Portfolio Choice and Asset Prices with Similarity Considerations — A Case-Based Approach

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I analyze the influence of similarity considerations on portfolio choices in an individual portfolio-choice problem with fixed prices, as well as in an economy with endogenous prices. In an individual problem only the similarity among portfolios is taken into account. Investors with concave similarity function diversify only if their aspiration level is low enough and choose an undiversified portfolio, else. Moreover, case-based decision makers with concave similarity fail to learn to choose the optimal alternative in the limit, even when using the adaptation rule of Gilboa and Schmeidler (1996). In a market environment similarity between price-portfolio pairs is considered. The result that only investors with low aspiration levels diversify obtains in this case as well. Investors with high aspiration levels switch between the undiversified portfolios, causing cycles in the economy.

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1 Introduction

In 1995 Gilboa and Schmeidler proposed a new theory of decision-making under uncertainty — the case-based decision theory. In contrast to the expected utility theory it does not assume that decision makers have beliefs about states of nature and probability distributions over state-contingent payoffs. Instead, people are supposed to learn from experience. They evaluate an alternative according to its performance in a number of past cases observed and use an aspiration level as a bench-mark in the decision process. The alternative which has performed best in the past is then chosen.

I have recently applied the case-based decision theory to decision making in financial markets, see Guerdjikova (2001, 2003 a, 2003 b). In this work, however, I have assumed, that an asset or a portfolio is evaluated only according to its own performance, but independently from the performance of other assets or portfolios. It is, however, plausible that an investor could consider investment alternatives as similar. In this case, the past performance of one investment possibility will influence the evaluation of another one, considered similar to the first.

The theory of case-based decisions allows to incorporate such kind of considerations into a formal model. Similarity is, however, a relatively new concept in economics. Hence, it is not a priori clear how similarity should be operationalized in different contexts. Therefore, the first question this paper should ask is what does ”similar” mean in the context of financial markets.

The case-based decision theory works with similarity functions which allow a numerical representation of perceived similarity. Still, there has been little research into how such a similarity function should look like and what characteristics it should have. This is the next issue to be discussed in this paper.

Once the two former issues have been discussed, it will be possible to present a model of individual portfolio choice, in which similarity considerations will be integrated. By using this model the impact of the form of the similarity function on the individual decision making will be discussed, by comparing the results to those obtained without similarity considerations and to the results of the standard theory, which assumes a fully informed expected utility maximizer.

The individual portfolio choice problem can afterwards be integrated into a market. Instead of considering similarities among assets or portfolios alone in this case, it seems reasonable to define similarity between problem – act / price – portfolio pairs. Again it will be interesting to know, how similarity considerations influence the portfolio choices and the price dynamics in a market with case-based decision makers.

2 Similarity

2.1 A Philosophical View

Similarity, without doubt, affects our judgements and therefore influences our decisions. Nevertheless, although ”there is nothing more basic to thought and language than our sense of similarity”, Quine (1969, p. 6), it seems to be very difficult to give a precise definition of similarity, which could be used in a formal model. Philosophical thought has been dwelling on this question for a long time and still, the problem does not seem to have a straightforward solution.

The problem is that similarity can be neither defined by means of sets, nor by means of logical structures. Indeed, suppose that one could divide the objects into sets, such that all objects, which are similar to each other are in one such set. One easily sees that as long as objects are compared with respect to all possible criteria, each object should either be placed into a separate set, or all objects will be considered similar in some sense and will build a unique set, thus leading the notion of similarity ad absurdum.

A representation of similarity by means of a logical structure is also problematic. Suppose you want to evaluate alternative α. If you know that alternative α' has performed well and you consider α and α' to be similar, you will probably think that α will also perform well. Now suppose that you also know that alternative α'' has performed badly and you consider α'' to be dissimilar to α. Will this make your believe that α will perform well stronger? Probably yes, if you feel that this confirms the (perceived) law that alternatives that perform poorly are not similar to α, therefore that alternatives which are similar to α cannot perform poorly. It is however paradoxical that an alternative, which is considered to be dissimilar to α should be used to predict the performance of α[2].

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[2] This argument is known as the Hempel’s puzzle. Its original version is, that since ”each black raven tends to confirm the law, that all ravens are black, so each green leave, being a non-black non-raven, should tend to confirm the law that all non-black things are non-ravens, that is, again, that al ravens are black”, Quine (1969, p. 5).

The main problem is, of course, that since mathematical concepts obey the laws of logic, one necessarily embeds this paradox into each mathematical model of similarity. And indeed, the theory of Gilboa and Schmeidler (1997 (a)) also bears this paradox in itself.

Although a philosophical definition of similarity is problematic, it still does not mean that similarity is a concept that couldn't be implemented in an economic model. In fact, as long as we are interested only in individual perception of similarity, it is enough to elicit the similarity function of an individual (by asking him questions of the type "is α similar to α'?" or "is α more similar to α' than to α''?"), in order to model his behavior. However this behavioristic modelling does not give any clues up to which similarity perceptions are sensible or what should be understood under similarity in economics.

2.2 Similarity in Economics

Similarity does not belong to the standard concepts of economic theory. Rubinstein (1988) suggests a theory of decision-making based on similarities between lotteries. The concept of similarity he uses is a binomial one. Two objects are either similar or not similar. Similarity is defined on intervals of real numbers and two numbers are considered similar, if and only if the difference between these two numbers is less than a given number s, and dissimilar else. This definition of similarity can be used to explain some of the experimentally observed behavior of choices between lotteries, Buschena and Zilberman (1995, 1999), as well as hyperbolic discounting, Rubinstein (2003). Since however the similarity relation is not transitive, it leads to preferences with intransitive indifference relations[3]. Of course, this intransitivity of preferences is due to the assumption of binomial similarity relation and can be avoided by introducing a "more similar than" relation.

Such a relation is proposed by Tversky (1977), who derives a similarity function from a set of axioms. Similarity is seen as possession of common attributes, still symmetry is not implied by the axioms and is generally violated empirically. The approach of Nehring and Puppe (2003) is similar to that of Tversky (1977). Their similarity concept is a trinomial relation, interpreted as "α is more similar to α'' than is α'". It satisfies reflexivity, symmetry and transitivity and expresses the idea that α and α'' have more common attributes than α' and α''. Moreover, this

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[3] Kajii (1996) provides a chracterization of preferences consistent with the similarity concept of Rubinstein.

trinomial relation is representable by a similarity function. However, the similarity relation is not complete, i.e. it is not defined for all triples of acts and therefore similarity judgements are not always possible.

An alternative theory is the case-based decision theory proposed by Gilboa and Schmeidler (1997 a). They assume a similarity function defined on problem – act pairs, which is also based on a ”more similar than” relation. They suggest that the cumulative utility of an act depends not only on the utility realizations observed when the same act was chosen in the past, but also on utility realizations of similar acts chosen in similar decision problems. An axiomatic representation of this functional form is provided. Nevertheless, the axiomatization does not give any clue about the characteristics a similarity function should have and their inpact on the decisions made.

Few models have used this concept of similarity up to now. Gilboa and Schmeidler (1997 b) show how similarity between goods can be interpreted in terms of complementarity and substitutability in a consumer choice problem. Blonski (1999) models social learning and represents social structures by similarity considerations. He shows that different similarity functions, associated with ”star”-communication structures, Δ-neighborhood structures and complete information, imply different stable states of the dynamical learning process. Gayer (2003) shows how similarity considerations may affect the perception of lotteries and lead to overestimation of low probabilities and underestimation of high probabilities. She proposes to measure the similarity between lotteries in terms of distance. In contrast to the models presented in this thesis, she assumes an endogenous similarity function, which becomes finer, as the memory grows. She shows that if the similarity function depends on the distance, its concavity insures that the decision-maker will be able to learn the correct distribution.

This research shows that the results obtained are very sensitive to the form of the similarity function assumed. It will turn out that the form of the similarity function plays a major role for the behavior of case-based investors in financial markets, as well.

2.3 Similarity in Financial Markets

Suppose that you ask an investor, who has lost a reasonable amount of money on dot.com assets, whether he would be ready to invest in a dot.com company now. You would probably receive a firm no, even, if the company you are proposing is a sound one. Not necessarily because the unlucky investor has looked up the performance of the company and has analyzed the pro and contras, but because his experience has taught him that dot.com assets can lead to significant losses and he obviously finds the investment you are proposing to be quite similar to the bad choice he has made last time. Thus, similarity can play major role in asset choice decisions.

The most natural concept of similarity (from the point of view of standard financial economics) is the concept of covariance. However, since the case-based decision theory presupposes no knowledge about state-contingent outcomes and their distribution, it does not seem appropriate to use the concept of covariance to model similarity in case-based decision-making[4].

Since the information available to a case-based decision-maker consists of a problem and a set of acts, it is reasonable to use the description of acts to elicit a notion of similarity. In this sense similarity may refer to the fact that assets of firms in the same industry are regarded as similar, as contrasted to firms from different industries. For instance, an investor may consider shares of BMW and Renault to be more similar to each other, than the shares of Renault and Telecom. Similarity perceptions might also include the nationality of an asset. Thus, an investor might find that Telecom is more similar to BMW than to Renault. Other characteristics, such as maturity or being derivatives of the same underlying asset can also influence similarity perceptions in financial markets.

The above discussed characteristics allow only for a comparison of individual assets, but not of portfolios. However, it is easy to imagine, how similarity might refer to the comparison of the structure of two portfolios. For instance, an investor may consider a portfolio, consisting of 20% risky assets and 80% bonds to be more similar to a riskless portfolio, consisting of 100% bonds, than a portfolio, consisting only of risky assets.

To formalize these ideas, note that each portfolio, consisting of at most K assets can be represented as a point in a K-dimensional simplex Δ^{K-1}. Now, similarity between portfolios can be modelled as a continuous decreasing function of the Euclidean distance between the points in this simplex:

$$s(\alpha; \alpha') \quad = \quad f(\|\alpha - \alpha'\|) \text{ with}$$

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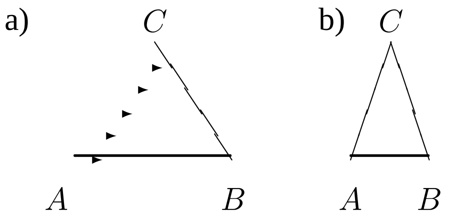
[4] There seems to exist a certain connection between the notions of similarity and correlation, as observed by Matsui (2000). Still, since his states of nature do not necessarilly correspond to the states of nature as considered in the standard financial economics, the interpretation of his result is ambiguous.

$$f'\left(\|\alpha-\alpha'\|\right) \quad < \quad 0.$$

In order to include similarity considerations between assets (such as maturity, or belonging to the same industry), it is enough to modify the simplex, allowing the distance between its vertices to vary with the degree of similarity. Figure 1 illustrates this.



a) A, B and C are mutually dissimilar.

b) A and B are similar, A and C, as well as B and C are mutually dissimilar.

Figure 1

Such a similarity function applies, of course, only to situations which are considered to be identical and in which only similarity between acts matter.

Still, the market situation might also influence the evaluation of different acts. Buying an asset in a market boom might be quite different from buying the same asset, when prices fall. The characteristics of a given decision situation are captured in the notion of a problem. In a financial market asset prices seem to bear the most important information about the decision situation and will, therefore, influence similarity perceptions.

In a model of case-based decision-making in financial markets these two aspects of similarity — similarity between problems and between acts — are captured in a single similarity function:

$$s\left(\left(\rho;\alpha\right);\left(\rho';\alpha'\right)\right),$$

which is to be interpreted as the degree of similarity of choosing act α in problem ρ to choosing act α' in problem ρ'. It has already been shown that the acts α can be situated on a metric space, depending on how similar they are perceived to one another. Since it seems to me that the major characteristic of a portfolio choice problem is represented by the prices in the economy, I propose to identify the problem with a price vector $(p_1...p_K)$ and to represent a problem - act pair in $\Delta^{K-1}\times\Delta^{K-1}$. Again, taking the Euclidean distance as measure of similarity, the similarity function can be written as:

$$s\left(\left(p;\alpha\right);\left(p';\alpha'\right)\right)=f\left(\left\|\left(\alpha;p\right)-\left(\alpha';p'\right)\right\|\right),$$

where $f\left(\cdot\right)$ is continuous and decreasing.

The axiomatization of Gilboa and Schmeidler (1997 (a)) implies that the similarity function is unique up to an affine-linear transformation, and therefore the similarity function can be normalized, so that it takes on only values between 0 and 1 where a value of 1 means that two objects are ”identical” or ”completely similar”:

$$s\left(\left(p;\alpha\right);\left(p;\alpha\right)\right)=1,$$

whereas 0 can be interpreted as ”having nothing in common”, or being ”completely different”, depending on the context.

Of course, the notion of similarity, as well as the specific similarity function used by an investor will influence his behavior in a financial market, since they will determine his evaluation of acts. Moreover, the similarity function may (as well as the aspiration level) evolve with the time, reflecting the fact that the decision-maker learns about (dis)similarities among alternatives, which he was not aware of, see Gilboa and Schmeidler (2001 (a), Chapter 19) and Gayer (2003).

I will assume that the similarity function remains constant over the time and will explore the influence of the form of the similarity function on investors' behavior.

3 Portfolio Choice with Similar Acts

In this section I analyze the individual portfolio choice problem for given asset prices, hence, for exogenous returns. I first consider the case of constant aspiration level and show how the aspiration level and the similarity function influence the limit behavior of a case-based decision maker. Next, I turn to the issue of learning. I study the question of whether the adaptation rule proposed by Gilboa and Schmeidler (1996) leads to optimal learning in presence of similarity and identify conditions necessary for this to be the case.

3.1 The portfolio choice problem

Suppose, that the set of problems P consists of one element p, described as follows: an investor has to invest one unit of wealth into a portfolio consisting of two assets A and B. The returns of the two assets, denoted by δ_A and δ_B, are identically and independently distributed on $\left[\underline{\delta}_A; \bar{\delta}_A\right]$ and $\left[\underline{\delta}_B; \bar{\delta}_B\right]$, respectively. $\mathfrak{A} = [0; 1]$ denotes the set of possible portfolios and $a \in [0; 1]$ stays for the share of wealth invested in the asset A. The possible utility realizations of a portfolio a, $\mathfrak{U}_a$ are given by:

$$u\left(a; \delta_A; \delta_B\right) = u\left(a\delta_A + (1-a)\,\delta_B\right)$$

If u is bounded and continuous on

$$\left[a\underline{\delta}_A + (1-a)\,\underline{\delta}_B; a\bar{\delta}_A + (1-a)\,\bar{\delta}_B\right]$$

for all $a \in \mathfrak{A}$ then the distribution $\left(F_a\right)_{a\in\mathfrak{A}}$ of $u\left(\cdot\right)$, for a given a is characterized by a finite mean, μ_a, finite variation and bounded support Δ_a. Assume, that the same investment decision has to be made in every period $t = 1, 2, ...$

Since all the problems are identical, the similarity function s can only record similarity between acts: $s\left(a; a'\right)$ for a, $a' \in \mathfrak{A}$. Assume, that the similarity function satisfies

$$\begin{aligned} s\left(a; a\right) &= 1 \\ s\left(a; a'\right) &= s\left(a'; a\right) \end{aligned}$$

$$s(0;1) \;=\; 0.$$

Let s further depend only on the distance between a and a' and assume that it is decreasing and concave in $|a - a'|$. Figure 2 gives an example of such a similarity function.

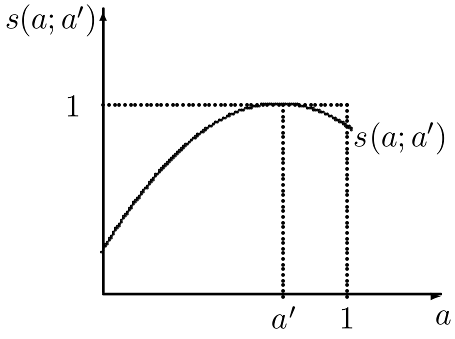


Figure 2

Note, that the decision maker is not assumed to know the distribution of the returns, nor their possible realizations, moreover he doesn't form beliefs about state-dependent outcomes. The only information, on which he bases his decision is his memory, a set of cases he has encountered in the past. A case is defined as a triple $(p_t; a_t; u_t(a_t))$, where $p_t \in P$ denotes the problem to be solved at time t, (and therefore $p_t = p$ for all p), $a_t \in \mathfrak{A}$ is the act chosen at time t, and $u_t(a_t) \in \mathfrak{U}_a$ is the result observed, after a has been chosen. The memory of the investor at time t is therefore represented by the set of cases:

$$M_t = \left((p_\tau; a_\tau; u_\tau(a_\tau))\right)_{\tau=1,2\ldots t}.$$

M_t is assumed to be endogenously determined at each time t, i.e. M_t contains only cases with portfolios actually chosen and utility realizations actually observed by the investor.

3.2 The case of constant aspiration level

Assume first that the aspiration level $\bar{u}$ of the investor is constant over the time. Given a problem

p and a memory of length t the decision rule of a case-based decision maker is specified as follows. Let $U(a)$ denote the cumulative utility of an act a:

$$U(a) = \sum_{\tau=1}^{t} s(a; a_\tau)\left[u_\tau - \bar{u}\right].$$

In each period t the investor calculates the cumulative utilities of all acts available and chooses the act with the highest cumulative utility. In the first period the memory is empty, therefore the cumulative utility of all acts is assumed to be equal to 0 and the investor chooses a portfolio at random.

Denote by S_0 the set of all possible decision paths:

$$S_0 = \left\{\omega = \left((\bar{u}_t; \alpha_t; u_t)\right)_{t=1,2...}\right\} \subset (\mathbb{R} \times \mathfrak{A} \times \Delta)^{\mathbb{N}}$$

with $\Delta = \cup_{a\in[0;1]}\Delta_a$. Let S denote the subset of paths on which the decision maker chooses the act with the highest cumulative utility in each period of time:

$$S = \left\{\omega \in S_0 \mid \alpha_t \in \arg\max_{\alpha\in\mathfrak{A}} U_t(\alpha)\ \forall t \geq 1\right\}$$

Let $a_1 = \bar{a}$ denote the act chosen in the first period. Assume as well, that $\bar{a}$ is a strictly diversified portfolio, i.e. $\bar{a} \in (0; 1)$. S_1 describes the set of possible paths on which the aspiration level remains constant over time:

$$S_1 = \{\omega \in S_1 \mid \bar{u}_t = \bar{u} \text{ for all } t = 1, 2, ...\}.$$

Let P denote a probability measure consistent with $(\Pi_a)_{a\in[0;1]}$ as in Gilboa and Schmeidler (1996, p. 11). It is easily seen that[5]:

Lemma 1 *If $\bar{u} < \mu_{\bar{a}}$, then the expected time, for which the investor will hold $\bar{a}$ is infinite. If $\bar{u} > \mu_{\bar{a}}$, then the investor will switch in finite time to a different act a, with $a = 0$, if $\bar{a} > \frac{1}{2}$ and $a = 1$, if $\bar{a} < \frac{1}{2}$.*

The interpretation of this lemma is simple: if the first alternative chosen is satisficing (in mean), then it is chosen for an infinitely long time (in expectation). However, if it is not satisficing, the investor switches to another one in finite time. The new portfolio chosen is the most dissimilar to the first one, or in other words, the one most far away from the initial $\bar{a}$.

The next proposition shows how an investor will behave, if his aspiration level is higher than the mean of the utility returns of the initially chosen portfolio.

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[5] All proofs are to be found in the appendix.

Proposition 2 *Let the similarity function $s(a;a')$ of an investor be concave in the distance between a and a' with $s(1;0) \in (0;1)$ and the memory of the investor be endogenous and contain all past cases up to the current period. If $\bar{u} > \mu_{\bar{a}}$ and*

- $\bar{u} > \max\{\mu_0;\mu_1\}$, *then*
  $$P\left\{\omega \in S_1 \mid \exists\pi(a) : [0;1] \to [0;1] \textit{ and } \frac{\pi(0)}{\pi(1)} = \frac{\mu_1 - \bar{u}}{\mu_0 - \bar{u}}, \pi(a) = 0 \textit{for } a \in \{0;1\}\right\} = 1;$$
- $\mu_0 > \bar{u} > \mu_1$, *then*
  $$P\{\omega \in S_1 \mid \exists\pi(a) : [0;1] \to [0;1] \textit{ and } \pi(0) = 1, \pi(a) = 0 \textit{for } a \neq 0\} = 1;$$
- $\mu_1 > \bar{u} > \mu_0$, *then*
  $$P\{\omega \in S_1 \mid \exists\pi(a) : [0;1] \to [0;1] \textit{ and } \pi(1) = 1, \pi(a) = 0 \textit{for } a \neq 1\} = 1;$$
- $\mu_1 > \bar{u}$ *and* $\mu_0 > \bar{u}$, *then*
  $$P\left\{\omega \in S_1 \mid \exists\pi(a) : [0;1] \to [0;1] \textit{ and } \begin{array}{lll} \pi(1) = 1, & \pi(a) = 0 & \text{for } a \neq 1 \\ \pi(0) = 1, & \pi(a) = 0 & \text{for } a \neq 0 \end{array} \textit{ or}\right\} = 1.$$

An investor, whose aspiration level exceeds the mean of the utility of the returns of the initially chosen portfolio will only diversify for a finite number of periods, afterwards he will either choose one of the undiversified portfolios forever, or switch between them, depending on whether he finds their mean utility satisficing or not. Note, that this result does not depend on the assumption, that the investor can remember all past cases. The statement of the proposition remains true, even if the investor can remember a finite number of cases only. It is essential, however, that the investor remembers only cases that really occurred, i.e. that his memory is endogenous. Moreover, the result of the proposition shows that the similarity between the corner acts does not influence the limit distribution. In other words, the symmetric Euclidean distance cannot capture simultaneously similarity with respect to the composition of the portfolios and similarity with respect to asset specific features such as nationality, maturity, etc.

