Updating Multiple-Prior Preferences

Eran Hanany* Peter Klibanoff†

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Preliminary

1 Introduction

In dynamic choice situations under uncertainty, preferences may need to be updated as new information is gathered. Updated preferences are viewed as governing choice upon the realization of a conditioning event. When multiple-prior preferences are updated, dynamic inconsistencies may occur. Consider Ellsberg's example, in which bets are made over the color of a ball drawn randomly from an urn with 90 balls, of which 30 are black (B) and the rest are red (R) or yellow (Y), with no further information on the distribution. Taking triples $(u_B, u_R, u_Y) \in \mathbb{R}^3$ to represent state contingent utilities of bets (utility acts), the typical preference $(1,0,0) \succ (0,1,0) \sim (0,0,1)$ and $(0,1,1) \succ (1,1,0)$ $\sim (1,0,1)$ has a max-min expected utility (EU) representation (Gilboa and Schmeidler [4]), e.g. $\min\{\frac{1}{3}u_B + q_R u_R + q_Y u_Y \mid q_R \geq 0 \ , q_Y \geq 0 \ , q_R + q_Y = \frac{2}{3}\}$, or equivalently, $\frac{1}{3}u_B + \frac{2}{3}\min\{u_R, u_Y\}$. Suppose that the utility acts $(1,3,4)$ $\succ (2,2,4)$ are offered and then it is revealed that the drawn ball is not yellow. Given this conditioning event, dynamic consistency requires that preference over

*Faculty of Engineering, Tel Aviv University, Tel Aviv 69978, Israel. Email: hananye@post.tau.ac.il

†Kellogg Graduate School of Management, Northwestern University, Evanston, IL 60208, USA. E-mail: peterk@kellogg.northwestern.edu

the two acts is maintained since both provide equal utility when the ball is yellow. However, assuming an update that applies Bayes rule to all probability measures and retains max-min EU, the conditional preference is represented by $\min\{q_B u_B + q_R u_R \mid q_B \in [\frac{1}{3}, 1], q_R = 1 - q_B\}$ and thus $(2,2,4)$ is now preferred. A similar preference reversal occurs when $(2,2,1) \succ (1,3,1)$ are offered and the preference is updated by maximum likelihood to $\frac{1}{3}u_B + \frac{2}{3}u_R$.

Nevertheless, dynamically consistent update rules do exist for multiple-prior preferences. Consider for example the following update rule for max-min EU preferences. Given a convex (in utility space) set of feasible acts and given a preference relation, choose an optimal act and one measure that is used by the preference representation to minimize the evaluation of this act; then define the conditional as an EU preference by applying Bayes rule only to that measure. Thus the conditional indifference curve through this act forms a separating hyperplane in utility space between the convex set of feasible acts and the convex set of preferred-to-the-optimal-but-infeasible acts, ensuring that the chosen optimal act remains optimal.

This extreme example, however, although ensuring dynamic consistency, violates an axiom which we claim to be elementary for any update rule. The axiom, entitled 'Generalized Bayesianism' (**GB**), requires that given an event E and a conditional certainty equivalent x of an act f, the unconditional indifference $f_E h \sim x_E h$ holds for some act h, where $f_E h$ represents an act that gives f on E and h otherwise[1]. Roughly speaking, the axiom requires that given a multiple-prior preference, an act f is evaluated conditionally as an average x of its values on the conditioning event E, according to the unconditional preference where some reference act h is taken outside E. Both the Ellsberg update

[1]To see that the update rule example violates **GB**, assume the feasible (utility) acts in $co\{(2,3,4), (2,1,4)\}$ and the preference represented by $I(u_1,u_2,u_3) = \min\{\sum_i q_i u_i \mid q \in co\{(0.2,0.6,0.2), (0.6,0.2,0.2)\}\}$. The optimal act $(2,3,4)$ is evaluated (minimized) by $q = (0.6,0.2,0.2)$, so using its Bayesian update, the conditional is represented by $0.75u_1 + 0.25u_2$. Thus a conditional certainty equivalent of the act $(2,1,4)$ has utility 1.75. However, any act $(2,1,u_3)$ is evaluated by $q = (0.2,0.6,0.2)$, so there exists no u_3 for which $I(2,1,u_3) = I(1.75,1.75,u_3)$.

examples given above satisfy this axiom. Although not defined explicitly before, the axiom **GB** is satisfied by previous update rules considered in the literature, e.g. Epstein and Le Breton [2], Gilboa and Schmeidler [5], Sarin and Wakker [7], Epstein and Schneider [3] and Siniscalchi [10].

This paper presents dynamically consistent update rules of multiple-prior preferences, which also satisfy the axiom **GB**. Attention is restricted to preferences that have a max-min EU representation or the analogue max-max EU representation. An update rule takes a preference relation and an event and assigns to them a preference relation conditional on the event. Update rules considered in this paper satisfy the following requirements. First, conditional preferences have multiple-prior representations, so that all max-min EU preferences are updated to max-min and similarly for max-max. Second, the complement of the event being conditioned on is null. Third, the ordering of constant acts (those with equal utility in all states) is preserved. Fourth, update rules are constrained by dynamic consistency, which is the requirement that an unconditional optimal act remains optimal also conditionally. Finally, these rules satisfy the axiom **GB**.

In order to achieve update rules that satisfy the requirements above, we allow non-consequentialist behavior through conditional preferences that may depend on ex-ante choices. These choices then become reference points for dynamic consistency, similarly to Machina [6], Segal [9] and Epstein and Le Breton [2]. However, contrary to the latter, update rules in this paper are not restricted to probabilistically sophisticated preferences. This is a result of the separation we allow between the definition of an update rule and the requirement of dynamic consistency, so that dynamic consistency does not define an update rule uniquely.

Moreover, these rules do not imply necessarily the Bayesian update of all measures, as in Sarin and Wakker [7], Epstein and Schneider [3] and Siniscalchi [10]. Nevertheless, exactly such updating is implied for the special case of rectangular preferences ([3]), i.e. when the representing set of measures for an unconditional preference satisfies stochastic independence given an event and given its complement. Thus update rules in this paper extend recursive multiple-priors and allow generalized Bayesian updating of all multiple-prior preferences.

The paper is organized as follows. Section 2 contains an exposition of the framework and axioms. Section 3 presents a family of update rules by first defining conditional certainty equivalents of acts and then showing that the resulting conditional preferences make up update rules that satisfy the axioms. Section 4 characterizes the set of measures that are used in the representation of conditional preferences, showing that these are Bayesian updates of a subset of the measures used to represent the unconditional preference.

