How to deal with partially analyzed acts? A proposal

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Abstract

In some situations, a decision is best represented by an incompletely analyzed act: conditionally to a given event A, the consequences of the decision on sub-events are perfectly known and uncertainty becomes probabilizable, whereas the plausibility of this event itself remains vague and the decision outcome on the complementary event A^c is imprecisely known. In this framework, we study an axiomatic decision model and prove a representation theorem. Resulting decision criteria aggregate partial evaluations consisting in: i) the conditional expected utility associated with the analyzed part of the decision; and ii) the best and worst consequences of its non-analyzed part. The representation theorem is consistent with a wide variety of decision criteria which allows for expressing various degrees of knowledge on (A, A^c) and various types of attitude towards ambiguity and uncertainty. This diversity is taken into account by specific models already existing in the literature. We take advantage of that fact and propose some particular forms of our model incorporating these models as sub-models and moreover expressing various types of beliefs concerning the relative plausibility of the analyzed and the non-analyzed events ranging from probabilities to complete ignorance that include capacities.

Keywords: Decision making under uncertainty, partially analyzed decisions

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1 Introduction

Public and private decision makers frequently encounter choice situations for which decision theory does not offer appropriate representations and choice criteria. In standard models, decisions are evaluated on the basis of the probabilities that they lead to such or such result. Now, whereas it is possible to estimate with precision some risks, such as automobile risks (thanks to actuarial data), the evaluation of others, such as natural disasters is a more difficult matter (statistics on frequencies and magnitudes are scarce). Such an evaluation becomes even problematic for environmental risks such as those resulting from global warming or from GMO and for some health risks such as epidemics. Although all these risks are perfectly identified, it seems difficult to assign probabilities to the relevant events as well as to evaluate precisely their effects.

On closer look, these last situations display a common, simple, pattern. Either the corresponding risk does not happen and the situation evolves "normally", i.e., available data allow one to associate precisely decision outcomes with specific events; or the risk happens, and such an association is no longer possible: feasible outcomes can only be located within some range of results. Besides, the risk likelihood is generally not known and moreover may depend on the decision considered.

Take for instance a situation of epidemic risk such as that faced by the World Health Organization at the time of the outbreak of SARS in February 2003. Various strategies can be encompassed for controlling a disease propagation. The efficiency of a given strategy depends on the values of several parameters which are initially unknown such as ways of spread, viral/bacterial origin of the infection etc. The introduction, for a given strategy, of the event associated with an efficient control and of the complementary event can always be made although generally it will not be possible to assess the relative likelihood of these two events (no quantitative information is attached to them). Then, conditionally to an efficient control, a routine sanitary situation is recovered and standard utility/probability evaluation can be performed; whereas, conditionally to a poor control, predictions that can be made are vague (for instance, a number of victims ranging from one thousand to one million) and, further, cannot be improved by having recourse to a finer analysis.

As another example, consider the question of the use of genetically modified organisms (GMO) in agriculture. Without GMO, farmers' incomes depend basically on climatic and market variables. Available data allow one to estimate their probability distribution and their impact on income. With GMO, expected income remains assessable conditionally on the absence of cross-fertilization and contamination of other plants. However, neither the plausibility of the contamination, nor its consequences on the farmers' income, can be precisely evaluated.

A suitable representation of these situations by a formal model must allow for ambiguity on the events and uncertainty on the consequences. The literature on decision making under uncertainty contains many theories with ambiguous events such as CEU (Schmeidler 1989) and GPT (Wakker, Tversky 1993) and few with uncertain consequences (Ghirardato 2001) but none combining both in the required way. This is our motivation for proposing a new axiomatic model which attempts to formalize such situations and to justify adapted decision criteria.

The paper is organized as follows. Section 2 explains the distinction between the analyzed and the non-analyzed parts of a decision and expresses them formally. Sections 3 and 4 are devoted to conditional preferences on the analyzed part. Preferences on the non-analyzed part are modelled in section 5. A representation theorem is achieved in section 6. Finally, the model is discussed in section 7.

2 The model

2.1 Decisions

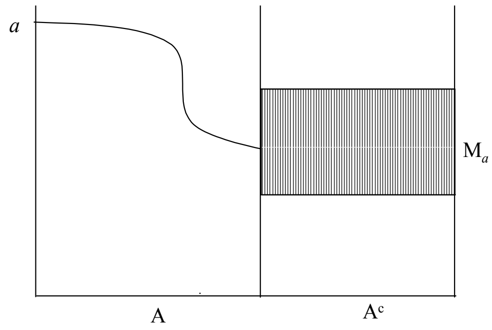
Consider: Ω, set of states of nature; $\mathcal{E}$, σ-algebra of events; $\mathcal{C}$, a set of consequences; $\mathcal{G}$, σ-algebra of subsets of $\mathcal{C}$ containing singletons. A decision problem involves a particular set of decisions, $\mathcal{D}$, which are (measurable) acts in the sense of Savage[1], i.e., mappings $(\Omega, \mathcal{E}) \longrightarrow (\mathcal{C}, \mathcal{G})$. However, in the decision model below, these acts are not completely known by the decision maker. Specifically, the decisions are only partially analyzed, i.e., for any decision $a \in \mathcal{D}$ there is an event A such that the restriction of a to A - the analyzed part of a - denoted $a|_A$ is exactly known but the only information about $a|_{\bar{A}}$ - the non-analyzed part of a - is its range $M_a = a(\bar{A})$. Thus, preferences will depend on pairs $(a|_A\,,\ M_a)$.

A specific feature of the model is that $\mathcal{D}$ is not assumed to contain all conceivable pairs $(a|_A\,,\ M_a)$. The reason is that decision makers cannot be expected to meaningfully evaluate non realistic decisions. Thus the range M on an "unfavorable" event (such as a natural catastrophe) should not include any blissful consequence. Similarly, in some situations, major ignorance about the effects of an event will necessarily imply much uncertainty about outcomes i.e., a wide consequence range M on this event.

Completely analyzed decisions, denoted by $(a|_\Omega\,, \cdot)$, can exist. In particular, for evaluation purposes, we shall assume the existence of completely analyzed $\mathcal{R}$ - measurable acts, where subalgebra $\mathcal{R}$ of $\mathcal{E}$ can be interpreted as events associated with sequences of heads and tails (see Savage 1954 p. 38-39 and de Finetti 1974 p.199-202).