With two assets only, similarity between the two undiversified portfolios can be normalized to 0, hence only similarity perceptions with respect to portfolio structure pay a role for the limit behavior. This feature is due to the indeterminacy of similarity function with respect to affine-linear transformations. To discuss the role of similarity between assets, the introduction of a third asset is necessary. Hence, suppose that besides assets A and B, a third asset $\hat{A}$ is present in the economy. Its dividend payments are denoted by $\delta_{\hat{A}}$, distributed on an interval $\left[\underline{\delta}_{\hat{A}};\bar{\delta}_{\hat{A}}\right]$. A portfolio consisting of the three assets A, B and $\hat{A}$ is now described by two variables $(a;\hat{a}) \in \mathfrak{A} = [0;1]^2$, denoting the proportion of A and of $\hat{A}$ in the portfolio. Thus, choosing a portfolio $(a;\hat{a})$ results in a utility realization of

$$u\left(a\delta_A + \hat{a}\delta_{\hat{A}} + (1-a-\hat{a})\,\delta_B\right)$$

If u is well-defined, bounded and continuous on

$$\left[a\underline{\delta}_A + \hat{a}\underline{\delta}_{\hat{A}} + (1-a-\hat{a})\,\underline{\delta}_B; a\bar{\delta}_A + \hat{a}\bar{\delta}_{\hat{A}} + (1-a-\hat{a})\,\bar{\delta}_B\right]$$

for all $(a;\hat{a}) \in \mathfrak{A}$, then the distribution $(\Pi_a)_{a\in\mathfrak{A}}$ of $u(\cdot)$ for a given a is characterized by a finite mean, $\mu_{a;\hat{a}}$, finite variance and bounded supports.

One can imagine the portfolios consisting of A, B and $\hat{A}$ situated on a triangle. The similarity between portfolios is then represented by the distance between the corresponding points. Because of the uniqueness of the similarity function with respect to linear affine transformations, it is possible to normalize the distance between A and $\hat{A}$ so that $s\left((1;0);(0;1)\right) = 0$. I assume as above that s depends only on the Euclidean distance between two points, that it is strictly decreasing and concave in the Euclidean distance.

Suppose that the initially chosen portfolio is strictly diversified with $(a_0;\hat{a}_0) \in (0;1)^2$ and note that in analogy to lemma 1 the strict monotonicity of the similarity function in the Euclidean distance implies that if $\mu_{a_0;\hat{a}_0} < \bar{u}$ (where $\bar{u}$ denotes the constant aspiration level), then $(a_0;\hat{a}_0)$ is abandoned almost surely in finite time and one of the undiversified portfolios $(0;1)$, $(1;0)$ or $(0;0)$ is chosen. The concavity of the similarity function further implies that one of the undiversified portfolios will be chosen in each period of time afterwards. In order to focus on the influence of similarity between the corner portfolios on limit behavior, I will therefore concentrate on the choices between the three corner portfolios and neglect the diversified ones. Hence, it is convenient to denote the three undiversified portfolios by A, B and $\hat{A}$ and the similarity by

$$\begin{aligned} s\left((1;0);(0;0)\right) &= s(A;B) = s_{AB} \\ s\left((0;1);(0;0)\right) &= s\left(\hat{A};B\right) = s_{\hat{A}B}. \end{aligned}$$

Denote as above, the set of all possible paths on which case-based decision are made by S_1. Let $\bar{S}$ denote a subset of S_1, on which the first chosen portfolios is an undiversified one and the aspiration level is constant and exceeds the maximal mean utility of an undiversified portfolio:

$$\bar{S} = \left\{ \omega \in S_1 \mid \begin{array}{l} \bar{u}_t = \bar{u} > \max\left\{\mu_A;\mu_{\hat{A}};\mu_B\right\} \text{ for all } t = 1,2,... \\ \text{and } (a_0;\hat{a}_0) \in \left\{A;\hat{A};B\right\} \end{array} \right\}.$$

Let P again denote a probability distribution on $\bar{S}$ which is consistent with $(\Pi_a)_{a\in[0;1]^2}$.

Proposition 3 • *If*

$$s_{AB} + s_{\hat{A}B} \geq 1,$$

then on $\bar{S}$, the frequencies with which the portfolios are chosen in the limit satisfy almost surely with respect to P:

$$\frac{\pi(A)}{\pi\left(\hat{A}\right)} = \frac{\mu_{\hat{A}} - \bar{u}}{\mu_A - \bar{u}}$$
$$\pi(a;\hat{a}) = 0, \text{ else.}$$

- *If*

$$s_{AB} + s_{\hat{A}B} < 1,$$

then on $\bar{S}$, the frequencies with which the corner portfolios are chosen in the limit satisfy almost surely with respect to P:

$$\frac{\pi(A)}{\pi\left(\hat{A}\right)} = \frac{(1 - s_{\hat{A}B})(\mu_{\hat{A}} - \bar{u})}{(1 - s_{AB})(\mu_A - \bar{u})}$$
$$\frac{\pi(B)}{\pi\left(\hat{A}\right)} = \frac{(\mu_{\hat{A}} - \bar{u})}{(1 - s_{\hat{A}B})(\mu_B - \bar{u})}$$
$$\frac{\pi(B)}{\pi(A)} = \frac{(\mu_A - \bar{u})}{(1 - s_{AB})(\mu_B - \bar{u})},$$
$$\pi(a;\hat{a}) = 0, \text{ else.}$$

The proposition shows that depending on the relationship between s_{AB} and $s_{\hat{A}B}$, similarity between the corner portfolios can have different effects on the limit choice of the investor. As long as their sum is larger than 1, the negative impact portfolios A and $\hat{A}$ exhibit on the cumulative utility of B is so large that B is never chosen after some period $\bar{t}$. If, however, their sum is lower than 1, B is still an optimal choice during a positive fraction of time. The frequencies with which A and $\hat{A}$ are chosen depend on how similar they are to B. Especially, the portfolio which is more similar to B is chosen less frequently, since its cumulative utility suffers more from the negative net utility realizations of B. Clearly, for $s_{\hat{A}B} = s_{AB}$, this effect disappears. Still, as long as $0 < s_{\hat{A}B} = s_{AB}$ holds, the utility realizations of B negatively affect the cumulative utilities of A and $\hat{A}$ and their frequencies are therefore lower than in the case in which no similarity between A and B and $\hat{A}$ and B is perceived.

3.3 Learning and similarity

Gilboa and Schmeidler (1996) pose the question of whether a case-based decision maker is able to learn to choose the optimal (expected utility maximizing) act if the same problem is repeated an infinite number of times. They find a rule for adapting the aspiration level that indeed leads to optimal behavior. The idea is as follows: start with an aspiration level $\bar{u}_1$ and update it towards the maximum of the average utility achieved by some act till time t, to be denoted by X_t. In some rare periods, however, increase the aspiration level by a constant h. Formally:

$$\begin{aligned} &\bar{u}_1 = \bar{u}_1 \\ &\bar{u}_t = \alpha \bar{u}_{t-1} + (1-\alpha) X_t \text{ for } t \geq 2,\, t \notin N_C \\ &\bar{u}_t = \bar{u}_{t-1} + h \text{ for } t \in N_C, \end{aligned} \tag{1}$$

where $N_C \subset \mathbb{N}$ is infinite but sparse and α is a constant between 0 and 1. Write

$$S_2(\bar{u}_1; \beta) = \left\{ \omega \in S \mid \begin{array}{l} \bar{u}_1 = \bar{u}_1 \text{ and} \\ \bar{u}_t = \beta \bar{u}_{t-1} + (1-\beta) X_t \text{ for } t \geq 2 \end{array} \right\}$$

This combination of realism and ambitiousness leads to optimal choice in the limit, formally, on almost all possible paths of dividend realizations, the frequency of choosing one of the alternatives $a \in \arg\max \{\mu_a \mid a \in [0;1]\}$ is 1. The theorem, assumes a very specific similarity function for which

$$\begin{aligned} s(a;a') &= 1, \text{ if } a = a' \\ s(a;a') &= 0, \text{ if } a \neq a'. \end{aligned}$$

In other words, two acts are only considered similar if they are identical.

It is interesting to know, whether this adaptation rule also works for more general similarity functions. Suppose, as above that the similarity function is concave. Let P be a probability measure on $S_2(\bar{u}_1; \beta)$, which is consistent with $(\Pi_a)_{a \in [0;1]}$, as in Gilboa and Schmeidler (1996, p.11). Suppose again that the first act chosen is $a_1 = \bar{a} \in (0;1)$. It is clear that an investor following updating rule (1) will switch at some time to act 1 or 0 (this is implied by the fact that the aspiration level is increased by h in an infinite number of periods). Suppose this happens at time $\bar{t}$. Denote by $X_{\bar{t}}(\bar{a})$ the achieved average utility of $\bar{a}$ till time $\bar{t}$.

Proposition 4

Proposition 5 *Suppose that s is concave and strict monotonically decreasing in the Euclidean distance between acts. For all $\bar{u}_1 \in \mathbb{R}$ and all $\beta \in (0;1)$ the following results apply:*

1. *On* $\tilde{S}$ *such that*

$$\tilde{S} = \{\omega \in S_2 \mid X_{\bar{t}}(\bar{a}) \leq \max\{\mu_0; \mu_1\}\},$$
$$P\left\{\omega \in \tilde{S} \mid \exists\pi\left(\arg\max_{a\in\{0;1\}} \mu_a\right) = 1\right\} = 1$$

holds.

2. *On* $S_2 \backslash \tilde{S}$

$$P\left\{\omega \in S_2\backslash\tilde{S} \mid \begin{array}{ll} \text{for each } a \in A & \exists\pi(a) \text{ such that} \\ \frac{\pi_0}{\pi_1} = \frac{\mu_1 - X_{\bar{t}}(\bar{a})}{\mu_0 - X_{\bar{t}}(\bar{a})} & \text{and} \\ \pi(a) = 0 & \text{for } a \notin \{0;1\} \end{array}\right\} = 1,$$

holds.

The concavity of the similarity function implies that similarity decreases relatively slowly, as the distance between the acts increases near 0. The utility realization of each chosen act therefore significantly influences the evaluation of those (unchosen) acts which are close to it. With increasing distance the similarity decreases faster, making the acts 0 and 1 most attractive, given a negative realization of an interior act. Although the decision-maker has thus never tried any of the acts $a \in (0;1)$, $a \neq \bar{a}$, his evaluation of an interior act is always lower than that of acts 0 and 1. His choices consist therefore only of the corner acts. Since the investor observes only the utility realizations of the acts 0 and 1, he can only learn to choose the better of these two acts, as long as one of them is satisficing. If however the initial average performance of $\bar{a}$ is too high, neither 0, nor 1 will be able to outperform it on average, so that both will seem unsatisficing in the limit.

Since the proof of proposition 4 heavily relies on the concavity of the similarity function, I now explore, how results change, if the similarity function is convex over some range[6]. Make the following assumptions:

Assumption 1: Suppose, that $s(a;a')$ is concave on $\left[a' - \frac{1}{l}; a' + \frac{1}{l}\right]$ for some $l > 1$ and all $a' \in [0;1]$ and $s(a;a') = 0$ outside this interval. Moreover, assume that $s(a;a')$ is continuous, so that $s\left(a' - \frac{1}{l}; a'\right) = s\left(a' + \frac{1}{l}; a'\right) = 0$ for all a'.

Assumption 2: Let $a_1 = 0$ and let the investor choose the alternative, which is next to the alternative he chose last, if indifferent. (In other words, suppose he chose $a_1 = 0$ last and the cumulative utility of a_1 has fallen below 0. Since the similarity between 0 and $a \geq \frac{1}{l}$ is 0, the

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[6] Note, that a continuous similarity function that has a maximum at $s(a;a) = 1$ cannot be convex everywhere.

investor is indifferent among all $a \geq \frac{1}{l}$ and by the above assumption he should choose $a = \frac{1}{l}$, the alternative next to 0.)

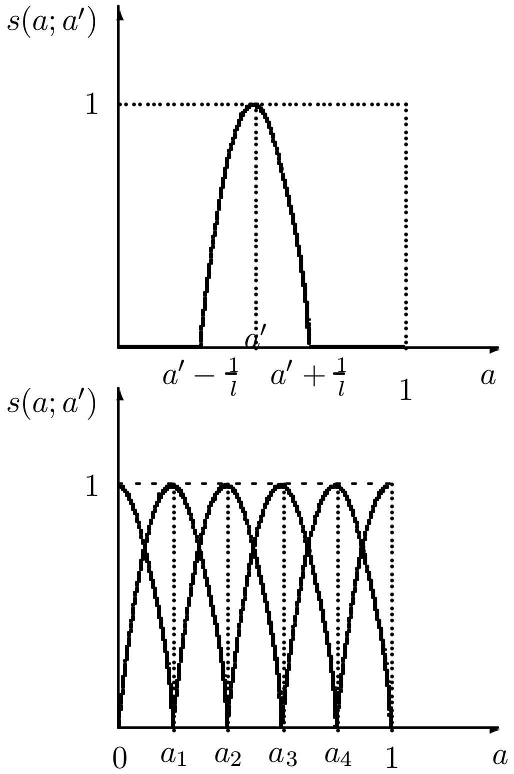

Figure 3

Figure 3 illustrates these two assumptions.

Proposition 6 *Suppose that assumptions 1 and 2 hold. For all* $\bar{u}_1 \in \mathbb{R}$ *and all* $\beta \in (0;1)$ *on almost all possible paths of* S_2*, an investor, who updates his aspiration level according to (1) will learn to choose the act*

$$\tilde{a} = \arg\max_a \left\{ \mu_a \mid a \in \left\{ 0; \frac{1}{l}; \frac{2}{l} ... \frac{l-1}{l}; 1 \right\} \right\}$$

with frequency 1.

It seems that convexities of the similarity function improve the learning process. Instead of just learning the better one of the two corner acts 0 and 1, the investor can now learn to choose the best of the $l + 1$ acts (including 0 and 1). Therefore, he can do better, the more intervals l he distinguishes. In the limit, $l \to \infty$ and the intervals on which the similarity function is positive shrink to single points, which leads again to the result of Gilboa and Schmeidler (1996) that optimal learning is possible for similarity functions of the type $s(a; a') = 1$, if $a = a'$ and $s(a; a') = 0$, else.

Assumption 2 however is not in the spirit of the result derived by Gilboa and Schmeidler (1996), since they assume no specific rule for choice under indifference. I therefore drop assumption 2 and denote by $a_1 ... a_l$ the first l portfolios chosen by the investor. It is easy to see that the similarity between any two of these l portfolio should be 0, else the decision-maker would violate the case-based decision theory.

Remark 1 *$s(a_i; a_j) = 0$ holds for any two distinct portfolios of the set $\{a_1 ... a_l\}$. Hence, $|a_i - a_j| \geq \frac{1}{l}$.*

It is therefore possible to assume without loss of generality that[7]:

$$\begin{aligned} a_1 &\in \left[0; \frac{1}{l}\right]; \\ a_2 &\in \left[\frac{1}{l}; \frac{2}{l}\right] \\ &\ldots \\ a_l &\in \left[\frac{l-1}{l}; 1\right]. \end{aligned}$$

Proposition 7 *Suppose that Assumption 1 holds. For all $\bar{u}_1 \in \mathbb{R}$ and all $\beta \in (0; 1)$, on no possible path on S_2, an investor who updates his aspiration level according to (1) will choose an act different from 0; $\frac{1}{l}$; $\frac{2}{l}$ .... $\frac{l-1}{l}$; 1; a_1; $a_1 + \frac{1}{l}$, $a_1 + \frac{2}{l}$... $a_1 + \frac{l-1}{l}$; $a_2 - \frac{1}{l}$, $a_2 + \frac{1}{l}$,... $a_2 + \frac{l-2}{l}$... $a_l - \frac{1}{l}$, ...$a_l - \frac{l-1}{l}$.*

The decision-maker will not necessarily choose all of the listed acts, hence it cannot be guaranteed that he will learn the optimal one even on this finite set. Still, this set is obviously bigger than the one obtained with concave similarity functions (except for $l = 1$). Note that the initial l choices are random (besides the constraints imposed by $|a_i - a_j| \geq \frac{1}{l}$). In order to choose the optimal portfolio in the limit, the decision-maker will have to choose the first l portfolios in such a way that the optimal portfolio belongs to the set of potentially chosen portfolios, formulated in proposition 7. But the set of actually chosen portfolios is finite and has a measure 0 relative to the set of available portfolios. Therefore the probability for choosing the "right" l portfolios in the beginning is 0. Hence, the proposition can be understood as a negative result

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[7] The order in which the acts are chosen does not play any role, from period t, in which the utility of all l acts becomes negative for the first time.

— the adaptation rule proposed by Gilboa and Schmeidler (1996) almost surely does not lead to an optimal portfolio choice in the limit, if non-trivial similarity considerations are included in the model.

4 Similarity in Asset Markets

After having discussed how similarity considerations influence the portfolio choice problem of a single investor, I turn to describing an asset market with case-based decision makers and exploring how similarity considerations based on problem - act / price - portfolio - pairs influence the price dynamics.

4.1 OLG-Model with representative consumers

Consider an economy, consisting of a continuum of investors, uniformly distributed on the interval $[0;1]$. There are two types of investors $i \in \{1;2\}$ with constant aspiration levels $\bar{u}^1$ and $\bar{u}^2$. The shares of these two types are denoted by θ_1 and $\theta_2 = 1 - \theta_1$ and remain constant over the time.

Each investor lives for two periods. The preferences of the investors are assumed to be such, that they wish to consume only in the second period of their life. The preferences about the consumption in the second period are represented by a utility function $u(\cdot)$, which is identical for all investors, independently of their aspiration level. I assume, as usual that u is strict monotonically increasing and continuous in consumption in period two. There is one consumption good in the economy with a price normalized to 1. The initial endowment of the investors consists of one unit of the consumption good in the first period and is 0 in the second period.

There are two possible ways to transfer consumption between two periods: either using a riskless storage technology B, or investing in a risky asset A with exogenous supply. The storage technology delivers $(1+r)$ units of consumption good in period t for each unit of the consumption good, stored in period $t-1$. It is available to every young consumer at no costs[8].

Suppose that the dividend paid by the risky asset δ_t is identically and independently distributed according to a probability distribution Q on the interval $\left[\underline{\delta};\bar{\delta}\right]$. Let $g(\cdot)$ denote the density of the

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[8] B can be interpreted as a riskless bond available in perfectly elastic supply at a price of 1 in each period, as in the model of De Long, Shleifer, Summers and Waldmann (1990).

distribution Q. The supply of the risky asset A is fixed at 1. No short sales are allowed. New emissions are not considered, since I am interested in the behavior of prices on the secondary asset market only.

Let a_t^i denote the share of A in the portfolio of an investor of type i at time t $(a_t \in [0;1])$ and let p_t be the price of asset A at time t. The similarity function of an investor is defined on pairs of $(p;a)$, i.e. the investor only considers the price of the risky asset as relevant for the description of the market situation, when he has to make his decision. Of course, the assumption that only prices influence the perception of an investment problem is very simplifying, still since in this model all other factors are constant over the time, this seems to be the natural description of a problem at least in the context of the model presented.

Since short sales are not allowed, and since the initial endowment of the economy is fixed at 1, it follows, that the price of the asset A, p_t, can only take on values between $[0;1]$. Since the portfolio share of A can also vary between $[0;1]$, it follows that the $(p;a)$-pairs can be represented on a square with side length one. The Euclidean distance on this square can therefore be taken as a measure of similarity. So, assume that

$$s\left((p;a)\,;(p';a')\right) = f\left(\|(p;a) - (p';a')\|\right),$$

where $f(\cdot)$ is strictly decreasing. $s(\cdot)$ is assumed not to depend on the type of investors.

Assume that the memory of the investors is endogenous, i.e. they can only remember cases $(p;a;u(a))$ that really occurred in the economy. Moreover, each investor of type $i \in \{1;2\}$ can only observe past cases experienced by investors of his own type, i.e. cases of the type $(p_t;a_t^i;u(a_t^i))$. Let m denote the number of past cases that an investor can remember. Since in period $t=0$ the memory of the investors is empty, let $a_0 \in (0;1)$ be the alternative chosen (at random) in period 0 by both types and $p_0 = a_0$ be the equilibrium price at $t=0$.