2 Axioms

Consider an Anscombe-Aumann framework [1], where X is the set of all simple (finite) lotteries over a set of consequences Z, S is a set of states of nature endowed with an algebra Σ of events and $\mathcal{A}$ is the set of all acts, i.e. Σ-measurable simple (finite-valued) functions $f : S \to X$. Abusing notation, elements of X are also used to denote the constant acts, for which $\forall s \in S$, $f(s) = x$ for some $x \in X$. Let $\mathcal{P}^{MMEU}$ denote the set of max-min EU preference relations over $\mathcal{A}$ (Gilboa and Schmeidler [4]), let $\mathcal{P}^{MXEU}$ and $\mathcal{P}^{EU}$ denote max-max EU and EU preference relations respectively and let $\mathcal{P}^{MEU}$ denote their union (multiple-prior preferences).

For a preference $\succsim \in \mathcal{P}^{MEU}$, let u, I denote a utility function and a representation functional for $\succsim$, so that $\forall f, g \in \mathcal{A}$, $f \succsim g \iff I[u \circ f] \geq I[u \circ g]$. Let C denote the set of probability measures corresponding to $\succsim$. Thus for $\succsim \in \mathcal{P}^{MMEU}$ and any act $f \in \mathcal{A}$, $I[u \circ f] = \min_{q \in C} \int (u \circ f) dq$ and similarly for $\succsim \in \mathcal{P}^{MXEU}$, $I[u \circ f] = \max_{q \in C} \int (u \circ f) dq$. As usual, $\sim$ and $\succ$ denote the symmetric and asymmetric parts of $\succsim$. Let $\mathcal{N}(\succsim)$ denote the set of events $E \in \Sigma$

for which $\forall q \in C$, $q(E) > 0$. Let $\mathcal{B}$ denote the set of all subsets $B \subseteq \mathcal{A}$ such that $u \circ B$ is convex. Elements of $\mathcal{B}$ are considered sets of feasible acts and their convexity in utility space reflects independent randomization that produces act mixtures.

Assume a multiple-prior preference $\succsim \in \mathcal{P}^{MEU}$, a non-singleton event $E \in \mathcal{N}(\succsim)$ and an act $g \in \mathcal{A}$ assumed to have been chosen before the realization of E. Denote by $\mathcal{T}$ the set of all such triples $(\succsim, E, g)$. An update rule is a function $U : \mathcal{T} \to \mathcal{P}^{MEU}$, producing a conditional multiple-prior preference denoted by $\succsim_{E,g}$. Such a conditional preference is viewed as governing choice in dynamic situations, upon the realization of the conditioning event E. We look for sets $\mathcal{U}$ of update rules, which satisfy the following axioms. The first axiom requires that the unconditional and the conditional preferences belong to the same family, thus preserving the overall attitude, either aversion or appeal, towards uncertainty.

Axiom 1 ***CL*** *(Closure). For any* $U \in \mathcal{U}$, $(\succsim, E, g) \in \mathcal{T}$ *and* $\succsim_{E,g} = U(\succsim, E, g)$, *the following holds:* $\succsim \in \mathcal{P}^{MMEU} \Rightarrow \succsim_{E,g} \in \mathcal{P}^{MMEU}$, $\succsim \in \mathcal{P}^{MXEU} \Rightarrow \succsim_{E,g} \in \mathcal{P}^{MXEU}$ *and* $\succsim \in \mathcal{P}^{EU} \Rightarrow \succsim_{E,g} \in \mathcal{P}^{EU}$.

The second axiom states that a preference conditional on an event E should not depend on the consequences outside of E, a basic requirement on conditional preferences. Next, an axiom is provided that requires the preservation of the ordering of constant acts, in conjunction with a proper separation between attitudes towards risk, reflected within this unchanged ordering, as opposed to updated attitudes towards uncertainty.

Axiom 2 ***NC*** *(Null Complement). For any* $U \in \mathcal{U}$, $(\succsim, E, g) \in \mathcal{T}$, $\succsim_{E,g} = U(\succsim, E, g)$ *and* $f, h \in \mathcal{A}$, $f \sim_{E,g} f_E h$.

Axiom 3 ***RP*** *(Risk Preference). For any* $U \in \mathcal{U}$, $(\succsim, E, g) \in \mathcal{T}$, $\succsim_{E,g} = U(\succsim, E, g)$ *and* $x, y \in X$, $x \succsim y \Leftrightarrow x \succsim_{E,g} y$.

The fourth axiom is dynamic consistency. A set of feasible acts B is included in the axiom in order to identify optimal acts g. According to the axiom, for each such set B, an update rule must exist that guarantees that for any preference and any conditioning event E, if g is optimal within B, it remains optimal also conditionally.

Axiom 4 ***DC*** *(Dynamic Consistency). For any* $B \in \mathcal{B}$*, there exists* $U \in \mathcal{U}$ *such that for any* $(\succsim, E, g) \in \mathcal{T}$ *and* $\succsim_{E,g} = U(\succsim, E, g)$*, if [*$g \in B$ *and* $\forall f_E g \in B$*,* $g \succsim f_E g$*], then for any* $f_E g \in B$*,* $g \succsim_{E,g} f$*.*

Optimality of g for the unconditional preference is checked with respect to acts that are identical to g on E^c, the complement of E. The justification for this prescription is threefold. First, dynamic consistency is relevant only *ceteris paribus*, i.e. when exactly the same consequences occur on E^c. Second, the specific choice of g on E^c makes it a reference point for sequential decisions, in conjunction with our non-consequentialist approach. Third, E and E^c are usually thought of as the known events in two nodes of a decision tree, so that $f, g \in B \Longrightarrow f_E g \in B$.

Furthermore, the axiom does not define an update rule uniquely. This is contrary to some of the definitions in the literature (e.g. Epstein and Le Breton [2]), which are justified by arguments about closure of the update rule, that we require separately. Moreover, some definitions require optimality implications in both directions, i.e. require also that conditional optimality given an event and its complement implies unconditional optimality. This other direction is needed to ensure non-negative value of information in cases where sequential decisions are made using backward induction, a procedure that does not fit our non-consequentialist assumption. Thus for the update rules we consider, negative value of information cannot obtain since dynamic consistency ensures that a conditional preference always agrees with the original choice.

Finally, we assume generalized Bayesianism. As explained in the introduction, roughly speaking the axiom requires that given a multiple-prior preference, an act f is evaluated conditionally as an average x of its values on the conditioning event E, according to the unconditional preference where some reference act h is taken outside E.