A decision a analyzed on an event A is called an A-act. It generates a σ-algebra of subsets of $A:\ \left\{a|_A^{-1}(G),\ G \in \mathcal{G}\right\}$, which we embed into a richer one, the σ-algebra $\mathcal{A}_a$ of subsets of A generated by $\left\{a|_A^{-1}(G) \cap R,\ G \in \mathcal{G},\ R \in \mathcal{R}\right\}$.

[1]More precisely, we use Savage's remark (§ 3.4., p. 42) that the results in his model remain valid with events, consequences and acts defined in the present way.



A partially analyzed act

We denote by $\mathcal{F}_a$ the set of all pairs $g = (g|_A\,,\ M_a)$ where $g|_A$ is any conceivable Savagean act $(A,\ \mathcal{A}_a) \longrightarrow (\mathcal{C},\ \mathcal{G})$. Thus, $g \in \mathcal{F}_a$ implies $M_g = M_a$. There is a set $\mathcal{F}_a$ corresponding to each $a \in \mathcal{D}$ and the union of these sets is denoted by $\mathcal{F}$. We denote by $\mathcal{A}_\mathcal{F}$ the set of all events A such that $\mathcal{F}$ contains at least one A-act.

Note that the fact that two acts a' and a'' are both A-acts, i.e., are analyzed on the same event A, does not imply the identity of $\mathcal{A}_{a'}$ and $\mathcal{A}_{a''}$, nor that of $\mathcal{F}_{a'}$ and $\mathcal{F}_{a''}$.

Example 1 *Acts a, a', a'' characterize various oil field management strategies in the same country. Political risk (event $\bar{A}$) may imply partial or complete loss of the investment. Act a' involves the same investment level I as a but concerns the exploitation of a different oil field, whereas act a'' corresponds to a more intensive exploitation of the same field as a. Thus, it is likely that $M_{a'} = M_a = [0, -I]$ but $\mathcal{A}_{a'} \neq \mathcal{A}_a$ (oil yields depend on different events), whereas $M_{a''} = [0, -I''] \neq M_a$ and $\mathcal{A}_{a''} = \mathcal{A}_a$. Hence, although the three acts are analyzed on the same event A, $\mathcal{F}_a$, $\mathcal{F}_{a'}$ and $\mathcal{F}_{a''}$ all differ from one another.*

2.2 Preferences

Preferences on $\mathcal{F}$ are expressed by a binary relation $\succsim$. We assume:

Axiom 1 *$\succsim$ is a weak order on $\mathcal{F}$.*

We want to endow $\succsim$ with standard properties and, moreover, to establish links between its restrictions $\succsim_a$ to the various $\mathcal{F}_a$. For this, we need in particular an appropriate version of Savage's Sure Thing Principle.

Due to the partial information on the decisions, the common part $Com(a, b)$ of two acts a and b analyzed on events A and B, respectively, is naturally defined as

$$Com(a, b) = \begin{cases} \{\omega \in A \cap B : a(\omega) = b(\omega)\} \text{ if } M_a \neq M_b \\ \{\omega \in A \cap B : a(\omega) = b(\omega)\} \cup \left(\bar{A} \cap \bar{B}\right) \text{ if } M_a = M_b \end{cases}$$

Thus, on any state of nature belonging to $Com(a, b)$, either the precise consequences of both events are known, in which case they must be the same, or they are not known, in which case the possible consequence range of the two decisions must be the same.

Axiom 2 *(Sure Thing Principle for partially analyzed decisions)*

Let $a, \widehat{a}, b, \widehat{b} \in \mathcal{F}$ *where* $\widehat{a}$ *results from* a *and* $\widehat{b}$ *from* b *by a common modification in the sense that* $Com(a, b) = Com(\widehat{a}, \widehat{b})$.

Then $a \succsim b \Longleftrightarrow \widehat{a} \succsim \widehat{b}$.

Note that the feasible common modifications of a given pair of acts are strongly limited by the fact that the modified acts must still belong to $\mathcal{F}$ (which makes Axiom 3 a rather weak form of the Sure Thing Principle).

The axiom can nonetheless apply to quadruples $a, \widehat{a}, b, \widehat{b}$ with different $\mathcal{F}_a$, $\mathcal{F}_{\widehat{a}}$, $\mathcal{F}_b$, $\mathcal{F}_{\widehat{b}}$.

Example 2 *Suppose there are three countries:* $\mathbb{A}, \mathbb{B}$ *and* $\mathbb{C}$*. Country* $\mathbb{A}$ *(resp.* $\mathbb{B}$*) may possibly face an economic crisis (event* $\bar{A}$ *(resp.* B^c*)) which however is unlikely in country* $\mathbb{C}$*. A firm has to take a decision concerning a productive investment of amount* I*. The decision* a *of investing* I *in country* $\mathbb{A}$ *will generate sales shared out among countries* $\mathbb{A}, \mathbb{B}$ *and* $\mathbb{C}$ *in proportions 45% in country* $\mathbb{A}$*, 5% in country* $\mathbb{B}$ *and 50% in country* $\mathbb{C}$*, unless an economic crisis (event* $\bar{A}$*) happens in* $\mathbb{A}$ *in which case* I *may be partially or completely lost, independently of a crisis occurring or not in country* $\mathbb{B}$*. See Figure 1.*

On the other hand, consider a' *with the same amount of investment in* $\mathbb{A}$ *as* a *but generating a different sales sharing, namely 70%, 30% and 0% respectively in countries* $\mathbb{A}, \mathbb{B}$ *and* $\mathbb{C}$ *if there is no economic crisis. With this investment decision, the firm may loose up to* I *if crisis occurs only in* $\mathbb{A}$*, but is sure to loose the investment completely if the crisis takes place simultaneously in* $\mathbb{A}$ *and* $\mathbb{B}$ *(event* $\bar{A} \cap B^c$*).*

Decisions b *and* b' *have similar characteristics with the roles of countries* $\mathbb{A}$ *and* $\mathbb{B}$ *exchanged. We assume moreover that the countries are "similar", in the sense that the return from sales is the same in* $\mathbb{A}$ *as in* $\mathbb{B}$*, that is* $a|_A = c$ *and* $b|_B = c$ *with* $c \in \mathcal{C}$*. See Figure 2.*

Thus, a *and* b *are respectively an* A*-act and a* B*-act with*

$$Com(a, b) = (A \cap B) \cup \left(\bar{A} \cap \bar{B}\right)$$

and $M_a = M_b = [0, -I]$.