4.2 Equilibrium paths

Given the initial allocation a_0 and the initial price $p_0 = a_0$, an equilibrium path of the economy is defined as a vector of asset prices $(p_t^*)_{t=0,1,\ldots}$ and a vector of portfolios $(a_t^*)_{t=0,1,\ldots}$ chosen by the young investors at t (with $a_0^* = a_0$, $p_0^* = p_0$), such that:

(i) young investors make case-based decisions in each period:

$$\begin{aligned} a_t^* &\in \arg\max_{a\in[0;1]} U_t^i(a) = \\ &= \arg\max_{a\in[0;1]} \sum_{\tau=t-m}^{t-1} s\left(\left(p_\tau; a_\tau^i\right); \left(p_t; a_t^i\right)\right) \cdot \\ &\cdot \left[u\left(\left(\frac{p_{\tau+1}+\delta_{\tau+1}}{p_\tau}\right) a_\tau^i + (1+r)\left(1-a_\tau^i\right)\right) - \bar{u}^i\right] \end{aligned}$$

(ii) The market for the risky asset is cleared in each period:

$$\begin{aligned} \frac{a_t^{1*}(p_t^*) + a_t^{2*}(p_t^*)}{p_t^*} &= 1, \text{ if } a_t^{1*}(0) + a_t^{2*}(0) \neq 0 \\ p_t^* &= 0, \text{ if } a_t^{1*}(0) + a_t^{2*}(0) = 0. \end{aligned}$$

I will not discuss the question of existence of an equilibrium path in general. For a concave similarity function and for $m = 1$ and $m = t$, it will be shown that such paths exist, by analyzing the price dynamics. Note, that the market clearing condition allows for degenerate equilibria, in which no one holds asset A and its price falls to 0. Since it has been shown, that case-based decision makers with concave similarity functions do not diversify, if their aspiration level is relatively high, it is natural to expect that such degenerate equilibria occur for high values of $\bar{u}$. That is why I will assume that the aspiration level of investors of type 1 is low enough, so that they never switch from their initially chosen portfolio. This will insure that the price of the risky asset remains positive for all possible paths of dividend realizations.

Assumption 3: Suppose that $\bar{u}^1 < u\left(\frac{\theta_1 a_0 + \underline{\delta}}{1-\theta_1(1-a_0)} + (1-a_0)(1+r)\right)$.

Assumption 3 insures, that even if all investors of type 2 hold $a = 1$ at some time t, whereas the investors of type 1 hold a_0 (hence $p_t = 1 - \theta_1(1-a_0)$), and if all investors of type 2 switch to $a = 0$ at $(t+1)$, causing the price to fall to $p_{t+1} = \theta_1 a_0$ and if the dividend of the risky asset is the lowest possible at $(t+1)$, the investors of type 1 are still satisfied by the return of their portfolio a_0. Given this condition on $\bar{u}^1$, the investors of type 1 will hold a_0 forever, no matter how long their memory is and independently of the price and dividend realizations and of the portfolio choices of type 2.To avoid the discussion of multiple cases I assume

$$1 > \bar{\delta} > r > a_0\underline{\delta} \geq 0.$$

4.3 Price Dynamics with One-Period Memory

Consider first the case of one-period memory, hence, the investors only remember the last case observed. Since the aspiration level of type 1 is fixed in such a way that they never switch from their initially chosen portfolio, only the aspiration level of type 2 needs to be considered. If this aspiration level is relatively low, then type 2 is always satisfied with the return of his initial portfolio a_0, given that everyone in the economy continues to hold a_0. Therefore, the following proposition obtains:

Proposition 8 *Suppose that assumption 3 holds and let* $\bar{u}^2 \leq u\left(1 + \underline{\delta} + (1 - a_0)\, r\right)$. *Then, there is an equilibrium path on which* $a_t^{1*} = a_0$, $a_t^{2*} = a_0$ *and* $p_t^* = p_0$ *for each* $t = 0, 1, ...$. *Hence,* $(a^1 = a_0; a^2 = a_0; p = p_0)$ *is a stationary state of the economy.*

Now, let the aspiration level be such that the portfolio a_0 is not satisficing for type 2 if the risky asset pays a dividend lower than $\tilde{\delta} \in \left(\underline{\delta}; \bar{\delta}\right)$ even if the price of A remains unchanged. Hence, let

$$\bar{u}^2 = u\left(1 + \tilde{\delta} + (1 - a_0)\, r\right)$$

for some $\tilde{\delta} \in \left(\underline{\delta}; \bar{\delta}\right)$. As long as the utility from the return of the riskless storage technology exceeds $\bar{u}^2$, (i.e. if $\tilde{\delta} < r$) the state in which the investors of type 2 hold portfolio $a^2 = 0$ in each period is a stationary state of the economy.

Proposition 9 *Suppose that assumption 3 holds and let* $\bar{u}^2 \in \left(u\left(1 + \underline{\delta} + (1 - a_0)\, r\right); u\left(1 + r\right)\right)$. *Then, on almost all paths of dividend realizations* ω, *there is an equilibrium path, such that* $a_t^{1*} = a_0$, $a_t^{2*} = 0$ *and* $p_t^* = \theta_1 a_0$ *for all* $t \geq \bar{t}\,(\omega)$, *for some* $\bar{t}\,(\omega)$. $(a^1 = a_0; a^2 = 0; p = \theta_1 a_0)$ *is, thus, a stationary state of the economy.*

Since the proof of this proposition demonstrates how a bubble can endogenously emerge and burst in an economy populated by case-based decision-makers, I include part of it into the main text.

Proof of Proposition 9

Since assumption 3 guarantees that the investors of type 1 never switch away from their initially chosen portfolio, only the behavior of the investors of type 2 needs to be considered.

Observe that since

$$u\left(1 + \delta + (1 - a_0)\, r\right) \leq \bar{u} < u\left(1 + r\right)$$

for $\delta < \tilde{\delta}$, it follows that

$$1 + \delta + (1 - a_0)\, r < 1 + r$$

and therefore that for each $\delta < \tilde{\delta}$

$$\delta < a_0 r < r.$$

Note further that for $\delta \geq \tilde{\delta}$

$$u\,(1 + \delta + (1 - a_0)\, r) \geq \bar{u},$$

hence, the return of the investors of type 2 is satisfactory for them if the young investors continue to hold a_0 and, therefore, by the argument of proposition 8 there is an equilibrium, such that $a_t^{2*} = a_0$ and $p_t^* = p_0$ for all t, such that $\delta_\tau \geq \tilde{\delta}$ for all $\tau \leq t$. Let $t' = \min \left\{ t \mid \delta_t < \tilde{\delta} \right\}$. t' is finite on almost all paths of dividend realizations ω, but its value depends on the path chosen[9]. In period t' the utility realization of a_0 is at most $u\,(1 + \delta_{t'} + (1 - a_0)\, r) < \bar{u}^2$ if the portfolio holdings remain unchanged. Therefore, the cumulative utility of a_0 is negative for the investors of type 2. Since the similarity function is decreasing in the distance between two portfolios, given a price p, it follows that the investors of type 2, who take the price as given, choose the portfolio furthest away from a_0. Hence $a_{t'}^2 = 1$ if $a_0 < \frac{1}{2}$ and $a_{t'}^2 = 0$ if $a_0 > \frac{1}{2}$.

Suppose first that $a_0 > \frac{1}{2}$. Then, $p_{t'}^* = \theta_1 a_0$ is the equilibrium price corresponding to $a_{t'}^{2*} = 0$ and one easily checks that $a^2 = 0$ indeed maximizes the cumulative utility of type 2 in this case. Once the portfolio consisting only of bonds has been chosen, the utility realization becomes $u\,(1 + r)$ in each period, independently of the price p_t and since $u\,(1 + r) > \bar{u}$, it follows that the state $(a^1 = a_0; a^2 = 0; p = \theta_1 a_0)$ is stationary.

Now consider the case $a_0 < \frac{1}{2}$. Given $p_{t'}$, the investors of type 2 choose the portfolio which is furthest away from a_0, i.e. $a = 1$. However, if $a_{t'} = 1$ is chosen, the price $p_{t'}$ rises to $\theta_1 a_0 + (1 - \theta_1)$ and the utility achieved by type 2 increases to

$$u\left(\frac{\theta_1 a_0 + \delta_{t'} + (1 - \theta_1)}{p_0} a_0 + (1 - a_0)\,(1 + r)\right).$$

If this is still smaller than $\bar{u}^2$, then the cumulative utility is indeed maximized at $a^2 = 1$, given $p_{t'} = 1$.

However, if

$$u\left(\frac{\theta_1 a_0 + \delta_{t'} + (1 - \theta_1)}{p_0} a_0 + (1 - a_0)\,(1 + r)\right) > \bar{u}^2,$$

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[9] Similarly, all period numbers introduced hereafter depend on the realized dividend path ω. I neglect this dependence in the notation for simplicity.

then the cumulative utility of a_0 is positive at $p_{t'} = \theta_1 a_0 + (1 - \theta_1)$ and, therefore, $a^2 = 1$ is not optimal given $p_{t'} = 1$. Should this be the case, choose $a_{t'}^{2*}$ in such a way that

$$u\left(\frac{p_{t'}^* + \delta_{t'}}{p_0} a_0 + (1 - a_0)(1 + r)\right) = \bar{u}^2,$$

where $p_t^{*'}$ clears the market, given that $a_{t'}^{2*}$ is chosen by type 2, whereas type 1 still holds a_0:

$$p_{t'}^* = \theta_1 a_0 + (1 - \theta_1) a_{t'}^{2*}.$$

Since $u(\cdot)$ is continuous and strictly increasing, such portfolio and equilibrium price exist by the intermediate value theorem and are unique. Note further that $1 > a_{t'}^{2*} > a_0$ and

$$\theta_1 a_0 + (1 - \theta_1) > p_{t'}^* > p_0$$

must hold. Moreover, the cumulative utility of a_0 given $p_{t'}^*$ is

$$U_{t'}^2(a_0) = u\left(p_{t'}^* + \delta_{t'} + (1 - a_0)(1 + r)\right) - \bar{u}^2 = 0 = U_{t'}^2(a)$$

for all $a \in [0; 1]$. Hence, at $p_{t'}^*$, the investors of type 2 are indifferent among all available portfolios and, therefore, $a_{t'}^{2*}$ is an optimal choice.

Again, two cases can occur: either $a_{t'}^{2*} > \frac{1}{2}$ and the investors of type 2 switch to $a_{t''}^{2*} = 0$ at time $t'' = \min\left\{t > t' \mid \delta_t < \tilde{\delta}\right\}$, as shown above, or $a_{t'}^{2*} < \frac{1}{2}$ holds. In the latter case, construct $a_{t''}^{2*}$ in the same manner as $a_{t'}^{2*}$. Again, $1 > a_{t''}^{2*} > a_{t'}^{2*} > a_0$ must hold. Repeat the same procedure n times as long as $a_{t^n}^{2*} < \frac{1}{2}$ holds. Now note that since

$$u\left(\frac{p_{t^k}^* + \delta_{t^k}}{p_{t^{k-1}}^*} a_{t^{k-1}}^{2*} + (1 - a_{t^{k-1}}^*)(1 + r)\right) = \bar{u}^2$$

and

$$p_{t^{k-1}}^* = \theta_1 a_0 + (1 - \theta_1) a_{t^{k-1}}^{2*},$$

it follows that the price at time t^k is given by

$$p_{t^k} = p_{t^{k-1}} \frac{\varpi - (1 + r)}{p_{t^{k-1}} - \theta_1 a_0} p_{t^{k-1}} (1 - \theta_1) + (1 + r) p_{t^{k-1}} - \delta_{t^k} \tag{2}$$

for all $k = 1...n$, where $\varpi = u^{-1}(\bar{u}^2)$ denotes the return which yields a utility exactly equal to the aspiration level of the investors of type 2. It has to be shown that the sequence defined recursively by (2) satisfies

$$p_{t^n}^* \geq \frac{1 - \theta_1 + 2\theta_1 a_0}{2} = \theta_1 a_0 + \frac{1}{2}(1 - \theta_1)$$

after a finite number of periods t^n, hence after a finite number of iterations $(n - 1)$, the critical value being computed as the price necessary to render $a_{t^n}^{*2} \geq \frac{1}{2}$. The demonstration of this is deferred to the appendix. But once this value of $p_{t^n}^*$ is reached,

$$a_{t^n}^{2*} = \frac{p_{t^n}^* - \theta_1 a_0}{1 - \theta_1} \geq \frac{\frac{2\theta_1(1-\theta_1)a_0 + (1-\theta_1)}{2} - \theta_1 a_0}{1 - \theta_1} = \frac{1}{2}$$

obtains and from this time on the investors of type 2 switch to asset B and hold it forever. Setting $t^n = \bar{t}(\omega)$ for the respective path of dividend realizations, proves the statement of the proposition.■

Note, however, that during the $(n-1)$ iterations the price of A rises. Moreover, it rises in those periods in which the dividend paid by the risky asset is 0. Imagine, therefore, that the risky asset has a fundamental value of 0 (either $\delta = 0$ or $q = 0$). In this case, the case-based decision-makers holding a small initial share of the risky asset steadily increase the share of their wealth invested in A, until it exceeds $\frac{1}{2}$. Hence, they cause a bubble. At the time when the critical value of p_{t^n} is reached the bubble bursts and never reemerges again.

If the aspiration level of type 2 exceeds $u(1+r)$, the economy starts to evolve in a cycle:

Proposition 10 *Let* $\bar{u}^2 \in \left(u(1+r); u\left(1+\frac{\bar{\delta}}{1-\theta_1(1-a_0)}\right)\right)$. *Then on almost all paths of dividend realizations* ω, *there is a time* $\bar{t}(\omega)$, *such that for all* $t \geq \bar{t}(\omega)$ *the economy evolves according to a stochastic cycle with two states:*

$$\begin{aligned} h\text{, with } a_h^1 &= a_0,\ a_h^2 = 1\text{, and } p_h = 1-\theta_1(1-a_0) \\ &\text{and} \\ l\text{, with } a_h^1 &= a_0,\ a_l^2 = 0 \text{ and } p_l = \theta_1 a_0. \end{aligned}$$

The frequencies of the two states are given by:

$$\begin{aligned} \pi_h &= \frac{1}{2-q} \\ \pi_l &= \frac{1-q}{2-q}, \end{aligned}$$

where

$$q = \int_{\hat{\delta}}^{\bar{\delta}} g(\delta)\, d\delta$$

denotes the probability of a dividend payment higher than $\hat{\delta}$, *defined as:*

$$\bar{u}^2 = u\left(1+\frac{\hat{\delta}}{1-\theta_1(1-a_0)}\right)$$

according to Q.

If the aspiration level is set even higher, so that even $u\left(1+\frac{\bar{\delta}}{1-\theta_1(1-a_0)}\right)$ is not satisficing, then the investors of type 2 switch between the two corner portfolios in each period:

Proposition 11 *Let* $\bar{u}^2 > u\left(1+\frac{\bar{\delta}}{1-\theta_1(1-a_0)}\right)$. *Then, on almost all paths of dividend realizations* ω, *there is a time* $\bar{t}(\omega)$ *such that for all* $t \geq \bar{t}(\omega)$ *the economy evolves in a deterministic cycle of period* 2 *with two states* h *and* l, *as described in proposition 10.*

The results of this section show that investors with short memory and a strictly decreasing similarity function diversify only for a finite number of periods, unless their aspiration level is relatively low. Note that to prove this, the assumption of a concave similarity function was not necessary. This is due to the fact that with one period memory only one utility realization at a time is observed. Since the similarity function obtains its maximum for identical problem- act pairs, the investor either retains his initially chosen portfolio (given a utility realization exceeding his aspirations) or chooses one of the corner portfolios, since they are most dissimilar to the initial one. It is possible to show that these results still hold with long memory, given that the similarity function is concave.

4.4 Price Dynamics with Long Memory

Now assume that the investors can remember the whole history of the economy from time 0 on. Suppose that the similarity function of the investors is concave in the Euclidean distance:

$$\left\| (p; a) ; (p'; a') \right\| .$$

The result that in an economy with two types of agents only those investors with relatively low aspiration level will diversify holds here as well. The introduction of a long memory further allows to consider learning effects. An investor who can only remember the last case realized is not able to learn much about the possible dividend and price realizations. In contrast, making observations for a long time might allow the investors to gather enough information so as to be able to choose the optimal portfolio from the point of view of the standard theory in the limit.

Denote by $Eu\left[a \mid p\right]$ the expected utility from holding portfolio $a \in [0; 1]$ at time t, given that the price of a remains constant at $p = p_t = p_{t+1}$. For instance for $p_t = p_{t+1} = p_0$,

$$Eu\left[a_0 \mid p_0\right] = \int_{\underline{\delta}}^{\bar{\delta}} u\left(1 + \delta + (1 - a_0)\, r\right) g\left(\delta\right) d\delta$$

obtains. To simplify matters, assume that the following inequality holds:

$$Eu\left[a_0 \mid p_0\right] < u\left(1 + r\right) < Eu\left[a = 1 \mid p = 1 - \theta_1 \left(1 - a_0\right)\right]. \tag{3}$$

Remark 2 *A necessary condition for the inequality (3) to hold is*

$$1 > \bar{\delta} > \frac{r\left[1 - \theta_1\left(1 - a_0\right)\right]}{\theta_1}.$$

Since the investors of type 1 are constructed in such a way that they hold a_0 in each period, independently of how the economy evolves, the analysis concentrates on the behavior of the investors of type 2, who will determine the evolution of the asset price. Note that since now their memory consists of all observed cases, in the long-run the mean of the observed utility realizations of an act determine its evaluation. Since however the behavior of type 2 has an influence on the market price, this mean utility shall be constructed for the respective equilibrium price which obtains, given the portfolio chosen by type 2. Should an expected utility of a portfolio be satisfactory at a constant equilibrium price, then the expected time for which this portfolio is held is infinity. Alternatively, if the expected utility of a portfolio lies below $\bar{u}^2$, then the investors of type 2 switch away from this portfolio in finite time. The first result is that the investors of type 2 only consider the utility realizations of three portfolios: a_0, $a = 1$ and $a = 0$. The inequality (3), therefore, assumes one possible ordering of the expected utilities of these three portfolios in order to avoid considering multiple cases.

Note further that as long as the investors of type 2 hold a_0 the price of a remains $p_0 = a_0$. If type 2 holds $a = 0$, $p = \theta_1 a_0$ obtains and in the case that the choice of type 2 is $a = 1$, $p = 1 - \theta_1 (1 - a_0)$ is the equilibrium price of the risky asset.

Proposition 12 *Suppose that the probability distribution* Q *on* $[\underline{\delta}; \overline{\delta}]$ *has a density function which is strictly bounded away from* 0 *on the interval* $\left[\hat{\delta} - \zeta; \hat{\delta} + \zeta\right]$ *for some* $\zeta > 0$ *and* $\hat{\delta}$ *such that*

$$u\left(\frac{1 - \theta_1 (1 - a_0) + \hat{\delta}}{1 - \theta_1 (1 - a_0)}\right) = \bar{u}^2.$$

1. *If* $\bar{u}^2 < Eu[a_0 \mid p_0]$, *the expected time for which the investors of type* 2 *hold* a_0 *is infinite.*
2. *If* $\bar{u}^2 \in (Eu[a_0 \mid p_0]; u(1 + r))$, *the investors of type* 2 *hold either* $a = 1$ *or* $a = 0$ *with frequency* 1 *almost surely in the limit.*
3. *If* $\bar{u}^2 \in (u(1 + r); Eu[1 \mid 1 - \theta_1 (1 - a_0)])$, *the investors of type* 2 *hold* $a = 1$ *with frequency* 1 *almost surely in the limit.*
4. *If* $\bar{u}^2 > Eu[1 \mid 1 - \theta_1 (1 - a_0)]$, *the investors of type* 2 *hold* $a = 1$ *and* $a = 0$ *with strictly positive frequencies* π_h *and* π_l, *respectively, almost surely in the limit, whereas the frequencies of all other acts are* 0. *These frequencies almost surely satisfy*
$$\frac{\pi_h}{\pi_l} = \frac{\bar{u}^2 - u(1 + r)}{\bar{u}^2 - \mu_1^r},$$
where μ_1^r *denote the mean utility of choosing asset* a *as observed by the investors of type* 2.

Comparing proposition 2 to proposition 12, one easily sees the analogy: if the aspiration level of the investors of type 2 is relatively low, the initially chosen portfolio is considered satisfactory. Hence, the initial allocation and price prevail infinitely long in expectations. If the investors of type 2 consider a_0 as unsatisfactory, they sooner or later switch to an undiversified portfolio and never diversify again, due to the concavity of their similarity function. Now, they have to choose between the two undiversified portfolios. If at least one of these portfolios is found to be satisfactory, then it is held forever. If, however the expected utility of none of these portfolios exceeds the aspiration level $\bar{u}^2$, then the investors of type 2 switch infinitely often between them, causing the price of A to fluctuate in a stochastic way.