Axiom 5 ***GB*** *(Generalized Bayesianism). For any* $U \in \mathcal{U}$*,* $(\succsim, E, g) \in \mathcal{T}$*,* $\succsim_{E,g} = U(\succsim, E, g)$*,* $f \in \mathcal{A}$ *and* $x \in X$ *such that* $f \sim_{E,g} x$*, there exists* $h \in \mathcal{A}$ *such that* $f_E h \sim x_E h$*.*

3 A Family of Update Rules

We construct a family of update rules satisfying the axioms. The construction is based on defining for each act f, a conditional certainty equivalent x^f using the Bayesian update of a measure that is used unconditionally to evaluate some act which is identical to f on E. More precisely, for each act $f \in \mathcal{A}$, we define a constant $x^f \in X$ satisfying $u(x^f) = \int_E (u \circ f) d(\frac{q}{q(E)})$ for a measure $q \in C$ for which $\int u \circ (f_E \overline{f}) dq = I[u \circ (f_E \overline{f})]$ for some specifically chosen act $\overline{f} \in \mathcal{A}$ that depends on f. We then show that such an approach is consistent with all the required axioms.

The update rules are described below for max-max EU preferences, but analogue constructions hold for max-min EU preferences when the orders $\succsim$, $\succ$, $\geq$ and $>$ are reversed (as well as reversing max to min and sup to inf).

Fix $\succsim \in \mathcal{P}^{MXEU}$, $E \in \mathcal{N}(\succsim)$ and $g \in \mathcal{A}$. Axioms **CL** and **DC** allow that a conditional certainty equivalent of g can be evaluated by the Bayesian update of a measure $q \in C$ for which $\int (u \circ g) dq = I[u \circ g]$. So for $f = g$, we set also $\overline{f} = g$. Observe that for any act $f \in \mathcal{A}$ and for any $\succsim' \in \mathcal{P}^{EU}$ defined by u and $C' = \{q\}$ for some $q \in C$, $I[u \circ f] = \int (u \circ f) dq$ if, and only if, $\forall h \in \mathcal{A}$, $h \succsim' f$ $\Rightarrow h \succsim f$. Indeed, this is the case since the indifference curve of $\succsim'$ through f forms a supporting hyperplane at f for the convex set $\{u \circ h \mid f \succsim h\}$. Using this fact we define a conditional certainty equivalent x^g of g.

Definition 1 *Let $x^g \in X$ such that there exists $\succsim^g \in \mathcal{P}^{EU}$ where*
(1) $\forall h \in \mathcal{A}$, $h \succsim^g g \Rightarrow h \succsim g$
and (2) $g \sim^g (x^g)_E g$.

Note that $u(x^g) = \int_E (u \circ g) d(\frac{q_g}{q_g(E)})$ for $q_g \in C$ such that $I[u \circ g] = \int (u \circ g) dq_g$. The following lemma shows order relations between conditional certainty equivalents and is needed in order to define the update rules. For any act $f \in \mathcal{A}$, let $y^f \in X$ be a conditional certainty equivalent of the act f that is achieved by updating all measures in C, defined by $f_E y^f \sim y^f$ (Siniscalchi [10]).

Lemma 1 *Let $f \in \mathcal{A}$, $w, x, z \in X$ and $p, r \in C$ such that $f \sim x_E f$, $I[u \circ f] = \int (u \circ f) dp$, $I[u \circ (x_E f)] = \int u \circ (x_E f) dr$, $u(w) = \int_E (u \circ f) d(\frac{p}{p(E)})$ and $u(z) = \int_E (u \circ f) d(\frac{r}{r(E)})$. Then $y^f \succsim w \succsim x \succsim z$ and there exist $q \in C$ for which $u(x) = \int_E (u \circ f) d(\frac{q}{q(E)})$.*

Proof. (1) $u(x) \geq u(z)$ since $r(E)u(x) + \int_{E^c} (u \circ f) dr = I[u \circ (x_E f)] = I[u \circ f] \geq \int (u \circ f) dr$, thus $x \succsim z$.
(2) $w \succsim x$ since $\int (u \circ f) dp = I[u \circ f] = I[u \circ (x_E f)] \geq p(E)u(x) + \int_{E^c} (u \circ f) dp$.
(3) The existence of q follows from convexity of the set of conditionals of measures in C and $w \succsim x \succsim z$.
(4) $y^f \succsim w$ since for any $q' \in C$, specifically for $q' = p$, $q'(E)u(y^f) + [1 - q'(E)]u(y^f) = u(y^f) = I[u \circ (f_E y^f)] \geq \int u \circ (f_E y^f) dq'$. ■

Letting $x_g \in X$ satisfy $g \sim (x_g)_E g$, it follows that $y^g \succsim x^g \succsim x_g$ by the lemma above.

We now extend the definition of x^g to any act $f \in \mathcal{A}$. Conditional certainty equivalents x^f are defined so that the indifference curve through g in utility space is 'tangent' to the corresponding unconditional indifference curve when g is fixed on E^c (see figure 1 below). All other indifference curves are defined accordingly for positive affinely related acts on E in utility space, so that a conditional in $\mathcal{P}^{MXEU}$ is produced. In the definition below, W^f is a set of candidate conditional certainty equivalents for f. The chosen act g is taken as a reference point by considering acts that are identical to g on E^c and indifferent to g according to the unconditional preference $\succsim$ (condition (2)). Among these acts, one is chosen for each subset of acts that are non-negative affinely related on E in utility space (condition (1)). Each such act is chosen so that the corresponding conditional certainty equivalent (conditions (3), (4)) is maximal according to $\succsim$ but not better than x^g (condition (5)). We show later that such a choice produces a conditional preference in $\mathcal{P}^{MXEU}$. For each act f, the act g^f takes the place of $\overline{f}$ that is described at the beginning of this section.

Definition 2 *For any $f \in \mathcal{A} \setminus \{g\}$, define the set W^f of all $w \in X$ such that $\exists f' \in \mathcal{A}$, $w' \in X$, $\succsim' \in \mathcal{P}^{EU}$ for which:*
(1) $\exists z \in X$, $\lambda \in [0,1]$ such that $f' = \lambda f + (1-\lambda)z$ or $f = \lambda f' + (1-\lambda)z$
and (2) $f'_E g \sim g$
and (3) $\forall h \in \mathcal{A}$, $h \succsim' f'_E g \Rightarrow h \succsim f'_E g$
and (4) $f_E g \sim' w_E g$ and $f'_E g \sim' w'_E g$
and (5) $x^g \succsim w'$.
Define $x^f, z^f \in X$ such that $x^f \sim \sup^{\succsim}\{w \in W^f\}$ and $z^f \sim \sup^{\succsim}\{w'$ corresponding to $w \in W^f\}$.
If $\sup^{\succsim}\{w \in W^f\} \notin W^f$ then define $g^f = y^f$. Otherwise, take any f', z, λ corresponding to x^f and let $g^f \in \mathcal{A}$ such that $f'_E g = \lambda(f_E g^f) + (1-\lambda)z$ or $(f_E g^f) = \lambda(f'_E g) + (1-\lambda)z$.
Let $y^f, x_f \in X$ such that $f_E y^f \sim y^f$ and $f_E g^f \sim (x_f)_E g^f$.