a' *and* b' *are* $(A \cap B) \cup \left(\bar{A} \cap \bar{B}\right)$*-acts resulting from* a *and* b *by a modification of their common part. More precisely,*

$$a|_{A\cap B} = b|_{A\cap B} = a'|_{A\cap B} = b'|_{A\cap B} \,,$$
$$M_{a'} = M_{b'} = [0, -I] \text{ and } a'|_{\bar{A}\cap\bar{B}} = b'|_{\bar{A}\cap\bar{B}} = -I.$$

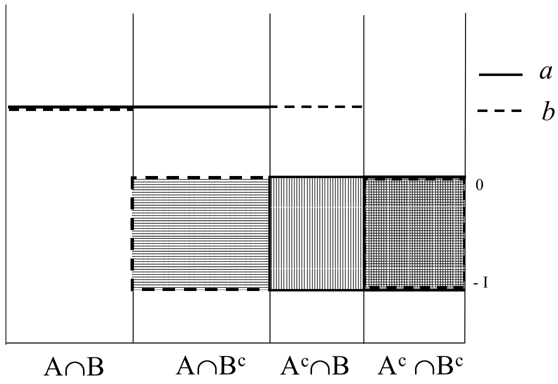


Figure 1:

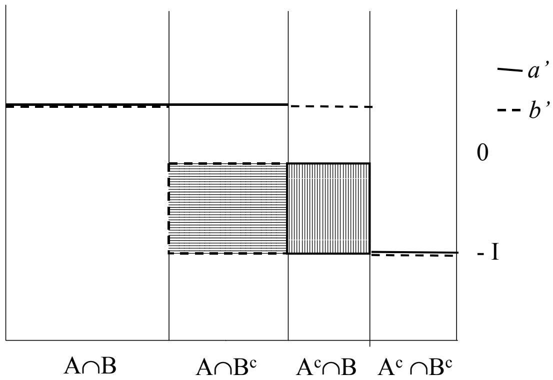


Figure 2:

3 Preferences on analyzed events and SEU

From preferences $\succsim_a$ on $\mathcal{F}_a$, we can now derive, "à la Savage", $\succsim_a^E$, conditional preferences given event E, where $E \in \mathcal{A}_a$, by

$$g \succsim_a^E h \Leftrightarrow g' \succsim_a h' \text{ where } g'|_E = g|_E\,,\; h'|_E = h|_E \text{ and } g'|_{A\backslash E} = h'|_{A\backslash E}\,.$$

Axiom 2 ensures that the ordering of g' and h' is independent from their common values on $A\backslash E$.

Note that $\succsim_a^A$ is the same as $\succsim_a$.

More generally, given an A-act, $a \in \mathcal{D}$ and a B-act, $b \in \mathcal{D}$ where $B \in \mathcal{A}_a$ (hence $B \subset A$), orderings $\succsim_a^B$ on $\mathcal{F}_a$ and $\succsim_b$ on $\mathcal{F}_b$ are related as shown by the following lemma.

Lemma 1 *Let* $a' = (a'|_A\,,\; M_a)\,,\; a'' = (a''|_A\,,\; M_a)$ *with* $a', a'' \in \mathcal{F}_a$. *Suppose that,* *for some* $B \in \mathcal{A}_a$, b *is a* $B-$*act and* $b' = \left(a'|_B\,,\ M_b\right), b'' = \left(a''|_B\,,\ M_b\right) \in \mathcal{F}_b$. *Then*

$$a' \succsim_a^B a'' \Leftrightarrow b' \succsim_b b''.$$

Proof. Consider g' and g'' resulting from b' and b'' by the common modification consisting in giving them a constant common consequence $g'(\omega) = g''(\omega) = c$ for $\omega \in A \backslash B$ and the same range M_a on $\bar{A}$. By Axiom 2, $g' \succsim g'' \Leftrightarrow b' \succsim_b b''$. Moreover, g' and g'' also belong to $\mathcal{F}_a$ and can be obtained by modifying a' and a'' on $A \backslash B$ and giving them constant value c on this event. By definition, $a' \succsim_a^B a'' \Leftrightarrow g' \succsim g''$. Hence $a' \succsim_a^B a'' \Leftrightarrow b' \succsim_b b''$. ■

As a direct consequence of Lemma 1, conditional preferences given E are intrinsic in the sense that they do not depend on which $\mathcal{A}_a$ containing E (hence on which a in $\mathcal{D}$) is considered, and can be defined by

$g \succsim^B h \Leftrightarrow$ there is $a \in \mathcal{D}$ such that $g \succsim_a^E h$.

We also need slightly modified versions of Savage's other definitions and axioms.

A constant A-act f_a^c in $\mathcal{F}_a$ is defined by: $f_a^c(A) = \{c\}$ with $c \in \mathcal{C}$ and $f_a^c(\bar{A}) = M_a$.

Savage's P3 becomes:

Axiom 3 *For* $c',\ c'' \in \mathcal{C}$, *let* $f_a^{c'},\ f_a^{c''}$ *be constant A-acts in* $\mathcal{F}_a$ *and* $f_b^{c'},\ f_b^{c''}$ *be constant B-acts in* $\mathcal{F}_b$.

Then $f^{c'} \succsim f^{c''} \Longleftrightarrow f_b^{c'} \succsim f_b^{c''}$.

Preferences among consequences can now be defined by

$c' \succeq_{\mathcal{C}} c'' \Longleftrightarrow$ there exist $a \in \mathcal{D}$ and constant A-acts $f_a^{c'},\ f_a^{c''}$ in $\mathcal{F}_a$ such that $f_a^{c'} \succsim f_a^{c''}$.

Since $\mathcal{C}$ can always be replaced by its quotient w. r. t. the symmetric part of $\succeq_{\mathcal{C}}$, we henceforth assume w.l.o.g. that $\succeq_{\mathcal{C}}$ is an order (i.e., is antisymmetric) which justifies the use of symbol $\succeq_{\mathcal{C}}$.

We now require Savage's P4 (irrelevance of the values of the prizes on the events) in every $\mathcal{F}_e$, where $e \in \mathcal{D}$ is an E-act.