The frequencies with which the two undiversified portfolios are chosen by the investors of type 2, provided that their aspiration level is relatively high as in case 4., depend not only on the mean returns of the two assets, but also on the perceived similarity between $(p = 1 - \theta_1 (1 - a_0); a = 1)$ and $(p = \theta_1 a_0; a = 0)$.

Note that if the aspiration level of the investors of type 2 is appropriately chosen, i.e. if

$$\bar{u}^2 \in (u (1 + r); Eu [1 \mid 1 - \theta_1 (1 - a_0)]), \tag{4}$$

these investors learn to choose the best among the three acts a_0, $a = 1$ and $a = 0$, namely $a = 1$ in the limit. Still, their choice might not be optimal from the point of view of an expected utility maximizer, since they only observe realizations of at most three portfolios.

Since the dynamic of the economy is predetermined solely by the behavior of the investors of type 2, it can easily be derived from proposition 12. The following corollary obtains:

Corollary 13 *Suppose that the probability distribution Q on $\left[\underline{\delta}; \bar{\delta}\right]$ has a density function which is strictly bounded away from 0 on the interval $\left[\hat{\delta} - \zeta; \hat{\delta} + \zeta\right]$ for some $\zeta > 0$ and $\hat{\delta}$ such that $u\left(\frac{1-\theta_1(1-a_0)+\hat{\delta}}{1-\theta_1(1-a_0)}\right) = \bar{u}^2$.*

1. *Let $\bar{u}^2 < Eu[a_0]$. Then, the expected time which the economy spends in the state $(a^1 = a_0; a^2 = a_0; p = p_0)$ is infinite.*
2. *Let $\bar{u}^2 \in (Eu[a_0]; u (1 + r))$. Then, with probability 1 in the limit, the economy remains either in state $(a^1 = a_0; a^2 = 0; p = \theta_1 a_0)$ with frequency 1 or in state $(a^1 = a_0; a^2 = 1; p = 1 - \theta_1 (1 - a_0))$ with frequency 1.*

3. *Let* $\bar{u}^2 \in (u(1+r); Eu[1 \mid 1-\theta_1(1-a_0)])$. *Then, with probability* 1 *in the limit, the economy remains in state* $(a^1 = a_0; a^2 = 1; p = 1-\theta_1(1-a_0))$ *with frequency* 1.
4. *Let* $\bar{u}^2 > Eu[1 \mid 1-\theta_1(1-a_0)]$. *Then, in the limit, the economy almost surely evolves according to a stochastic cycle with two states* h *and* l*, as described in proposition 10.*

The results are similar to those derived for an economy in which investors do not take similarity between acts and problems into account and in which diversification is not allowed, Guerdjikova (2003 b). Investors with low aspiration levels induce stable prices and portfolio allocations. However, the portfolio held by the investors in a stationary state needs not coincide with the optimal portfolio in an economy with a representative investor, implying that the case-based decision makers do not make optimal decisions, given the market price. Hence, arbitrage possibilities can emerge in such a market. If, for instance, the share of A in the initial portfolio, a_0, is relatively high, implying a high p_0, the riskless asset B may dominate the risky one. Neverethe-less, $(a_0; a_0; p_0)$ represents a stationary state, in which all investors hold a strictly dominated portfolio. The relatively low aspiration levels lead to a lack of incentives to experiment and choose different acts with possibly higher payoffs.

If at least some of the investors in the economy have a relatively high aspiration level, then the economy evolves according to a cycle with two states — a low-price and a high-price state. The investors with relatively high aspiration levels and concave similarity functions hold underdiversified portfolios. In general, they trade too much buying the risky asset in period in which its price is high and abandoning it in periods in which its price is low. This behavior causes excessive price volatility which contradicts the result of the standard theory. Indeed, since the economy described in this model is dynamically efficient, it follows that the price of the risky asset should remain constant under rational expectations. Moreover, the fluctuation of the price has a greater amplitude, the higher the value of $1-\theta_1$, i.e. the mass of the investors of type 2 in the economy.

The identical and independent distribution of the dividends further implies that current dividend payments cannot be used to predict future prices. Nevertheless, an external observer would be able to predict a price increase from the past dividend realizations and prices. For instance, with short memory, an external observer would know that a price increase will occur in the period following a price decrease. In contrast, price decreases are impredictable, since they depend on the probability of low dividends.

5 Conclusion

I have analyzed the impact of similarity considerations on the behavior of investors in a financial market. In the individual portfolio-choice problem only similarity considerations among acts are taken into account, whereas problems are assumed to be identical. It can be shown, that an investor with a concave similarity function among portfolios will only hold a diversified portfolio, if his aspiration level is sufficiently low. Moreover, his portfolio will coincide with the initially chosen and probably suboptimal one. If the aspiration level of the investor is chosen relatively high, then he will choose only undiversified portfolios from some finite period t on. He will either hold one of the undiversified portfolios forever, or if his aspiration level is even higher than the expected utility from each of the undiversified portfolios, he will switch infinitely often between the undiversified portfolios in the limit.

It could further be shown, that the method for adapting the aspiration level proposed by Gilboa and Schmeidler (1996), which insures that a case-based decision maker learns to choose the act with maximal expected utility in a stationary environment, does not work with concave similarity functions. In this case the decision maker only learns to choose the act with the highest expected utility among the undiversified ones. Introducing convexities into the similarity function leads to more successful learning, the limit being the case of no similarity considerations analyzed by Gilboa and Schmeidler (1996).

In a market environment not only similarity among portfolios (acts), but among problem-act / price-portfolio pairs are considered. Investors with one period memory hold a diversified portfolio, only if their aspiration level is sufficiently low and choose only undiversified portfolios from some finite time on, else. This result is independent on the curvature of the similarity function. For the case of long memory the results are similar, but rely on the assumption of concavity of the similarity function. The investors can learn to choose the best alternative among the undiversified portfolios, if their aspiration level is appropriately chosen. Still, they only learn to choose the best portfolio in the limit, if their utility function is linear or convex, hence if no diversification is indeed optimal for them from the expected utility theory point of view.

Appendix

Proof of Lemma 1 Suppose first that $\bar{u} < \mu_{\bar{a}}$. The cumulative utility of $\bar{a}$, as long as the investor holds it, is then a random walk with differences

$$\bar{a}\delta_a + (1 - \bar{a})\,\delta_b - \bar{u}.$$

Since the expected value of the difference is $\mu_{\bar{a}} - \bar{u} > 0$ and the process starts at 0, the expected time until the first period in which the process reaches 0 is ∞. But, as long as $U_t\,(\bar{a}) > 0$, $U_t\,(a) = s\,(a; \bar{a})\,U_t\,(\bar{a}) \geq U_t\,(\bar{a})$, since $s\,(a; \bar{a}) \in [0; 1]$ and therefore $\bar{a}$ is chosen in every period.

Now suppose that $\bar{u} > \mu_{\bar{a}}$. Then, the expected increments of $U\,(\bar{a})$ are negative. Therefore, when the process starts at 0, it will cross any finite barrier below 0 in finite time. Let t be the first period, at which $U_t\,(\bar{a}) < 0$. Then $U_t\,(a) = s\,(a; \bar{a})\,U_t\,(\bar{a}) < 0$. Since $s = (a; a')$ is strictly decreasing in the distance between the acts, $U_t\,(a)$ has a maximum either at 0 or at 1. Moreover, $s\,(1; \bar{a}) > s\,(0; \bar{a})$, iff $\bar{a} > \frac{1}{2}$ and since $U_t\,(\bar{a}) < 0$, the act lest similar to $\bar{a}$ is chosen. It follows that

$$a_{t+1} = \left\{ \begin{array}{ll} 1, & \text{if } \bar{a} < \frac{1}{2} \\ 0, & \text{if } \bar{a} > \frac{1}{2}. \end{array} \right\} \blacksquare$$

Proof of Proposition 2

It has already been shown, that for $\mu_{\bar{a}} - \bar{u} < 0$ the investor switches in finite time to $a = 1$ or to $a = 0$. Suppose, that $\bar{a} > \frac{1}{2}$ and therefore $a = 0$ is chosen at some time $\bar{t}$, such that $\bar{t} = \min\{t \mid U_t\,(\bar{a}) < 0\}$. Two cases are possible: either $\mu_0 - \bar{u} < 0$ or $\mu_0 - \bar{u} > 0$. Let $C_t\,(a)$ denote the set of periods in which act $a \in [0; 1]$ has been chosen up to time t. Define $V_t\,(a)$ as:

$$V_t\,(a) = \sum_{\tau \in C_t(a)} [u_\tau\,(a_\tau) - \bar{u}_t]\,.$$

Then at time $t > \bar{t}$ the cumulative utility of an act a can be written as:

$$U_t\,(a) = V_t\,(\bar{a})\,s\,(a; \bar{a}) + V_t\,(0)\,s\,(a; 0)\,.$$

As long as $V_t\,(0) > 0$,

$$U_t\,(0) = V_t\,(\bar{a})\,s\,(0; \bar{a}) + V_t\,(0) > V_t\,(\bar{a})\,s\,(a; \bar{a}) + V_t\,(0)\,s\,(a; 0) = U_t\,(a)\,,$$

where the second inequality stems from the fact, that $V_t\,(\bar{a})\,s\,(0; \bar{a}) \leq V_t\,(\bar{a})\,s\,(a; \bar{a}) \leq V_t\,(\bar{a}) < 0$ and $0 < V_t\,(0)\,s\,(a; 0) < V_t\,(0)$. If $\mu_0 - \bar{u} > 0$ holds, then $V_t\,(0) > 0$ infinitely long in expectation. If however $\mu_0 - \bar{u} < 0$ then $V_t\,(0) < V_t\,(\bar{a})\,(s\,(1; \bar{a}) - s\,(0; \bar{a})) < 0$ obtains in finite time. Let now $\bar{t}'$ denote $\bar{t}' = \min\{t \mid V_t(0) < V_t(\bar{a})\, s(1;\bar{a})\}$. Note that at $\bar{t}'$ the cumulative utility of $a = 1$ is:

$$U_{\bar{t}'}(1) = V_t(\bar{a})\, s(1;\bar{a}) + V_t(0)\, s(1;0) = V_t(\bar{a})\, s(1;\bar{a}).$$

Moreover, since now $V_t(\bar{a}) < 0$, $V_t(0) < 0$ and s is concave, it follows that at every $\bar{t}'$ $U_t(a)$ is concave for every $a \in [0;1]$. Therefore the optimal alternative is either 1 or 0. Moreover:

$$U_{\bar{t}'}(1) = V_{\bar{t}'}(\bar{a})\, s(1;\bar{a}) > V_{\bar{t}'}(\bar{a})\, s(0;\bar{a}) + V_{\bar{t}'}(0) = U_{\bar{t}'}(0),$$

so that $a_{\bar{t}'+1} = 1$ is chosen.

Again, if $\mu_1 - \bar{u} > 0$, then $a = 1$ will be held infinitely long in expectation, whereas if $\mu_1 - \bar{u} < 0$, then the cumulative utility of $a = 1$ becomes less then the cumulative utility of $a = 0$ in finite time. Repeated use of the above argument shows, that the investor will choose only the portfolios $a = 0$ and $a = 1$.

Note that if one of the two acts, say $a = 0$ were chosen only for a finite number of times, the cumulative utility of the other act, $a = 1$, would converge almost surely to $-\infty$, since $\mu_1 - \bar{u} < 0$. Let $\bar{t}''$ denote the last period in which $a = 0$ is chosen (depending on the path of dividend realizations). Then, it would be possible to find a period $t > \bar{t}''$ such that

$$V_t(1) < V_{\bar{t}''}(0) + V_{\bar{t}'}(\bar{a}) \frac{[s(0;\bar{a}) - s(1;\bar{a})]}{1-s}$$

holds and, therefore,

$$U_t(1) = V_t(1) + sV_{\bar{t}''}(0) + V_{\bar{t}'}(\bar{a})\, s(1;\bar{a}) < sV_t(1) + V_{\bar{t}''}(0) + V_{\bar{t}'}(\bar{a})\, s(0;\bar{a}) = U_t(0).$$

Hence, the assumption that act 0 is chosen only for a finite number of times leads to a contradiction on almost all possible paths of dividend realizations. Analogous reasoning shows that (because of the assumption $\mu_2 - \bar{u} < 0$), act $a = 1$ cannot be chosen for a finite number of times on a set of paths with a positive probability measure.

Now, consider the following process:

$$\begin{array}{rcl} \varepsilon_{\bar{t}'}(1;0) & = & V_{\bar{t}'}(\bar{a}) \dfrac{[s(1;\bar{a}) - s(0;\bar{a})]}{1-s} \\ \varepsilon_{t+1}(1;0) & = & \begin{array}{ll} \varepsilon_t + u(\delta_{A_t}) - \bar{u}, & \text{if } \varepsilon_t \geq 0 \\ \varepsilon_t + u(\delta_{B_t}) - \bar{u}, & \text{if } \varepsilon_t < 0 \end{array} \end{array}.$$

$(1-s)\,\varepsilon_t(1;0)$ represents the difference between the cumulative utilities of the acts $a = 1$ and $a = 0$ after period $\bar{t}'$. To see this note that

$$U_t(1) - U_t(0) \quad = \quad [V_t(1) + sV_t(0) + V_{\bar{t}'}(\bar{a})\, s(1;\bar{a})] - [sV_t(1) + V_{\bar{t}''}(0) + V_{\bar{t}'}(\bar{a})\, s(0;\bar{a})] =$$

$$= \quad (1-s)\left[V_t(1) - V_t(0)\right] + V_{\bar{t}'}(\bar{a})\left[s(1;\bar{a}) - s(0;\bar{a})\right]$$

and

$$\varepsilon_t(1;0) = V_t(1) - V_t(0)\,.$$

As long as act $a = 1$ is chosen, this difference represents a random walk on the half line with negative expected increment. Define $\tilde{\varepsilon}_t(1;0)$ to be equal to $\varepsilon_t(1;0)$ in the periods in which $a = 1$ is chosen and 0, else. Such a random walk has an accessible atom at 0[10]. Moreover, each set of the type $[0;c]$ is regular, see Meyn and Tweedie (1996, p. 278). This means that the state 0 is reached in finite expected time, starting from each set of the type $[0;c]$ and especially, starting from the set $[0;\bar{u} - \min\{u(\underline{\delta}_A);u(\underline{\delta}_B)\}]$. Denote the supremum of these expected times by $\mathcal{N}$ and observe that it is finite according to the definition of regular sets. Note that $\bar{u} - \min\{u(\underline{\delta}_A);u(\underline{\delta}_B)\}$ equals the maximal possible value of $\varepsilon_t(1;0)$ in a period, in which the decision-maker switches to from $a = 0$ to $a = 1$. Observe as well that since the probability that $\varepsilon_t(1;0) = 0$ is 0 (for atomless distributions of δ_A and δ_B), it follows that $\tilde{\varepsilon}_t(1;0) = 0$ coincides with $\varepsilon_t(1;0) < 0$. Hence, the decision-maker switches away from $a = 1$ in the first period, in which $\tilde{\varepsilon}_t(1;0) = 0$ is reached. It follows that the expected time for which an arbitrary act a is held in a row is finite and bounded from above.

It can be shown that $\varepsilon_t(1;0)$ is bounded above on almost each path of dividend realizations. At times at which $a = 1$ is chosen ε_t never falls below

$$u(\underline{\delta}_A) - \bar{u},$$

since this would contradict choosing the act with highest cumulative utility in each period. Suppose, therefore that there is a set of paths on which there exists a sequence of periods (dependent on the path) t', t''..., such that $\varepsilon_{t'}(1;0)$, $\varepsilon_{t''}(1;0)$... grows to infinity. In other words, suppose that on each such path for each $\mathcal{M} > 0$ there is a k, such that $\varepsilon_{t^n}(1;0) > \mathcal{M}$ for all $n > k$. Since $U_t(1)$ has negative expected increments, it follows (as shown above) that $a = 0$ is chosen infinitely many times on almost each path of dividend realizations. But each time that the act $a = 0$ is chosen, the difference $\varepsilon_t(1;0)$ falls below 0. If $\varepsilon_{t^n}(1;0) > \mathcal{M}$, the time needed to return to the origin is at least $\frac{\mathcal{M}}{u(\underline{\delta}_A)-\bar{u}}$, which grows to infinity, as ε_{t^n} and, hence, $\mathcal{M}$ become very large. However, as has been explained above, the expected time for return to the origin 0 of $\tilde{\varepsilon}_t(1;0)$ is finite and bounded above by $\mathcal{N}$. The Law of Large Numbers then implies that for

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[10] See Meyn and Tweedie (1996, p. 105) for a definition of an accessible atom.

each $\kappa > 0$ on almost each path of dividend realizations there is a period $\mathcal{K}$, such that

$$\frac{\sum_{i=1}^{n} \tau_i}{n} \leq \mathcal{N} + \kappa$$

for all $n \geq \mathcal{K}$, where τ_i denotes the time needed to reach the origin, once $a = 1$ has been chosen. On the other hand, the assumption that $\varepsilon_{t^n}(1;0) \to \infty$ implies that the stopping times τ_i become infinitely large as the time grows on a set of paths with a positive probability measure—a contradiction. Hence, on almost each path of dividend realizations, each sequence $\varepsilon_{t'}(1;0)$, $\varepsilon_{t''}(1;0)$... (where t', t''... denote periods at which a is chosen) is bounded from above. A symmetric argument can be used to establish that $\varepsilon_t(1;0)$ is almost surely bounded from below. It follows that on almost each path $\omega \in S_1$

$$\lim_{t\to\infty} \frac{U_t(1)}{U_t(0)} = \lim_{t\to\infty} \frac{U_t(1) + (1-s)\,\varepsilon_t(1;0)}{U_t(0)} = 1$$

holds.

$$\lim_{t\to\infty} \frac{U_t(1)}{U_t(0)} = 1.$$

Hence,

$$\lim_{t\to\infty} \frac{\left[V_t(1) + sV_t(0) + V_{\bar{t}'}(\bar{a})\,s(1;\bar{a})\right]}{\left[V_t(0) + sV_t(1) + V_{\bar{t}'}(\bar{a})\,s(0;\bar{a})\right]} = 1$$

$$\lim_{t\to\infty} \frac{\left[|C_t(1)| \sum_{\tau\in C_t(1)} \frac{[u(\delta_{A_\tau})-\bar{u}]}{|C_t(1)|} + s\,|C_t(0)| \sum_{\tau\in C_t(0)} \frac{[u(\delta_{B_\tau})-\bar{u}]}{|C_t(0)|} + V_{\bar{t}'}(\bar{a})\,s(1;\bar{a})\right]}{\left[s\,|C_t(1)| \sum_{\tau\in C_t(1)} \frac{[u(\delta_{A_\tau})-\bar{u}]}{|C_t(1)|} + |C_t(0)| \sum_{\tau\in C_t(0)} \frac{[u(\delta_{B_\tau})-\bar{u}]}{|C_t(0)|} + V_{\bar{t}'}(\bar{a})\,s(0;\bar{a})\right]} = 1.$$

Since $|C_t(1)| \to \infty$ and $|C_t(0)| \to \infty$ on almost each path, it follows according to the Law of Large Numbers that

$$\lim_{t\to\infty} \frac{\sum_{\tau\in C_t(1)} [u(\delta_{A_\tau}) - \bar{u}]}{|C_t(1)|} = \mu_1 - \bar{u}$$

$$\lim_{t\to\infty} \frac{\sum_{\tau\in C_t(0)} [u(\delta_{B_\tau}) - \bar{u}]}{|C_t(0)|} = \mu_0 - \bar{u}$$

obtain almost surely in the limit. Hence,

$$\lim_{t\to\infty} \frac{\left[|C_t(1)|(\mu_1-\bar{u}) + s\,|C_t(0)|(\mu_0-\bar{u}) + V_{\bar{t}'}(\bar{a})\,s(1;\bar{a})\right]}{\left[s\,|C_t(1)|(\mu_1-\bar{u}) + |C_t(0)|(\mu_0-\bar{u}) + V_{\bar{t}'}(\bar{a})\,s(0;\bar{a})\right]} = 1.$$

$$\begin{aligned}
&\lim_{t\to\infty} \frac{\left[|C_t(1)|(\mu_1-\bar{u}) + s\,|C_t(0)|(\mu_0-\bar{u}) + V_{\bar{t}'}(\bar{a})\,s(1;\bar{a})\right]}{\left[s\,|C_t(1)|(\mu_1-\bar{u}) + |C_t(0)|(\mu_0-\bar{u}) + V_{\bar{t}'}(\bar{a})\,s(0;\bar{a})\right]} \\
= \; &\lim_{t\to\infty} \frac{\left[\frac{|C_t(1)|}{|C_t(0)|}(\mu_1-\bar{u}) + s(\mu_0-\bar{u}) + \frac{V_{\bar{t}'}(\bar{a})s(1;\bar{a})}{|C_t(0)|}\right]}{\left[s\frac{|C_t(1)|}{|C_t(0)|}(\mu_1-\bar{u}) + (\mu_0-\bar{u}) + \frac{V_{\bar{t}'}(\bar{a})s(0;\bar{a})}{|C_t(0)|}\right]} = \\
= \; &\lim_{t\to\infty} \frac{\frac{|C_t(1)|}{|C_t(0)|}(\mu_1-\bar{u}) + s(\mu_0-\bar{u})}{s\frac{|C_t(1)|}{|C_t(0)|}(\mu_1-\bar{u}) + (\mu_0-\bar{u})} = 1
\end{aligned}$$

almost surely holds (since $V_{\bar{t}'}(\bar{a})$ is finite on almost all paths, it does not influence the limit behavior). Therefore, the limit frequencies π_1 and π_0 satisfy

$$\frac{\pi_1}{\pi_0} = \lim_{t \to \infty} \frac{|C_t(1)|}{|C_t(0)|} = \frac{\mu_0 - \bar{u}}{\mu_1 - \bar{u}}. \blacksquare$$