The following lemma extends lemma 1 (a proof is provided in the appendix).

Lemma 2 *For all $f \in \mathcal{A}$, x^f, z^f are well defined and $y^f \succsim x^f \succsim x_f$. Moreover, if $x^g \succ z^f$ or $x^f \notin W^f$, then $x^f \sim y^f$.*

Using the definitions above we can now construct an update rule. We first define the act h that appears in the definition of **GB**.

Definition 3 *Let $h^{\succsim,E,g} : \mathcal{A} \to \mathcal{A}$ such that for any $f \in \mathcal{A}$, $f_E h^{\succsim,E,g}(f) \sim (x^f)_E h^{\succsim,E,g}(f)$.*

The existence of $h^{\succsim,E,g}$ is shown as follows. For any $k \in \mathcal{A}$, let $w^k \in X$ such that $f_E k \sim (w^k)_E k$. Then $w^{y^f} \sim y^f$ and $w^{g^f} \sim x_f$, since $f_E y^f \sim y^f$, $f_E g^f \sim (x_f)_E g^f$. Define $h^{\succsim,E,g}(f)$ such that $w^{h^{\succsim,E,g}(f)} \sim x^f$, the existence of which is ensured by $y^f \succsim x^f \succsim x_f$ and sup-norm continuity of $u(w^k)$.

Now the conditional preference $\succsim_{E,g}$ can be defined as follows.

Definition 4 *$f \succsim_{E,g} f'$ if $\exists x, x'$ so that $f_E h^{\succsim,E,g}(f) \sim x_E h^{\succsim,E,g}(f)$, $f'_E h^{\succsim,E,g}$ $(f') \sim x'_E h^{\succsim,E,g}(f')$ and $x \succsim x'$.*

Clearly, for any $f \in \mathcal{A}$, $f \sim_{E,g} x^f$. Thus update rules can be defined by $U(\succsim, E, g) = \succsim_{E,g}$, depending on the choice of x^g in definition 1. Define $\mathcal{U}^{MEU}$ as the set of all update rules U thus defined. Clearly, the set of rules $\mathcal{U}^{MEU}$ satisfies **GB**. Axiom **NC** is satisfied since x^f does not depend on the definition of f on E. The definition also implies that the order of constant acts is preserved (axiom **RP**). Before we show that the definition also satisfies the axioms **CL** and **DC**, the following is a 3-state example of an update rule in $\mathcal{U}^{MEU}$.

Example 1 *(1) Let $S \equiv \{1, 2, 3\}$ and let $\succsim$ be a Choquet preference (Schmeidler [8]) with concave capacity (non-additive measure) $v(\cdot)$ such that $v(\{1\}) = 0.3$, $v(\{2\}) = 0.8$, $v(\{3\}) = 0.4$, $v(\{1, 2\}) = 0.9$, $v(\{1, 3\}) = 0.5$, $v(\{2, 3\}) = 0.9$. Therefore $\succsim \in \mathcal{P}^{MXEU}$, where*

$$\begin{aligned} C &= \{q \mid \forall E \in \Sigma, q(E) \leq v(E)\} \\ &= co\{(0.1, 0.5, 0.4), (0.1, 0.8, 0.1), (0.3, 0.5, 0.2), (0.3, 0.6, 0.1)\}. \end{aligned}$$

Let $u \circ g = (1, 20, 18)$ and $E = \{1, 2\}$. Then $u \circ g$ is evaluated (maximized) unconditionally by $(0.1, 0.8, 0.1)$, the Bayesian update of which is $(\frac{1}{9}, \frac{8}{9}, 0)$, so $u(x^g) = 17.889$. This updated measure is used to evaluate conditionally all utility acts that are positive affinely related (p.a.r.) on E to $u \circ g$ (points above the diagonal in figure 1). Note that $u(x_g) = 17.833 < u(x^g) = u(y^g)$. All other non-constant utility acts (points below the diagonal) are p.a.r. to $u \circ f = (20, 16.6, 18)$, which is evaluated unconditionally by $(0.3, 0.5, 0.2)$ and $f_E g \sim g$. The Bayesian update of this measure is $(\frac{3}{8}, \frac{5}{8}, 0)$, according to which $u \circ f$ has a *conditional expectation* 17.875. *This value is* $u(x^f)$ *since it is maximal amongst all conditional expectations* $u(w') < u(x^g)$ *computed this way for some utility act which is p.a.r. on* E *to* $u \circ f$. *Thus the conditional preference* $\succsim_{E,g} \in \mathcal{P}^{MXEU}$, *where*

$$C_{E,g} = co\{(\frac{1}{9}, \frac{8}{9}, 0), (\frac{3}{8}, \frac{5}{8}, 0)\}.$$

Note that all measures are updated, so $\succsim_{E,g}$ *is defined by the certainty equivalents* y^f *for all acts* f.

(2) Now let $u \circ g = (17.833, 17.833, 18)$. *Then* $u \circ g$ *is evaluated unconditionally by* $(0.1, 0.5, 0.4)$ *and* $u(x^g) = 17.833$. *For all acts* $f_E g \sim g$, *the corresponding* w' *satisfies* $u(w') \geq u(x^g)$, *so all acts are evaluated conditionally by* $(\frac{1}{6}, \frac{5}{6}, 0)$, *the Bayesian update of* $(0.1, 0.5, 0.4)$. *Thus the conditional preference is in* $\mathcal{P}^{EU}$ *with* $C_{E,g} = \{(\frac{1}{6}, \frac{5}{6}, 0)\}$, *i.e. only a strict subset of the original set of measures is updated.*

(3) Consider again the Ellsberg examples given in the introduction, where $\succsim \in \mathcal{P}^{MMEU}$ *with* $C = co\{(\frac{1}{3}, 0, \frac{2}{3}), (\frac{1}{3}, \frac{2}{3}, 0)\}$. *The conditional preference is generated by the Bayesian update of a subset of measures, depending on* g. *For* $u \circ g = (1, 3, 4)$, *which is evaluated unconditionally by* $(\frac{1}{3}, \frac{2}{3}, 0)$, *we have* $C_{E,g} = \{(\frac{1}{3}, \frac{2}{3}, 0)\}$. *For* $u \circ g = (2, 2, 1)$, *evaluated unconditionally by* $(\frac{1}{3}, 0, \frac{2}{3})$, *we get* $C_{E,g} = \{(1, 0, 0)\}$. *For* $u \circ g = (1, 3, 1)$ *all measures are updated. Finally if* $u \circ g = (1, 2, 2)$ *and* x^g *is chosen such that* $u(x^g) = \frac{3}{2}$, *then* $C_{E,g} = co\{(\frac{1}{3}, \frac{2}{3}, 0), (\frac{1}{2}, \frac{1}{2}, 0)\}$.