Axiom 4 *Let* $A, B \in \mathcal{A}_e$ *where* $e \in \mathcal{D}$ *is an E-act; let* $c_1, c_1', c_2, c_2' \in \mathcal{C}$ *be such that* $c_1 \succ_{\mathcal{C}} c_1'$ *and* $c_2 \succ_{\mathcal{C}} c_2'$. *Define acts* $f, f', g, g' \in \mathcal{F}_e$ *by:*

i) $f(E^c) = f'(E^c) = g(E^c) = g'(E^c) = M_e$;

ii) $f(\omega) = c_1,\quad f'(\omega) = c_1'$, *for* $\omega \in A$;

$f(\omega) = c_2,\quad f'(\omega) = c_2'$, *for* $\omega \in E \backslash A$;

iii) $g(\omega) = c_1,\quad g'(\omega) = c_1'$, *for* $\omega \in B$;

$g(\omega) = c_2,\quad g'(\omega) = c_2'$, *for* $\omega \in E \backslash B$;

then $f \succsim g \Leftrightarrow f' \succsim g'$.

Whenever $f \succsim g$ holds for $f,\ g$ defined as in Axiom 4, we can write $A \succsim_e^E B$. However, if $A, B \in \mathcal{A}_{e^*}$ for some other $e^* \in \mathcal{D}$ which is also an $E-$act, it results from Axiom 2 ($f(\bar{E}) = f'(\bar{E}) = g(\bar{E}) = g'(\bar{E}) = M_e$ above can be replaced by

$f(\bar{E}) = f'(\bar{E}) = g(\bar{E}) = g'(\bar{E}) = M_{e^*})$ that: $A \succsim_e^E B \Leftrightarrow A \succsim_{e^*}^E B$. We can therefore drop the subscript e and simply write $A \succsim^E B$ and read "event A is qualitatively more probable than B conditionally to event E".

The next axiom assumes both that $\succeq_{\mathcal{C}}$ is not trivial and that worst and best consequences exist.

Axiom 5 *There exists a pair $\overline{c}$, $\underline{c} \in \mathcal{C}$ such that $\overline{c} \succ_{\mathcal{C}} \underline{c}$ and $\overline{c} \succeq_{\mathcal{C}} c \succeq_{\mathcal{C}} \underline{c}$ for all $c \in \mathcal{C}$.*

We also introduce a version of Savage's P6. It makes it clear that one of the roles of the coin-toss related subalgebra of events $\mathcal{R}$ is to make all $(\mathcal{A}_a, \succsim_a)$ atomless.

Axiom 6 *Let $f, g \in \mathcal{F}_a$, where $a \in \mathcal{D}$ is an A-act, with $f \succ g$ and $c \in \mathcal{C}$. There exists a partition of A, consisting of events $R \cap A$, $R \in \mathcal{R}$, such that if f (resp. g) is modified on any element of the partition and given constant outcome c on this element, then the modified act f' (resp. g') also satisfies $f' \succ g$ (resp. $f \succ g'$).*

We also need Savage's P7 for each $\succsim_a$.

Axiom 7 *Let $f, g \in \mathcal{F}_a$, where $a \in \mathcal{D}$ is an A-act and let $E \in \mathcal{A}_a$. If $f \succsim_a^E$ (resp. $\precsim_a^E$) $g(\omega)$ for all $\omega \in E$, then $f \succsim_a^E$ (resp. $\precsim_a^E$) g.*

Axioms 1-7 imply the validity of Savage's P1-7 in every $\mathcal{F}_a$, where thus his main result holds: preferences in $\mathcal{F}_a$ can be represented by a subjective expected utility (SEU) criterion with respect to an atomless probability on $\mathcal{A}_a$.

Moreover, due to the explicit introduction of σ-algebra $\mathcal{R}(A) = \{A \cap R, R \in \mathcal{R}\}$ in the statement of Axiom 6, it is clear that this result still holds if $\mathcal{F}_a$ is replaced by its restriction to $\mathcal{R}(A)$ - measurable acts. We can thus state:

Proposition 1 *For every $a \in \mathcal{D}$ there exist a bounded utility u_a and an additive probability P_a such that*

$$f \succsim g \Leftrightarrow \int_A u_a \circ f \, dP_a \geq \int_A u_a \circ g \, dP_a, \quad \forall f, g \in \mathcal{F}_a$$

where

u_a is unique up to an affine transformation;

P_a is unique and for every $\rho \in [0,1]$ there exists $B \in \mathcal{A}_a$ such that $P_a(B) = \rho$.

Moreover, these existence and unicity statements are also valid when $\mathcal{F}_a$ is replaced by its restriction to $\mathcal{R}(A)$ - measurable acts and thus $\mathcal{A}_a$ by $\mathcal{R}(A)$.

NB In the rest of the paper, we shall simply write "probability" for "additive probability".

4 Intrinsic utility and probability consistency

It is well known that Savage's axioms do not imply the existence of certainty equivalents for the acts. However, this property is easily acceptable for sufficiently rich consequence sets (for instance when $\mathcal{C}$ is a real interval) and, although not necessary, will be technically helpful later in the paper. So, we assume:

Axiom 8 *For any $a \in \mathcal{D}$ there exist $c \in \mathcal{C}$ such that the constant A-act f_a^c satisfies $f_a^c \sim_a a$.*

The next assumption and the lemma that follows assert that coin-toss related events are "qualitatively" independent and thus "quantitatively" independent from events in $\mathcal{E}$.

Axiom 9 *For every $A, B \in \mathcal{A}_{\mathcal{F}}$ conditional preferences on events $\succsim^A$ and $\succsim^B$ satisfy, for all $R', R'' \in \mathcal{R}$:*

$$A \cap R' \succsim^A A \cap R'' \Longleftrightarrow B \cap R' \succsim^B B \cap R''.$$

Lemma 2 *Let $a, b \in \mathcal{D}$. For every $R \in \mathcal{R}$, $P_a(A \cap R) = P_b(B \cap R)$.*

Proof. For any $R', R'' \in \mathcal{R}$, $P_a(A \cap R') \geq P_a(A \cap R'') \Leftrightarrow A \cap R' \succsim^A A \cap R'' \Longleftrightarrow B \cap R' \succsim^B B \cap R'' \Leftrightarrow P_b(B \cap R') \geq P_b(B \cap R'')$. Thus, the mapping $\mathcal{R}(A) \longmapsto [0, 1]$ defined by $A \cap R \longmapsto P_b(B \cap R)$ is a probability measure representing $\succsim^A$ which however is uniquely represented by P_a. Therefore $P_a(A \cap R) = P_b(B \cap R)$ for every $R \in \mathcal{R}$. ■

Whenever $A \cap R' \succsim^A A \cap R''$ holds for $R', R'' \in \mathcal{R}$ and some $A \in \mathcal{A}_{\mathcal{F}}$, we shall simply write $R' \succsim^{\mathcal{R}} R''$ and read "event R' is qualitatively more probable than R''". Qualitative probability $\succsim^{\mathcal{R}}$ is uniquely represented by probability $P_{\mathcal{R}}$ defined by $P_{\mathcal{R}}(R) = P_a(A \cap R)$ for some A.