Proof of proposition 3

First, I show that a diversified portfolio is never chosen. Indeed, suppose that up to time $\bar{t}$ only undiversified portfolios have been chosen. Then the cumulative utility of any portfolio $(a;\hat{a})$ is given by

$$U_{\bar{t}}(a;\hat{a}) = V_{\bar{t}}(A)\, s(A;(a;\hat{a})) + V_{\bar{t}}\left(\hat{A}\right) s\left(\hat{A};(a;\hat{a})\right) + V_{\bar{t}}(B)\, s(B;(a;\hat{a})).$$

Note that if an act has been chosen in the past at least once and is not chosen at time $\bar{t}$, then its $V_{\bar{t}}$ must be negative, else it would not have been abandoned for another act. Indeed, if $(a';\hat{a}') \neq \left(a'_{t-1};\hat{a}'_{t-1}\right)$ is chosen at time $\tilde{t}$, then

$$U_{\tilde{t}}(a;\hat{a}) \leq U_{\tilde{t}}(a';\hat{a}')$$

for all $(a;\hat{a}) \in [0;1]^2$ must hold at this time. As long as $V_t - V_{\tilde{t}}$ is positive,

$$U_t(a;\hat{a}) \leq U_t(a';\hat{a}')$$

still holds for all $(a;\hat{a}) \in [0;1]^2$. The investor can switch to a different act, only if $V_t - V_{\tilde{t}} < 0$ holds. Since when $(a';\hat{a}')$ is chosen for the first time, $V_{\tilde{t}}(a';\hat{a}') = 0$, it follows that the first switch away from $(a';\hat{a}')$ occurs at $V_t(a';\hat{a}') < 0$. When at time $\tilde{t}'$, $(a';\hat{a}')$ is chosen again, $V_{\tilde{t}'} < 0$ and, therefore, the next switch away from $(a';\hat{a}')$ occurs at t such that $V_t < V_t - V_{\tilde{t}'} < 0$. Hence, if $(a';\hat{a}')$ is not chosen at $\bar{t}$ but has been chosen in the past, $V_{\bar{t}}(a';\hat{a}') < 0$ must hold. Else, its $V_{\bar{t}}$ is 0. Hence, at $\bar{t}$ exactly one of the corner acts can have a positive $V_{\bar{t}}$, namely the one chosen at time $\bar{t}$. If $V_{\bar{t}}(a_t;\hat{a}_t) > 0$, then

$$(a_{t+1};\hat{a}_{t+1}) = \arg \max_{(a;\hat{a}) \in [0;1]^2} U_{\bar{t}}(a;\hat{a}) = (a_t;\hat{a}_t)$$

and therefore an undiversified act is chosen again. If, on the other hand, $V_{\bar{t}}(a_t;\hat{a}_t) < 0$, then the function $U_{\bar{t}}(a;\hat{a})$ is a sum of convex functions and has therefore a corner optimum. Hence, again an undiversified act is chosen. It follows that starting with an undiversified portfolio, the investor never diversify and, hence,

$$\pi(a;\hat{a}) = 0 \text{ for all } (a;\hat{a}) \notin \left\{A;\hat{A};B\right\}.$$

Suppose now that

$$s_{AB} + s_{\hat{A}B} \geq 1$$

holds and assume that the investor has chosen to hold the undiversified portfolio consisting only of asset B in the first period,

$$(a_0; \hat{a}_0) = B.$$

Without loss of generality, assume

$$s_{AB} < s_{\hat{A}B}$$

Since $\mu_B < \bar{u}$,

$$U_{\bar{t}}(B) = V_{\bar{t}}(B) < 0$$

obtains almost surely in finite time. Hence, at $\bar{t}+1$, $(a; \hat{a}) = A$ is chosen, since

$$U_{\bar{t}}(A) = s_{AB} V_{\bar{t}}(B) > s_{\hat{A}B} V_{\bar{t}}(B) = U_{\bar{t}}\left(\hat{A}\right) > V_{\bar{t}}(B) = U_{\bar{t}}(B).$$

As long as A is chosen, the cumulative utility of the portfolios consisting only of B and only of $\hat{A}$ is given by:

$$\begin{aligned} U_t(B) &= V_{\bar{t}}(B) + s_{AB} V_t(A) \\ U_t\left(\hat{A}\right) &= s_{\hat{A}B} V_{\bar{t}}(B), \end{aligned}$$

since the similarity between $\hat{A}$ and A is 0. Hence,

$$U_t\left(\hat{A}\right) > U_t(B)$$

at each such t and especially, in the first period $\bar{t}'$ such that

$$\begin{aligned} U_{\bar{t}'}(A) &< U_{\bar{t}'}(B) \text{ or} \\ U_{\bar{t}'}(A) &< U_{\bar{t}'}\left(\hat{A}\right) \end{aligned}$$

holds. Clearly, $\bar{t}'$ is almost surely finite, since

$$\begin{aligned} U_t(A) - U_t(B) &= (s_{AB} - 1)\left(V_{\bar{t}}(B) - V_t(A)\right) \\ U_t(A) - U_t\left(\hat{A}\right) &= \left(s_{AB} - s_{\hat{A}B}\right) V_{\bar{t}}(B) + V_t(A) \end{aligned}$$

and $V_t(A)$ has negative expected increments $\mu_A - \bar{u} < 0$. Hence, in period $\bar{t}'$ act $\hat{A}$ is chosen.

As long as the investor holds portfolios $\hat{A}$, the cumulative utilities of the three portfolios satisfy:

$$\begin{aligned} U_t(B) &= V_{\bar{t}}(B) + s_{AB} V_{\bar{t}'}(A) + s_{\hat{A}B} V_t\left(\hat{A}\right) \\ U_t\left(\hat{A}\right) &= s_{\hat{A}B} V_{\bar{t}}(B) + V_t\left(\hat{A}\right) \\ U_t(A) &= s_{AB} V_{\bar{t}}(B) + V_{\bar{t}'}(A) \end{aligned}$$

Note that as long as $V_t\left(\hat{A}\right) > 0$ holds, act $\hat{A}$ is chosen, since $V_{\bar{t}}(B) < 0$ and $V_{\bar{t}'}(A) < 0$ hold. Once, however $V_t\left(\hat{A}\right) < 0$ obtains,

$$s_{AB} + s_{\hat{A}B} \geq 1$$

implies that

$$s_{AB}V_{\bar{t}'}\left(A\right)+s_{\hat{A}B}V_{t}\left(\hat{A}\right)<\max\left\{V_{t}\left(\hat{A}\right);V_{\bar{t}'}\left(A\right)\right\}.$$

To see this, assume without loss of generality, $V_t\left(\hat{A}\right) > V_{\bar{t}'}\left(A\right)$ and note that

$$\begin{aligned}&\left(s_{AB}+s_{\hat{A}B}\right)V_{t}\left(\hat{A}\right)+s_{\hat{A}B}\left[V_{\bar{t}'}\left(A\right)-V_{t}\left(\hat{A}\right)\right]\\ <\;&V_{t}\left(\hat{A}\right)+s_{\hat{A}B}\left[V_{\bar{t}'}\left(A\right)-V_{t}\left(\hat{A}\right)\right]<V_{t}\left(\hat{A}\right)\end{aligned}$$

holds since both $V_t\left(\hat{A}\right)$ and $V_{\bar{t}'}\left(A\right)$ are negative.

If portfolio A or portfolio $\hat{A}$ is the first one chosen, then the term $V_{\bar{t}'}\left(A\right)=0$ and the above argument applies as well.

Hence, at period $\bar{t}''$ when the next switch occurs, the investor chooses again A. Applying this argument inductively and noting that in each period of time, $V_t\left(a;\hat{a}\right)>0$ can hold for at most one portfolio at a time implies that B is never chosen again after period $\bar{t}$. Hence, its limit frequency is almost surely 0.

In contrast, A and $\hat{A}$ must be chosen for an infinite number of periods each by an argument analogous to that in the proof of proposition 2. Moreover, since the similarity between these two assets is 0 and since the finite $V_{\bar{t}'}\left(A\right)$ does not influence the limit behavior, it follows that the frequencies with which $\hat{A}$ and A are chosen are also determined analogously to the proof of proposition 2 and are given by

$$\frac{\pi\left(A\right)}{\pi\left(\hat{A}\right)}=\frac{\mu_{\hat{A}}-\bar{u}}{\mu_{A}-\bar{u}}.$$

Suppose now that

$$s_{AB}+s_{\hat{A}B}<1$$

holds. Assume that portfolio B is chosen only for a finite number of times. Denote the last period in which B is chosen by $\bar{t}'$. As in the first case it can be shown that A and $\hat{A}$ must be chosen infinitely often almost surely and that the difference of their cumulative utilities is almost surely bounded above and below. Now consider the difference between the cumulative utilities of B and $\hat{A}$:

$$\begin{aligned}U_{t}\left(\hat{A}\right)-U_{t}\left(B\right) &= \left[V_{t}\left(\hat{A}\right)-V_{\bar{t}}\left(B\right)\right]\left(1-s_{\hat{A}B}\right)-V_{t}\left(A\right)s_{AB}=\\ &= \left[V_{t}\left(\hat{A}\right)-V_{t}\left(A\right)\right]s_{AB}-V_{\bar{t}}\left(B\right)\left(1-s_{\hat{A}B}\right)\\ &\quad +V_{t}\left(\hat{A}\right)\left(1-s_{\hat{A}B}-s_{AB}\right).\end{aligned}$$

Whereas $\left[V_t\left(\hat{A}\right) - V_t(A)\right]$ has the same limit properties as $U_t\left(\hat{A}\right) - U_t(A)$ and is therefore bounded above and below,

$$\left[V_t\left(\hat{A}\right) - V_{\bar{t}}(B)\right] \to -\infty$$

almost surely since $\hat{A}$ is chosen infinitely often and the expected increments of $V_t\left(\hat{A}\right)$ are negative, $\mu_{\hat{A}} - \bar{u} < 0$. Combined with

$$s_{AB} + s_{\hat{A}B} < 1,$$

this implies that

$$U_t\left(\hat{A}\right) - U_t(B) \to -\infty$$

almost surely. In the same way, it can be shown that

$$U_t(A) - U_t(B) \to -\infty$$

But then on almost each path, it would be possible to find a period $\tilde{t}$, such that

$$\begin{aligned} U_t(A) &< U_t(B) \\ U_t\left(\hat{A}\right) &< U_t(B) \end{aligned}$$

and still act B is not chosen. This obviously contradicts the case-based decision rule. Hence, act B must be chosen infinitely often on almost all paths.

Assuming that act $\hat{A}$ is chosen for only a finite number of times with $\bar{t}'$ being the last period in which $\hat{A}$ is chosen, whereas the other two portfolios are chosen infinitely often, would imply

$$\begin{aligned} U_t\left(\hat{A}\right) - U_t(B) &= \left[V_{\bar{t}'}\left(\hat{A}\right) - V_t(B)\right]\left(1 - s_{\hat{A}B}\right) - V_t(A)\, s_{AB} = \\ &= \left[V_t(B) - V_t(A)\right] s_{AB} + V_{\bar{t}'}\left(\hat{A}\right)\left(1 - s_{\hat{A}B}\right) \\ &\quad - V_t(B)\left(1 - s_{\hat{A}B} + s_{AB}\right), \end{aligned}$$

hence

$$U_t\left(\hat{A}\right) - U_t(B) \to \infty,$$

which is in contradiction with the case-based decision rule. The same argument applies to the case when act A is chosen for a finite number of times, whereas acts B and A are chosen infinitely often.

Now suppose that two acts are chosen for a finite number of periods each. Obviously, these cannot be acts B and $\hat{A}$, since then

$$U_t(A) \to -\infty,$$

whereas $U_t\left(\hat{A}\right)$ remains finite, implying a contradiction to the case-based decision rule. In the same way, it cannot be the acts B and A. Hence, suppose that A and $\hat{A}$ are chosen for a finite number of times each. Then

$$U_t\left(\hat{A}\right)-U_t(B)=\left[V_t\left(\hat{A}\right)-V_t(B)\right](1-s_{\hat{A}B})-V_t(A)\,s_{AB}$$

and since $V_t\left(\hat{A}\right)$ and $V_t(A)$ are finite, whereas $V(B)$ has negative expected increments, it follows that

$$U_t\left(\hat{A}\right)-U_t(B)\rightarrow\infty$$

and still act B is always chosen after some period $\bar{t}'$, which again contradicts the case-based decision rule. Hence, each of the three acts must be chosen for an infinite number of times on almost each path.

Now write the differences between the cumulative utilities as:

$$\begin{aligned}\varepsilon_t\left(A;\hat{A}\right) &= V_t(A)-V_t\left(\hat{A}\right)+V_t(B)\,(s_{AB}-s_{\hat{A}B})\\ \varepsilon_t\left(B;\hat{A}\right) &= (1-s_{\hat{A}B})\left[V_t(B)-V_t\left(\hat{A}\right)\right]+s_{AB}V_t(A)\\ \varepsilon_t(A;B) &= (1-s_{AB})\,[V_t(A)-V_t(B)]+s_{\hat{A}B}V_t\left(\hat{A}\right).\end{aligned}$$

As long as $\hat{A}$ is chosen,

$$\begin{aligned}\varepsilon_t\left(A;\hat{A}\right) &\leq \bar{u}-u\left(\underline{\delta}_{\hat{A}}\right)\\ \varepsilon_t\left(B;\hat{A}\right) &\leq (1-s_{\hat{A}B})\left[\bar{u}-u\left(\underline{\delta}_{\hat{A}}\right)\right]\end{aligned}$$

hold. If the investor switches from $\hat{A}$ to A, $\varepsilon_t\left(A;\hat{A}\right)$ behaves as a random walk on the half line with negative expected increments, as long as A is chosen. Hence, the expected time since its return to 0 is almost surely finite and uniformly bounded above for initial values on the interval $\left[0;\bar{u}-u\left(\underline{\delta}_{\hat{A}}\right)\right]$. In the same way, if the investor switches from $\hat{A}$ to B, $\varepsilon_t\left(B;\hat{A}\right)$ behaves as a random walk on the half line with negative expected increments, as long as B is chosen. Hence, the expected time since its return to 0 is almost surely finite and uniformly bounded above for initial values on the interval

$$\left[0;(1-s_{\hat{A}B})\left[\bar{u}-u\left(\underline{\delta}_{\hat{A}}\right)\right]\right].$$

Analogous arguments apply for the other two portfolios B and A. Hence, the argument of the proof of proposition 2 can be used to show that the differences ε_t are bounded above and below almost surely. It follows that

$$\lim_{t\rightarrow\infty}\frac{U_t(A)}{U_t(B)} = 1 \qquad (5)$$

$$
\begin{aligned}
\lim_{t\to\infty}\frac{U_t\left(\hat{A}\right)}{U_t\left(B\right)} &= 1\\
\lim_{t\to\infty}\frac{U_t\left(A\right)}{U_t\left(\hat{A}\right)} &= 1
\end{aligned}
$$

holds on almost each path.

Because of $\left|C_t\left(a;\hat{a}\right)\right|\to\infty$ on almost all paths, it follows that

$$\lim_{t\to\infty}\frac{\sum_{\tau\in C_t(a;\hat{a})}\left[u_\tau\left(a;\hat{a}\right)-\bar{u}\right]}{\left|C_t\left(a;\hat{a}\right)\right|}=\mu_{a;\hat{a}}-\bar{u}$$

holds with probability 1 for all non-diversified portfolios. Hence, (5) can be rewritten as

$$
\begin{aligned}
\lim_{t\to\infty}\frac{C_t\left(A\right)\left(\mu_A-\bar{u}\right)+s_{AB}C_t\left(B\right)\left(\mu_B-\bar{u}\right)}{s_{AB}C_t\left(A\right)\left(\mu_A-\bar{u}\right)+C_t\left(B\right)\left(\mu_B-\bar{u}\right)+s_{\hat{A}B}C_t\left(\hat{A}\right)\left(\mu_{\hat{A}}-\bar{u}\right)} &= 1\\
\lim_{t\to\infty}\frac{C_t\left(\hat{A}\right)\left(\mu_{\hat{A}}-\bar{u}\right)+s_{\hat{A}B}C_t\left(B\right)\left(\mu_B-\bar{u}\right)}{s_{AB}C_t\left(A\right)\left(\mu_A-\bar{u}\right)+C_t\left(B\right)\left(\mu_B-\bar{u}\right)+s_{\hat{A}B}C_t\left(\hat{A}\right)\left(\mu_{\hat{A}}-\bar{u}\right)} &= 1\\
\lim_{t\to\infty}\frac{C_t\left(A\right)\left(\mu_A-\bar{u}\right)+s_{AB}C_t\left(B\right)\left(\mu_B-\bar{u}\right)}{s_{\hat{A}B}C_t\left(B\right)\left(\mu_B-\bar{u}\right)+C_t\left(\hat{A}\right)\left(\mu_{\hat{A}}-\bar{u}\right)} &= 1.
\end{aligned}
$$

Since $\frac{\pi(a;\hat{a})}{\pi(a';\hat{a}')}=\lim_{t\to\infty}\frac{C_t(a;\hat{a})}{C_t(a';\hat{a}')}$ by definition, it follows that

$$
\begin{aligned}
\frac{\pi\left(A\right)\left(\mu_A-\bar{u}\right)+s_{AB}\pi\left(B\right)\left(\mu_B-\bar{u}\right)}{s_{AB}\pi\left(A\right)\left(\mu_A-\bar{u}\right)+\pi\left(B\right)\left(\mu_B-\bar{u}\right)+s_{\hat{A}B}\pi\left(\hat{A}\right)\left(\mu_{\hat{A}}-\bar{u}\right)} &= 1\\
\frac{\pi\left(\hat{A}\right)\left(\mu_{\hat{A}}-\bar{u}\right)+s_{\hat{A}B}\pi\left(B\right)\left(\mu_B-\bar{u}\right)}{s_{AB}\pi\left(A\right)\left(\mu_A-\bar{u}\right)+\pi\left(B\right)\left(\mu_B-\bar{u}\right)+s_{\hat{A}B}\pi\left(\hat{A}\right)\left(\mu_{\hat{A}}-\bar{u}\right)} &= 1\\
\frac{\pi\left(A\right)\left(\mu_A-\bar{u}\right)+s_{AB}\pi\left(B\right)\left(\mu_B-\bar{u}\right)}{s_{\hat{A}B}\pi\left(B\right)\left(\mu_B-\bar{u}\right)+\pi\left(\hat{A}\right)\left(\mu_{\hat{A}}-\bar{u}\right)} &= 1.
\end{aligned}
$$

Simplifying and solving for $\frac{\pi(B)}{\pi(A)}$, $\frac{\pi(B)}{\pi(\hat{A})}$ and $\frac{\pi(A)}{\pi(\hat{A})}$ one obtains the relationships stated in the proposition.■

Proof of proposition 4

At $\bar{t}$

$$U_{\bar{t}}\left(\bar{a}\right)=V_{\bar{t}}\left(\bar{a}\right)<0.$$

Hence, $\bar{u}_{\bar{t}}>X_{\bar{t}}\left(\bar{a}\right)$. Moreover, (1) implies $\bar{u}_t\geq X_{\bar{t}}\left(\bar{a}\right)$ for all $t>\bar{t}$. Therefore, $V_t\left(\bar{a}\right)\leq 0$ as long as $a_\tau\neq\bar{a}$ for all $t>\tau>\bar{t}$.

At $\bar{t}$

$$U_{\bar{t}}(a) = V_{\bar{t}}(\bar{a})\, s(a;\bar{a}) < 0,$$

and since $s(a;\bar{a})$ is strictly decreasing,

$$\arg\max_{a\in[0;1]} U_{\bar{t}}(a) \in \{0;1\}.$$

Assume without loss of generality that $a_{\bar{t}+1} = 0$. As long as $a_\tau = 0$ is chosen for all $t > \tau > \bar{t}+1$

$$U_t(a) = s(a;\bar{a})\, V_{\bar{t}}(\bar{a}) + s(a;0)\, V_t(0)$$

holds. If $V_t(0) > 0$, $a = 0$ is chosen:

$$U_t(a) = s(a;\bar{a})\, V_{\bar{t}}(\bar{a}) + s(a;0)\, V_t(0) \leq U_t(0).$$

If $V_t(0) < 0$ obtains, the function

$$U_t(a) = s(a;\bar{a})\, V_{\bar{t}}(\bar{a}) + s(a;0)\, V_t(0)$$

becomes convex (since now $V_{\bar{t}}(\bar{a}) < 0$ and $V_t(0) < 0$ hold and s is concave). Therefore,

$$\arg\max_{a\in[0;1]} U_t(a) \in \{0;1\}.$$

Suppose that $a = 1$ is chosen at some time $\bar{t}' + 1$. As long as $a_\tau \neq \bar{a}$ for all $t > \tau > \bar{t}'$,

$$U_t(a) = s(a;\bar{a})\, V_{\bar{t}}(\bar{a}) + s(a;0)\, V_{\bar{t}'}(0) + s(a;1)\, V_t(1).$$

$V_{\bar{t}'}(0) < 0$, is implied by the usage of the case-based decision rule at $\bar{t}' + 1$. Applying the same argument as above, $V_t(0) < 0$ as long as $a = 0$ is not chosen again.

