$$

Fixed point theorem.....?

FIX THIS! .... Given a representation W', it is straightforward to establish the necessity of Axioms 1-5. Necessity of Axiom 7 is as in Lemma

7. Before considering the remaining axioms, we note that the representation implies

$$\begin{aligned} \forall \mu, \eta \text{ and } t &> 1, \\ \mu &\gtrapprox_t \eta \Longleftrightarrow \mu \gtrapprox_1 \eta. \end{aligned} \tag{36}$$

Hence, in the CT model, preferences reverse tomorrow, if at all, that is,

$$\forall \mu, \eta, \;\; \tau(\mu, \eta) \leq 1.$$

It follows that for any pair (μ, η) and any sequence (μ_n, η_n) that converges to (μ, η),

$$\limsup \tau(\mu_n, \eta_n) < \infty.$$

Therefore, we do not need to check .... The observation (36) establishes Reversal..... Independence is obvious.

$\Longrightarrow$: In the proof of sufficiency of Theorem 1, Preference for Early Choice is used only in Lemmas 30 and 31, that is, after a preference $\succsim$ over Z is derived from $\mathcal{C}(\cdot)$ and shown to satisfy Order*, Independence*, Continuity*, Set-Betweenness*, Stationarity*, Separability* and Indifference to Timing*. Furthermore, Preference for Early Choice is not needed to show that $\succsim$ is nondegenerate and generates $\mathcal{C}(\cdot)$. Hence, we start by assuming the existence of a $\succsim'$ with all these properties. By [15, Lemma ?], this is represented by W' such that

$$W'(x) = \max_{\mu \in x} \int_{C \times Z} \left(u(c) + \delta W'(y) + v(c) + \widehat{V}(y) \right) d\mu(c, y) - \max_{\eta \in x} \int_{C \times Z} \left(v(c) + \widehat{V}(y) \right) d\eta(c, y),$$

where $\widehat{V}$ is linear and continuous.

Lemma 40 $\{(c, x)\} \succsim' \{(c, y)\} \Longrightarrow \{(c, x)\} \sim' \{(c, x), (c, y)\}$

Proof. Follows from Menus Do Not Tempt, Reversal, and Lemma 2. ■

Lemma 41 *There exists* $\gamma \geq 0$ *and* θ *such that for all* x,

$$\widehat{V}(x) = \gamma W'(x) + \theta.$$

Proof. There are two possibilities. First, $\widehat{V}$ may be constant (equal to some θ). In that case, the result follows with $\gamma = 0$. Secondly, $\widehat{V}$ may be nonconstant. Suppose by way of contradiction that $\widehat{V}$ is not ordinally equivalent to W'. That is, there exists x and y such that

$$\begin{aligned} \widehat{V}(x) &\geq \widehat{V}(y) \text{ and } W'(x) < W'(y), \\ \text{or } \widehat{V}(x) &> \widehat{V}(y) \text{ and } W'(x) \leq W'(y). \end{aligned}$$

In either case, nonconstancy of $\widehat{V}$ and W' implies the existence of menus for which both inequalities are strict. We prove this for the case that $\widehat{V}(x) = \widehat{V}(y)$ and $W'(x) < W'(y)$. The same argument can be applied to the other case, that is, $\widehat{V}(x) > \widehat{V}(y)$ and $W'(x) = W'(y)$. So suppose $\widehat{V}(x) = \widehat{V}(y)$ and $W'(x) < W'(y)$. There are two possibilities to consider. First, there is z such that $\widehat{V}(x) > \widehat{V}(z)$.[21] If $W'(x) < W'(z)$, there is nothing to prove. If $W'(z) \leq W'(x)$, then $W'(z) < W'(y)$, and since $\widehat{V}(y) > \widehat{V}(z)$, the assertion is proved for this case as well. Second, there is z such that $\widehat{V}(z) > \widehat{V}(x)$. The argument is similar. Since $\widehat{V}$ is nonconstant, one of the two possibilities must be true, and hence we are done.

