Reference-dependent Preferences: Rationality, Mechanism and Welfare Implications

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Abstract

In this paper, we ask three questions about reference-dependent preferences (RDP): to what extent can they be said to be irrational? What is the mechanism that underlies reference dependence? How to design welfare improving policies when preferences are reference-dependent? As to the first question, we characterize three notions of rationality to assess the rationality of RDP and show that there is a sense in which they are rational. As to the second, we show that the effect of a shifting reference point is two-sided: first modifying the relevant criteria for choice and second modifying the desirability of an option. As to the third question we define a welfare ordering based on the comparison on the status quo strength and show how to relate it to the representation of preferences.

Keywords: Reference-dependent preferences, status quo, rationality, welfare ordering, reference point shifting, implicit criteria, partial orders.

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1 Introduction and Overview of the Results

It is now a well-documented fact that people's preferences depend on their reference point. This has been repeatedly shown in experiments (Camerer [1], Samuelson and Zeckhauser [19], Kahneman, Knetsch and Thaler [10]). Samuelson and Zeckhauser [19] have specifically identified what they have called the status quo bias, namely the tendency to prefer remaining in a given state only because it is the initial state.

Compared to the considerable empirical literature on the phenomenon of reference-dependence, there are relatively few theoretical works that study it axiomatically[1]. This is probably due to the fact that reference-dependence has a flavor of irrationality that is at odds with the normative nature of most axiomatic approaches. Besides the seminal paper by Kahneman and Tversky [23], the main recent works include Sugden [21], Munro and Sugden [15], Masatlioglu and Ok [13] and Sagi [18]. All these papers, except Masatlioglu and Ok [13], start, like Kahneman et Tversky, with a family of preference relation $\{\succsim_r\}_{r\in X}$ defined over a set X (consisting of commodity bundles for Kahneman and Tversky [23] and Munro and Sugden [15], of Savage acts for Sugden [21] and of lotteries for Sagi [18]), and derive axiomatically a representation of these preference relation. Masatlioglu and Ok [13] take a more foundational approach by considering choice functions, and axiomatically characterizing the existence of a status quo bias in this context.

In light of these contributions, we shall adopt in this paper the approach of representing the fact that an agent observable preferences are reference-dependent by assuming that his or her behavior can be represented by a family of preference relations $\{\succsim_r\}_{r\in\mathcal{R}}$ defined over a subset X_r of some set X. This shall be called a family of reference-dependent preferences on X, RDP for short. An element r of the index set $\mathcal{R}$ shall be said to be a reference point. The general interpretation of such a reference point is that it indexes the context, in the broad sense, in which the decision is made. Now this context can be described in different ways. The simplest case is when $\mathcal{R} \subseteq X$ and each r stands for the reference consumption, be it the initial endowment or the consumption to which the individual is used or the one he or she aspires to. Other cases include all types of circumstances that may define the context of decision: time, state of nature, state of mind or a subset of the set of alternatives. As can be seen from the latter case it is quite natural to consider that the reference point determines the set of alternatives. For instance, in an exchange economy, the initial endowment determines the set of feasible commodity bundles; similarly, the set of feasible commodity bundles does not necessarily remain the same

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[1] We do not consider here the literature on sign dependence, which does not concern reference-dependence *per se*, but in a sense uses it to explain other phenomena

throughout time.

We propose in this paper to answer three questions about reference-dependent preferences:

1. Is there a suitable sense in which reference-dependent preferences can be said to be rational or rationalizable?
2. What kind of decision process is at the root of reference-dependence?
3. How to deduce from observable reference-dependent preferences a preference relation representing the welfare of the individual in a reference-independent way?

1.1 The Rationality of Reference-Dependent Preferences

The first question is motivated by the already alluded to concern that reference-dependent preferences correspond to an irrational behavior. This concern can sound compelling if one makes the philosophical and methodological choice of behaviorism, that is of regarding as the ultimate primitive the agent's observable behavior. At this level, reference-dependent preferences may seem irrational because they seem to be inconsistent. However, the behavior observed through these preferences may be regarded as resulting from the implementation, in a concrete decision situation, of abstract principles, determined *prior* to the specification of the reference point, which can be seen as a constraint on the achievement of the aims posited by these principles. In moral philosophy, such principles are called the *maxims* of behavior. They are the rules of behavior from the application of which stems the observed behavior. Building on this notion, it is possible to define two concepts of rationality. A decision maker shall be said:

- *behaviorally rational*, if his or her observed behavior in every concrete decision situation is consistent,
- *teleological rationality*, if his or her observed behavior results from the application, in a concrete situation, of non-contradictory maxims.

Now, one can wonder whether it is possible to say that an agent exhibiting reference-dependent observable preferences can be said to be rational in one of these senses. As an answer to this question, we show in the paper that *a decision maker with reference-dependent preferences is behaviorally rational if and only if he or she is teleologically rational.* Specifically, we show that, for a family of reference-dependent preference relations $\{\succsim_r\}_{r\in\mathcal{R}}$ defined over a set X, each preference relation in this family is reflexive and transitive (hence behaviorally rational) if and only if there exists a reflexive and transitive (hence non-contradictory) preference relation $\succsim$ defined over the set

$$\Theta := \{(x, r) \in X \times \mathcal{R} \mid x \in X_r\},$$

such that: $\forall r \in \mathcal{R}, \forall x, y \in X_r$,

$$x \succsim_r y \Leftrightarrow (x, r) \succsim (y, r),$$

so that $\succsim$ can be seen as representing the maxims of behavior — hence teleological rationality. In addition, each $\succsim_r$ is complete if and only if there exists a complete $\succsim$ over Θ satisfying the previous property.

Now this relation $\succsim$ bearing on objects of the form (x, r) may seem difficult to interpret, and even to accept, given its partially unobservable character and because it seems to demand a lot from the decision maker's ability to express preferences. A general line of interpretation is to consider this relation as representing *ex ante* preferences, that is the plans, contingent on the realization of the state represented by the reference point, formed by the decision-maker under a veil of ignorance as to the circumstances — the reference point — with which he or she will gave to cope[2]. In this interpretation, an object (x, r) may be regarded as the object x received by the agent when the context is the one indexed by r, where the notion of context has been discussed above. So the general interpretation of the statement "the pair (x, r) is preferred to the pair (x', r')" is that it is the answer to the question: "Would you rather receive object x when you are in state r or object x' when you are in state r'?" This rather abstract question can take a very concrete form: "Would you rather receive \$100 while being rich or \$1000 while being poor?". Specifically, in the case where the reference point is the initial endowment, it would amount to answer for instance the following question: "Would you rather play a gamble that yields - 100 and 1000 with probability 1/2 when you are endowed with 10 or a gamble that yields - 10 and 100 with probability 1/2 while you are endowed with 100?" or "Would you rather eat fish after having eaten eggs or meat after having eaten a soup?". More generally: "Would you rather be in state x when you were in state r or in state x' when you were in state r'?". Other possible translations include the statements "Would you rather receive an apple today or two tomorrow?", " Would you rather go to the movies when you are happy or go the theater when you are not?" or "Would you rather watch a drama (x) on television (r) or a blockbuster (x') at the movies (r')"?

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[2] We thank Jacob Sagi for suggesting this "rawlsian" interpretation.

1.2 The Mechanism underlying Reference-Dependent Preferences

As far as the second question is concerned, our answer is contained in the preference functional that we axiomatize. As one of the main result of this paper, we show that, under certain conditions, for each reference point there exist:

- a family $\mathcal{V}_r$ of utility functions or criteria,
- some function a_r defined over X and with values in $[0, 1]$,

such that each preference profile $\{\succsim_r\}_{r\in\mathcal{R}}$ can be represented by functions $\{u_r\}_{r\in\mathcal{R}}$ defined over X_r in the following way:

$$u_r(x) = a_r(x) \inf_{v\in\mathcal{V}_r} v(x) + (1 - a_r(x)) \sup_{v\in\mathcal{V}_r} v(x).$$

It can be shown that this representation means, roughly speaking, that a given reference point determines both a list of relevant criteria and, for each pair of objects x and y, the (measure-theoretic) size of the set of decisive criteria, that is the criteria such that, if x is better than y for all of these criteria, then $x \succ_r y$. The function a_r is responsible for the second aspect of the effect of r.

An intuition of the mechanism that underlies this functional can be grasped by examining the case where $\mathcal{R} \subseteq X$ and $y = r$, i.e. the initial endowment. In this case, the reference point determines both the set of relevant criteria and the number of criteria among these for which a given alternative x must beat the endowment for the decision-maker to forego it. Hence, if $a_r(x) > a_r(y)$, then it will be more likely that y is preferred to r when r is the status-quo than that x be preferred to r. The status quo bias associated to a reference point r will be the stronger the larger the number of decisive criteria necessary to forego it (i.e. the bigger $a_r(x)$ is for all x).

Consider the following example: you wish to move from your accommodation A. Suppose that, in general, the relevant criteria for the choice of an accommodation are: location in town, proximity of stores, elevator, ancient or modern style, view, number of rooms and price. Suppose accommodation A does not have an elevator, and that this has got you used to climbing the stairs. We can thus assume that this criterion is of no importance to you, that it can be deleted from the set of criteria. Notice that this is possible only because you are endowed with an accommodation without an elevator. This might no longer be true if the initial endowment were different. Suppose now that on the whole you feel comfortable in your present accommodation, and that you wish to move only for some change in your life. In this case, you will probably be more demanding on the conditions that must be met by the potential alternatives. If, on the contrary, you are poorly satisfied with your accommodation, for other reasons than the absence of elevator, for instance noise on the street you live on, you will probably be far less demanding with respect to the features of the possible alternatives.

The model that we propose has the advantage of allowing for a more flexible analysis of reference-dependent decision-making than Sagi [18]'s model, which is formally the model closest to ours. Indeed, this model corresponds, in Sagi's specification, to the case where $a_r(x) = 1$ for all x, that is, roughly speaking, to a uniformly maximal status quo bias[3].

In order to derive the functional introduced above, we stick to the methodology of Giraud [8], which is inspired by Ghirardato, Maccheroni, Marinacci [7], and apply it to convex spaces. This richer structural setting allows greater precision in the definition of the set $\mathcal{V}$ found in Giraud [8], and deeper results.

1.3 Status Quo Bias and Welfare Measurement

As far as the third and last question is concerned, it arises in the particular case where $\mathcal{R} \subseteq X$. Indeed, if the preferences of an individual depend on his or her endowment, then it becomes difficult to assess his or her welfare. This is in particular problematic for a social planner who is to allocate wealth in a welfare-improving way: if for instance, given two possible endowments ω and ω' such that $\omega' \succ_\omega \omega$ and $\omega \succ_{\omega'} \omega'$, then it is impossible for the social planner to decide what to give to the individual. Specifically, we want to address the problem of deducing from reference-dependent preferences a meaningful reference-independent preference relation over X. By meaningful we mean a preference relation that has a natural interpretation in terms of welfare.