Therefore, as long as $V_t(1) > 0$, $a_{t+1} = 1$. As soon as $V_t(1) < 0$ obtains, the cumulative utility function becomes convex and obtains it maximum at 0 or 1. Hence,

$$P\{\omega \in S_2 \mid a_t \in \{0;1\} \text{ for each } t \geq \bar{t}' + 1\}.$$

Suppose that $a = 0$ and $a = 1$ are chosen infinitely often. Then claim 7.8 of Gilboa and Schmeidler (2001, p. 172-173) shows that for almost each $\omega \in S_2$,

$$\lim_{t\to\infty} (\bar{u}_t - X_t) = 0$$

and

$$\lim_{t\to\infty} X_t(a) - \mu = 0$$

for each a chosen infinitely often. Therefore, if $X_{\bar{t}}(\bar{a}) < \max\{\mu_0;\mu_1\}$, then there is a period T (depending on the path ω) such that for all but a sparse set of periods after T

$$|\bar{u}_t - \max\{\mu_0;\mu_1\}| < \zeta$$

for any initially chosen ζ on $\omega \in \tilde{S}$. According to claim 7.9 in Gilboa and Schmeidler (2001, p.

173), acts 0 and 1 are indeed chosen in an infinite number of periods. Therefore, $X_t(0) \to \mu_0$ and $X_t(1) \to \mu_1$. Claims 7.4-7.7 in Gilboa and Schmeidler (2001, p. 166-170) then show that on paths ω such that

$$\lim_{t\to\infty}(\bar{u}_t - X_t) = 0,$$
$$\pi\left(\arg\max_{a\in[0;1]}\mu_a\right) = 1$$

almost surely obtains.

If, however, $X_{\bar{t}}(\bar{a}) > \max\{\mu_0;\mu_1\}$, then $\bar{u}_t \to X_{\bar{t}}(\bar{a})$. Hence, there is a period T (depending on the path ω) such that

$$|\bar{u}_t - X_{\bar{t}}(\bar{a})| < \zeta$$

for each $t > T$ on $\omega \in S_2\backslash\tilde{S}$ except on a sparse set of periods for an arbitrary chosen ζ. Again, claim 7.9 of Gilboa and Schmeidler (2001, p. 173) assures that each the acts 0 and 1 are chosen an infinite number of times in the limit. But since now

$$\lim_{t\to\infty}\bar{u}_t = X_{\bar{t}}(\bar{a}) > \max\{\mu_0;\mu_1\},$$

$\pi(0) > 0$ and $\pi(1) > 0$ obtain. $\pi(0)$ and $\pi(1)$ satisfy

$$\frac{\pi(0)}{\pi(1)} = \frac{\mu_1 - X_{\bar{t}}(\bar{a})}{\mu_0 - X_{\bar{t}}(\bar{a})},$$

as shown in Gilboa and Pazgal (2001).■

Proof of proposition 6

Since $a_1 = 0$ and the aspiration level is raised by h in some periods,

$$U_{\bar{t}}(0) = V_{\bar{t}}(0) < 0$$

for some finite $\bar{t} > 0$. Since $s(0;a) > 0$ only for $a \in \left(0;\frac{1}{l}\right)$, it follows that

$$U_{\bar{t}}(a) = 0 > U_{\bar{t}}(a') \text{ for all } a \geq \frac{1}{l} \text{ and } a' < \frac{1}{l}.$$

Therefore, $a_{\bar{t}+1} = \frac{1}{l}$ is chosen by assumption 2. Note that

$$U_t(0) = V_t(0) < 0$$

for all $t > \bar{t}$, such that $a_\tau \neq 0$ for all $\bar{t} < \tau < t$ by the argument in the proof of proposition **4.** Since

$$U_{\hat{t}}\left(\frac{1}{l}\right) < 0$$

obtains for a finite $\hat{t} > \bar{t}$,

$$U_{\hat{t}}(a) = 0 > U_{\hat{t}}(a') \text{ for all } a \geq \frac{2}{l} \text{ and } a' < \frac{2}{l}.$$

Hence $a_{\hat{t}+1} = \frac{2}{l}$, etc.

Once each of the acts $\left\{0;\frac{1}{l};\frac{2}{l}...\frac{l-1}{l};1\right\}$ has been chosen at least once,

$$\begin{aligned} U_t(a) &= \sum_{i=0}^{l} V_t\left(\frac{i}{l}\right) s\left(a;\frac{i}{l}\right) \\ &= V_t\left(\frac{k}{l}\right) s\left(a;\frac{k}{l}\right) + V_t\left(\frac{k-1}{l}\right) s\left(a;\frac{k-1}{l}\right), \end{aligned}$$

for $a \in \left[\frac{k-1}{l};\frac{k}{l}\right]$, since $s(a;a') > 0$ only for $a' = \frac{k}{l}$ and $a' = \frac{k-1}{l}$ out of $\left\{0;\frac{1}{l};\frac{2}{l}...\frac{l-1}{l};1\right\}$. According to the argument stated in the proof of proposition **4** either $V_t\left(\frac{k}{l}\right) > 0$ can hold only for one $k \in \{0;...l\}$. If

$$V_t\left(\frac{k}{l}\right) > 0,$$

then $a_{t+1} = a_t = \frac{k}{l}$. If

$$\max\left\{V_t\left(\frac{k}{l}\right);V_t\left(\frac{k-1}{l}\right)\right\} < 0,$$

then $U_t(a)$ is convex, because $s(a;\cdot)$ is concave. Hence,

$$\arg\max_{a\in\left[\frac{k-1}{l};\frac{k}{l}\right]} U_t(a) \in \left\{\frac{k}{l};\frac{k-1}{l}\right\}$$

Hence,

$$\arg\max_{a\in[0;1]} U_t(a) \in \left\{0;\frac{1}{l};...\frac{k}{l};...1\right\} \text{ for every } t.$$

Therefore, the

$$\lim_{t\to\infty} \bar{u}_t = \max_a \left\{\mu_a \mid a \in \left\{0;\frac{1}{l};\frac{2}{l}...\frac{l-1}{l};1\right\}\right\}$$

almost surely, see claims 7.4 – 7.7 in Gilboa and Schmeidler (2001, p. 166-170). Hence, only

$$a^* = \arg\max_{a\in[0;1]} \mu_a$$

is satisficing in the limit and $\pi(a^*) = 1$ almost surely obtains on S_2, as shown in claim 7.8 and 7.9 in Gilboa and Schmeidler (2001, p. 172-174).■

Proof of proposition 7

Obviously, $U_t(a) < 0$ obtains for a finite t for all $a \in \{a_1...a_l\}$. At t:

$$U_t(a) = \sum_{i=1}^{l} s(a;a_i)\, V_t(a_i).$$

Then, since $U_t(a)$ is convex,

$$\begin{aligned} s\left(\arg\max_{a\in[0;1]} U_t(a);a\right) &> 0 \\ s\left(\arg\max_{a\in[0;1]} U_t(a);a'\right) &> 0 \end{aligned}$$

implies $a = a'$, as in the proof of proposition 6. Hence,

$$a_{t+1} \in [0;1] \cap \{a_i;a_i+l;a_i-l;0;1\}_{i=1...l}.$$

Analogously, at t', such that $a_{t'+1} \neq a_t$, $a_\tau = a_t$ for all $t < \tau < t' + 1$,

$$a_{t+1} \in [0;1] \cap \{a_i; a_i + l; a_i - l; a_{t+1}; a_{t+1} + l; a_{t+1} - l; 0; 1\}_{i=1\dots l}.$$

Applying this argument inductively, gives the result of the proposition.■

Proof of Proposition 8

The aspiration level of the investors of type 1 has been chosen low enough, so as to guarantee that they never switch away from their initial choice. Hence, only the behavior of the investors with high aspiration level has to be considered. Note that

$$u\left(1 + \underline{\delta} + (1 - a_0)\, r\right) = u\left(\frac{p_0 + \underline{\delta}}{p_0} a_0 + (1 - a_0)(1 + r)\right)$$

is the utility realization of portfolio a_0, given that the dividend of the risky asset is 0 (the lowest possible) and still all young investors choose a_0. Since for all $\delta_1 \in \left[\underline{\delta}; \bar{\delta}\right]$,

$$\bar{u}^2 < u\left(1 + \underline{\delta} + (1 - a_0)\, r\right) \leq u\left(1 + \delta_1 + (1 - a_0)\, r\right)$$

holds by the strict monotonicity of $u(\cdot)$, type 2 observes a cumulative utility of an act a given by:

$$\begin{array}{rcl} U_1^2(a) & = & s\left((p_0; a_0); (p_{t+1}; a_{t+1})\right) \cdot \\ & & \cdot \left[u\left(\left(\dfrac{p_1}{p_0}\right) a_0 + (1 + r)(1 - a_0)\right) - \bar{u}^2\right]. \end{array}$$

If $a_1^{2*} = a_0^{2*}$, then $p_1 = p_0$ and therefore $U_1^2(a_0) > U_1^2(a)$ for all $a \neq a_0$. Therefore if every young investor chooses a_0, a_0^{2*} is indeed the optimal choice of type 2. Hence, in equilibrium $p_1 = p_0$ obtains for each $\delta_1 \in \left[\underline{\delta}; \bar{\delta}\right]$.

By induction, the same result holds for each period of time t, hence $(a_0; a_0; p_0)$ is a stationary state of the economy.■

Proof of Proposition 9 (continued from the main text):

To show that

$$p_{t^n}^* \geq \frac{1 - \theta_1 + 2\theta_1 a_0}{2} = \theta_1 a_0 + \frac{1}{2}(1 - \theta_1)$$

obtains almost surely in finite time, first compute the difference between two subsequent members of the sequence $p_{t^k}^*$:

$$p_{t^k}^* - p_{t^{k-1}}^* = \frac{p_{t^{k-1}}^* (1 - \theta_1)}{p_{t^{k-1}}^* - \theta_1 a_0} \left[\varpi - (1 + r) + \left(r - \frac{\delta_{t^k}}{p_{t^{k-1}}^*}\right) a_{t^{k-1}}\right]. \tag{6}$$

Note that

$$p_{t^k}^* - p_{t^{k-1}}^* > 0$$

holds for any k, since

$$\varpi - (1+r) + \left(r - \frac{\delta_{t^k}}{p^*_{t^{k-1}}}\right) a_{t^{k-1}} = \varpi - (1 - a_{t^{k-1}})(1+r) - \left(\frac{p^*_{t^{k-1}} + \delta_{t^k}}{p^*_{t^{k-1}}}\right)$$

and

$$\varpi - (1 - a_{t^{k-1}})(1+r) - \left(\frac{p^*_{t^{k-1}} + \delta_{t^k}}{p^*_{t^{k-1}}}\right) > \varpi - (1 - a_0)(1+r) - 1 - \delta_{t'} > 0$$

where the first inequality obtains from

$$\delta_{t^k} \leq \tilde{\delta} < r$$

and the second follows from the assumption that

$$u(\varpi) = \bar{u}^2 = u\left((1 - a_0)(1+r) + 1 + \tilde{\delta}\right)$$

and the fact that $\delta_{t'} < \tilde{\delta}$ and the fact that $a_{t^{k-1}} > a_0$. Hence, the term in square brackets in (6) is bounded away from 0.

At the same time,

$$\frac{p^*_{t^{k-1}}(1 - \theta_1)}{p^*_{t^{k-1}} - \theta_1 a_0} > (1 - \theta_1) > 0$$

holds and therefore the difference $p^*_{t^k} - p^*_{t^{k-1}}$ is bounded away from 0 for every k, since

$$p^*_{t^k} - p^*_{t^{k-1}} > (1 - \theta_1)\left[\varpi - (1 - a_0)(1+r) - 1 - \delta_{t'}\right] > 0.$$

Therefore, the definition of Cauchy for divergent sequences implies that $p^*_{t^k}$ can become arbitrary large in finite time, especially

$$p^*_{t^n} \geq \frac{1 - \theta_1 + 2\theta_1 a_0}{2}$$

will hold after $n - 1$ steps, where n is finite. ■

Proof of Proposition 10

It has been shown above that for $u(1 + \underline{\delta} + (1 - a_0) r) < \bar{u}^2$ the investors of type 2 keep their initial portfolio a_0 for a finite number of periods. Afterwards, these investors switch either to $a^{2*}_t = 0$, implying that the price $p^*_t = \theta_1 a_0$, or to $a^{2*}_t = 1$, implying $p^*_t = 1 - \theta_1(1 - a_0)$. Call the period t^n derived in the proof of proposition 9 $\bar{t}(\omega)$. It remains to show that after the first time, in which $a^{2*}_t = 0$ or $a^{2*}_t = 1$ obtains, a cycle emerges.

Consider first the case of $a^{2*}_t = 0$ and $p^*_t = \theta_1 a_0$. Since the last (and only) case observed now by type 2 is $(\theta_1 a_0; 0; u(1+r))$ and since $u(1+r) < \bar{u}$, it follows that the optimal act at p_{t+1} is $a^2_{t+1} = 1$. But if $a^2_{t+1} = 1$, the price becomes $p^*_{t+1} = 1 - \theta_1(1 - a_0)$ and since the similarity function is strictly decreasing in the Euclidean distance between problem-act pairs, it is easily seen that $a^{2*}_{t+1} = 1$ is indeed the optimal choice.

Alternatively, if $a_t^{2*} = 1$ and $p_t^* = 1 - \theta_1 (1 - a_0)$ hold and the dividend realization at time $(t+1)$ is $\delta_{t+1} < \hat{\delta}$, then the investors of type 2 observe a utility realization of at most $u\left(1 + \frac{\delta_{t+1}}{1-\theta_1(1-a_0)}\right) < \bar{u}^2$. They are, therefore, unsatisfied with $a = 1$ even at the highest price that might obtain in period $(t+1)$. From the fact that the similarity function is strictly decreasing in the Euclidean distance, it follows that the investors of type 2 choose $a_{t+1}^{2*} = 0$. The equilibrium price is computed as $p_{t+1}^* = \theta_1 a_0$ and it is easily verified that at p_{t+1}^* $a_{t+1}^{2*} = 0$ is indeed optimal.

If at time $(t+1)$ at which the investors of type 2 hold the risky asset, its dividend realization is higher than $\hat{\delta}$, then the utility they obtain exceeds $\bar{u}^2$ as long as the price remains unchanged at

$$p_t^* = 1 - \theta_1 (1 - a_0).$$

But in this case the investors of type 2 are satisfied with $a = 1$ and

$$\arg \max_{a \in [0;1]} U_{t+1}^2 (a) = 1$$

holds, since the similarity function obtains its maximum if the problem-act pairs are identical. Therefore $a_{t+1}^{2*} = 1$ and $p_{t+1}^{2*} = 1 - \theta_1 (1 - a_0)$ obtains in an equilibrium.

The argument above shows that the evolution of prices and portfolio choices follows a Markov process with a transition matrix:

$$\bar{P} = \left(\begin{array}{|c|c|c|} \hline & p_{t+1}^* = 1 - \theta_1 (1 - a_0) & p_{t+1}^* = \theta_1 a_0 \\ \hline p_t^* = 1 - \theta_1 (1 - a_0) & q & 1 - q \\ \hline p_t^* = \theta_1 a_0 & 1 & 0 \\ \hline \end{array} \right).$$

The frequencies π_h and π_l can now be computed as the invariant probability distribution of this Markov process, i.e.:

$$\begin{pmatrix} \pi_h \\ \pi_l \end{pmatrix} = \begin{pmatrix} \pi_h \\ \pi_l \end{pmatrix}' \bar{P},$$

which simplifies to

$$q \pi_h = (2 - \pi_h).$$

It follows that the invariant probabilities satisfy:

$$\begin{aligned} \pi_h &= \frac{1}{2-q} \\ \pi_l &= \frac{1-q}{2-q}. \end{aligned}$$

These probabilities are obviously strictly positive for $q \in (0;1)$ and therefore the Markov chain described by P is positive recurrent. Since any positive recurrent chain on a countable space is also positive Harris recurrent, see Meyn and Tweedie (1996, p. 208), it follows that the Law of Large Numbers applies for this chain. Hence, let ι_h denote the indicator function for state h.

Hence,

$$\iota_h(t) = \left\{ \begin{array}{ll} 1, & \text{if } p_t^* = 1 - \theta_1 (1 - a_0) \\ 0, & \text{if } p_t^* = \theta_1 a_0 \end{array} \right\}.$$

According to theorem 17.1.7 in Meyn and Tweedie (1996, p. 425),

$$\lim_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t} \iota_h(t) = \int \iota_h(t)\, d\pi = \pi_h$$

almost surely for any initial distribution. Since $\frac{1}{t}\sum_{\tau=0}^{t} \iota_h(t)$ describes the mean time up to period t that the economy spends in state h, it follows that the frequency of state h equals π_h almost surely in the limit. Analogous arguments show that the frequency of state l equals π_l on almost each path ω.■

Proof of Proposition 11

It follows from the proof of proposition 10 that the investors of type 2 hold a_0 only for a finite number of periods. Afterwards, the economy cycles between the states h and l. Moreover, the investors of type 2 switch to $a_{t+1}^{2*} = 0$, if the last period price satisfies $p_t^* = p_h$ and the dividend is lower than $\hat{\delta}$, causing the price to fall to $p_{t+1}^* = p_l$. Conversely, given that the last period price of A is low ($p_t^* = p_l$, $a_t^{2*} = 0$), the investors of type 2 are not satisfied with $u(1+r)$ and switch to $a_{t+1}^{2*} = 1$, causing the price to rise to p_h.

It remains only to consider periods t, such that $p_t^* = p_h$, $a_t^{2*} = 1$ and $\delta_{t+1} \geq \hat{\delta}$ hold. Since $u\left(1 + \frac{\bar{\delta}}{p_h}\right) < \bar{u}^2$, it follows that the cumulative utility of $a = 1$ is negative at $(t+1)$ for the investors of type 2 and, thus, the optimal act at any price $p_t \leq p_h$ is $a_t^{2*} = 0$. But for $a_t^{2*} = 0$, the price $p_t = p_l < p_h$ must hold and, therefore, $a_t^{2*} = 0$ and $p_t^* = p_l$ obtain as equilibrium at time t.

To summarize, the investors of type 2 choose $a_t^{2*} = 1$ in each period t, such that $a_{t-1}^{2*} = 0$ and they choose $a_t^{2*} = 0$ in each period t, such that $a_{t-1}^{2*} = 1$. Therefore, the result of the proposition obtains.■

Proof of Proposition 9 (continued from the main text):

To show that

$$p_n^* \geq \frac{1 - \theta_1 + 2\theta_1 a_0}{2} = \theta_1 a_0 + \frac{1}{2}(1 - \theta_1)$$

indeed obtains in finite time, first compute the difference between two subsequent members of the sequence $p^*_{t^k}$:

$$\Delta_{k-1} =: p^*_{t^k} - p^*_{t^{k-1}} = \frac{p^*_{t^{k-1}}(1-\theta_1)}{p^*_{t^{k-1}} - \theta_1 a_0}\left[\varpi - (1+r) + r\frac{p^*_{t^{k-1}} - \theta_1 a_0}{1-\theta_1}\right].$$

Note that

$$p^*_{t^k} - p^*_{t^{k-1}} > 0$$

holds for any k, since

$$\varpi - (1+r) + r\frac{p^*_{t^{k-1}} - \theta_1 a_0}{1-\theta_1} = \varpi - (1 + r - ra^*_{t^{k-1}}) > \varpi - (1 + r - ra_0) > 0,$$

where the first inequality follows from $a^*_{t^{k-1}} > a_0$ and the second from the assumption that

$$u(\varpi) = \bar{u}^2 > u(1 + r - ra_0).$$

If it could be shown, that this difference is bounded away from 0 for any k, then using the definition of Cauchy for divergent sequences, it can be shown that $p^*_{t^k}$ can become arbitrary large in finite time[11].