The following proposition proves closure of update rules in $\mathcal{U}^{MEU}$ directly by showing that for any act f, the conditional certainty equivalent x^f maximizes the expected utility of f over a set of measures. This set has measures corresponding to all $\succsim' \in \mathcal{P}^{EU}$, which are used to define conditional certainty equivalents in definition 2.

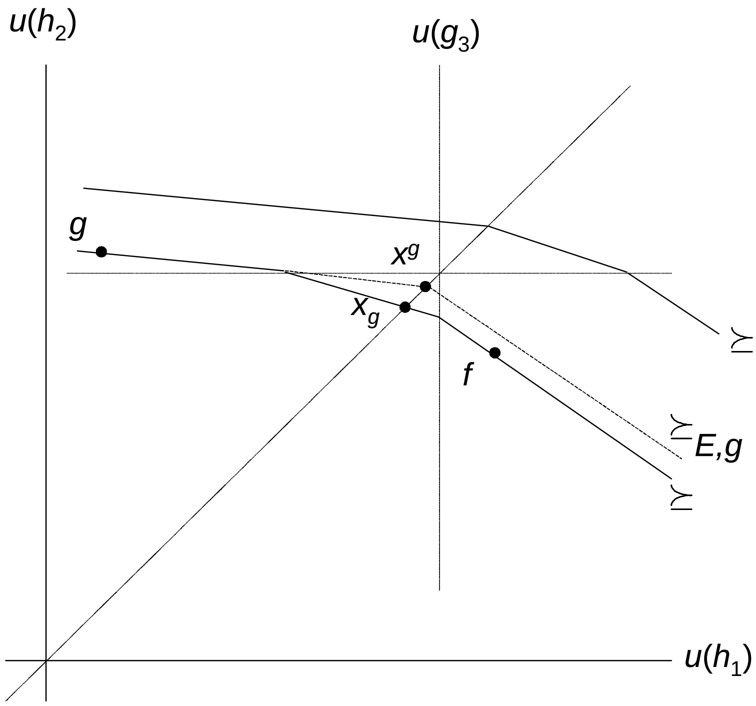

Figure 1: Indifference curves in utility space of a 3-state example

Proposition 1 *Let $h \in \mathcal{A}$, $w_h \in W^h$, $h' \in \mathcal{A}$, $w'_h \in X$ and $\succsim'_h \in \mathcal{P}^{EU}$ correspond to w_h as in definition 2 and let $f \in \mathcal{A}$, $v \in X$ such that $f_E g \sim'_h v_E g$. Then $x^f \succsim v$.*

Proof. If $x^g \succ z^f$ or $x^f \notin W^f$, then $x^f \sim y^f$ by lemma 2 and thus $x^f \succsim v$ by lemma 1. Suppose $x^g \sim z^f$ and $x^f \in W^f$, then $z^f \succsim w'_h$. Let $f' \in \mathcal{A}$ and $\succsim'_f \in \mathcal{P}^{EU}$ correspond to x^f as in definition 2. Let $q_f, q_h \in C$ be the measures corresponding to $\succsim'_f, \succsim'_h$ respectively and let $v' \in X$ such that $f'_E g \sim'_h v'_E g$. Then

$$\begin{aligned}\int u \circ ((z^f)_E g) dq_h &\geq \int u \circ ((w'_h)_E g) dq_h \\ &= \int u \circ (h'_E g) dq_h = I[u \circ (h'_E g)] = I[u \circ g] \\ &= I[u \circ (f'_E g)] \geq \int u \circ (f'_E g) dq_h = \int u \circ (v'_E g) dq_h.\end{aligned}$$

Thus $z^f \succsim v'$ and so $x^f \succsim v$ by definition 2. ■

We close this section with the following proposition, which establishes dynamic consistency of $\mathcal{U}^{MEU}$. Note that the orders $g \succsim f_E g$ and $g \succsim_{E,g} f$ that appear in the definition of **DC** are not reversed for max-max and max-min preferences, so the proof is separated for the two cases. In fact, this proposition is the unique result which is obtained for the two cases without reversing orders.

Proposition 2 *The set of rules $\mathcal{U}^{MEU}$ satisfies **DC**.*

Proof. Let $B \in \mathcal{B}$.
For max-max preferences. Choose any $U \in \mathcal{U}^{MEU}$. Let $(\succsim, E, g) \in \mathcal{T}$ such that $[g \in B$ and $\forall f_E g \in B$, $g \succsim f_E g]$ and let $\succsim_{E,g} = U(\succsim, E, g)$. Let $f_E g \in B$, so $g \succsim f_E g$. Let $w \in W^f$, so $\exists f' \in \mathcal{A}$, $w' \in X$ and $\succsim' \in \mathcal{P}^{EU}$ for which (1)-(5) of definition 2 hold, where $q' \in C$ corresponds to $\succsim'$. $g \succsim f_E g$ implies $f'_E g \succsim f_E g$ since $f'_E g \sim g$ by (2), so

$$\int u \circ (f'_E g) dq' = I[u \circ (f'_E g)] \geq I[u \circ (f_E g)] \geq \int u \circ (f_E g) dq'$$

since $q' \in C$. Thus $f'_E g \succsim' f_E g$, so $w'_E g \succsim' w_E g$ since $f'_E g \sim' w'_E g$ and $f_E g \sim' w_E g$. Hence $w' \succsim' w$ by monotonicity of $\succsim'$, so $w' \succsim w$ and therefore $x^g \succsim w$ since $x^g \succsim w'$ by (5). It follows that $x^g \succsim w$ for all $w \in W^f$, thus $x^g \succsim x^f$ and therefore $g \succsim_{E,g} f$.

For max-min preferences. Given $(\succsim, E, g) \in \mathcal{T}$, if $[g \in B$ and $\forall f_E g \in B$, $g \succsim f_E g]$, then the conditional certainty equivalent x^g and the corresponding $\succsim^g$ that appear in definition 1 can be chosen such that for any $f_E g \in B$, $g \succsim^g f_E g$. Indeed, this is ensured by convexity of $u \circ B$ and convexity of $\{u \circ f \mid f \succsim g\}$ by uncertainty aversion of $\succsim$. Thus there exists a corresponding $U \in \mathcal{U}^{MEU}$ satisfying this condition for any $(\succsim, E, g) \in \mathcal{T}$. Let $(\succsim, E, g) \in \mathcal{T}$ such that $[g \in B$ and $\forall f_E g \in B$, $g \succsim f_E g]$ and let $\succsim_{E,g} = U(\succsim, E, g)$. Let $f_E g \in B$, so $g \succsim^g f_E g$. Let $v \in X$ such that $f_E g \sim^g v_E g$. Then, $v \succsim x^f$ by proposition 1 with reversed orders, so

$$x^g_E g \sim^g g \succsim^g f_E g \sim^g v_E g \succsim^g x^f_E g.$$

Thus $g \succsim_{E,g} f$ since $x^g \succsim x^f$. ∎

4 Updating Probabilities

In this section we study the implication of updating preferences, as presented above, in terms of updating the set of measures which is used to represent these preferences. As a corollary of definition 2 and proposition 1 from the previous section, the conditional preference $\succsim_{E,g}$ is represented with the following set of Bayesian updated measures.