Thus, Axiom 9 ensures the existence of an intrinsic probability $P_{\mathcal{R}}$ on $\mathcal{R}$.

We shall use this result to derive properties of utilities. That far, all we know about the $u_a, a \in \mathcal{D}$ is that they represent the same ordering $\succeq_{\mathcal{C}}$ and are therefore increasing transforms from one another. We would like functions u_a to be identical (after calibration).

According to Proposition 1 for every triple $c' \succeq_{\mathcal{C}} c \succeq_{\mathcal{C}} c''$, with $c' \succ_{\mathcal{C}} c''$, there is an event $R \in \mathcal{R}$ such that act $g \in \mathcal{F}_a$ with $g(\omega) = c'$, for $\omega \in A \cap R$, and $g(\omega) = c''$ for $\omega \in A \cap R^c$ is indifferent to the constant $A-$act f_a^c in $\mathcal{F}_a$. In other terms, there is $R \in \mathcal{R}$ such that $P_a(A \cap R)$ satisfies:

$$u_a(c) = P_a(A \cap R)u_a(c') + (1 - P_a(A \cap R))u_a(c''), \tag{1}$$

hence, according to the definition that follows Lemma 2

$$u_a(c) = P_{\mathcal{R}}(R)u_a(c') + (1 - P_{\mathcal{R}}(R))u_a(c'').$$

Thus, all we need is an axiom ensuring that the event R in (1) only depends on c.

Axiom 10 *For every triple $c' \succeq_{\mathcal{C}} c \succeq_{\mathcal{C}} c''$, with $c' \succ_{\mathcal{C}} c''$, there exist an event $R \in \mathcal{R}$ such that for every $a \in \mathcal{D}$, act $g \in \mathcal{F}_a$ with $g(\omega) = c'$, for $\omega \in A \cap R$, and $g(\omega) = c''$ for $\omega \in A \cap R^c$ is indifferent to the constant $A-$act f_a^c in $\mathcal{F}_a$.*

If follows immediately that:

Proposition 2 *Utilities u_a $(a \in \mathcal{D})$ are affine transforms from one another.*

Thus, after calibration u_a's are identical and we will write from now on u instead of u_a. Note that u is a utility function representing $\succeq_{\mathcal{C}}$.

Next proposition guarantees the existence of intrinsic conditional probabilities in the sense that they are independent from the context in which they are evaluated.

Proposition 3 *Let $a, b \in \mathcal{D}$ be analyzed on A and B, respectively, with $B \in \mathcal{A}_a$ and let moreover $E \in \mathcal{A}_b$ (hence $E \subset B \subset A$). Then $P_a(E/B) = \dfrac{P_a(E)}{P_a(B)} = \dfrac{P_b(E)}{P_b(B)} = P_b(E)$.*

Proof. By Proposition 2, there exists $R \in \mathcal{R}$ such that $R \cap B \sim_b E$, and thus, by Lemma 1, $R \cap B \sim_a^B E$, implying

$$P_b(R \cap B) = P_b(E) \text{ and } P_b(R \cap B/B) = P_a(E/B). \tag{2}$$

Moreover, by applying Lemma 1 to acts offering prizes on events $R' \cap B$ and $R'' \cap B$, where $R', R'' \in \mathcal{R}$, we get $R' \cap B \succsim_b R'' \cap B \Leftrightarrow R' \cap B \succsim_a^B R'' \cap B$. Thus, the same ordering (say $\succsim_b$) on set of events $\{R \cap B,\ R \in \mathcal{R}\}$ is representable by (restrictions of) probabilities P_b and $P_a(./B)$; by unicity of such a representation (see Proposition 2), $P_b(R \cap B) = P_a(R \cap B/B)$, for all $R \in \mathcal{R}$. Then according to (2) $P_b(E) = P_a(E/B)$. ■

Thus, as for conditional preferences, intrinsic conditional probabilities can be defined by $P(E/B) = P_a(E/B)$ where E, $B \in \mathcal{A}_a$ and $E \subset B$.

5 Preferences on non-analyzed events

Let us now turn to the non-analyzed part of the decisions.

Denote by $\mathcal{M}$ the set of ranges corresponding to all the decisions:

$\mathcal{M} = \{M_a,\ \exists\, a \in \mathcal{D}$ such that $a = (a|_A\,,\ M_a)\}$.

The following assumption is easy to admit and technically useful.

Axiom 11 *Every $M \in \mathcal{M}$ has a $\succeq_{\mathcal{C}}$-greatest and a $\succeq_{\mathcal{C}}$-lowest consequence, respectively denoted $g(M)$ and $l(M)$.*

We define a partial preference relation over $\mathcal{M}$. For this, two axioms are needed: Axiom 12 ensures the existence of the relation and Axiom 13 its transitivity.

Axiom 12 *Let a', a'' be $A-$acts such that $a' = (a'|_A\,,\ M_{a'})\,,\ a'' = (a''|_A\,,\ M_{a''})$ with $a'|_A = a''|_A$ and let b', b'' be $B-$acts such that $b' = (b'|_B\,,\ M_{b'})\,,\ b'' = (b''|_B\,,\ M_{b''})$ with $b'|_B = b''|_B\,,\ M_{b'} = M_{a'}$ and $M_{b''} = M_{a''}$. Then*

$$a' \succsim a'' \Leftrightarrow b' \succsim b''.$$

Preferences among ranges can now be defined by the transitive closure $\succsim_{\mathcal{M}}$ of the relation $\succsim^0_{\mathcal{M}}$ given by:

$M' \succsim^0_{\mathcal{M}} M'' \Longleftrightarrow$ there exist $A-$ acts $a', a'' \in \mathcal{D}$ such that $M_{a'} = M'$, $M_{a''} = M''$, $a'|_A = a''|_A$ and $a' \succsim a''$.

$\succsim_{\mathcal{M}}$ is automatically a partial order if:

Axiom 13 *$\succsim^0_{\mathcal{M}}$ is acyclic i.e. there is no sequence M^i, $i = 1..n$ in $\mathcal{M}$ such that $M^i \succsim^0_{\mathcal{M}} M^{i+1}$, $i = 1..n-1$ and $M^n \succ^0_{\mathcal{M}} M^1$.*

Let's now turn to the representation of the preference relation $\succsim_{\mathcal{M}}$.