One way of doing this is to say that an object x welfare-dominates an object y if it is preferred to y when y is the reference point ($x \succsim_y y$). This definition, which appears in Sagi [18], Munro and Sugden [15] and Matloglu and Ok [13] as the most natural way of defining a reference-independent preference, has the drawback of yielding a relation — to be denoted $\succsim_G$ — that is not transitive in general but only under very special conditions and which, under this conditions, is very partial. We propose to replace this welfare ordering by another which is consistent with it and which, in a case where $\succsim_G$ is transitive, extends it. The idea is to compare two options when they are the reference point, and to compare the strength of the status quo bias in each case. Recall that the status quo bias (SQB) corresponds

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[3] However, this implication does not seem to be formally provable in our specification, as there seems to exist a discontinuity in behavior as our model "converges" to Sagi's.

to the reluctance to forego the reference point, that is to exchange r for x when r is the reference point. In other words, the existence of a SQB means that the situation $x \succ_r$ is less frequent than the situation $r \succsim_r x$. When r is the initial endowment, the SQB corresponds to Samuelson-Zeckauser [19]'s *endowment effect*. When r is the endowment resulting from a previous decision, the SQB corresponds to the *confirmatory bias*, that is the tendency to ignore the information that suggests that the first period decision was a bad decision, and thus to stick to that decision, only because it has already been made once (see for instance Rabin and Schrag [17] or Yariv [25]).

In our approach, therefore, the option for which the SQB is the strongest, *i.e.* the one the decision maker is more reluctant to forego, will be assumed to the one that provides him with the greatest welfare. This kind of approach is absent from the studies mentioned in our brief survey. Indeed, the possibility for the strength of the SQB to be variable is not mentioned in general, except by Munro and Sugden [15]. It is always maximal in Sagi [18] and Masatlioglu and Ok [13]. The nature of the reference point does not seem to play any rŹle in the decision process. A symptom of this fact is that, in these models, the utility value of the status quo can always be normalized to 0. This may be considered a limitation of these models, as it is perfectly possible to imagine that the strength of the SQB depends on the intrinsic value attached to the status quo and the greater the latter the stronger the former. However, when Masatlioglu and Ok move from the theory of SQB to the task of modeling the endowment effect — that they distinguish from the former by insisting on the fact that the latter bears essentially on the *value* attributed to the status quo —, the possibility of distinguishing different status quo from one another appear in the fact that, in this model, an object x's utility is $u(x)$ if it is not owned and $u(x) + \varphi(x)$ if it is. This sheds light on the double nature of the endowment's value. As for Munro and Sugden [15], they mention the variable strength of the status quo, but this does not lead them to our concept of welfare ordering.

1.4 Outline of the Paper

The structure of the paper closely follows the questions we asked at the outset. Section 2 deals with the issue of rationalizability of reference-dependent preferences. Section 3 axiomatizes the preference functional introduced above. Section 4 discusses the issue of welfare measurement. Proofs appear in section 5. In fact, most of the results in section 3 are applications of the theory of Decision Problems with Normative Equivalence (see Giraud [8]) in the case of convex spaces of a normed vector space.

2 On the Rationalization of Observable Preferences: Ends and Means

Let X and $\mathcal{R}$ be two sets of objects. The decision-maker is assumed to have reference-dependent preferences (RDP) over the set X, the set $\mathcal{R}$ being the set of reference points. To model this fact, we shall introduce a family $\mathcal{P} = \{\succsim_r\}_{r \in \mathcal{R}}$ of binary relations defined over $X_r \subseteq X$. This family defines, for each reference point r, the associated preference relation (and the set on which it is defined), and shall be called a *preference profile.*

The relations in the preference profile need not coincide *a priori.* They may therefore be regarded as modeling inconsistent, or even irrational, behavior. However, as already noted in the introduction, this judgment of irrationality may be deemed accurate if and only if one adopts a strictly behaviorist position, meaning by that the idea that only observable behavior matters in assessing the rationality thereof. But it is possible to address this issue from another angle. In Kant's moral philosophy, the only way of assessing the morality of some behavior is to study the abstract principle from which it results, *i.e.* the rule of behavior the application of which leads to the this particular behavior. Kant calls it the *maxim* of the action. By analogy, one can look for the rationality of the behavior observed through these preferences by inferring the rule of behavior that lies behind it and by studying the rationality of this rule. may be regarded as the implementation of determinate means to achieve determinate ends. From this principle, one can define two notions of rationality:

Definition 1 (Informal Definition of Rationality). *A decision maker will be said to be:*

- behaviorally rational *if his or her observable behavior is consistent,*
- teleologically rational, *if his or her observable behavior results from the application of non contradictory maxims.*

Now, one can wonder whether it is possible to say that an agent exhibiting reference-dependent observable preferences can be said to be rational in one of these senses. The remaining part of this section is devoted to answering this question.

Notice first that, from preference profile $\mathcal{P}$, one can induce a relation on $\mathcal{R}$: given $r, r' \in \mathcal{R}$, define $r \curvearrowright r'$ by:

$$X_{r'} \subseteq X_r \text{ and } \forall x, x' \in X_{r'}, [x' \succsim_{r'} x \Rightarrow x' \succsim_r x].$$

It is easy to verify that the relation $\curvearrowright$ is reflexive and transitive, hence a preorder. To interpret it, suppose that for all $r \in \mathcal{R}$, $\succsim_r$ is complete. Then,

$$r \curvearrowright r' \Leftrightarrow X_{r'} \subseteq X_r \text{ and } \forall x, x' \in X_{r'}, [x \succ_r x' \Rightarrow x \succ_{r'} x'],$$

where $\succ_r$ denotes strict preference. Thus, whenever $r \curvearrowright r'$, a strict preference of x over x' when viewed from reference point r imposes a strict preference of x over x' when viewed from the reference point r'. For this reason, and because during most of this paper the relations $\succsim_r$ will be assumed to be complete, we shall say that r *directs* r'. Notice that this relation should not be interpreted as indicating, in some sense, that r is preferred to r'. It only represents the internal structure of the preference profile, *i.e.* the relationship between its elements. In particular, let $\circlearrowright$ be the symmetric part of this relation. Clearly:

$$r \circlearrowright r' \Leftrightarrow \succsim_r = \succsim_{r'} \text{ and } X_r = X_{r'}.$$

Hence, the equivalence classes of this relation gather the reference points or contexts having the same effect on the DM's preferences.

The preference profile $\mathcal{P}$ may be seen as modeling the DM's observable behavior, in the sense that it specifies what he or she does given a reference point. Now, teleological rationality stipulates that a behavior is rational if it stems from the application of non contradictory abstract principles. This shall lead us to ask the question of the rationality of $\mathcal{P}$ in this sense. But answering this question requires a more formal definition of teleological rationality.

Definition 2 (Teleological Rationality). *Let $\mathcal{P}$ be a preference profile and let*

$$\Theta := \{(x, r) \in X \times \mathcal{R} \mid x \in X_r\}.$$

Let R be a binary relation defined over Θ. We shall say that R rationalizes *$\mathcal{P}$ if:*

(i) $\forall r \in \mathcal{R}$, $\forall x, y \in X_r$,

$$x \succsim_r y \Leftrightarrow (x, r) R (y, r),$$

(ii) $\forall r, r' \in \mathcal{R}$,

$$r \curvearrowright r' \Rightarrow \forall x \in X_r \cap X_{r'}, (x, r') R (x, r).$$

A preference profile $\mathcal{P}$ is rationalizable *if there exists a relation R over Θ that rationalizes $\mathcal{P}$.*

A preference profile $\mathcal{P}$ is teleologically rational *if it is rationalizable by a preorder.*

It is easy to see why condition (i) corresponds to the intuition according to which the behavior represented by $\mathcal{P}$ results in the application of an abstract rule of behavior, represented by R, to a concrete decision situation: as noted in the introduction, the interpretation of R is that it represents the plans formed by the DM prior to the determination of the reference point, under a veil of ignorance as to the circumstances in which he or she will have to implement his or her plans.Once these circumstances are determined, they act as a constraint on the realization of these plans that the DM ought to take into account. This is the meaning of condition (i): the DM's behavior is completely determined by the relation R once the reference point is selected.

This condition is not, however, is not strong enough to determine R. There exists indeed a host of binary relations satisfying condition (i). It suffices to consider any reflexive relation $S_{\mathcal{R}}$ over $\mathcal{R}$ and to define the relation $R(S_{\mathcal{R}})$ by letting:

$$(x, r)R(S_{\mathcal{R}})(x', r') \Leftrightarrow X_r \subseteq X_{r'} \text{ and } x \succsim_{r'} x' \text{ and } r' S_{\mathcal{R}} r.$$

$R(S_{\mathcal{R}})$ is readily seen to satisfy (i) by construction. Hence condition (ii) allows to select in the set of binary relations satisfying condition (i) relations reflecting more closely the internal structure of profile $\mathcal{P}$. Given a relation R over Θ, let R^U be the relation defined over $\mathcal{R}$ by

$$rR^U r' \Leftrightarrow \forall x \in X_r \cap X_{r'}, (x, r)R(x, r').$$

Let I^U be the symmetric part of this relation. Then condition (ii) means that there are less equivalence classes for I^U than for $\circlearrowright$. In other words, the equivalence classes of I^U respect the equivalence classes of $\circlearrowright$, so that if $\succsim_r = \succsim_{r'}$, then $rI^U r'$. In this sense can it be said that R respects or reflects the structure of profile $\mathcal{P}$. We shall see that this condition strongly determines R.

Consider now, given a preference profile $\mathcal{P}$, the relation $\succsim_{\mathcal{P}}$ defined over Θ by:

$$(x, r) \succsim_{\mathcal{P}} (x', r') \Leftrightarrow [x \succsim_{r'} x' \text{ and } r' \curvearrowright r].$$

Notice that this relation is well-defined, as $x \in X_r$ and $r' \curvearrowright r$ together imply $x \in X_{r'}$. It is easy to see that $\succsim_{\mathcal{P}}$ is nothing but $R(\curvearrowright)$. Call this relation the *rational closure* of $\mathcal{P}$. This term can be justified by the following theorem:

Theorem 1

Let $\mathcal{P} = \{\succsim_r\}_{r \in \mathcal{R}}$ be a preference profile. The following statements are equivalent:

(i) For all $r \in \mathcal{R}, \succsim_r$ is reflexive and transitive;

(ii) $\succsim_{\mathcal{P}}$ *is the smallest pre-order that rationalizes* $\mathcal{P}$.