Definition 5 *The set of measures $C_{E,g}$ corresponding to $\succsim_{E,g}$ is the convex closure of $\{\left\langle \frac{q(F\cap E)}{q(E)} \right\rangle_{F\in\Sigma} \mid q \in \arg\max_{q'\in C} \int u \circ (f_E g) dq'$ for some $f_E g \in \mathcal{A}$ such that $f_E g \sim g$ and $\frac{1}{q(E)} \int_E (u \circ f) dq \leq u(x^g)\}$.*

The following proposition shows the set $C_{E,g}$ in the special case of rectangular preferences (Epstein and Schneider, [3]), i.e. when the representing set of measures C satisfies stochastic independence given an event and given its complement. For these preferences, each act is evaluated by maximizing separately the conditional expectation on E and on E^c and then maximizing the expectation of these conditional expectations. As a result, the conditional preference is defined using the Bayesian update of all measures ($\forall f \in \mathcal{A}$, $x^f \sim y^f$).

Proposition 3 *Suppose $E, E^c \in \mathcal{N}(\succsim)$ and C is rectangular with respect to $\{E, E^c\}$, i.e. $\forall q^1, q^2, q^3 \in C$, $\exists q^4 \in C$ such that $\forall F$, $q^4(F) = q^3(E)\frac{q^1(F\cap E)}{q^1(E)} + q^3(E^c)\frac{q^2(F\cap E^c)}{q^2(E^c)}$. Then $\forall f \in \mathcal{A}$, $y^f \sim x^f \sim x_f$.*

Proof. By rectangularity, for any $h \in \mathcal{A}$, if $q_h \in \arg\max_{q\in C} \int (u \circ h) dq$ then $q_h \in \arg\max_{q\in C}[\frac{1}{q(E)} \int_E (u \circ h) dq]$. Otherwise, for any $p \in \arg\max_{q\in C}[\frac{1}{q(E)} \int_E (u \circ h) dq]$, $\exists q' \in C$ such that $q'(F) = q_h(E)\frac{p(F\cap E)}{p(E)} + q_h(F \cap E^c)$, so

$$\begin{aligned}
\int (u \circ h) dq_h &= q_h(E)\frac{1}{q_h(E)} \int_E (u \circ h) dq_h + \int_{E^c} (u \circ h) dq_h \\
&< q_h(E)\frac{1}{p(E)} \int_E (u \circ h) dp + \int_{E^c} (u \circ h) dq_h = \int (u \circ h) dq',
\end{aligned}$$

contradicting $q_h \in \arg\max_{q\in C} \int (u \circ h)dq$.

Given $f \in \mathcal{A}$, let $w \in W^f$ and $f', \succsim', w'$ corresponding to w as in definition 2. Let $q_{f'_E g} \in \arg\max_{q\in C} \int u \circ (f'_E g)dq$, then $u(w') = \frac{1}{q_{f'_E g}(E)} \int_E (u \circ f')dq_{f'_E g}$ $= \max_{q\in C}[\frac{1}{q(E)} \int_E (u \circ f')dq]$. Let $q_{f'_E w'} \in \arg\max_{q\in C} \int u \circ (f'_E w')dq$, then

$$\begin{aligned} \int u \circ (f'_E w')dq_{f'_E w'} &= q_{f'_E w'}(E) \max_{q\in C}[\frac{1}{q(E)} \int_E (u \circ f')dq] + q_{f'_E w'}(E^c)u(w') \\ &= u(w'). \end{aligned}$$

Thus $f'_E w' \sim w'$, so $w' \sim y^{f'}$. By c-independence of $\succsim$, $w \sim y^f$, so $x^f \sim y^f$.

To show $x^f \sim x_f$, let $z \in X$ such that $f'_E g \sim z_E g$ and $q_{z_E g} \in \arg\max_{q\in C} \int u \circ (z_E g)dq$. Denote $b \equiv \max_{q\in C}[\frac{1}{q(E^c)} \int_{E^c} (u \circ g)dq]$, so

$$\begin{aligned} q_{f'_E g}(E)u(w') + q_{f'_E g}(E^c)b &= \int u \circ (f'_E g)dq_{f'_E g} = \int u \circ (z_E g)dq_{z_E g} \\ &= q_{z_E g}(E)u(z) + q_{z_E g}(E^c)b. \end{aligned}$$

Since $E \in \mathcal{N}(\succsim)$, then $q_{f'_E g}(E), q_{z_E g}(E) > 0$, so $u(w') \geq b \iff u(z) \geq b$ and $u(w') \leq b \iff u(z) \leq b$. Therefore, by rectangularity, if $u(w') \geq u(b)$ then $q_{f'_E g}(E) = q_{z_E g}(E) = \max_{q\in E} q(E)$ and if $u(w') \leq u(b)$ then $q_{f'_E g}(E) = q_{z_E g}(E) = \min_{q\in E} q(E)$, otherwise contradicting arg-maximality of either $q_{f'_E g}$ or $q_{z_E g}$. It follows that $u(z) = u(w')$, thus $z \sim w' \sim y^{f'}$ and so $x_f \sim y^f$ by c-independence of $\succsim$. ■

Next we show, as an illustration, how to calculate the set of updated measures $C_{E,g}$ when S is finite and E^c is a singleton. Let p be a measure that is used by $\succ$ to evaluate g and where $u(x^g)$ is the evaluation of g using the Bayesian update of p. The proposition shows that only all measures in either $\{q \in C \mid q(E) \geq p(E)\}$ or $\{q \in C \mid q(E) \leq p(E)\}$ are updated. For max-min EU preferences, the inequalities $\leq, \geq$ are reversed.