The following requirement will allow us to extend a result of Barbera, Barrett, Pattanaik (1984).

Axiom 14 *(1) For all $M \in \mathcal{M}$ and $c \in \mathcal{C}$, there exist A and two $A-$acts a', a'' such that $M_{a'} = M$ and $M_{a''} = M \cup \{c\}$;*

(2) Let $c_1, c_2 \in \mathcal{C}$ be such that $c_1 \succ_{\mathcal{C}} c_2$. Then, for any $M_0 \in \mathcal{M}$ such that $c_1, c_2 \notin M_0$,

$$\{c_1\} \cup M_0 \succsim_{\mathcal{M}} \{c_1, c_2\} \cup M_0 \succsim_{\mathcal{M}} \{c_2\} \cup M_0.$$

Moreover, if $c \succ_{\mathcal{C}} c_2$ for all $c \in M_0$, then:

$$\{c_1\} \cup M_0 \succ_{\mathcal{M}} \{c_1, c_2\} \cup M_0$$

and if $c_1 \succ_{\mathcal{C}} c$ for all $c \in M_0$, then

$$\{c_1, c_2\} \cup M_0 \succ_{\mathcal{M}} \{c_2\} \cup M_0.$$

Note that Axiom 17 makes both existence and comparability requirements. In particular, it implies that $M \in \mathcal{M} \Longrightarrow M \cup \{c_1, ..., c_k\} \in \mathcal{M}$. Moreover, for $M_0 = \emptyset$, we get $\{c_1\} \succ_{\mathcal{M}} \{c_1, c_2\} \succ_{\mathcal{M}} \{c_2\}$.

Lemma 3 *(i) For all finite $M \in \mathcal{M}$ such that $g(M) \succ_{\mathcal{C}} l(M)$,*
$M \sim_{\mathcal{M}} \{g(M), l(M)\}$.
(ii) For finite $M', M'' \in M$:

$$\left.\begin{array}{c} g(M') \succeq_{\mathcal{C}} g(M'') \\ l(M') \succeq_{\mathcal{C}} l(M'') \end{array}\right\} \Rightarrow M' \succsim_{\mathcal{M}} M''. \tag{3}$$

Moreover,

$$\left.\begin{array}{c} g(M') \succ_{\mathcal{C}} g(M'') \\ l(M') \succ_{\mathcal{C}} l(M'') \end{array}\right\} \Rightarrow M' \succ_{\mathcal{M}} M''. \tag{4}$$

Proof. *(i)* For $c \in M \backslash \{g(M), l(M)\}$, by Axiom 14, $g(M) \succ_{\mathcal{C}} c$ implies $M \backslash \{c\} \succsim_{\mathcal{M}} M$ (take $M_0 = M \backslash \{g(M), c\}$) and symmetrically $c \succ_{\mathcal{C}} l(M)$ implies $M \succsim_{\mathcal{M}} M \backslash \{c\}$; hence $M \sim_{\mathcal{M}} M \backslash \{c\}$.

Let $M = \{g(M), c_1, c_2, ..., c_n, l(M)\}$ where $g(M) \succ_{\mathcal{C}} c_1 \succ_{\mathcal{C}} c_2 \succ_{\mathcal{C}} ... \succ_{\mathcal{C}} c_n \succ_{\mathcal{C}} l(M)$. Then, by repeated application of last relation:
$M \sim_{\mathcal{M}} M \backslash \{c_1\} \sim_{\mathcal{M}} M \backslash \{c_1, c_2\} \sim_{\mathcal{M}} ...$
$\sim_{\mathcal{M}} M \backslash \{c_1, c_2, ..., c_n\} = \{g(M), l(M)\}$.

(ii) From (i) of the Lemma, we have $M' \sim_{\mathcal{M}} \{g(M'), l(M')\}$ and $M'' \sim_{\mathcal{M}} \{g(M''), l(M'')\}$. $\succsim_{\mathcal{M}}$ being transitive (Axiom 13), we just need to prove that $\{g(M'), l(M')\} \succsim_{\mathcal{M}} \{g(M''), l(M'')\}$. Assume that in the left side of (3) there is at least one strict preference, for instance $g(M') \succ_{\mathcal{C}} g(M'')$ (if it is not the case, the result is straightforward). By Axiom 14 (point (2)) with $M_0 = \{l(M')\}$, we have $\{g(M'), l(M')\} \succsim_{\mathcal{M}} \{g(M''), l(M')\}$. If $l(M') \succ_{\mathcal{C}} l(M'')$, by the same Axiom with $M_0 = \{g(M'')\}$, $\{g(M''), l(M')\} \succsim_{\mathcal{M}} \{g(M''), l(M'')\}$. Else ($l(M') \sim_{\mathcal{C}} l(M'')$), from the proof of (i)

$$\{g(M''), l(M'), l(M'')\} \sim_{\mathcal{M}} \{g(M''), l(M')\} \sim_{\mathcal{M}} \{g(M''), l(M'')\}.$$

The proof of the second part of (ii) is similar and uses the second part of point (2) in Axiom 14 (strict inequalities). ■

Lemma 3 directly implies that, for a finite sequence $(M_i)_{i=1}^n$ of finite M_i with $g(M_i)$ and $l(M_i)$ independent of i, $\cup_{j=1}^n M_j \sim_{\mathcal{M}} M_i$, $i = 1..n$. We extend this property to infinite unions in the following axiom.

Axiom 15 *For any family $(M_i)_{i \in I}$, of finite $M_i \in \mathcal{M}$ such that $g(M_i)$ and $l(M_i)$ are independent of i, $\cup_{j \in I} M_j \sim_{\mathcal{M}} M_i$, $i \in I$.*

We can then prove the following proposition:

Proposition 4 *For all $M \in \mathcal{M}$ such that $g(M) \succ_{\mathcal{C}} l(M)$, $M \sim_{\mathcal{M}} \{g(M), l(M)\}$.*

Proposition 5 *There exists a mapping $v : \mathcal{M} \to \mathbb{R}$ such that*

$$M' \succ_{\mathcal{M}} M'' \Rightarrow v(M') > v(M'')$$
$$M' \sim_{\mathcal{M}} M'' \Rightarrow v(M') = v(M'')$$

with $M \mapsto v(M) = \varphi(g(M), l(M))$ and

$$\left.\begin{array}{c} g(M') \succ_{\mathcal{C}} g(M''), l(M') \succeq_{\mathcal{C}} l(M'') \\ or \\ g(M') \succeq_{\mathcal{C}} g(M''), l(M') \succ_{\mathcal{C}} l(M'') \end{array}\right\} \Rightarrow v(M') > v(M'').$$