Therefore we can assume without loss of generality that

$$W'(x) > W'(y) \text{ and } \widehat{V}(x) < \widehat{V}(y).$$

Observe that

$$W'\{(c,x)\} = u(c) + \delta W'(x) > u(c) + \delta W'(y) = W'\{(c,y)\}.$$

If $\delta W'(x) + \widehat{V}(x) > \delta W'(y) + \widehat{V}(y)$, then

$$W'\{(c,x)\} = u(c)+\delta W'(x) > u(c)+\delta W'(x)+\widehat{V}(x)-\widehat{V}(y) = W'(\{(c,x),(c,y)\}).$$

This contradicts Lemma 40. On the other hand, if $\delta W'(x)+\widehat{V}(x) \leq \delta W'(y)+\widehat{V}(y)$, then

$$W'\{(c,x)\} = u(c) + \delta W'(x) > u(c) + \delta W'(y) = W'\{((c,x),(c,y)\},$$

again contradicting Lemma 40. Thus we establish that $\widehat{V}$ is ordinally equivalent to W'. By [15, Lemma ?], there exists $\gamma \geq 0$ and θ such that for all x,

$$\widehat{V}(x) = \gamma W'(x) + \theta,$$

[21] A $\widehat{V}-$best and worst menu exists since $\widehat{V}$ is continuous and Z is compact.

as was to be shown. ■

Uniqueness in GP... implies we can set $\theta = 0$. Therefore,

$$\begin{aligned} W'(z) &= \max_{\mu \in z} \int_{C \times Z} (u(c) + \delta W'(x) + v(c) + \gamma W'(x))\, d\mu(c,x) \\ &\quad - \max_{\eta \in z} \int_{C \times Z} (v(c) + \gamma W'(y))\, d\eta(c,y), \end{aligned}$$

which completes the proof.

H Appendix: Proof of Theorem 4

I Appendix: Proof of Theorem 6

By Theorem B (Appendix B), normative preference $\succsim^*$ is well-defined and satisfies the vNM axioms. Axiom A5 leads to non-triviality of $\succsim^*$. To see this, note that by $\mu > \eta$, $\mu <_{\tau(\mu,\eta)} \eta$ and the continuity of each $\gtrapprox_t$, for any sequence $\{(\mu_n, \eta_n)\}$ that converges to (μ, η), there exists N such that $\mu_n > \eta_n$ and $\mu_n <_{\tau(\mu,\eta)} \eta_n$ for all $n \geq N$. Hence, for all $n \geq N$, $\tau(\mu_n, \eta_n) \leq \tau(\mu, \eta)$. It follows that $\limsup_{n \to \infty} \tau(\mu_n, \eta_n) \leq \tau(\mu, \eta)$, and hence $(\mu, \eta) \in \Omega$. The non-triviality of $\succsim^*$ follows from $\mu <_{\tau(\mu,\eta)} \eta$ and Lemma 4.

Note that

$$\begin{aligned} \Lambda &\equiv \{(\mu, \eta) : \mu >_{\tau(\mu,\eta)} \eta \text{ and } (\mu, \eta) \notin \Omega\} \\ &= \Omega^c - \{(\mu, \eta) : \mu \approx_{\tau(\mu,\eta)} \eta\}, \end{aligned}$$

and define

$$\Psi = \Omega^c \cup \{(\mu, \eta) : \mu \approx_{\tau(\mu,\eta)} \eta\}.$$

Clearly, $\Lambda \subset \Psi$.

Step 1: $(\mu, \eta) \in \Psi \Longleftrightarrow \mu \sim^* \eta$.
Follows from Lemma 4.