This theorem thus shows that the fact that each relation in the preference profile $\mathcal{P}$ is a pre-order is a necessary and sufficient condition for this preference profile to be rationalizable by a pre-order, *i.e.* to satisfy teleological rationality. A quick inspection of the proof will tell two things: first, that in fact a weaker result also holds, namely that the reflexivity of each relation in $\mathcal{P}$ is a necessary and sufficient condition for $\mathcal{P}$ to be rationalizable by $\succsim_{\mathcal{P}}$, which in this case is only reflexive but no longer transitive, and is included in every transitive relation that rationalizes $\mathcal{P}$; second, that, as $\succsim_{\mathcal{P}}= R(\curvearrowright)$, the equivalence classes for the relation $\sim^U_{\mathcal{P}}= I(\curvearrowright)^U$ are exactly the same as the equivalence classes for $\circlearrowright$. The relation $\succsim_{\mathcal{P}}$ is thus optimal in the sense that it reflects exactly the structure of $\mathcal{P}$.

As far as the first question we asked is concerned, if one defines the non-contradictions of the principles or maxims by the transitivity of the relation that rationalizes $\mathcal{P}$, and consistency of the observable behavior by the transitivity of each preference relation in $\mathcal{P}$, then theorem 1 provides the following answer: *a decision maker with reference-dependent preferences is behaviorally rational if and only if he or she is teleologically rational.* In this sense, usual and simple requirements on the behavior of the decision maker *given a reference point* completely characterize the fact that he or she is rational in a time-honored sense.

Among the classical assumptions on preference relations, we have not, until now, dealt with the completeness axiom. We do this in the following corollary:

Corollary 1. *Let* $\mathcal{P} = \{\succsim_r\}_{r\in\mathcal{R}}$ *be a preference profile. The following statements are equivalent:*

(i) For all $r \in \mathcal{R}, \succsim_r$ *is a weak order;*

(ii) There exists a weak order $\succsim$ *that rationalizes* $\mathcal{P}$.

Moreover, if $\succsim'$ *is another weak order that rationalizes* $\mathcal{P}$*, then*

$$\succsim_{\mathcal{P}}\subseteq (\succsim \cap \succsim').$$

This corollary shows that the fact that each $\succsim_r$ is a weak order is a necessary and sufficient condition for the preference profile to be rationalizable by a weak order. Thus, if one adds completeness to both definitions of rationality, the equivalence between notions of rationality still holds. However, a weak order that rationalizes a given preference profile is not unique nor canonically defined, in contrast with what happens with rationalization by a pre-order, and an inspection of the proof of this corollary further reveals that it cannot be completely elicited from the preference profile. Part of it remains hidden in the depths of the decision maker's mind, so to speak. Therefore, any assumption bearing directly on this weak order is necessarily an assumption that goes beyond the individual's observable behavior. We tried to show in the introduction why this is legitimate.

We recall from what was said in the introduction that we may interpret the relation $\succsim$ as representing what the decision-maker *intends* to do when a reference point is selected, *before* knowing which reference point has been selected. Therefore, it may be regarded as modeling his or her hedonic or contemplative preferences which express his or her desires, or norms. We shall hereafter consider that these desires or norms are the true primitives of our theory. We shall thus assume that he or she knows how to rank contingent objects, that is objects of the form (x, r), where r indexes a context and x is an object of choice. The idea is to say that what really matters are the maxims of the decision maker, not their concrete implementation, as long as one can infer them from observable preferences.

3 The Model

3.1 Primitives of the model

From now on, we shall assume that X and $\mathcal{R}$ are convex subsets of the vector spaces E_X and $E_{\mathcal{R}}$. One may think of X as a set of lotteries over a prize space Z, or of Savagian acts with values in $\mathbb{R}^n$, for instance a set of commodity bundles. As to $\mathcal{R}$, it may be a convex subset of X, or a real interval denoting continuous time or a connected set of states of nature, for instance levels of wealth.

As to preferences, in line with what we said in the previous section, we shall make the following assumption:

Assumption 1 (Rationality) *(i) There exists a binary relation $\succsim$ defined over a convex subset Θ of $X \times \mathcal{R}$ representing the hedonic preferences of the DM;*

(ii) This relation rationalizes its associated observable preference profile $\mathcal{P}(\succsim) = \{\succsim_r\}_{r \in \mathcal{R}}$ defined by:

$$X_r := \{x \in X \mid (x, r) \in \Theta\}$$

and, for all $x, y \in X_r$,

$$x \succsim_r y \Leftrightarrow (x, r) \succsim (y, r).$$

(iii) For all $x \in X$, the set

$$\mathcal{R}_x := \{r \in \mathcal{R} \mid x \in X_r\}$$

has at least two elements and, if $\mathcal{R} \subseteq X$ then, for all $r \in \mathcal{R}$, $r \in X_r$.

We recall that point (ii) means that

$$r \curvearrowright r' \Rightarrow (x, r') \succsim (x, r)$$

for all $x \in X_r \cap X_{r'}$. Point (iii) is a non-triviality condition.

Given $\theta \in \Theta$, we shall denote x_θ and r_θ the projections of θ on X and $\mathcal{R}$ respectively.

Corollary 1 leads to assume the following axiom

AXIOM 1 (WEAK ORDER)

(i) The relation $\succsim$ is a weak order;

(ii) the relation $\succsim$ is non-trivial: there exists $\theta_1, \theta_2 \in \Theta, \theta_1 \succ \theta_2$.

According to corollary 1, we know that part (i) of this axiom is equivalent to the statement that all observable preferences of the decision maker, elements of his or her preference profile $\{\succsim_r\}_{r \in \mathcal{R}}$, are weak orders. It is thus not very restrictive with respect to usual requirements in decision theory. It is a classical axiom. However, here the completeness assumption may seem stronger than usual because $X \times \mathcal{R}$ may seem a conceptually larger set than X. But, on the one hand, the relation $\succsim$ is defined over a *subset* of $X \times \mathcal{R}$, weakening the cognitive demands of this axiom, and, on the other hand we have seen that requiring completeness of $\succsim$ is tantamount to requiring it of each relation $\succsim_r$. Thus there is no extra complexity cost involved in this assumption.

Notice, moreover, that endowed with the binary relation $\succsim$ satisfying axiom 1, Θ possess a structure of Decision Problem with Normative Equivalence (Giraud [8]) when the normative equivalence relation $\approx$ is defined by:

$$\theta \approx \theta' \Leftrightarrow x_\theta = x_{\theta'}.$$

The equivalence class of $\theta \in \Theta$ is the set:

$$[\theta] = \{x_\theta\} \times \mathcal{R}_{x_\theta}.$$

3.2 Existence of a Value Function

We shall need the following topological assumption:

ASSUMPTION 2 (TOPOLOGICAL ASSUMPTION)

(i) E_X and $E_\mathcal{R}$ are normed vector spaces;

(ii) Θ *is a compact subset of* $E := E_X \times E_{\mathcal{R}}$.

This assumption, which will be always implicitly made from now on, allows us to posit the following axiom:

AXIOM 2 (CONTINUITY) *For all* $\theta \in \Theta$, *the sets* $\{\theta' \in \Theta \mid \theta' \succsim \theta\}$ *and* $\{\theta' \in \Theta \mid \theta \succsim \theta'\}$ *are closed in* Θ.

This axiom is classical. It implies in particular the continuity of each weak order $\succsim_r$. Because Θ is a compact, hence separable, metric space, we know that the previous axioms imply that there exists a continuous (hence bounded) function representing the relation $\succsim$ over Θ :

Lemma 1 (Utility Function (Debreu [3])). *The relation* $\succsim$ *satisfies Weak Order and Continuity iff there exists a continuous function* $u : \Theta \to \mathbb{R}$ *such that:*

$$\forall \theta, \theta' \in \Theta, \theta \succsim \theta' \Leftrightarrow u(\theta) \geq u(\theta').$$

A similar result holds for each relation $\succsim_r \in \mathcal{P}(\succsim)$:

Lemma 2 (Utility Functions for the Preference Profile). *If* $\succsim$ *satisfies Weak Order and Continuity, then for each* $r \in \mathcal{R}$, *there exists a continuous function* $u_r : X_r \to \mathbb{R}$ *such that:*

$$\forall x, y \in X_r, x \succsim_r y \Leftrightarrow u_r(x) \geq u_r(y).$$

3.3 The Absolute Preference Relation

Generally speaking, the notion reference-dependent preference implies that observable preferences depend on the reference point, in the sense that there exists $r, r' \in \mathcal{R}, x, y \in X$ such that $x \succ_r y$ and $y \succ_{r'} x$. Now, if one considers that, from a normative point of view, preferences should not be reference-dependent (based on a money pump argument, for instance), it is interesting to wonder to what extent preferences diverge from this ideal situation. This leads us to consider the part of the hedonic preferences that is reference-independent, that is the part that describes the *ex ante* plans that do are not affected by the selection of the reference point:

Definition 3 (Absolute Preference). *Let* $\theta, \theta' \in \Theta$. *Say taht* θ *is* absolutely preferred to θ' *— denoted* $\theta \succsim^a \theta'$ *— if:*

$$\begin{cases} \tilde{\theta} \succsim \tilde{\theta}', \forall \tilde{\theta} \in [\theta], \tilde{\theta}' \in [\theta'] & \textit{if } \theta \neq \theta' \\ \theta = \theta' & \textit{otherwise.} \end{cases}$$

Denote $\succ^a$ and $\sim^a$ the asymmetric and symmetric part of this relation and $\|^a$ the non-comparability relation of this relation, which is partial when preferences are truly reference-dependent.

As explained in Giraud [8], whenever $\succsim$ is a pre-order, the relation $\succsim^a$ is also a pre-order and it is the maximal restriction of $\succsim$ satisfying the reference-independence property:
$\forall\theta, \theta' \in X^2, \theta \neq \theta', \forall r, r' \in \mathcal{R}_{x_\theta} \times \mathcal{R}_{x_{\theta'}}$,

$$\begin{cases} \theta \succ^a \theta' \Leftrightarrow (x_\theta, r) \succ^a (x_{\theta'}, r') \\ \theta \sim^a \theta' \Leftrightarrow (x_\theta, r) \sim^a (x_{\theta'}, r'). \end{cases}.$$

Any relation satisfying this property will be said to be $\mathcal{R}$-independent.