Proof. Let the elements of $\mathcal{C}$ be indexed as $c_1 \succ_{\mathcal{C}} c_2 \succ_{\mathcal{C}} ... \succ_{\mathcal{C}} c_N$ and mapping φ defined:

$$\text{for } i < j \text{ by } \varphi(c_i, c_j) = \sum_{(r,s) \in E_{ij}} \frac{1}{2^{r+s}},$$
$$\text{where } E_{ij} = \{(r, s) : r < s \text{ and } \{c_i, c_j\} \succ_{\mathcal{M}} \{c_r, c_s\}\}$$
$$\text{for } i = j \text{ by } \varphi(c_i, c_i) = \sum_{(r,s) \in F_i} \frac{1}{2^{r+s}},$$
$$\text{where } F_i = \{(r, s) : r < s \text{ and } \{c_i\} \succ_{\mathcal{M}} \{c_r, c_s\}\}$$

Then, v defined by $v(M) = \varphi(g(M), l(M))$ has the required properties since if $g(M) \succ_{\mathcal{M}} l(M)$ then $M \sim_{\mathcal{M}} \{c_i, c_j\}$ for some $c_i = g(M)$ and $c_j = l(M)$ and if $g(M) = l(M)$ $M \sim_{\mathcal{M}} \{c_i\}$ for $c_i = g(M)$. ■

Note that v is not unique and, in general, not even ordinal (not unique up to a strictly increasing transformation).

6 Preferences representation

6.1 Representation theorem

We now want to construct a utility representation of preferences $\succsim$ in $\mathcal{F}$ that incorporates the results obtained so far concerning its restrictions $\succsim_a$ to the various $\mathcal{F}_a$ as well as those concerning $\succsim_{\mathcal{M}}$.

This construction will be based on the existence of certainty equivalents for the acts which is directly required by the following axiom, justified as Axiom 8 by the richness of consequence set $\mathcal{C}$. We denote by f^k the constant act $f^k(\Omega) = \{k\}$.

Axiom 16 *For any act $a \in \mathcal{D}$ there exists $k \in \mathcal{C}$ such that $f^k \sim a$.*

Theorem 1 *The weak order $\succsim$ on $\mathcal{F}$ is representable by a utility function V :*

- *For an A-act a with $A \neq \Omega$ and with $\succeq_{\mathcal{C}}$-greatest and $\succeq_{\mathcal{C}}$-lowest consequences in M_a denoted respectively $g(M_a)$, $l(M_a)$:*

$$a = \left(a|_A\,,\ M_a\right) \longmapsto V(a) = \Phi\left(A,\ \int\limits_A u \circ a\ dP_a,\ g(M_a),\ l(M_a)\right)$$

  where P_a is a subjective additive probability on the σ-algebra $\mathcal{A}_a$;

  and Φ is increasing with $\displaystyle\int_A u \circ a\ dP_a,\ g(M_a),\ l(M_a)$.

- *Otherwise, for $A = \Omega$,*

$$a = \left(a|_\Omega\,,\ \cdot\right) \longmapsto V(a) = \Psi\left(\int\limits_\Omega u \circ a\ dP_a\right)$$

  with Ψ increasing with its argument.

- *Moreover, probabilities $\{P_a,\ a \in \mathcal{D}\}$ possess the following consistency property: for $a, b \in \mathcal{D}$ with $B \in \mathcal{A}_a$ and $E \in \mathcal{A}_b$: $P_a(E/B) = P_b(E)$.*

Proof. Any a in $\mathcal{F}$ has a certainty equivalent k in $\mathcal{C}$ (by Axiom 16) and $\succeq_{\mathcal{C}}$ is representable by utility function u. A priori consequence k, hence number $u(k)$, depends on all the elements characterizing a namely $A,\ \mathcal{A}_a,\ a|_A$ and $\mathcal{M}_a$.

Since, by Axiom 8, there exist c in $\mathcal{C}$ such that $a \sim_a f^c_a$, then

$$a \sim \left(A,\ \mathcal{A}_a,\ f^c_a|_A\,,\ M_a\right). \tag{5}$$

The constant A-act f^c_a being measurable with respect to any σ-algebra $\mathcal{A}_a$ of subsets of A, we have, for any A-acts a', a'' such that $M_{a'} = M_{a''}$ and $f^c_{a'} = f^c_{a''}$, $a' \sim a''$. Thus, the preference between a' and a'' does not explicitly depend on $\mathcal{A}_{a'}$ and $\mathcal{A}_{a''}$ and (5) becomes:

$$a \sim \left(A,\ f^c_a|_A\,,\ M_a\right). \tag{6}$$

Moreover, the certainty equivalent k depends on $a|_A$ only through $\displaystyle\int_A u \circ a\ dP_a$ (by Proposition 1) and on M_a only through $g(M_a)$, $l(M_a)$ (by Proposition 4). ■

6.2 Some particular criteria

The representation theorem is consistent with a wide variety of decision criteria which allows for expressing various degrees of knowledge on $(A,\ \bar{A})$ and various types of attitude towards ambiguity and uncertainty. This diversity is taken into account by specific models already existing in the literature. We can take advantage of that fact and propose some particular forms of our model incorporating these models as submodels and moreover expressing various types of beliefs concerning the relative plausibility of the analyzed and the non-analyzed events ranging from probabilities $(P(A)+P(\bar{A})=1)$ to complete ignorance that include capacities $(v(A)+v(\bar{A})\neq 1)$.