Proposition 4 *Suppose S is finite and E^c is a singleton. Let $p \in \arg\max_{q'\in C}$ $\int (u \circ g)dq'$ such that $\frac{1}{p(E)} \int_E (u \circ g)dp = u(x^g)$. Then $C_{E,g}$ has as members the*

Bayesian updates $\left\langle \frac{q(F\cap E)}{q(E)} \right\rangle_{F\in\Sigma}$ *of all* $q \in C$ *satisfying:*
(1) $q(E) \geq p(E)$ *if* $g \succ g(E^c)$
(2) $q(E) \leq p(E)$ *if* $g(E^c) \succ g$
(3) No further restrictions if $g \sim g(E^c)$

Proof. We first show the following property, establishing that measures q that violate (1)-(3) are not updated. Let $f_E g \in \mathcal{A}$ such that $f_E g \sim g$ and let $q \in \arg\max_{q'\in C} \int u \circ (f_E g) dq'$; if $g \sim g(E^c)$, then $\frac{1}{q(E)} \int_E (u \circ f) dq = u(x^g)$ $= u(y^g)$; if $g \succ g(E^c)$, then $\frac{1}{q(E)} \int_E (u \circ f) dq \leq u(x^g) \iff q(E) \geq p(E)$; and if $g(E^c) \succ g$, then $\frac{1}{q(E)} \int_E (u \circ f) dq \leq u(x^g) \iff q(E) \leq p(E)$.
To prove this, note that $g \succsim g(E^c)$ implies $0 \leq \int (u \circ g) dp - u[g(E^c)] = \int_E (u \circ g) dp - p(E) u[g(E^c)]$, so $u(x^g) \geq u[g(E^c)]$ since $\frac{1}{p(E)} \int_E (u \circ g) dp = u(x^g)$. Adding $f_E g \sim g$, we have

$$\begin{aligned}
0 &= I[u \circ (f_E g)] - I[u \circ g] \\
&= [q(E)(\frac{1}{q(E)} \int_E (u \circ f) dq - u[g(E^c)]) + u[g(E^c)]] \\
&\quad -[p(E)(u(x^g) - u[g(E^c)]) + u[g(E^c)]],
\end{aligned}$$

so $q(E)(\frac{1}{q(E)} \int_E (u \circ f) dq - u[g(E^c)]) = p(E)(u(x^g) - u[g(E^c)]) \geq 0$. Thus, if $g \sim g(E^c)$, then $\frac{1}{q(E)} \int_E (u \circ f) dq = u[g(E^c)] = u(x^g)$ and $g(E^c) \sim y^g$, so $u(x^g) = u(y^g)$. If $g \succ g(E^c)$ then all inequalities are strict and thus $\frac{1}{q(E)} \int_E (u \circ f) dq \leq u(x^g) \iff q(E) \geq p(E)$. The proof is similar in the case $g(E^c) \succ g$.

We now show that measures q that satisfy (1)-(3) are updated. First, if $g \sim g(E^c)$, then by the property above $\forall f \in \mathcal{A}$, $x^f \sim y^f$ and so $\forall q \in C$, $\left\langle \frac{q(F\cap E)}{q(E)} \right\rangle_{F\in\Sigma} \in C_{E,g}$, thus establishing (3). To prove (1), suppose $g \succ g(E^c)$. Let δ_{E^c} be the degenerate measure for which $\delta_{E^c}(E^c) = 1$, then $\delta_{E^c} \notin C$ since $E \in \mathcal{N}(\succsim)$. For any $q \in C$ such that $q(E) \geq p(E)$, if $\exists \lambda > 1$ such that $\tilde{q} \equiv \lambda q + (1-\lambda)\delta_{E^c} \in C$, then $\tilde{q}(E) = \lambda q(E) > q(E) \geq p(E)$ and $\left\langle \frac{\tilde{q}(F\cap E)}{\tilde{q}(E)} \right\rangle_{F\in\Sigma} = \left\langle \frac{q(F\cap E)}{q(E)} \right\rangle_{F\in\Sigma}$. Thus it is sufficient to show that $\left\langle \frac{q(F\cap E)}{q(E)} \right\rangle_{F\in\Sigma} \in C_{E,g}$ for $q \in C$ which is also a boundary point of $co\{C \cup \{\delta_{E^c}\}\}$, so

$$q(E) \geq p(E) \text{ and } \forall \lambda > 1,\, \lambda q + (1-\lambda)\delta_{E^c} \notin C.$$

Suppose such q, then $\exists f' \in \mathcal{A}$ such that $q \in \arg\max_{q' \in co\{C \cup \{\delta_{E^c}\}\}} \int (u \circ f')dq'$, so $\int (u \circ f')dq \geq u[f'(E^c)]$ since $\delta_{E^c} \in co\{C \cup \{\delta_{E^c}\}\}$. By closure of $C_{E,g}$, it is sufficient to consider the case of strict inequality, so $0 < \int_E (u \circ f' - u[f'(E^c)])dq$ $= b(I[u \circ g] - u[g(E^c)])$ for some $b > 0$. Let $f_E g \in \mathcal{A}$ such that $u \circ (f_E g)$ $= \frac{1}{b}(u \circ f' - u[f'(E^c)])$ $+u[g(E^c)]$, thus $q \in \arg\max_{q' \in C} \int u \circ (f_E g)dq'$ and $\int u \circ (f_E g)dq = I[u \circ g]$, so $f_E g \sim g$. Then $\frac{1}{q(E)} \int_E (u \circ f)dq \leq u(x^g)$ by $q(E) \geq p(E)$ and the property initially proved, so $\left\langle \frac{q(F \cap E)}{q(E)} \right\rangle_{F \in \Sigma} \in C_{E,g}$.

The proof of (2), where $g(E^c) \succ g$, is similar with the change that q satisfies $q(E) \leq p(E)$ and $\forall \lambda \in [0,1)$, $\lambda q + (1-\lambda)\delta_{E^c} \notin C$ and $co\{C \cup \{\delta_{E^c}\}\}$ is replaced by $C' \equiv \{\lambda q'' + (1-\lambda)\delta_{E^c} \mid q'' \in C, \lambda \geq 1\}$. Thus $\exists f'$ such that $q \in \arg\max_{q' \in C'} \int (u \circ f')dq'$, so $\int (u \circ f')dq < u[f'(E^c)]$ since f' separates C' from $\delta_{E^c} \notin C'$ and then $0 > \int_E (u \circ f' - u[f'(E^c)])dq = b(I[u \circ g] - u[g(E^c)])$ for some $b > 0$. ■

5 Appendix

Proof. (Lemma 2). Let $f \in \mathcal{A}$, then W^f is non-empty since it has as a member $w \in X$ for which $f' = w' = z = x_g$ and $\lambda = 0$ hold in the definition of W^f. Assuming that x^f is well defined, lemma 1 may be used with f replaced by $f_E g^f$ to show that $y^f \succsim x^f \succsim x_f$.