The definition of $\succsim^a$ implies that, for all $x, y \in X$ such that $x \neq y$, if there exists $r_1, r_2 \in X$ such that $(x, r_1) \succ^a (y, r_2)$, then : for all $r \in \mathcal{R}_x \cap \mathcal{R}_y$,

$$x \succsim_r y.$$

Remark 1. *In the definition of absolute preference, we impose the condition $(x, r) \neq (y, r')$. This condition is needed for the relation $\succsim^a$ to be a pre-order (hence a reflexive relation) without being equal to $\succsim$ (in which case the problem would be trivial or $\succsim^a$ would not be the right tool). This precludes deducing from relation $\succsim^a$ a relation R^a over the projection of θ onto X being a pre-order too by setting xR^ay if and only if $(x, r) \succsim^a (y, r')$. In fact, this relation would in general be ill-defined, as one could have $(x, r) \succ (x, r') \sim (x, r'')$, so that the relation would neither be reflexive nor irreflexive. However, with a slight abuse of language, in the sequel we may say "x is absolutely preferred to y", whenever $(x, r) \succsim^a (y, r')$ with $x \neq y$ for some r, r'.*

AXIOM 3 (ABSOLUTE PREFERENCE INDEPENDENCE (API)) $\forall x, y, z \in X, \forall\alpha \in [0, 1]$, *if*

$$\forall(r, r') \in \mathcal{R}_x \times \mathcal{R}_y, (x, r) \succsim (y, r'),$$

then

$$\forall(r, r') \in \mathcal{R}_{\alpha x+(1-\alpha)z} \times \mathcal{R}_{\alpha y+(1-\alpha)z}, (\alpha x + (1-\alpha)z, r) \succsim (\alpha y + (1-\alpha)z, r').$$

Imposing this axiom is tantamount to imposing that the absolute preference relation satisfies the classical independence axiom:

Lemma 3. *The relation $\succsim$ satisfies (API) iff the relation $\succsim^a$ satisfies the independence axiom: for all $\theta, \theta', \theta'' \in \Theta$, for all $\lambda \in [0,1]$,*

$$\theta \succsim^a \theta' \Rightarrow \lambda\theta + (1-\lambda)\theta'' \succsim^a \lambda\theta' + (1-\lambda)\theta''.$$

Although this axiom is now well known to be descriptively inadequate when applied to the primitive relation, it still has some normative appeal, for instance because it permits dynamic consistency (see Mongin [14]). Therefore, as long as the behavior modeled by the absolute preference relation enjoys a strong rationality property, as it represents the part of preferences that complies with the normative ideal of reference-independence, it seems legitimate to consider that it is a strong rationality kernel of preferences. This allows to impose to this relation to satisfy other normative properties, and in particular the independence axiom. Notice that axiom (API) bears on the options to evaluate, that is elements of X, and not on the reference point, which is immaterial when we consider the absolute preference.

When the preference relation $\succsim$ is $\mathcal{R}$-independent, $\succsim$ and $\succsim^a$ coincide. Therefore, in this case, if the absolute preference relation satisfies axiom 3, $\succsim$ satisfies independence. This means that in our model, if preferences do not satisfy the independence axiom, it must be because they are not $\mathcal{R}$-independent. In decision theory, the independence axiom is considered to be a property that preferences ought to satisfy if everything conformed to some ideal of rationality. Its failure is often associated to aspects related to decision under risk (when the perception of probabilities by the DM leads to a distortion of them) or under uncertainty (when the DM perceives ambiguity in the decision situation he or she faces). Our model reinforces the normative status of the independence axiom, by stipulating that it is a necessary condition for other normative properties to be satisfied, namely those implied by $\mathcal{R}$-independence: context-independence, no endowment effect, patience, etc. The independence axiom becomes the minimal axiom of rationality (beyond those implied by the maximization of utility). However, preferences can satisfiy the independence axiom without being $\mathcal{R}$-independent. Indeed, in this case $\succsim^a \subsetneq \succsim$ and $\succsim$ satisfies the independence axiom.

The continuity of a pre-order is an important condition to obtain a representation for it. I the case of the (generally incomplete) relation $\succsim^a$, it is necessary, in the abstract setting we are working in, to postulate a strong continuity condition. To state it, define the *domination cone* of $\succsim^a$, denoted $\mathcal{C}^a$, by:

$$\mathcal{C}^a := \{\lambda(\theta - \theta') \mid \lambda > 0, \theta \succsim^a \theta'\}.$$

The continuity assumption is the following:

Assumption 3 (Absolute Preference Continuity (APC)) *$\mathcal{C}^a$ is closed in E.*

This assumption yields a first representation for $\succsim^a$:

Proposition 1 (Shapley et Baucells [20], Dubra, Maccheroni et Ok [5])
Assume (CPA). Then the relation $\succsim$ satisfies Weak Order, Continuity and (API) iff $\mathcal{C}^a$ is a convex cone containing 0 that represents $\succsim^a$, i.e. for all $\theta, \theta' \in \Theta$,

$$\theta \succsim^a \theta' \Leftrightarrow \theta - \theta' \in \mathcal{C}^a.$$

Dubra, Maccheroni and Ok [5] showed that assumption (CPA) is satisfied when lorsque Θ is the set of Borel probability measures over a compact metric space Z. Let us give now an example of this situation:

Example 1. *Let Z be a finite set, of cardinality $n \in \mathbb{N}$. Assume that preferences depend on the probability that some prize $z_0 \in Z$ obtains (for instance, the probability to have a lethal disease). Denoting $r \in [0,1]$ this probability, the $\mathcal{R}$ can be identified with $[0,1]$. Given $r \in [0,1]$, define*

$$X_r := \{x \in [0,1]^{n-1} \mid \sum_{i=1}^{n-1} x_i = 1 - r\}.$$

X_r can be seen as the set of lotteries conditional on the fact that z_0 does not obtain (divide by $1-r$). Then, $\Theta = \{(x,r) \mid x \in X_r\}$ is isomorphic to $\Delta(Z)$, the set of lotteries on Z. In this case, $\mathcal{C}^a$ will always be closed when the axioms of the model are satisfied

3.4 A Representation Theorem for RDP

Wa shall now provide a representation theorem for RDP shedding light on the mechanism underlying them. Before stating it, let us give some definitions and notations.

Notation 1. *Given a convex subset C of a topological vector space espace vectoriel topologique, denote $\mathcal{A}(C)$ the set of continuous affine functionals $f : C \to \mathbb{R}$, i.e. for all $c, c' \in C, \alpha \in [0,1]$, $f(\alpha c + (1-\alpha)c') = \alpha f(c) + (1-\alpha) f(c')$. We denote E^* the set of linear functionals over E.*

Remark 2. *To each $f \in \mathcal{A}(\Theta)$ corresponds, for all r, an element of $\mathcal{A}(X_r)$. Indeed, let $f \in \mathcal{A}(\Theta)$. Then, for all $r \in \mathcal{R}$,$x, y \in X_r$,$\lambda \in [0, 1]$,*

$$\begin{aligned} f(\lambda x + (1-\lambda)y, r) &= f(\lambda x + (1-\lambda)y, \lambda r + (1-\lambda)r) \\ &= f(\lambda(x, r) + (1-\lambda)(y, r)) \\ &= \lambda f(x, r) + (1-\lambda) f(y, r). \end{aligned}$$

Similarly, for all x, the function $f(x, .)$ belongs to $\mathcal{A}(\mathcal{R}_x)$.

We then have the following representation theorem, which proof is based on Dubra, Maccheroni and Ok [5]:

REPRESENTATION THEOREM FOR RDP

Assume (CPA). Then the relation $\succsim$ satisfies Weak Order, Continuity and (API) iff there exists

- *a weak*-closed convex set $\mathcal{V} \subseteq \mathcal{A}(\Theta)$, with at least one non-constant functional and such that for all $\theta \in \Theta$,*

  $$-\infty < \inf_{v \in \mathcal{V}} v(\theta) \leq \sup_{v \in \mathcal{V}} v(\theta) < +\infty$$

  with strict inequality whenever $\succsim^a$ is different from $\succsim$;
- *a fonction $a : \Theta \to [0, 1]$ satisfying*

  $$\theta \sim^a \theta' \Rightarrow a(\theta) = a(\theta'),$$

such that:

(i) $\forall \theta, \theta' \in \Theta$,

$$\theta \succsim^a \theta' \Leftrightarrow \forall v \in \mathcal{V}, v(x, r) \geq v(y, r');$$

(ii) $\forall x, y \in X, x \neq y \Rightarrow \forall r, r' \in \mathcal{R}_x \cap \mathcal{R}_y$,

$$(\forall v \in \mathcal{V}, v(x, r) \geq v(y, r)) \Leftrightarrow (\forall v \in \mathcal{V}, v(x, r') \geq v(y, r'));$$

(iii) $\forall \theta \in \Theta$,

$$u(\theta) = a(\theta) \inf_{v \in \mathcal{V}} v(\theta) + (1 - a(\theta)) \sup_{v \in \mathcal{V}} v(\theta);$$

(iv) si $\mathcal{R} \subseteq X$, there exists $v \in \mathcal{V}$,$x, y \in \mathcal{R}$ such that $v(x, x) \neq v(y, y)$.

Moreover, if Θ is finite-dimensional, a is continuous.

Remark 3. *Condition (iv) implies in particular that it is not the case that* $v(r,r)=0$ *for all* $r\in X, v\in\mathcal{V}$. *In particular, it is impossible to have*

$$v(x,y)=\psi(x)-\psi(y)$$

for some function $\psi\in\mathcal{L}(X)$. *Nevertheless, the form*

$$v(x,y)=\varphi(x)-\psi(y),$$

with $\varphi\neq\psi$ *is possible.*

The following corollary provides a similar representation for the family $\{u_r\}_{r\in\mathcal{R}}$ representing $\mathcal{P}(\succsim)$:

Corollary 2. *Under the conditions of the representation theorem, for all* $r\in\mathcal{R}$, *there exists:*

- *a convex set* $\mathcal{V}_r\subseteq\mathcal{A}(X_r)$;
- *a function* $a_r : X_r\to[0,1]$

such that

$$u_r(x)=a_r(x)\inf_{v_r\in\mathcal{V}_r}v_r(x)+(1-a_r(x))\sup_{v_r\in\mathcal{V}_r}v_r(x).$$

Proof.

It suffices to apply the preceding theorem and to set $a_r(x)=a(x,r)$ and $v_r(x)=v(x,r)$. □

We shall dedicate the following section to the precise interpretation of the representation theorem and its corollary. Before that, let us make two important points.

Remark 4. *When* Θ *is finite-dimensional, the theorem holds without assumption (CPA). In this case, one car rely on theorems by Shapley and Baucells [20] and, as in finite-dimensional settings every linear map is continuous, the techniques used to prove the theorem go through.*

Remark 5. *(Reference-Dependence under Risk) An interesting special case of our setting is the case where* X *and* $\mathcal{R}$ *are space of lotteries. Let* Z *be a compact metric space,* $C(Z)$ *be the set of continuous (hence bounded) functions from* Z *to* $\mathbb{R}$, *endowed with the sup norm,* $\mathcal{P}(Z)$ *the set of Borel probability measures over* Z *and* $ca(Z)$ *the* *vector space spanned by $\mathcal{P}(Z)$, endowed with the weak convergence topology*[4]*. There exists a duality between $C(Z)$ and $ca(Z)$ defined by:*

$$< w, \mu >= \int w \mathrm{d}\mu,$$

denoted $\mathbb{E}_\mu(w)$.