- *Case of $P(A)$ precisely known:*

$$V(a)=P(A)\int_A u\circ a\ dP_a+(1-P(A))\,\varphi(g(M_a),\ l(M_a))$$

where u and φ express respectively attitudes towards risk and ambiguity. The particular form of φ due to Hurwicz (1951) only involves a pessimism index α which leads to:

$$V(a)=P(A)\int_A u\circ a\ dP_a+(1-P(A))\left[\alpha u\left(l(M_a)\right))+(1-\alpha)\,u\left(g(M_a)\right)\right].$$

- *Case of $P(A)$ imprecisely known.* The representation of the imprecision on $(A,\ \bar{A})$ by a capacity v together with a rank dependent criterion (Schmeidler 1989, Gilboa, Schmeidler 1989) leads to the following form for V :

- $\int_A u\circ a\ dP_a \leqslant \varphi(g(M_a),\ l(M_a))$,

$$V(a)=\left(1-v\left(\bar{A}\right)\right)\int_A u\circ a\ dP_a+v\left(\bar{A}\right)\varphi(g(M_a),\ l(M_a))$$

- $\int_A u\circ a\ dP_a \geqslant \varphi(g(M_a),\ l(M_a))$,

$$V(a)=v\left(A\right)\int_A u\circ a\ dP_a+(1-v\left(A\right))\,\varphi(g(M_a),\ l(M_a)).$$

- *Case of complete ignorance on $(A,\ \bar{A})$.* According to models proposed by Arrow, Hurwicz (1972), Cohen, Jaffray (1980) and Barbera, Barett, Pattanaik (1984) only best and worst evaluations on A or $\bar{A}$ are relevant which leads to:

$$V(a)=\varphi(\max\{u(g(M_a)),\ \int_A u\circ a\ dP_a\},\ \min\{u(l(M_a)),\ \int_A u\circ a\ dP_a\}).$$

Despite its generality, our model has normative implications. It rules out some well established criteria and suggests alternative ones, as in the following example.

Example 3 *A common practice in international borrowing consists in classifying countries into various groups according to their degree of insolvency risk. The rating is generally based on a check-list of economic indicators through a multiple criteria decision model; probability evaluations are rarely involved (Saini, Bates 1984). A given country is then allowed to borrow money at an interest rate equal to the LIBOR, i, plus a risk spread Δi, which depends on its group. Thus, the net expected present value of a one period investment I with expected return ER is*

$$EV = -I + \frac{ER}{(1 + i + \Delta i)} \tag{7}$$

which, by neglecting a second order term in $i \times \Delta i$, yields equivalence

$$EV \approx -I + \frac{ER}{(1 + i)} - \frac{\Delta i \times ER}{(1 + i + \Delta i)}.$$

Thus, whenever two projects, characterized respectively by (I_1, R_1) and (I_2, R_2) are such that

$$I_1 > I_2 \text{ and } ER_1 > ER_2 \text{ with } -I_1 + \frac{ER_1}{(1 + i)} = -I_2 + \frac{ER_2}{(1 + i)},$$

then $EV_1 < EV_2$, which means that the criterion favors low investement/low return projects over high investment/high return ones in high risk countries, which has no apparent economic justification.

Now, our model (see Proposition 6), applies to investment projects with $\bar{A}$ as the insolvency event, $\int_A u \circ a \; dP_a = ER$, $g(M_a) = 0$, and $l(M_a) = -L \geq -I$; thus, a particular, additive, instance of this model would evaluate a project according to formula:

$$V^* = -I + \frac{ER}{(1 + i)} - k(A) \times L \tag{8}$$

i.e., would require the risk premium to be proportional to the maximal possible loss, here L (and independent from ER), which appears as reasonable. Of course, weight $k(A)$ will only depend on the investment country through its insolvency risk group.

Note that preference for low investment/low return projects is not excluded but now only prevails among projects for which all the investment can be lost ($L = I$).

7 Discussion and conclusion

The model is consistent with various generalizations of SEU. For instance partially analyzed acts are a special case of multivalued acts; once restricted to this special class, the criteria of Ghirardato's (2001) model, become a subfamily of ours. More generally, the family of criteria described by the representation theorem is rather wide and various behavioural assumptions could be added and lead to more specific criteria such as those exhibited above.

It should also be noted that the building blocks of the model, SEU for the analyzed part and "(max, min)" for the non-analyzed one could easily be replaced by other theories: for instance, the analyzed part would still be probabilizable but Rank Dependent Utility (Quiggin 1982) would replace EU, or information on the non-analyzed part of the acts would not be quantified in terms of consequence sets but according to symbolic categories.

On the other hand, the dichotomy analyzed part/ non analyzed part is a fundamental feature of the model and should not be modified. For instance, the generalization consisting in specifying more precise outcome intervals on subsets of $\bar{A}$ is, in our opinion, illusory. The reason is that, due to the fact that complete ignorance will necessarily prevail on the subsets of $\bar{A}$, decisions will depend only on the best and the worst outcome on all subsets which is to say on the best and worst ones on the whole of $\bar{A}$ itself.

To conclude, the model proposed was intended to be particularly suited for decision making in situations involving potential risks and will, hopefully, prove to be so. At any rate, should alternative models be considered, the technical solutions developed here (insuring the consistency of conditional preferences and beliefs as well as the merging of submodels) can be adapted for the axiomatic justification of these models.

References

[1] Arrow K., Hurwicz L. (1972) "An optimality criterion for decision making under ignorance" in *Uncertainty and Expectations in Economics*, ed. by C. F. Carter and J. L. Ford., Oxford: Basil Blackwell & Mott Ltd.

[2] Barbera S, Barrett C. R., Pattanaik P.K. (1984) "On Some Axioms for Ranking Sets of Alternatives", *Journal of Economic Theory* 32-2, 301-308.

[3] Cohen M., Jaffray J.Y. (1980), "Rational Behavior under Complete Ignorance", *Econometrica*, 48, 1281-1299.

[4] de Finetti B. (1974) *Theory of Probability* vol. 1, Wiley.

[5] Ghirardato P. (2001) "Coping with Ignorance: Unforeseen Contingencies and Non-Additive Uncertainty", *Economic Theory* 17, 247-276

[6] Gilboa I., Schmeidler D. (1989), "Maxmin expected utility with a non-unique prior", *Journal of Mathematical Economics*, 18, 141-153.

[7] Hurwicz L. (1951) "Optimality criteria for decision making under ignorance", *Cowles Commission discussion paper.* Statistics n°370.

[8] Quiggin J. (1982) “A Theory of Anticipated Utility”, *Journal of Economic Behavior and Organization*, 3, 324-343.

[9] Saini K. G., Bates P. S. (1984) ”A survey of the quantitative approaches to country risk analysis”, *Journal of Banking and Finance*, 8-2, 341-356.

[10] Savage L. J. (1954) *The Foundations of Statistics*, Wiley, New York.

[11] Schmeidler D. (1989), “Subjective Probability and Expected Utility without Additivity”, *Econometrica*, 57, 571-587.

[12] Wakker P., Tversky A. (1993), “An Axiomatization of Cumulative Prospect Theory”, *Journal of Risk and Uncertainty*, 7, 147-176.