Step 2: $\overline{\Lambda} \subset \Psi$.
Take any sequence $\{(\mu_n, \eta_n)\} \subset \Lambda$ that converges to (μ, η). Suppose by way of contradiction that $(\mu, \eta) \notin \Psi$. Then, by Step 1, $\mu \succ^* \eta$. By continuity

of $\succsim^*$ there exists N such that for all $n \geq N$, $\mu_n \succ^* \eta_n$. But then, by Step 1, $(\mu_n, \eta_n) \notin \Psi$ for all $n \geq N$, and in particular, $(\mu_n, \eta_n) \notin \Lambda$, a contradiction.

Step 3: Ψ^c is dense in $\Delta \times \Delta$.

By non-triviality of $\succsim^*$, there exists $\rho, \nu \in \Delta$ such that $\rho \succ^* \nu$. Since $\succsim^*$ satisfies the vNM axioms,

$$\mu \succsim^* \eta \Longrightarrow \mu\alpha\rho \succ^* \eta\alpha\nu, \text{ for all } \alpha \in (0,1).$$

Hence, for any $(\mu, \eta) \in \Delta \times \Delta$, there exists a sequence $\{(\mu_n, \eta_n)\}$ that converges to (μ, η) such that for all n, $\mu_n \not\prec^* \eta_n$. By Step 1, $\{(\mu_n, \eta_n)\} \subset \Psi^c$. Therefore, Ψ^c is dense in $\Delta \times \Delta$.

This establishes that Ψ, and hence $\overline{\Lambda}$, has empty interior. It follows that Λ is no-where dense.

References

[1] Ainslie, G. (1992): *Picoeconomics*, Cambridge University Press.

[2] Aliprantis, C. and K. Border (1994): *Infinite Dimensional Analysis: a Hitchhiker's Guide*, 2nd Edition, Springer Verlag.

[3] Benartzi, S. and R. Thaler (2004): 'Save More Tomorrow™: Using Behavioral Economics to Increase Employee Savings', *Journal of Political Economy*, Vol. 112, No. 1, pp. S164-S187.

[4] Brown, A. and C. Pearcy (1995): *An Introduction to Analysis*, Springer, New York.

[5] Fedderson, T. and A. Sandroni (2002) 'A Theory of Participation in Elections', Working Paper.

[6] Fehr, E. and K. Schmidt (1999): 'A Theory of Fairness, Competition, and Cooperation', *Quarterly Journal of Economics,* 817-68.

[7] Gul, F. and W. Pesendorfer (2001): 'Temptation and Self-Control', *Econometrica* 69, 1403-1435.

[8] ———— (2003): 'Self-Control and the Theory of Consumption', *Econometrica*, 72, 119-158.

[9] Herstein, I. N. and J. Milnor (1953): 'An Axiomatic Approach to Measurable Utility ', *Econometrica* 21, *291-297.*

[10] Kannai, Y. (1970): 'Continuity Properties of the Core of a Market', *Econometrica* 38, 791-815.

[11] Karni, E. and Z. Safra (2002): 'An Individual Sense of Justice: A Utility Representation', *Econometrica* 70, 263-284.

[12] Krusell, P., B. Kurusçu and A. Smith, Jr. (2002): 'Time Orientation and Asset Prices', *Journal of Monetary Economics,* Vol 49, Issue 1, pp 107-135.

[13] Laibson, D. (1997): 'Golden Eggs and Hyperbolic Discounting', *Quarterly Journal of Economics* 112, 443-77.

[14] Mas-Collel, A., M. D. Whinston and J. R. Green (1995): *Microeconomic Theory*, New York: Oxford University Press.

[15] Noor, J. (2004): 'Future Temptation and Self-Control', University of Rochester Working Paper.

[16] O'Donoghue, T., and M. Rabin (1999): 'Doing it Now or Later', *American Economic Review*, 89, 103-24.

[17] Parthasarathy, K. (1970): *Probability Measures on Metric Spaces*, New York. Academic Press.

[18] Strotz R. (1955): 'Myopia and Inconsistency in Dynamic Utility Maximization', *Review of Economic Studies*, XXIII, pp 165-180.