The following corollary of the representation theorem holds:

Corollary 3. *When $\mathcal{R} = \mathcal{P}(Z_\mathcal{R})$, $X = \mathcal{P}(Z_X)$, Z_X countable, under the conditions of the theorem, for all $r \in \mathcal{P}(Z)$, the preferences are represented for all $p \in \mathcal{P}(Z_X)$ by the function:*

$$u_r(p) = a_r(p) \inf_{w \in \mathcal{W}} \mathbb{E}_{p \otimes r}(w) + (1 - a_r(p)) \sup_{w \in \mathcal{W}} \mathbb{E}_{p \otimes r}(w),$$

where $\mathcal{W} \subseteq C(Z_X \times Z_\mathcal{R})$ and $p \otimes r$ denotes the product measure of p and r.

Sagi [18] has axiomatized, in the case where $Z_X = Z_\mathcal{R} = Z$, a functional that may be rewritten:

$$u_r(p) = \inf_{\psi \in \Psi} \mathbb{E}_{p \otimes r(z,z')}(\psi(z) - \psi(z')),$$

where $\Psi \subseteq C(Z)$. This functional may be considered, in spirit, as a special case of the above functional. Nevertheless, because of remark 3, the function w may not be written as the difference of two equal functions. However, one can consider functions of the following kind: $w(z, z') = f(g(z) - h(z'))$, $f : \mathbb{R} \to \mathbb{R}$ and $g, h \in C(Z)$, with $g \neq h$.

3.5 Interpretation of the Representation Theorem

The representation theorem and its corollary allows to figure out that, in the decision process leading to RDP, the selection of a reference point affects preferences in two ways.

First, given $r \in \mathcal{R}$, a set $\mathcal{V}_r$ is selected, which may, similarly to Sagi [18], be interpreted as a set of implicit criteria with respect to which an element x is compared

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[4] According to Ouvrard [16], this topology is normable. Indeed, as Z is compact, $C(Z)$ is separable. Let therefore $\{f_n\}_{n \in \mathbb{N}}$ be a dense sequence in $C(Z)$. We then let, for all $\mu \in ca(Z)$:

$$\|\mu\| := \sum_{n=1}^{+\infty} \frac{1}{2^n \|f_n\|_\infty} \Big| \int_Z f_n \mathrm{d}\mu \Big|.$$

to an element y. For all $v_r \in \mathcal{V}_r$, one can indeed define a weak order over X:

$$xR_v^r y \Leftrightarrow v_r(x) \geq v_r(y).$$

Hence, to each $r \in \mathcal{R}$ corresponds a family of preference relations $\{R_v^r\}_{v \in \mathcal{V}}$ which is the true family of criteria. The first effect of the reference point is thus to modify the way the decision maker envisions the world, by determining which aspects of it it will consider to be relevant.

Second, suppose $a_r(x)$ is close to 1 and $a_r(y)$ close to 0. Then, for $x \succ_r y$ to hold, it is necessary that $v_r(x) > v_r(y)$ for almost all $v_r \in \mathcal{V}_r$, whereas for $y \succ_r x$ to hold, it suffices that $v_r(y) > v_r(x)$ for a small number of criteria. Hence, it is more likely that that $y \succ_r r$. More generally, if $a_r(x) > a_r(y)$, it is more likely that $y \succ_r x$, as shown by the following proposition:

Proposition 2

For all $r \in \mathcal{R}$, for all $x, y \in X_r$ such that $a_r(x) \neq 1$ and $a_r(y) \neq 1$, there exists $m_r(x, y) \in [0, 1]$ such that:

$$a_r(x) < m_r(x, y) \Rightarrow x \succ_r y \text{ end } a_r(y) < m_r(x, y) \Rightarrow y \succ_r x.$$

This proposition shows that if $a_r(x) > a_r(y)$, $y \succ_r x$ is more likely than $x \succ_r y$. Therefore, if the reference point belongs to X, $y \succ_r r$ is more likely than $x \succ_r r$. The coefficient $a_r(x)$ thus measures, roughly speaking, the measure-theoretic size of the set of decision criteria for the choice of x over y. It can be seen, moreover, that the exact value of $a_r(x)$ is not informative by itself: this parameter is of an ordinal nature, as shown by the proposition. It measures, like a utility function, the desirability of x relative to y, from the point of view of r.

This can be intuitively understood by noticing that, in fact, by the function a_r, the decision maker is going to select one criterion among the relevant one to assess the value of an option. One may think of a_r as a *salience* function, that puts forward one attribute of the object considered. The higher $a_r(x)$ is, the lower the utility value of x for the criterion selected with respect to the other criteria, hence the less desirable x is. As this function depends on the reference point, a different reference point will lead to a different salient attribute. Thus the second effect of the reference point is to modify the desirability of a determinate object.

For concreteness, consider the following story: Professor Cosine is currently tenured at University A. Suppose that in general the quality of a job at a university can be summarized by four attributes: salary, international prestige, quality of the students and location. Suppose Prof. Cosine is forced to abandon his job for some reason and that he has two offers, one at university B, the other at university C. We shall distinguish two situations: $A1$ and $A2$.

	A1	A2	B	C
Salary	*	****	***	***
Prestige	****	*	***	***
Location	***	*	****	*
Quality	*	***	*	****

Table 1: Prof. Cosine's choices.

Table 1 describes how each job fares with respect to each criterion. Consider first situation $A1$: Prof. Cosine's current employment is very prestigious and located in a fairly nice place, but has very low wage and students of very poor quality. Therefore, on these last two attributes, things can only improve. For this reason, these attributes are irrelevant. So we can just delete the corresponding lines. When we do that, job B appears clearly superior to job C (there is dominance), so we have $B \succ_{A1} C$. By the same reasoning, we have $C \succ_{A2} B$. Thus we see with this simple example how the mechanism described in the representation theorem may lead to preference reversal.

4 RDP and Welfare

4.1 Status Quo Bias and Welfare

We turn now to the third question that we asked in the introduction, namely the question of assessing the welfare of an individual whose preferences are reference-dependent. The main problem is: which preference relation in the profile should we choose for that purpose? The problem becomes particularly acute when the set of reference points is X, a situation we will consider from now on. Suppose indeed that there exists two endowments x and x' such that $x' \succ_x x$ and $x \succ_{x'} x'$. Then it is impossible to compare x and x' from the point of view of welfare, as one does not know which relation to favor. The question is this to construct a reference-independent preference ordering on which to base the allocation of resources.

The relation usually used in the literature to solve this problem is a relation based on the notion of gain: x os a gain with respect to y if x is preferred to y when the reference point is y. Formally[5]:

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[5] In the sequel, to simplify notation we shall assume that $X_r = X$ for all $r \in X$.

Definition 4.

- *x is a gain w.r.t. y (denoted $x \succsim_G y$) if $x \succsim_y y$;*
- *x is a strict gain w.r.t.y (denoted $x \succ_{SG} y$) if $x \succ_y y$.*

The relation $\succsim_G$ is generally incomplete. Moreover, it is generally non transitive. Different conditions were introduced in the literature to cope with this absence of transitivity. For instance, Masatlioglu and Ok [13] proposed, in the context of choice functions and of revealed preferences axioms that imply that this relation is the revealed preference and is transitive. Munro and Sugden [15] and Sagi [18] suggested to introduce as rationality axioms for RDP no-cycling axioms. Munro and Sugden [15] thus propose a weak acyclicity condtion :

WEAK ACYCLICITY (MUNRO AND SUGDEN [15]) : *There does not exist a finite sequence $\{x_1, \dots, x_n\}$ such that $x_1 \succsim_G x_2 \succsim_G \dots \succsim_G x_n$ and $x_n \succ_{SG} x_1$.*

This condition implies that $\succsim_G$ is acyclic. Sagi [18] proposed a stronger acyclicity axiom:

STRONG ACYCLICITY (SAGI [18]) *For all $x, y \in X$,*

$$x \succsim_y (\succ_y) y \Rightarrow \forall r \in X, x \succsim_r (\succ_r) y.$$

One may nevertheless be reluctant to accept this axiom. First of all, it considerably reduces the reference-dependence phenomenon. Moreover, in the setting that we chose, we already have a notion of rationality for reference-dependent preferences that holds without this axiom, the hedonic preferences, which relate the various instances of observable preferences, satisfying the classical rationality axioms, and hence being immune to money-pumps (as long as one accepts them for the true preferences). To add this axiom constitutes a strong theoretical choice and one may deem interesting to see what happens when we do not make this choice, and even more so because the kind of cycles that are ruled out may be considered one of the main interests of the phenomenon of reference-dependence.

However, if one refuses to impose this axiom, the problem of the assessment of welfare remains unsolved. To remedy for that, we shall tackle the issue by building on the following intuition: if an option x provides a greater welfare than another option y, then one will be more reluctant to forego x when x is the endowment than to forego y when y is the endowment. Thus, the fact that the status quo bias (SQB) is stronger in favor of x than in favor of y will be a symptom of the fact that x is truly preferred to y in terms of welfare. Formally, this corresponds to the following definition:

Definition 5.

- r is a better status quo than r' *(denoted $r \succsim_{SQ} r'$) if:*

$$\forall x \in X, x \succ_r r \Rightarrow x \succ_{r'} r'.$$

- r is a strictly better status quo than r' *(denoted $r \succsim_{SSQ} r'$) if :*

$$\forall x \in X, x \succsim_r r \Rightarrow x \succ_{r'} r'.$$

This definition of the comparative quality of a status quo clearly formalizes the intuition we gave: if r gives me more welfare than r' and if I am willing to forego r for x, then *a fortiori* will I be willing to forego r' for x and, if the preference of r over r' is strict, there will exist an option x for which I am willing to forego r' but not r.

The main properties of this relation are gathered in the following proposition:

PROPOSITION 3
In what follows, the quantifier: for all $r, r' \in X$ is always implied.

1. $\succsim_{SQ}$ *is reflexive and transitive;* $\succ_{SSQ}$ *is asymmetric and transitive.*
2. $r \succ_{SSQ} r' \Rightarrow r \succsim_{SQ} r' \Rightarrow \neg(r' \succ_{SG} r) \Rightarrow \neg(r' \succ_{SSQ} r)$;
3. *If* $\succ_{SG}$ *is transitive,* $r \succ_{SG} r' \Rightarrow r \succsim_{SQ} r'$;
4. *If strong acyclicity is true, then,* $r \succsim_G r' \Rightarrow r \succsim_{SQ} r'$ *and* $r \succ_{SG} r' \Rightarrow r \succ_{SSQ} r'$
5. *If* $r \succsim_G r' \Rightarrow r \succsim_{SQ} r'$, *then,* $[r \succsim_G r' \Rightarrow \neg(r' \succ_{SG} r)]$.
6. *If* $\succsim_r = \succsim_{r'}$, *for all* $r, r' \in X$, *then*

$$r \succsim_{SQ} r' \Leftrightarrow r \succsim_G r' \Leftrightarrow \neg(r' \succ_{SG} r) \Leftrightarrow \neg(r' \succ_{SSQ} r).$$

   Moreover, each of these relations is complete;

7. $r \curvearrowright r'$ *and* $r \succsim_G r' \Rightarrow r \succsim_{SQ} r'$;
8. $(r, r) \succsim (r', r')$ *and* $[\forall x \in X, (x, r') \succsim (x, r)] \Rightarrow r \succsim_{SQ} r'$.