To do so differentiate Δ_{k-1} with respect to $p^*_{t^{k-1}}$ to obtain:

$$\frac{\partial \Delta_{k-1}}{\partial p^*_{t^{k-1}}} = r\frac{p^*_{t^{k-1}}}{p^*_{t^{k-1}} - \theta_1 a_0} - \frac{\theta_1 a_0 (1-\theta_1)}{\left(p^*_{t^{k-1}} - \theta_1 a_0\right)^2}\left[\varpi - (1+r) + r\frac{p^*_{t^{k-1}} - \theta_1 a_0}{1-\theta_1}\right].$$

Δ_{k-1} obtains an optimum at a point, in which $\frac{\partial \Delta_{k-1}}{\partial p^*_{t^{k-1}}} = 0$, or

$$r\left(p^*_{t^{k-1}}\right)^2 - 2\theta_1 a_0 r p^*_{t^{k-1}} - \theta_1 a_0 (1-\theta_1)\varpi + \theta_1 a_0 (1-\theta_1)(1+r) - \theta_1^2 (a_0)^2 = 0$$

It is not necessary to solve this equation. Denote by $\hat{p}$ the bigger of the two roots of this equation, provided that it exists[12]. Δ_{k-1} obtains its local minimum at $\hat{p}$, whereas the global minimum either coincides with the local one or is obtained at p_0, the initial value of p. Note, that if $p_0 > \hat{p}$, then the difference Δ is increasing for all $p > p_0$ and therefore its minimum is in p_0 and is positive, as shown above, since

$$\varpi - (1 + r - ra_0) > 0.$$

If $p_0 < \hat{p}$, then the minimum is either at $\hat{p}$ or at p_0 and it is in both cases positive, since

$$\varpi - \left(1 + r - ra^{2*}(\hat{p})\right) > \varpi - (1 + r - ra_0) > 0$$

holds, where $a^{2*}(\hat{p})$ is determined according to:

$$\hat{p} = \theta_1 a_0 + (1-\theta_1)\, a^{2*}(\hat{p})$$

and $a^{2*}(\hat{p}) > a_0$, since $\hat{p} > p_0$.

Since the minimal difference between two succeeding members of the sequence is positive, and

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[11] I thank Dmitri Vinogradov for suggesting this method of proof.

[12] If no solution exists, then Δ_{k-1} is increasing for all p_{t^k} and its minimum is obtained at p_0.

therefore the difference is bounded away from 0, the sequence will reach each predetermined value in finite time, especially

$$p_{n+1}^{*} \geq \frac{1-\theta_1+2\theta_1 a_0}{2}$$

will hold after n steps, where n is finite. ■

Proof of Proposition 10

It has been shown above, that for $u\left(1+\left(1-a_0\right)r\right) < \bar{u}^2$ the investors of type 2 keep their initial portfolio a_0 for a finite number of periods. Afterwards these investors switch either to $a_t^{2*}=0$, implying that the price $p_t^{*}=\theta_1 a_0$, or to $a_t^{2*}=1$, which leads to $p_t^{*}=1-\theta_1\left(1-a_0\right)$.

Consider first the case of $a_t^{2*}=0$ and $p_t^{*}=\theta_1 a_0$. Since the last (and only) case observed now is $\left(\theta_1 a_0; 0; u\left(1+r\right)\right)$ and since $u\left(1+r\right) < \bar{u}$, it follows, that the optimal act, given p_{t+1} is $a_{t+1}^{2}=1$. But if $a_{t+1}^{2}=1$, the price becomes $p_{t+1}^{*}=1-\theta_1\left(1-a_0\right)$ and since the similarity function is strictly decreasing in the Euclidean distance between problem-act pairs, it is easily seen, that $a_{t+1}^{2*}=1$ is indeed the optimal choice.

On the other hand, if $a_t^{2*}=1$ and $p_t^{*}=1-\theta_1\left(1-a_0\right)$ hold and the dividend realization is low, then the investors of type 2 observe a utility realization of at most $u\left(1\right) < \bar{u}^2$. They are therefore unsatisfied with $a=1$, even at the highest price that might obtain in period $\left(t+1\right)$. From the fact that the similarity function is strictly decreasing in the Euclidean distance, it follows that the investors of type 2 choose $a_{t+1}^{2*}=0$. The equilibrium price is computed as $p_{t+1}^{*}=\theta_1 a_0$ and it is easily verified, that given p_{t+1}^{*} $a_{t+1}^{2*}=0$ is indeed optimal.

If at time $\left(t+1\right)$ at which the investors of type 2 hold the risky asset, its dividend realization is high, then the utility they realize is $u\left(1+\frac{\delta}{1-\theta_1\left(1-a_0\right)}\right) > \bar{u}^2$, as long as the price remains unchanged at $\left(t+1\right)$. But in this case the investors of type 2 are satisfied with $a=1$ and

$$\arg\max_{a\in[0;1]} U_{t+1}^{2}\left(a\right)=1$$

holds, since the similarity function obtains its maximum if the problem-act pairs are identical. Therefore $a_{t+1}^{2*}=1$ and $p_{t+1}^{2*}=1-\theta_1\left(1-a_0\right)$ obtains in an equilibrium.

The argument above shows, that the evolution of prices and portfolio choices follows a Markov process with a transition matrix:

$$P=\left(\begin{array}{|c|c|c|}\hline & p_{t+1}^{*}=1-\theta_1\left(1-a_0\right) & p_{t+1}^{*}=\theta_1 a_0 \\ \hline p_t^{*}=1-\theta_1\left(1-a_0\right) & q & 1-q \\ \hline p_t^{*}=\theta_1 a_0 & 1 & 0 \\ \hline\end{array}\right).$$

The invariant probability distribution of the states $p_h = 1 - \theta_1 (1 - a_0)$; $a_h^1 = a_0$; $a_h^2 = 1$ and $p_l = \theta_1 a_0$; $a_l^1 = a_0$; $a_l^2 = 0$ is now easily computed from:

$$(\pi_h; \pi_l) P = (\pi_h; \pi_l)',$$

which implies

$$\begin{aligned} \pi_h &= \frac{1}{2-q} \\ \pi_l &= \frac{1-q}{2-q}. \end{aligned}$$

These probabilities are obviously strictly positive for $q \in (0; 1)$ and therefore the Markov chain described by P is positive recurrent. Since any positive recurrent chain on a countable space is also positive Harris recurrent, see Meyn and Tweedie (1996, p. 208), it follows that the Law of Large Numbers applies for this chain. Hence, let ι_h denote the indicator function for state h:

$$\iota_h (t) = \left\{ \begin{array}{ll} 1, & \text{if the state of the economy is } h \\ 0, & \text{if the state of the economy is } l \end{array} \right\}.$$

According to theorem 17.1.7 in Meyn and Tweedie (1996, p. 425),

$$\lim_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t} \iota_h (t) = \int \iota_h (t) \, d\pi = \pi_h$$

holds almost surely for any initial distribution over the states h and l. Since $\frac{1}{t} \sum_{\tau=0}^{t} \iota_h (t)$ describes the mean time up to period t that the economy spends in state h, it follows that the frequency of state h equals π_h almost surely in the limit. Analogous arguments show that the frequency of state l equals π_l on almost each path.■

Proof of Proposition 11

It follows from the proof of proposition 10, that type two will only hold a_0 for a finite number of periods and afterwards the economy will cycle along the states h and l. Moreover the investors of type 2 will switch to $a_{t+1}^{2*} = 0$, if the last period price $p_t^* = p_h$ and the dividend is 0, making the price $p_{t+1}^* = p_l$. Conversely, given that the last period price of A is low ($p_t^* = p_l$, $a_t^{2*} = 0$), the investors of type 2 are not satisfied with $u(1+r)$ and switch to $a_{t+1}^{2*} = 1$, causing the price to rise to 1. It remains only to consider periods t, such that $p_{t-1}^* = p_h$, $a_{t-1}^{2*} = 1$ and $\delta_t = \delta$ hold. Since $u\left(1 + \frac{\delta}{p_h}\right) < \bar{u}^2$, it follows, that the cumulative utility of $a = 1$ is negative in t for the investors of type 2, and so the optimal act, given any price $p_t \leq p_h$ is $a_t^{2*} = 0$. But for $a_t^{2*} = 0$, the price $p_t = p_l < p_h$ must hold, and therefore $a_t^{2*} = 0$ and $p_t^* = p_l$ obtain as equilibrium at time t.

To summarize, the investors of type 2 choose $a_t^{2*} = 1$ in each period t, such that $a_{t-1}^{2*} = 0$ and they choose $a_t^{2*} = 0$ in each period t, such that $a_{t-1}^{2*} = 1$. Therefore the result of the proposition obtains.■

Proof of proposition 12

1. If $\bar{u}^2 < Eu[a_0]$, then the cumulative utility of a_0 for the investors of type 2 is given by:
$$U_t^2\left(a_0\right) = \sum_{\tau=1}^{t} \left[v_\tau\left(a_0\right) - \bar{u}^2\right],$$
as long as they hold a_0. Since
$$E\left[v_t\left(a_0\right)\right] = Eu\left[a_0 \mid p_0\right] > \bar{u}^2$$
$U_t\left(a_0\right)$ behaves as a random walk on $\mathbb{R}^+$ with positive expected increment. According to theorem 9.5.1 in Main and Tweedie (1996, p. 228) such random walks are transient, hence the expected time until their first return to 0 is infinite.

2. If $\bar{u}^2 \in \left(Eu[a_0]; u\left(1+r\right)\right)$, then the process
$$\tilde{U}_t^2\left(a_0\right) = \begin{array}{ll} U_t^2\left(a_0\right), & \text{if } U_t^2\left(a_0\right) \geq 0 \\ 0, & \text{else} \end{array}$$
describes the cumulative utility of a_0 for the investors of type 2 as long as it is positive. $\tilde{U}_t^2\left(a_0\right)$ is a random walk on $\mathbb{R}^+$, but with negative expected increments, since now
$$E\left[v_t\left(a_0\right)\right] = Eu\left[a_0 \mid p_0\right] < \bar{u}^2.$$
Since for such random walks all compact sets are regular, see proposition 11.4.1 in Meyn and Tweedie (1996, p. 278), it follows that
$$\tilde{U}_t^2\left(a_0\right) = 0$$
obtains in finite time with probability 1. Therefore, since the distribution Q is continuous, it follows that
$$U_t^2\left(a_0\right) < 0$$
obtains almost surely in finite time.
Once the cumulative utility of a_0 has become negative, apply the proof of proposition 10 to show that the investors of type 2 will choose either $a = 0$ or $a = 1$ in finite time. Note that this result can be applied, since the portfolios $a_{t^k}^{2*}$, $t^k \leq t^{n-1}$, constructed in this proof[13] have a cumulative utility of 0 and therefore do not influence the evaluation of any of the

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[13] In the case of long memory, however, the time periods t^k will not denote the *subsequent* periods in which

acts available, whereas the cumulative utility of the last chosen diversified portfolio $a^{2*}_{t^n}$ is negative. Once $a = 1$ or $a = 0$ has been chosen for the first time, its cumulative utility behaves as a random walk with positive expected increments, since

$$\bar{u}^2 < u\left(1 + r\right) < Eu\left[1 \mid 1 - \theta_1 \left(1 - a_0\right)\right].$$

Therefore, it remains positive infinitely long in expectations. Hence, the expected time during which the investors of type 2 hold $a = 1$ or $a = 0$ is infinity.

Note further that the cumulative utility of any portfolio a is given by:

$$\begin{aligned} U_t\left(a\right) = s\left(\left(p_t; a\right); \left(\left(1 - \theta_1\left(1 - a_0\right); 1\right)\right)\right) V_t\left(1\right) + s\left(\left(p_t; a\right); \left(\left(\theta_1 a_0; 0\right)\right)\right) V_t\left(0\right) \\ + s\left(\left(p_t; a\right); \left(\left(p^*_{t^n}; a^{2*}_{t^n}\right)\right)\right) V_t\left(a^{2*}_{t^n}\right), \end{aligned}$$

where the notation from the proof of proposition 2 is used. Now, if exactly one of the numbers $V_t\left(1\right)$, $V_t\left(0\right)$ and $V_t\left(a^{2*}_{t^n}\right)$ is positive, the act 1, 0 or $a^{2*}_{t^n}$ will be chosen, respectively. The rule of cumulative utility maximization precludes the case that two of these numbers are positive at some t[14]. But if all of them are negative, then $U_t\left(a\right)$ becomes a convex function in a, since the similarity function is concave in the Euclidean distance and since the distance only changes with respect to a for a given price p_t. Therefore, a corner solution obtains in each period of time. Hence, either $a = 1$ or $a = 0$ are chosen. On all paths, on which the cumulative utility of $a = 1$ never falls below the cumulative utility of $a = 0$, the frequency of $a = 1$ is 1. If, however, the investors of type 2 switch to $a = 0$ at some time T, then

$$u\left(1 + r\right) > \bar{u}^2$$

implies that the cumulative utility $a = 0$ exceeds the cumulative utility of $a = 1$ for each $t > T$. Hence, $a = 0$ is chosen in each period afterwards. On these paths, the limit frequency of $a = 0$ is, therefore, equal to 1.

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the dividend realization is lower than $\tilde{\delta}$, but those periods in which $\delta_{t^k} < \tilde{\delta}$ and

$$U_{t^k-1}\left(\alpha_0\right) + u\left(1 + \delta_{t^k} + \left(1 - \alpha_0\right)\left(1 + r\right)\right) - \bar{u}^2 < 0,$$

whereas $U_{t^k-1}\left(\alpha_0\right) \geq 0$ holds. The portfolio $\alpha^{2*}_{t^k}$ (and hence the price $p^*_{t^k}$) are then chosen in such a way that

$$U_{t^k-1}\left(\alpha_0\right) + u\left(\frac{p^*_{t^k} + \delta_{t^k}}{p^0}\alpha_0 + \left(1 - \alpha_0\right)\left(1 + r\right)\right) - \bar{u}^2 = 0$$

Hence, $U_{t^k}\left(\alpha\right) = 0$ for each $\alpha \in \left[0; 1\right]$ and therefore the choices till time t^k do not influence the evaluation of the available portfolios.

[14] Indeed, as long as the realized cumulative utility of some act α is positive, it is chosen again and again. The act α is abandoned when its cumulative utility becomes negative. But from this time on, $V_t\left(\alpha\right)$ remains unchanged and negative, until act α is chosen again. However, for this to be the case, it must be that the decision maker has become dissatisfied with all other acts, hence that $V_t\left(\alpha'\right) < 0$ holds for all $\alpha' \in \left[0; 1\right]$.

3. If $\bar{u}^2 \in (u(1+r); Eu[1 \mid 1-\theta_1(1-a_0)])$, then the investors of type 2 switch to $a=1$ or to $a=0$ in finite time, as shown in part 2 of this proof. If $a=1$ has been chosen, then its cumulative utility behaves like a random walk with positive expected increments and therefore $a=1$ is held infinitely long in expectation. If $a=0$ has been chosen, then it will be only held for a finite time, since its return is considered unsatisfactory. Moreover, since the similarity function is concave, once the cumulative utilities of $a=0$, a_{t^n} and hence also of $a=1$ have become negative, the optimal act will be a corner solution, as shown in part 2 of this proof. Therefore, a corner solution obtains and $a=1$ will be chosen in finite time and then held forever in expectation.

   Even, if the investors of type 2 should switch to $a=0$ at some time, the cumulative utility of this portfolio would become lower than the cumulative utility of $a=1$ in finite time. But since the events that the cumulative utility of $a=1$ becomes negative have a probability lower than 1 and are independent, the probability of the event that the cumulative utility of $a=1$ falls below 0 infinitely often is 0. Hence, in the limit, type 2 indeed holds $a=1$ with frequency 1.

4. Now, let $\bar{u}^2 > Eu[1 \mid 1-\theta_1(1-a_0)]$. In this case the investors of type 2 again switch to one of the corner acts in finite time. However, in this case neither $a=0$, nor $a=1$ are considered satisfactory in expectations. Therefore, their cumulative utilities behave as random walks with negative expected increments. Hence, they become lower than any given number (especially, than the cumulative utility of the alternative act) in finite time with probability 1. Note that if, say $a=0$ is chosen, then the negative increments of its cumulative utility influence negatively the cumulative utility of act $a=1$, as long as

   $$s((p=1-\theta_1(1-a_0); a=1); (p=\theta_1 a_0; a=0)) \in (0;1)$$

   holds. However, the cumulative utility of the unchosen act decreases more slowly than the cumulative utility of the act actually chosen and, therefore, the cumulative utility of $a=0$ eventually becomes lower than those of $a=1$ and vice versa for the case in which $a=1$ is chosen. Moreover, at this time the cumulative utilities of these acts will be negative. Once however, all three acts a_{t^n}, $a=1$ and $a=0$ have negative cumulative utilities, the concavity of the similarity function implies that a corner solution is chosen in each period of time. Hence, the investors of type 2 switch infinitely often between $a=1$ and $a=0$.

Denote by ε_t the following process:

$$\begin{aligned}
\varepsilon_0 &= \varepsilon_0 \in \left[\bar{u}^2 - u\left(\frac{\theta_1 a_0 + \underline{\delta}}{1-\theta_1\left(1-a_0\right)}\right); +\infty\right) \\
\varepsilon_t &= \left\{ \begin{array}{ll}
\varepsilon_{t-1} + u\left(1 + \frac{\delta_t}{1-\theta_1(1-a_0)}\right) - \bar{u}^2, & \text{if } \varepsilon_{t-1} \geq 0 \text{ and } \varepsilon_{t-1} + u\left(1 + \frac{\delta_t}{1-\theta_1(1-a_0)}\right) - \bar{u}^2 \geq 0 \\
\varepsilon_{t-1} + u\left(\frac{\theta_1 a_0 + \delta_t}{1-\theta_1(1-a_0)}\right) - \bar{u}^2, & \text{if } \varepsilon_{t-1} \geq 0 \text{ and } \varepsilon_{t-1} + u\left(1 + \frac{\delta_t}{1-\theta_1(1-a_0)}\right) - \bar{u}^2 < 0 \\
\varepsilon_{t-1} + u\left(1+r\right) - \bar{u}^2, & \text{if } \varepsilon_{t-1} < 0.
\end{array} \right\}.
\end{aligned}$$

The process $\left(1-s'\right)\varepsilon_t$ describes the evolution of the difference between the cumulative utilities of a and b for the investors of type 2 after period t^n, where

$$s' = s\left(\left(p = 1-\theta_1\left(1-a_0\right); a = 1\right); \left(p = \theta_1 a_0; a = 0\right)\right),$$

To see this, note that

$$\varepsilon_t = V_t^2\left(1\right) - V_t^2\left(0\right)$$

and

$$U_t^2\left(1\right) - U_t^2\left(0\right) = V_t^2\left(1\right) + s'V_t^2\left(0\right) - V_t^2\left(0\right) + s'V_t^2\left(1\right) = \left(1-s'\right)\varepsilon_t.$$

It is obvious that ε_t is a Markov chain, since δ_t is identically and independently distributed according to Q. Moreover, ε_t evolves on

$$\Psi' = \left[\bar{u}^2 - u\left(\frac{\theta_1 a_0 + \underline{\delta}}{1-\theta_1\left(1-a_0\right)}\right); +\infty\right),$$

since the greatest amount by which the cumulative utility of B can exceed the cumulative utility of A is $\bar{u}^2 - u\left(\frac{\theta_1 a_0 + \underline{\delta}}{1-\theta_1(1-a_0)}\right)$, whereas the cumulative utility of A can become very large, if $\delta > \hat{\delta}$ occurs for a long period of time. The idea of the proof consists in showing that ε_t is a stationary process with an invariant probability distribution π, as defined in the statement of the proposition. Since for positive ε_t the investors of type 2 choose asset A, whereas for negative ε_t, they choose B, the frequency with which A and B are chosen in the limit coincide with

$$\pi\left[0; +\infty\right)$$

and

$$\pi\left[\bar{u}^2 - u\left(\frac{\theta_1 a_0 + \underline{\delta}}{1-\theta_1\left(1-a_0\right)}\right); 0\right),$$

respectively.

Denote by G the interval $\left[0; \bar{u}^2 - u\left(1+r\right)\right]$. The following Lemma shows that the set G is a small set, i.e. that there exists a measure ν on the set

$$\Psi' = \left[\bar{u}^2 - u\left(\frac{\theta_1 a_0 + \underline{\delta}}{1-\theta_1\left(1-a_0\right)}\right); +\infty\right)$$

such that

$$P^K(\varepsilon; F) \geq \nu(F)$$

for any set $F \in \Psi'$ and any $\varepsilon \in G$, where $P^K(\varepsilon; F)$ denotes the probability to reach a set F starting from ε in K steps, see Meyn and Tweedie (1996, p. 111).

Lemma 14 *The set* $G = [0; \bar{u}^2 - u(1+r)]$ *is small.*

Proof of lemma 14:

The assumption about the probability distribution of δ and the continuity of the utility function $u(\cdot)$ implies that the net utility realizations

$$\tilde{u} = u\left(1 + \frac{\delta_t}{1 - \theta_1(1 - a_0)}\right) - \bar{u}^3$$

of A (as long as its cumulative utility remains positive) are distributed according to a probability distribution Q', such that Q' has an absolutely continuous part with respect to the Lesbegue measure on the real numbers. Moreover, there is a number ζ', such that the density of $\tilde{u}$ g' is bounded away from 0 on an interval $(-\zeta'; \zeta')$ for some ζ' satisfying $\bar{u}^2 - u(1+r) > \zeta' > 0$, i.e.:

$$g'(\tilde{u}) \geq \phi' > 0$$

for all $\tilde{u} \in (-\zeta'; \zeta')$ and for some ϕ'.