It is left to show that x^f, x^f are well defined. If there exists $w \in W^f$ for which a corresponding w' satisfies $x^g \sim w'$, then clearly z^f and x^f are well-defined and $x^g \sim z^f$. Suppose that for any $w \in W^f$ a corresponding w' satisfies $x^g \succ w'$. In this case, where $x^g \succ z^f$ or $x^f \notin W^f$, we show that $x^f \sim y^f$. Let $\beta : \mathbb{R}_+ \to \mathbb{R}$, $q : \mathbb{R}_+ \to C$ such that $\forall \alpha \geq 0$, $I[(\alpha(u \circ f) + \beta(\alpha))_E\,(u \circ g)] = I[(u \circ g)]$ and $q_\alpha \in \arg\max_{q \in C} \int (\alpha(u \circ f) + \beta(\alpha))_E\,(u \circ g)dq$. By the assumption above and the definition of W^f, $W^f = \cup_{\alpha \geq 0}\{w \in X \mid u(w) = \frac{1}{q_\alpha(E)}\int_E (u \circ f) dq_\alpha\}$. It is sufficient to show that $\sup_{\alpha \geq 0}\{\frac{1}{q_\alpha(E)}\int_E (u \circ f) dq_\alpha\}$ exists and equals $u(y^f)$. Start by showing that $\frac{1}{q_\alpha(E)}\int_E (u \circ f) dq_\alpha$ is non-decreasing with α. Let $\alpha_1 > \alpha_2$. Then

$$\begin{aligned}\int (\alpha_1(u \circ f) + \beta(\alpha_1))_E\,(u \circ g) dq_{\alpha_1} &= \int (\alpha_2(u \circ f) + \beta(\alpha_2))_E\,(u \circ g) dq_{\alpha_2} \\ &\geq \int (\alpha_2(u \circ f) + \beta(\alpha_2))_E\,(u \circ g) dq_{\alpha_1},\end{aligned}$$

so $(\alpha_1 - \alpha_2)\int_E (u \circ f) dq_{\alpha_1} \geq [\beta(\alpha_2) - \beta(\alpha_1)] q_{\alpha_1}(E)$. Similarly, $(\alpha_2 - \alpha_1)\int_E (u \circ f) dq_{\alpha_2} \geq [\beta(\alpha_1) - \beta(\alpha_2)] q_{\alpha_2}(E)$. Thus $\frac{1}{q_{\alpha_2}(E)}\int_E (u \circ f) dq_{\alpha_2} \leq \frac{\beta(\alpha_2)-\beta(\alpha_1)}{\alpha_1 - \alpha_2} \leq \frac{1}{q_{\alpha_1}(E)}\int_E (u \circ f) dq_{\alpha_1}$, establishing that $\frac{1}{q_\alpha(E)}\int_E (u \circ f) dq_\alpha$ is non-decreasing with α. Hence by adding that it is bounded, $\sup_{\alpha \geq 0}\{\frac{1}{q_\alpha(E)}\int_E (u \circ f) dq_\alpha\}$ exists and equals $\lim_{\alpha \to \infty}\{\frac{1}{q_\alpha(E)}\int_E (u \circ f) dq_\alpha\}$. By definition of $\beta(\alpha)$ and q_α, $\forall \alpha > 0$,

$$\begin{aligned}0 &= \frac{1}{\alpha q_\alpha(E)}[I[(\alpha(u \circ f) + \beta(\alpha))_E\,(u \circ g)] - I[(u \circ g)]] \\ &= \frac{1}{q_\alpha(E)}\int_E (u \circ f) dq_\alpha + \frac{\beta(\alpha)}{\alpha} + \frac{1}{\alpha q_\alpha(E)}[\int_{E^c} (u \circ g) dq_\alpha - I[(u \circ g)]],\end{aligned}$$

so $\lim_{\alpha \to \infty}\{-\frac{\beta(\alpha)}{\alpha}\} = \lim_{\alpha \to \infty}\{\frac{1}{q_\alpha(E)}\int_E (u \circ f) dq_\alpha + \frac{1}{\alpha q_\alpha(E)}[\int_{E^c} (u \circ g) dq_\alpha - I[(u \circ g)]]\} = \lim_{\alpha \to \infty}\{\frac{1}{q_\alpha(E)}\int_E (u \circ f) dq_\alpha\}$. By c-linearity, $\forall \alpha > 0$, $I[(u \circ f)_E \left(\frac{(u \circ g) - \beta(\alpha)}{\alpha}\right)]$ $= \frac{I[(u \circ g)] - \beta(\alpha)}{\alpha}$. Thus at the limit $\alpha \to \infty$, $I[u \circ f)_E(\lim_{\alpha \to \infty}\{-\frac{\beta(\alpha)}{\alpha}\})] = \lim_{\alpha \to \infty}\{-\frac{\beta(\alpha)}{\alpha}\}$, so $u(y^f) = \lim_{\alpha \to \infty}\{-\frac{\beta(\alpha)}{\alpha}\}$. Hence $u(x^f) = \sup_{\alpha \geq 0}\{\frac{1}{q_\alpha(E)}\int_E (u \circ f) dq_\alpha\} = u(y^f)$.

To prove that z^f is well defined it is sufficient to show that $\sup_{\alpha \geq 0}\{\frac{\alpha}{q_\alpha(E)}\int_E (u \circ f) dq_\alpha + \beta(\alpha)\}$ exists. This is the case since $\alpha \int_E (u \circ f) dq_\alpha + q_\alpha(E)\beta(\alpha) = I[(u \circ g)]$ $- \int_{E^c} (u \circ g) dq_\alpha$, so $\frac{\alpha}{q_\alpha(E)}\int_E (u \circ f) dq_\alpha + \beta(\alpha)$ is bounded. Moreover, let $\alpha_1 > \alpha_2$. By definition of $\beta(\alpha)$, q_α and the inequalities above,

$$\begin{aligned}\frac{\alpha_1}{q_{\alpha_1}(E)}\int_E (u \circ f) dq_{\alpha_1} + \beta(\alpha_1) &\geq \frac{\alpha_2}{q_{\alpha_1}(E)}\int_E (u \circ f) dq_{\alpha_1} + \beta(\alpha_2) \\ &\geq \frac{\alpha_2}{q_{\alpha_2}(E)}\int_E (u \circ f) dq_{\alpha_2} + \beta(\alpha_2),\end{aligned}$$

establishing that $\frac{\alpha}{q_\alpha(E)}\int_E(u\circ f)dq_\alpha+\beta(\alpha)$ is non-decreasing with α. Hence $\sup_{\alpha\geq 0}\{\frac{\alpha}{q_\alpha(E)}\int_E(u\circ f)dq_\alpha+\beta(\alpha)\}=\lim_{\alpha\to\infty}\{\frac{\alpha}{q_\alpha(E)}\int_E(u\circ f)dq_\alpha+\beta(\alpha)\}$. ■

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