Some words of commentary on these properties. The fact that the relations $\succsim_{SQ}$ and $\succ_{SSQ}$ are transitive is quite remarkable, though trivial, as this does not depend on any extra assumption, contrary to what happens with the relations $\succsim_G$ and $\succ_{SG}$. This is thus a positive feature of this relation.

The second property shows that $\succ_{SSQ}$ is a stronger ordering than $\succsim_{SQ}$. Moreover, it is rather natural to wonder what are the relationship between orderings based on the notion of SQB and the ones based on the notion of gain. More generally, it is interesting to wonder under what conditions they are extensions of one another. The various results gathered in this proposition bring some answers to these questions. It first shows a natural property of the relation $\succsim_{SQ}$: if r is a strongest status quo than r', then, whenever r is the reference point, r is weakly preferred to r', that is to say, r' is a loss with respect to r. This does not necessarily imply, however, that r is a gain with respect to r', due to the extra strength given to r' by its reference point status. Similarly, it shows that r cannot be a strictly better status quo than r' if it is not a strict gain.

The converse of these properties are obtained under various acyclicity conditions. Whenever one tries to improve the interest of relation $\succsim_G$ by imposing acyclicity conditions, it loses some of its interest by being extendable by $\succsim_{SQ}$. More precisely, under strong acyclicity conditions, $\succ_{SSQ}$ and $\succ_{SG}$ are equal and $\succsim_{SQ}$ extends $\succsim_G$. Both relation are in general incomparable. In a situation as the one described at the beginning of this section, where $x \succ_x x'$ and $x' \succ_{x'} x$, the relation $\succsim_G$ does not permit to conclude, but nothing precludes that x and x' be comparable by $\succsim_{SQ}$. The converse is true as well, but $\succsim_{SQ}$ offers the advantage, of being always transitive, as we just saw. The policies that it recommends are thus more stable in the long run.

The last property analyses the relationship between $\succsim_{SQ}$ and two other natural measures of welfare, rooted in the hedonic preference. The first one corresponds to the idea that, if, when I have an endowment, I must keep it forever, then I prefer this endowment to be r rather than r'. We will say that I *intrinsically prefer* r to r'. The second one is based on the following intuition in terms of preferences for state variations: suppose that, for all x, if I think that moving from state r to state x is an improvement, then I think that moving from state r' to state x is an even bigger gain, and if I think that moving from state r' to state x is welfare reducing, then I think that is even more so for moving from state r to state x. Then, a natural explanation for that is that I rank r higher than r' in some sense. We shall say that r *exchange-dominates* r'. What the last property shows is that the conjunction of these two orderings yields the status quo ordering.

Let us now come back to our welfare comparison concept $\succsim_{SQ}$. We intend now to relate this notion to the elements of the representation of preferences, in order to have a genuine measure of welfare, that is to say a numerical measure. The next theorem shows that the coefficient $a_r(x)$ measures the strength of the status quo bias when the endowment is r and the alternative is x, and allows to read the relation: "r is a stronger status quo than r' " in terms of the elements of the representation:

THEOREM 2

Assume that, for all $x, r \in X, a_r(x) \neq 1$. *Then, there exists a pseudo-distance*[6] *function* d *on* X *with values in* $[0,1]$ *such that:*

$$[\forall x \in X, a_r(x) - a_{r'}(x) \geq d(r,r')] \Rightarrow r \succsim_{SQ} r'.$$

Moreover, $r \succ_{GF} r' \Rightarrow a_r(r) - a_{r'}(r) < d(r,r')$

To understand the meaning of this theorem, suppose to begin with that $d(r,r') = 0$. Fix $x \in X$. Suppose for the moment that $a_r(x)$ is close to one and $a_{r'}(x) = 0$. Then, $(x,r) \succsim (x,r')$ requires $v(x,r) \geq v(x,r')$ for almost all $v \in \mathcal{V}$, whereas $(x,r') \succsim (x,r)$ requires that, for at least one $v \in \mathcal{V}$, $v(x,r') \geq v(x,r)$. Therefore, it is more likely that $(x,r') \succsim (x,r)$. Suppose now more generally that $a_r(x) \geq a_{r'}(x)$. Then, by an argument similar to proposition 2, it is more likely that $(x,r') \succsim (x,r)$. Now suppose this is true for all $x \in X$. Then it is more likely that r exchange-dominates r'. Now we see that if $d(r,r') = 0$, this implies $r \succsim_{SQ} r'$. Hence $d(r,r') = 0$ plays a role similar to the role of the condition $(r,r) \sim (r',r')$. Suppose now $d(r,r') > 0$ and is very close to 1. Then, for $a_r(x)$ to be greater to $a_{r'}(x) + d(r,r')$, $a_r(x)$ must be very close to 1 and $a_{r'}(x)$ very close to 0, which makes us come back to the first situation. Hence, $(x,r') \succsim (x,r)$ is much more likely than $(x,r) \succsim (x,r')$. Hence, the function d indicates how likely the event "r exchange-dominates r'" must be to yield $r \succsim_{SQ} r'$. If $d(r,r')$ is small, this event need not be very likely; if $d(r,r')$ is big, it must be very likely. So, as d is a pseudodistance function, one can say that it measures similarity: if r, r' are very close or very similar, then it is possible to conclude with less information about their relationship.

Finally, the theorem relates the interpretation of d to the notion of strict gain. Suppose $d(r,r') = 0$. Then, a necessary condition for r to be a strict gain w.r.t. r' is that the desirability of r rises as the reference point moves from r to r'. When $d(r,r') > 0$, this means that the difference in desirability must not move too much, and the greater the distance, the more likely it is. Hence, the closer they are, the less comparable r and r' are. The antinomy between the two orderings appears in this fact, as in proposition 3: if r and r' are comparable for one, they probably are not for the other.

Remark 6. *The pseudo-distance function* d *is concept derived from the preference relation and depends on the choice of* $\mathcal{V}$. *It is thus not an intrinsic concept. The notion of similarity is thus a subjective notion and depends on this choice of* $\mathcal{V}$. *It*

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[6] A function $d : X \times X \to \mathbb{R}_+$ is a pseudo-distance function if $d(x,x) = 0$, if it is symmetric and if it satisfies the triangle inequality. So it differs from the usual notion of distance in that $d(x,y) = 0$ does not necessarily imply $x = y$.

has the dimension of a utility function. However, the theorem does not depend on the choice of $\mathcal{V}$, as it relies on a comparative reasoning. Therefore, the criterion of comparability is always applicable.

Remark 7. *Intuitively, in the limit case where $a_r(x) = 1$ for all $x \in X$ and all $r \in X$, that is to say in a case very similar to the functional axiomatized by Sagi [18], we would have $d(r, r') = 0 = a_r(x) - a_{r'}(x)$. If the theorem was still valid, one would have $r \sim_{SQ} r'$ for all r, r'. This seems to be in line with the results in Sagi [18] that say that the status quo bias is maximal for all status quo, but these where proved with $v(x, r) = v(x) - v(r)$, a case wich is excluded in our model. So we conjecture that is true, but we were unable to prove it formally.*

4.2 Gains and Losses

In the discussion on welfare comparison, the notions of gains and losses pop up naturally. These notions are tightly related to discussions on reference-dependent preferences, as shown by the title of Kahneman and Tversky [23]'s seminal article on this topic[7]. The main idea that has been put forward in this literature is the one that gains and losses are asymmetric notions. In our setup, this notion can be formalized in the following way:

Definition 6 (Gain/Loss Asymmetry). *x is a net gain net w.r.t. y (denoted xGy) if*

$$x \succ_{SG} y \Rightarrow x \succ_x y.$$

There is Gain/Loss Asymmetry if xGy forall $(x, y) \in X^2$.

This condition means that if being in state x is a strict improvement (a gain) with respect to state y,then being in state y is a strict loss with respect to being in state x. There is a gain-loss asymmetry to the extent that the converse may not be true: if being in state x is a loss with respect to being in state y, being in state y does not have to be an improvement with respect to x. Moreover, the formulation of Gain/Loss Asymmetry is compatible with the idea that people are in general more sensitive to losses than to gains. Indeed, if the decision maker satisfies this condition, he or she will more easily feel that he or she has suffered a loss than feel that he or she has experienced a gain, as in this case all gains correspond to losses but all losses do not correspond to gains, so that there are “more" losses. This idea of extra-sensitivity to losses is often termed loss aversion. A way of formalizing this notion would be the following:

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[7] “Loss aversion in Riskless Choice: A Reference-dependent Model”

Loss Aversion (LA) *The family* $\{u_r\}_{r\in\mathcal{R}}$ *representing observable preferences* $\{\succsim_r\}_{r\in\mathcal{R}}$ *exhibits loss aversion if:*

$$\forall x, y \in X, u_x(x) - u_x(y) \geq u_y(x) - u_y(y).$$

It is easy to see that, thus formalized, Loss Aversion implies Gain/Loss asymmetry: if a decision-maker is averse to losses, this means that they mean more to him or her than do gains, so that he or she will perceive more losses than gains.

Notice that the notion of Gain/Loss Asymmetry is also an acyclicity axiom. Hence, a descriptive property like Loss Aversion implies a normative property, which is a noticeable feature. Thus, Loss Aversion may not be the result of irrationality, but, on the contrary, would stem from an underlying rationality principle.

In our setup, it is possible to relate Gain/Loss Asymmetry to the desirability of objects, as shown bu the following theorem:

THEOREM 3
Suppose $a(x, y) \neq 1$ *for all* $x, y \in X$. *Then, there exists a function* $\gamma : X \to [0, 1]$ *such that, for all* $x, y \in X$,

$$a_y(x) - a_x(x) \geq \gamma(x) \Rightarrow xGy.$$

As seen previously, if $a_r(x)$ is greater than $a_r(y)$, $y \succ_r x$ is more likely. This theorem says that, if $a_y(x)$ is significantly bigger than $a_x(x)$, *i.e.* if x becomes more desirable when the reference point moves from y to x, then x is a net gain net w.r.t. y. Indeed, if x is already better than y when y is the reference point, then x will be all the more when the reference point moves to x if its desirability rises. The function γ measures in some sense the repulsiveness of x: the bigger $\gamma(x)$, the less likely it is that x be a net gain w.r.t. y.

5 Proofs

Theorem 1.