Divide the set G into K sets, $G_1 ... G_K$ with length less than $\frac{\zeta'}{2}$. Fix an $\varepsilon \in G_i$ and suppose that $F \subset G_j$. Now, for each $0 < \xi < \frac{\zeta'}{2}$, there is a positive probability P_{ξ^+} that

$$\varepsilon_{t+1} \in \left(\varepsilon_t + \frac{\zeta'}{2} - \xi; \varepsilon_t + \frac{\zeta'}{2}\right)$$

and a positive probability P_{ξ^-} that

$$\varepsilon_{t+1} \in \left(\varepsilon_t - \frac{\zeta'}{2}; \varepsilon_t - \frac{\zeta'}{2} + \xi\right)$$

Moreover, because of the assumptions made on the probability distribution Q, these probabilities are bounded away from 0:

$$\begin{aligned} P_{\xi^+} &\geq \phi'\xi \\ P_{\xi^-} &\geq \phi'\xi. \end{aligned}$$

Now choose ξ such that $\xi(K-1) \leq \frac{\zeta'}{2}$ holds. It follows that after $(K-1)$ steps the process ε_t will be at a distance of at most ζ' away from the set G_j, of which F is a subset with probability of at least

$$[\phi'\xi]^{K-1}.$$

Therefore, at step K, there is a positive probability of at least:

$$P\left(\varepsilon_{t+K-1};F\right)=P\left(\tilde{u}\in\left(F-\varepsilon_{t+K-1}\right)\right)=\int_{F-\varepsilon_{t+K-1}} g'\left(\tilde{u}\right)d\tilde{u}\geq\phi'\mu^{Leb}\left(F\right).$$

Hence, the probability that set $F\subset G_j$ is reached after K steps starting at some ε_t is at least

$$P^K\left(\varepsilon_t;F\right)\geq\left[\phi'\xi\right]^{K-1}\phi'\mu^{Leb}\left(F\right)=\nu\left(F\right),$$

where ν is absolutely continuous with respect to the Lebesgue measure on the interval $[0;u\left(1+r\right)]$. The probability that a set F which is not a subset of any G_j is reached in K steps fulfills

$$P^K\left(\varepsilon_t;F\right)=\sum_{i=1}^{K}P^K\left(\varepsilon_t;F_i\right)\geq\left[\phi'\xi\right]^{K-1}\phi'\sum_{i=1}^{K}\mu^{Leb}\left(F_i\right)=\left[\phi'\xi\right]^{K-1}\phi'\mu^{Leb}\left(F\right),$$

where $\cup_{i=1}^{K}F_i=F$ and $F_i\subset G_i$, i.e. F_i is a partition of F into sets each of which is a (possibly empty) subset of some G_i. Since each set outside G is reached with a non-negative probability starting from G, it follows that the set G is a small set and the measure $\nu\left(F\right)$ is defined as

$$\begin{aligned}\nu\left(F\right)&=\left[\phi'\xi\right]^{K-1}\phi'\mu^{Leb}\left(F\right),\ F\subset G\\ \nu\left(F\right)&=0\text{, else.}\end{aligned}$$

Moreover, according to proposition 5.5.3 in Meyn and Tweedie (1996, p. 127), since each small set is a petite set, G is a petite set.■

The next Lemma demonstrates that the Markov chain defined by ε_t is φ-irreducible. φ-irreducibility is an analogue to the concept of irreducibility of Markov chains on countable sets, defined for Markov chains on general sets. It defines a measure φ, which assigns a strictly positive value only to subsets of the set Ψ' which are reached with strictly positive probability from every initial point ε_t, see Meyn and Tweedie (1996, p. 91).

Lemma 15 *Let φ be defined as the Lebesgue measure on the set $[0;\bar{u}^2-u\left(1+r\right)]$ and be 0 elsewhere. Then the Markov chain ε is φ-irreducible.*

Proof of lemma 15:

Obviously, φ assigns a positive probability only to subsets of the interval G. The statement of the lemma is therefore true if it can be shown that each of the subsets of this interval is reached with positive probability from any initial point. Since it has been shown that starting from any point in the interval G any subset of G is reached with positive probability, it remains to demonstrate that starting outside the interval $[0;\bar{u}^2-u\left(1+r\right)]$, a subset of this interval is reached with positive probability. Consider two cases: if $\varepsilon_t<0$, then ε_t grows by $\bar{u}^2-u\left(1+r\right)$ in each period, until $\varepsilon_{t+k}\geq0$ obtains for the first time. But at time $\left(t+k\right)$ $\varepsilon_{t+k}\in[0;\bar{u}^3-u\left(1+r\right)]$, hence the interval G is reached with probability 1, starting from a negative ε_t. If $\varepsilon_t > \bar{u}^2 - u\left(1+r\right)$ holds, then there is a positive probability that the next $\left\lceil 2\frac{\varepsilon_t}{\zeta'} \right\rceil$ steps are negative with realizations between $\left(-\frac{\zeta'}{2}; -\frac{\eta\zeta'}{2}\right)$ with $\varepsilon_t\eta < \bar{u}^2 - u\left(1+r\right)$, (hence, $\eta < 1$) and, therefore, the subset $\left[0; \varepsilon_t\eta\right]$ of G is reached with strictly positive probability in finite time from any initially chosen ε_t. Therefore, the Markov chain is φ-irreducible.

Since φ is finite, it follows according to proposition 4.2.2 in Meyn and Tweedie (1996, p. 92) that there exists a probability measure ψ on Ψ', which assigns a probability of 0 to a subset F of Ψ' if and only if

$$\psi\left(\varepsilon \mid \sum_{n=1}^{\infty} P^n\left(\varepsilon; F\right) > 0\right) = 0.$$

ψ is absolutely continuous with respect to φ, hence if $\varphi\left(F\right) > 0$, then $\psi\left(F\right) > 0$ holds as well. Denote by $\mathcal{B}\left(\Psi'\right)$ the Borel σ-algebra on Ψ'. Let $\mathcal{B}^+\left(\Psi'\right)$ denote the subset $\mathcal{B}\left(\Psi'\right)$, whose elements are assigned a strictly positive probability according to ψ:

$$\mathcal{B}^+\left(\Psi'\right) = \left\{F \in \mathcal{B}\left(\Psi'\right) \mid \psi\left(F\right) > 0\right\}.$$

Note that the petite set G satisfies $G \in \mathcal{B}^+\left(\Psi'\right)$.■

Part (ii) of theorem 10.4.10 in Meyn and Tweedie (1996, p. 254) combines the notion of petite set and irreducibility of a Markov chain with the notion of positive recurrence, which assures the existence of an invariant probability distribution π:

Proposition 16 *Suppose that a Markov chain is ψ-irreducible. Let τ_G denote the first hitting time of the set G. The chain is positive recurrent, if for some petite set $G \in \mathcal{B}^+\left(\Psi'\right)$*

$$\sup_{\varepsilon \in G} E_{\varepsilon}\left[\tau_G\right] < \infty.$$

Proof of proposition 16:

See Meyn and Tweedie (1996, p. 254).■

It has already been shown that the chain defined by ε_t is ψ-irreducible and that G is a petite set with $\psi\left(G\right) > 0$ (since $\varphi\left(G\right) > 0$). It remains, therefore, to show that the expected hitting time of the set G, starting from G, is bounded from above. To demonstrate this note that the process ε_t, constrained to its positive part, is a random walk on a half line with negative expected increment. Proposition 11.4.1 in Meyn and Tweedie (1996, p. 278) demonstrates that for such a random walk all compact sets are regular. A regular set F' has the property that

$$\sup_{\varepsilon \in F'} E_{\varepsilon}\left[\tau_F\right] < \infty$$

for all $F \in \mathcal{B}^+(\Psi')$, see Meyn and Tweedie (1996, p. 263). Since $G \in \mathcal{B}^+(\Psi')$, it follows that

$$\sup_{\varepsilon \in G} E_\varepsilon [\tau_G] < \infty$$

holds for the process ε_t reduced to a random walk on the half line. Moreover, since all compact sets are regular, it follows that

$$\sup_{\varepsilon \in F} E_\varepsilon [\tau_G] < \infty$$

holds for all compact sets $F \subset [0; +\infty)$.

Now, consider the unconstrained process ε_t. There are two possibilities: either ε_t remains nonnegative forever and in this case it behaves like a random walk on the half line and, therefore, G is regular, or ε_t eventually becomes negative. If $\varepsilon_t < 0$ at some t, then the expected time in which ε reaches G is at most

$$\frac{\bar{u}^2 - u\left(\frac{\theta_1 a_0 + \delta}{1 - \theta_1 (1 - a_0)}\right)}{\bar{u}^2 - u(1+r)},$$

which is finite. Therefore, the expected time that the process needs to reach the set G starting from set G is bounded from above. But then the condition of proposition 16 is satisfied and the Markov chain defined by ε_t is positive recurrent. Hence, there exists an invariant probability measure π for the process ε_t, see Theorem 10.0.1 in Meyn and Tweedie (1996, p. 238). Moreover, since

$$\sup_{\varepsilon \in \Psi'} E_\varepsilon [\tau_G] < \infty$$

holds, it follows that the process described by ε_t is a positive Harris chain, see Meyn and Tweedie (1996, p. 207) for a definition of a Harris chain.

It now remains to show that π_h and π_l as defined in the statement of the proposition are positive and satisfy

$$\frac{\pi_h}{\pi_l} = \frac{s'\left(-\mu_1^r + \bar{u}^2\right) - u(1+r) + \bar{u}^2}{\bar{u}^2 - \mu_1^r + s'\left(-u(1+r) + \bar{u}^2\right)}.$$

Note that according to the Strong Law of Large Numbers, the cumulative utility of $a = 1$ if it is chosen for an infinite number of times by the investors of type 2 satisfies:

$$\lim_{t \to \infty} U_t^2(1) = -\infty,$$

since the mean utility of a is lower than the aspiration level $\bar{u}^2$. Analogously, if b is chosen for an infinite number of periods,

$$\lim_{t \to \infty} U_t^2(0) = -\infty$$

obtains, since $u(1+r) < \bar{u}^2$ by assumption. The case-based decision rule implies that on almost each path of dividend realizations both portfolios will be held infinitely often by the investors of type 2.

Now consider the difference $U_t^2(1) - U_t^2(0) = (1-s')\,\varepsilon_t$. It can be shown that ε_t remains bounded above on almost each path of dividend realizations. At times at which $a = 1$ is chosen ε_t never falls below

$$u\left(\frac{\theta_1 a_0 + \underline{\delta}}{1-\theta_1(1-a_0)}\right) - \bar{u}^2,$$

since this would contradict choosing the act with the highest cumulative utility in each period. Suppose, therefore that there is a sequence of periods t', t''..., such that $\varepsilon_{t'}$, $\varepsilon_{t''}$... grows to infinity. In other words, suppose that for each $N > 0$ there is a k such that $\varepsilon_{t^n} > N$ for all $n > k$. Since $U_t(1)$ has negative expected increments, it follows (as shown above) that $a = 0$ is chosen infinitely many times on almost each path of dividend realizations. But each time that $a = 0$ is chosen, the difference ε_t falls below 0. If $\varepsilon_{t^n} > N$, the time needed to return to the origin is at least

$$\frac{N}{\bar{u}^2 - u\left(\frac{\theta_1 a_0 + \underline{\delta}}{1-\theta_1(1-a_0)}\right)},$$

which grows to infinity, as ε_{t^n} becomes very large. However, since the positive part of ε_t is a random walk on the half line, it follows from proposition 11.4.1 in Meyn and Tweedie (1996, p. 278) that the set G is regular for this process and, therefore, it is reached in finite expected time from each point in G. Moreover, the expected stopping time is uniformly bounded above by a number

$$\tilde{N} = \sup_{\varepsilon \in G} E_\varepsilon\left[\tau_G\right] < \infty.$$

The Law of Large Numbers then implies that for each $\kappa > 0$ on almost each path of dividend realizations, there is a period $\mathcal{K}$ such that

$$\frac{\sum_{i=1}^n \tau_{G_i}}{n} \le \tilde{N} + \kappa$$

for all $n \ge \mathcal{K}$. On the other hand, the assumption that $\varepsilon_{t^n} \to \infty$ implies that there is a time $\mathcal{K}'$ such that $\tau_{G_i} > \tilde{N} + \kappa$ for all $i \ge \mathcal{K}'$. It is therefore always possible to choose n large enough, so that

$$\frac{\sum_{i=1}^n \tau_{G_i}}{n} > \tilde{N} + \kappa,$$

a contradiction. Hence, almost each sequence $\varepsilon_{t'}$, $\varepsilon_{t''}$... (where t', t''... denote periods at which A is chosen) is bounded above and below.

Analogously, at times at which $a = 0$ is chosen, ε_t increases in each period of time by an amount

$$\bar{u}^2 - u\left(1 + r\right).$$

However, ε_t cannot exceed $\bar{u}^3 - u\left(1 + r\right)$, since this would again be in contradiction with the case-based decision rule. Hence, ε_t is bounded on almost all paths of dividend realizations.

But then it follows that

$$\lim_{t\to\infty} \frac{U_t^2\left(a\right)}{U_t^2\left(b\right)} = \lim_{t\to\infty} \frac{U_t^2\left(a\right)}{U_t^2\left(a\right) - \varepsilon_t} = 1 \tag{7}$$

holds with probability 1 in the limit. For a given set $\left[u\left(\frac{\theta_1 a_0 + \delta}{1-\theta_1\left(1-a_0\right)}\right) - \bar{u}^2; 0\right)$, the invariant probability π_l describes the mean time that the Markov chain defined by ε_t spends in this set between its visits to another set, $(0; +\infty]$, see theorem 10.4.9 in Meyn and Tweedie (1996, p. 253). Note that $a = 1$ is chosen in periods in which $\varepsilon_t \geq 0$ holds, whereas b is chosen in periods in which $\varepsilon_t < 0$ holds. Now define a function $\iota_h : \Psi' \to \{0; 1\}$ with

$$\iota_h\left(x\right) = \left\{ \begin{array}{ll} 1, \text{ if} & x \in [0; +\infty) \\ 0, \text{ if} & x \in \left[u\left(\frac{\theta_1 a_0 + \delta}{1-\theta_1\left(1-a_0\right)}\right) - \bar{u}^2; 0\right) \end{array} \right\}.$$

It is clear that $\iota_h \in L_1\left(\Psi'; \mathcal{B}\left(\Psi'\right); \pi\right)$, hence that ι_h has a finite expectation with respect to π on Ψ'. Moreover, it has been shown above that the process described by ε is positive Harris recurrent. Therefore, theorem 17.1.7 in Meyn and Tweedie (1996, p. 425) implies that

$$\lim_{t\to\infty} \frac{1}{t} \sum_{\tau=1}^{t} \iota_h\left(\varepsilon_\tau\right) = \int \iota_h d\pi$$

holds almost surely for any initial probability distribution. Note that $\frac{1}{t}\sum_{\tau=1}^{t} \iota_h\left(\varepsilon_\tau\right)$ represents the mean time that the system spends in state h. By the definition of $\iota_h\left(\varepsilon_\tau\right)$ it, therefore, follows that

$$\lim_{t\to\infty} \frac{1}{t} \sum_{\tau=1}^{t} \iota_h\left(\varepsilon_\tau\right) = \pi_h$$

almost surely. Hence, the frequency with which the investors of type 2 choose $a = 1$ in the limit equals π_h on almost all paths of dividend realizations. It follows that

$$\lim_{t\to\infty} \frac{\left|C_t\left(1\right)\right|}{\left|C_t\left(0\right)\right|} = \frac{\pi_h}{\pi_l}$$

holds with probability 1 as well. Substituting this into (7) implies:

$$\begin{aligned} \lim_{t\to\infty} \frac{U_t^2\left(1\right)}{U_t^2\left(0\right)} &= \lim_{t\to\infty} \frac{\left|C_t\left(1\right)\right| \left(\sum_{\tau\in C_t(a)} \frac{v_\tau(a)}{\left|C_t(a)\right|} - \bar{u}^3\right) + s' \left|C_t\left(0\right)\right| \left(u\left(1+r\right) - \bar{u}^3\right)}{\left|C_t\left(0\right)\right| \left(u\left(1+r\right) - \bar{u}^3\right) + s' \left|C_t\left(1\right)\right| \left(\sum_{\tau\in C_t(a)} \frac{v_\tau(a)}{\left|C_t(a)\right|} - \bar{u}^3\right)} = \\ &= \frac{\frac{\pi_h}{\pi_l} \lim_{t\to\infty} \left(\sum_{\tau\in C_t(a)} \frac{v_\tau(a)}{\left|C_t(a)\right|} - \bar{u}^3\right) + s' \left(u\left(1+r\right) - \bar{u}^3\right)}{\left(u\left(1+r\right) - \bar{u}^3\right) + \frac{\pi_h}{\pi_l} s' \lim_{t\to\infty} \left(\sum_{\tau\in C_t(a)} \frac{v_\tau(a)}{\left|C_t(a)\right|} - \bar{u}^3\right)} = 1 \end{aligned}$$

almost surely. It follows that the mean utility of $a = 1$, as observed by the investors of type 3 converges with probability 1 to a number μ_a^r, with

$$\mu_1^r = \lim_{t\to\infty} \sum_{\tau\in C_t(a)} \frac{u_\tau(a)}{|C_t(a)|}.$$

Hence,

$$\frac{\pi_h}{\pi_l} = \frac{\bar{u}^2 - u(1+r)}{\bar{u}^2 - \mu_1^r}.\blacksquare$$

References

Blonski, M. (1999). ”Social Learning with Case-Based Decision”, Journal of Economic Behavior and Organization 38: 59-77. Buschena, D. E., Zilberman, D. (1995). ”Performance of the Similarity Hypothesis Relative to Existing Models of Risky Choice.” Journal of Risk and Uncertainty 11: 233-263.

Buschena, D. E., Zilberman, D. (1999). ”Testing the Effects of Similarity on Risky Choice: Implications for Violations of Expected Utility.” Theory and Decision 46: 253-280.

De Long, J. B., Shleifer, A., Summers, L. H.,Waldmann, R. J. (1990). ”Noise Trader Risk in Financial Markets”, Journal of Political Economy 98: 703-738.

Gayer, G. (2003). ”Perception of Probabilities in Situations of Risk A Case Based Approach”, Tel Aviv University, mimeo.

Gilboa, I., Pazgal, A. (2001). ”Cumulative Discrete Choice”, Marketing Letters12: 118-130.

Gilboa, I. and Schmeidler, D. (1995). ”Case-Based Decision Theory”, Quarterly Journal of Economics 110: 605-639.

Gilboa, I. and Schmeidler, D. (1996). ”Case-Based Optimization”, Games and Economic Behavior 15: 1-26.

Gilboa, I., Schmeidler, D. (1997 a). ”Cumulative Utility Consumer Theory”, International Economic Review 38: 737-761.

Gilboa, I., Schmeidler, D. (1997 b). ”Act Similarity in Case-Based Decision Theory”, Economic Theory 9: 47-61.

Gilboa, I., Schmeidler, D. (2001 a). A Theory of Case-Based Decisions, Cambridge University Press, Cambridge.

Gilboa, I., Schmeidler, D. (2001 b). ”Reaction to Price Changes and Aspiration Level Adjustments”, Review of Economic Design 6: 215-223.

Guerdjikova, A. (2001). ”Case-Based Decision Theory — An Application for the Financial Markets”, University of Heidelberg, mimeo.

Guerdjikova, A. (2003 a). ”On the Definition and Existence of an Equilibrium in a Financial

Market Populated by Case-Based Decision Makers”, mimeo, University of Heidelberg.

Guerdjikova, A. (2003 b). ”Asset Pricing in an Overlapping Generations Model with Case-Based Decision Makers”, mimeo, University of Heidelberg.

Kajii, A. (1996). ”Rubinstein’s Similarity Consistent preferences — A Complete Characterization”, Discussion Paper 316, Australian National University.

Matsui, A. (2000). ”Expected Utility and Case-Based Reasoning”, Mathematical Social Science 39: 1-12.

Meyn, S. P., Tweedie, R. L. (1996). Markov Chains and Stochastic Stability, Springer Verlag, Berlin, Heidelberg.

Nehring, K., Puppe, C. (2003). ”Diversity and Dissimilarity in Lines and Hierarchies”, Mathematical Social Sciences 45: 167-183.

Rubinstein, A. (1988). ”Similarity and Decision-Making Under Risk”, Journal of Economic Theory, 46: 145-153.

Rubinstein, A. (2003). ”Economics and Psychology? The Case of Hyperbolic Discounting”, International Economic Review 44: 1207-1216.

Quine, W. V. (1969). ”Natural Kinds”, in: Essays in Honor of Carl G. Hempel, eds. Rescher, N., Reidel, D., Publishing Company Dodrecht, Boston.

Tversky, A. (1977). ”Features of Similarity”, Psychological Review 84: 327-352.