(i) $\Rightarrow$ (ii):

- $\succsim_{\mathcal{P}}$ rationalizes $\mathcal{P}$: for all $r \in \mathcal{R}$ and $x, x' \in X_r$,

$$(x, r) \succsim_{\mathcal{P}} (x', r) \Leftrightarrow [x \succsim_r x' \text{ and } r \curvearrowright r] \Leftrightarrow x \succsim_r x'$$

because $\curvearrowright$ is reflexive and, for all $r, r' \in \mathcal{R}$ and for all $x \in X_r \cup X_{r'}$,

$$(x, r) \succsim_{\mathcal{P}} (x, r') \Leftrightarrow [x \succsim_{r'} x \text{ and } r' \curvearrowright r] \Leftrightarrow r' \curvearrowright r$$

because $\succsim_r$ is reflexive.

- $\succsim_{\mathcal{P}}$ is reflexive:

  $$(x, r) \succsim_{\mathcal{P}} (x, r) \Leftrightarrow x \succsim_r x \text{ and } r \curvearrowright r$$

  .

- $\succsim_{\mathcal{P}}$ is transitive: for all $(x, r), (x', r'), (x'', r'') \in \Theta$, $(x, r) \succsim_{\mathcal{P}} (x', r')$ and $(x', r') \succsim_{\mathcal{P}} (x'', r'')$ imply

  $$x \succsim_{r'} x' \text{ and } r' \curvearrowright r \text{ and } x' \succsim_{r''} x'' \text{ and } r'' \curvearrowright r'.$$

  Now, first, by the definition of $\curvearrowright$, $r' \curvearrowright r$ and $r'' \curvearrowright r'$ imply $x \in X_{r''}$ and, second, $r'' \curvearrowright r'$ and $x \succsim_{r'} x'$ implies $x \succsim_{r''} x'$. But this, combined with $x' \succsim_{r''} x''$ yields $x \succsim_{r''} x''$ by transitivity of $\succsim_{r''}$. Finally, by transitivity of $\curvearrowright$, $r'' \curvearrowright r$ obtains. Hence, $(x, r) \succsim_{\mathcal{P}} (x'', r'')$.

- Let R be a pre-order that rationalizes $\mathcal{P}$. Then, for all $(x, r), (x', r') \in \Theta$, $(x, r) \succsim_{\mathcal{P}} (x', r')$ implies $x \succsim_{r'} x'$ and $r' \curvearrowright r$. But, because R rationalizes $\mathcal{P}$, this implies $(x, r')R(x', r')$ and $(x, r)R(x, r')$ and, by transitivity of R, this yields $(x, r)R(x', r')$. Thus $\succsim_{\mathcal{P}} \subseteq R$.

(ii) $\Rightarrow$ (i): trivial. □

Corollary 1.

(i) $\Rightarrow$ *(ii)*:

Because $\succsim_r$ is a pre-order for all $r \in \mathcal{R}$, by theorem 1 $\succsim_{\mathcal{P}}$ is a pre-order that rationalizes $\mathcal{P}$. Consider then a weak order compatible extension of $\succsim_{\mathcal{P}}$, i.e. a relation $\succsim$ such that $\succsim_{\mathcal{P}} \subseteq \succsim$ and $\succ_{\mathcal{P}} \subseteq \succ$. Such a relation exists by a theorem in Donaldson and Weymark [4] and Duggan [6]. Let us prove that $\succsim$ rationalizes $\mathcal{P}$.

Suppose $r \curvearrowright r'$. Then $(x, r') \succsim_{\mathcal{P}} (x, r)$ for all $x \in X_r \cap X_{r'}$, which implies $(x, r') \succsim (x, r)$ for all $x \in X_r \cap X_{r'}$. Hence condition *(ii)* of rationality is satisfied.

Suppose $x \succsim_r x'$. Then $(x, r) \succsim_{\mathcal{P}} (x', r)$, hence $(x, r) \succsim (x', r)$.

Suppose now that $x \succ_r x'$. We shall prove that $x \succ_r x' \Leftrightarrow (x, r) \succ_{\mathcal{P}} (x', r)$:

$$\begin{aligned}(x, r) \succ_{\mathcal{P}} (x', r') &\Leftrightarrow (x, r) \succsim_{\mathcal{P}} (x', r') \text{ and } \neg((x', r') \succsim_{\mathcal{P}} (x, r)) \\ &\Leftrightarrow x \succsim_{r'} x' \text{ and } r' \curvearrowright r \text{ and } [x \succ_r x' \text{ or } r \Rsh r'],\end{aligned}$$

where $\Rsh$ denotes the asymmetric part of $\curvearrowright$.

This implies that $(x, r) \succ (x', r)$ is equivalent to $x \succ_r x'$. Now, because $\succsim$ is a compatible extension of $\succsim_{\mathcal{P}}$, this implies that $(x, r) \succ (x', r)$.

This completes the proof that $x \succsim_r y$ if and only if $(x, r) \succsim (y, r)$, as $\succsim$ and $\succsim_r$ are complete.

$(ii) \Rightarrow (i)$: trivial.

As for uniqueness, if $\succsim$ and $\succsim'$ are weak orders that rationalize $\mathcal{P}$, then by theorem 1, they contain $\succsim_{\mathcal{P}}$.

□

Lemma 3.

⇒: Let $(\theta, \theta') \in \Theta^2$ be such that $\theta \succsim^a \theta'$ and $\theta'' \in \Theta, \alpha \in [0, 1]$. then, if $\theta = \theta'$, we are done. If this not the case, then, for all $r \in \mathcal{R}_{x_\theta}$, for all $r' \in \mathcal{R}_{x_{\theta'}}$,$(x_\theta, r) \succsim (x_{\theta'}, r')$. This implies, by (API),

$$(\alpha x_\theta + (1-\alpha)x_{\theta''}, t) \succsim (\alpha x_{\theta'} + (1-\alpha)x_{\theta''}, t')$$

for all $(t, t') \in \mathcal{R}_{\alpha x_\theta + (1-\alpha)x_{\theta''}} \times \mathcal{R}_{\alpha x_{\theta'} + (1-\alpha)x_{\theta''}}$, hence

$$(\alpha x_\theta + (1-\alpha)x_{\theta''}, t) \succsim^a (\alpha x_{\theta'} + (1-\alpha)x_{\theta''}, t').$$

This is true in particular for $t = \alpha r_\theta + (1-\alpha) r_{\theta''}$ et $t' = \alpha r_{\theta'} + (1-\alpha) r_{\theta''}$, which belong to $\mathcal{R}_{\alpha x_\theta + (1-\alpha)x_{\theta''}}$ and $\mathcal{R}_{\alpha x_{\theta'} + (1-\alpha)x_{\theta''}}$, as

$$(\alpha x_\theta + (1-\alpha)x_{\theta''}, \alpha r_\theta + (1-\alpha) r_{\theta''}) = \alpha\theta + (1-\alpha)\theta'' \in \Theta$$

et

$$(\alpha x_{\theta'} + (1-\alpha)x_{\theta''}, \alpha r_{\theta'} + (1-\alpha) r_{\theta''}) = \alpha\theta' + (1-\alpha)\theta'' \in \Theta.$$

Hence the result.

⇐: Suppose $\succsim^a$ satisfies the independence axiom. Let $(x, y) \in X^2$ be such that for all $(r, r') \in \mathcal{R}_x \times \mathcal{R}_y, (x, r) \succsim (y, r')$. Then, as $\mathcal{R}_x$ and $\mathcal{R}_y$ have at least two elements, there exists $r \in \mathcal{R}_x, r' \in \mathcal{R}_y$, $r \neq r'$ such that $(x, r) \succsim^a (y, r')$. this implies, by the independence axiom for $\succsim^a$, that, for all $z \in X, s \in \mathcal{R}_z, \alpha \in [0, 1]$,

$$\alpha(x, r) + (1-\alpha)(z, s) \succsim^a \alpha(y, r') + (1-\alpha)(z, s),$$

hence

$$(\alpha x + (1-\alpha)z, \alpha r + (1-\alpha)s) \succsim^a (\alpha y + (1-\alpha)z, \alpha r' + (1-\alpha)s),$$

hence, as $r \neq r'$, $(\alpha x + (1-\alpha)z, t) \succsim (\alpha y + (1-\alpha)z, t')$ for all $(t,t') \in \mathcal{R}_{\alpha x+(1-\alpha)z} \times \mathcal{R}_{\alpha y+(1-\alpha)z}$.

□

Proposition 1.

⇒: The arguments of Dubra, Maccheroni and Ok [5] only use the convex structure of the set on which they work. Hence, they go through here, as long as we prove that $\succsim^a$ satisfies the required continuity property, that stipulates that for all sequences $\{\theta_n\}_{n\in\mathbb{N}}$, $\{\theta'_n\}_{n\in\mathbb{N}}$ of elements of Θ converging to $\theta, \theta' \in \Theta$ respectively and such that $\theta_n \succsim^a \theta'_n$ for all $n \in \mathbb{N}$, it is the case that $\theta \succsim^a \theta'$. Let us show it.

Let there be such sequences. If $\theta = \theta'$ for infinitely many n, we are done. Suppose therefore $\theta = \theta'$ for finitely many n. Then, there exists $n \in \mathbb{N}, \theta_n \neq \theta'_n$. Consider then the subsequences $\{\theta_{n_p}\}_{p\in\mathbb{N}}$, $\{\theta'_{n_p}\}_{p\in\mathbb{N}}$ constructed by deleting all terms with $\theta_n = \theta'_n$. These subsequences converge as well to θ et θ'. Let $\tilde{\theta} \approx \theta, \tilde{\theta'} \approx \theta'$. Then, there exists $r \in \mathcal{R}_{x_\theta}, r' \in \mathcal{R}_{x_{\theta'}}$ such that $\tilde{\theta} = (x_\theta, r)$ and $\tilde{\theta'} = (x_{\theta'}, r')$. Let then be the sequences $\tilde{\theta}_p := (x_{\theta_{n_p}}, r)$ and $\tilde{\theta'}_p := (x_{\theta'_{n_p}}, r')$. It is the case that $\tilde{\theta}_p \approx \theta_{n_p}$, $\tilde{\theta'}_p \approx \theta'_{n_p}$, hence, as $\theta_{n_p} \succsim^a \theta'_{n_p}$ and $\theta_{n_p} \neq \theta'_{n_p}$, on a

$$\tilde{\theta}_p \succsim \tilde{\theta'}_p.$$

Hence, by Continuity and because $\tilde{\theta}_p$ and $\tilde{\theta'}_p$ converge to $\tilde{\theta}$ and $\tilde{\theta'}$, we have

$$\tilde{\theta} \succsim \tilde{\theta'}.$$

As this is true for all $\tilde{\theta} \approx \theta, \tilde{\theta'} \approx \theta'$, this implies $\theta \succsim^a \theta'$.

⇐: Trivial.

□

Representation Theorem.

The necessity of the axioms being easy to see, we shall concentrate on their sufficiency.

By proposition 1, when assumtion (CPA) holds, $\succsim^a$ satisfies Weak Order, Continuity and (API) iff $\mathcal{C}^a$ is a convex cone that represents $\succsim^a$.

Let then $\mathcal{W}$ be the polar of $\mathcal{C}^a$, *i.e.*:

$$\mathcal{W} := \{w \in E' \mid w(z) \geq 0, \forall z \in \mathcal{C}^a\}.$$

Let $\mathcal{W}$ be the polar of $\mathcal{W}$, *i.e.*

$$\mathcal{W} := \{z \in E \mid w(z) \geq 0, \forall w \in \mathcal{W}\}.$$

Clearly, $\mathcal{C}^a \subseteq \mathcal{W}$. If we show the converse inclusion, we shall have proved that o $\mathcal{W}$ represents $\succsim^a$, *i.e.*:

$$\theta \succsim^a \theta' \Leftrightarrow w(\theta) \geq w(\theta'), \forall w \in \mathcal{W}.$$

Suppose there exists $z_0 \in \mathcal{W}$ such that $z_0 \notin \mathcal{C}^a$. $\mathcal{C}^a$ being convex by proposition 1 and closed by assumption (APC) and because E is locally convex, by a classical separation theorem, there exists $T \in E', \alpha \in \mathbb{R}$,

$$\forall z \in \mathcal{C}^a, T(z) \geq \alpha > T(z_0).$$

But $\mathcal{C}^a$ is a cone, hence, for all $n \in \mathbb{N}^*$, for all $z \in \mathcal{C}^a$, $nz \in \mathcal{C}^a$, hence $nT(z) = T(nz) \geq \alpha$, which implies $T(z) \geq \frac{\alpha}{n}$ for all $n \in \mathbb{N}^*$, hence $T(z) \geq 0$ for all $z \in \mathcal{C}^a$: $T \in \mathcal{W}$. But, on the other hand, $0 \in \mathcal{C}^a$, hence $T(z_0) < \alpha \leq T(0) = 0$. Now this cannot be, if $z \in \mathcal{W}$. Hence a contradiction.

Suppose now $\succsim^a \neq \succsim$. Then, there exists $\theta_1, \theta_2 \in \Theta$ such that $\theta_1 \|^a \theta_2$. Let

$$\mathcal{W}_1 := \{w \in \mathcal{W} \mid w(\theta_1) > w(\theta_2)\}$$

and

$$\mathcal{W}_2 := \{w \in \mathcal{W} \mid w(\theta_1) \leq w(\theta_2)\}.$$

As $\theta_1 \|^a \theta_2$, $\mathcal{W}_1$ et $\mathcal{W}_2$ are both non-empty. Normalize now all functions in $\mathcal{W}_1$ so that they take their values in $[0,1]$ and all functions in $\mathcal{W}_2$ so that they take their values in $[-2,-1]$ (this is possible because these functions are continuous over a compact set). Let now

$$\mathcal{V} := \overline{\text{co}}\, \{\mathcal{W}_1 \cup \mathcal{W}_2\},$$

where the closure is for the topology of pointwise convergence. By construction, $\mathcal{V} \subseteq \mathcal{A}(\Theta)$ is convex, closed,

$$-\infty < \inf_{v \in \mathcal{V}} v(\theta) < \sup_{v \in \mathcal{V}} v(\theta) < +\infty$$

and represents $\succsim^a$. $\mathcal{V}$ contains at least one non-constant function as $\succsim$ is non-trivial. This proves point (i).

Point (ii) is an immediate consequence of the definition of $\succsim^a$.

Let now $\theta \in \Theta$. Normalize u (continuous over a compact set) so that it takes its values in $[-1,0]$. By construction, there exists $v_1 \in \mathcal{W}_1 \subset \mathcal{V}$, $v_2 \in \mathcal{W}_2 \subset \mathcal{V}$ such that $v_1(\theta) \geq 0$ and $v_2(\theta) \leq -1$. As a consequence, $u(\theta) \in [v_2(\theta), v_1(\theta)] \subseteq \mathcal{V}(\theta) := \{v(\theta) \mid v \in \mathcal{V}\}$, which is a non-degenerate interval because $\mathcal{V}$ is convex and

$$\inf_{v \in \mathcal{V}} v(\theta) < \sup_{v \in \mathcal{V}} v(\theta).$$

Let

$$a(\theta) := \tfrac{\sup_{v \in \mathcal{V}} v(\theta) - u(\theta)}{\sup_{v \in \mathcal{V}} v(\theta) - \inf_{v \in \mathcal{V}} v(\theta)}.$$

This proves point (iii).

As for point (iv), suppose $\mathcal{R} \subseteq X$. If, for all $v \in \mathcal{V}$, for all $x, y \in \mathcal{R}$, $v(x,x) = v(y,y)$, then $(x,x) \sim^a (y,y)$, hence, if $x \neq y$, $(x,r) \sim^a (y,r')$ for all $r, r' \in \mathcal{R}_x \times \mathcal{R}_y$, hence $\theta \sim \theta'$ for all $\theta, \theta' \in \Theta$, contrary to the non-triviality of $\succsim$.

Suppose now that Θ is finite-dimensional. Let $Z = \{z_1, \ldots, z_n\}$ be the set of extremal points of Θ. Suppose w.l.o.g. that Θ contains 0. Let F be the vector subspace generated by Z. Extend each $v \in \mathcal{V}$ to a linear map over F by letting, for all $y \in F$,

$$\hat{v}(y) = \sum_{i=1}^{n} v(z_i) y_i.$$

Then, the $\hat{v}$ are continuous and the functions $y \mapsto \sup_{v \in \mathcal{V}} \hat{v}(y)$ and $y \mapsto \inf_{v \in \mathcal{V}} \hat{v}(y)$ are respectively convex and concave, hence continuous. This implies that a is continuous as u est continue. $\square$

Corollary 3.

Let $f \in \mathcal{A}(\mathcal{P}(Z))$. Riesz representation theorem stipulates that there exists $w \in C(Z)$ such that for all $p \in \mathcal{P}(Z)$,

$$f(p) =< w, p >= \mathbb{E}_p(w).$$

Now let $v \in \mathcal{A}(\Theta)$. Fix $r \in \mathcal{P}(Z_{\mathcal{R}})$. By remark 2, the map $p \mapsto v(p,r)$ belongs to $\mathcal{A}(\mathcal{P}(Z_X))$. As a consequence, there exists $W_r \in C(Z_X)$ such that

$$v(p,r) = \mathbb{E}_p(W_r).$$

Fix now $z_0 \in Z_X$. Let us show that the map $\varphi_0 : r \mapsto W_r(z_0)$ is affine and continuous.

First,

$$\begin{aligned}\mathbb{E}_p(W_{\lambda r+(1-\lambda)r'}) &= v(p, \lambda r + (1-\lambda)r') \\ &= \lambda v(p,r) + (1-\lambda)v(p,r') \\ &= \lambda \mathbb{E}_p(W_r) + (1-\lambda)\mathbb{E}_p(W_{r'}) \\ &= \mathbb{E}_p(\lambda W_r + (1-\lambda)W_{r'}).\end{aligned}$$

This being true for all p,

$$\begin{aligned}\varphi_0(\lambda r + (1-\lambda)r') &= W_{\lambda r+(1-\lambda)r'}(z_0) \\ &= \mathbb{E}_{\delta_{z_0}}(W_{\lambda r+(1-\lambda)r'}) \\ &= \mathbb{E}_{\delta_{z_0}}(\lambda W_r + (1-\lambda)W_{r'}) \\ &= \lambda\varphi_0(r) + (1-\lambda)\varphi_0(r').\end{aligned}$$

Second, if $r_n \to r$, then, as v is continuous, for all $p \in X$, $v(p, r_n) \to v(p,r)$. Hence $\mathbb{E}_p(W_{r_n}) \to \mathbb{E}_p(W_r)$ forall p, hence in particular for $p = \delta_{z_0}$, which implies $W_{r_n}(z_0) \to W_r(z_0)$. φ_0 is therefore donc continuous.

Now, again by Riesz' theorem, there exists $w_{z_0} \in C(Z_{\mathcal{R}})$ such that

$$W_r(z_0) = \mathbb{E}_r(w_{z_0}).$$

Define $w \in C(Z_X \times Z_{\mathcal{R}})$ by $w(z,z') = w_z(z')$. Then $v(p,r) = \mathbb{E}_p(\mathbb{E}_r(w))$. We wish now to apply Fubini's theorem. For this, we must show that w is measurable w.r.t. the product σ-algebra. Let A be an open set of $\mathbb{R}$. Then,

$$\begin{aligned}w^{-1}(A) &= \{(z,z') \in Z_X \times Z_{\mathcal{R}} \mid w(z,z') \in A\} \\ &= \cup_{z\in Z_X}\{z\} \times \{z' \in Z_{\mathcal{R}} \mid w_z(z') \in A\} \\ &= \cup_{z\in Z_X}\{z\} \times w_z^{-1}(A).\end{aligned}$$

Now, w_z is continuous, hence $w_z^{-1}(A)$ is open, hence measurable. $\{z\}$ is closed, hence measurable. Therefore, $\{z\} \times w_z^{-1}(A)$ is measurable. As, moreover, Z_X is countable, $\cup_{z\in Z_X}\{z\} \times w_z^{-1}(A) = w^{-1}(A)$ is measurable: w is measurable. We can now apply Fubini, which gives

$$v(p,r) = \mathbb{E}_{p\otimes r}(w).$$

Il suffices now to inject this result into the functional of the representation theorem. □

Proposition 2.

We first prove the following lemma, of independent interest:

Lemma 4. *There exist a function* $b : \Theta \times \Theta \longrightarrow [0, 1]$ *such that, for all* $\theta, \theta' \in \Theta \times \Theta$, $b(\theta, \theta') \in [a(\theta), 1]$ *and, if* $a(\theta) \neq 1$*:*

$$\theta \succ \theta' \Leftrightarrow b(\theta, \theta') > a(\theta).$$

Proof.

The proof consists first, given $\theta, \theta' \in \Theta \times \Theta$, in showing the existence of a convex subset $V_{\theta,\theta'}$ of $\mathcal{V}$, and of a set function μ_θ defined over the power set of $\mathcal{V}$ such that

$$\theta \succ \theta' \Leftrightarrow \mu_\theta(V_{\theta,\theta'}) > 0.$$

We then show that there exists a number $b(\theta, \theta') \in [a(\theta), 1]$ such that

$$\mu_\theta(V_{\theta,\theta'}) = b(\theta, \theta') - a(\theta).$$

We proceed in several steps and to help understand the proof we highly recommend that the reader draw diagrams representing the interval

$$\mathcal{V}(\theta) = \{v(\theta) \mid v \in \mathcal{V}\}$$

and the respective positions of $u(\theta)$ and $u(\theta')$ in this interval.

Step 1: Let V be a convex subset of $\mathcal{V}$. Then, for all $\theta \in \Theta, V(\theta) = \{v(\theta) \mid v \in V\}$ is a bounded real interval. Indeed, let $\alpha, \beta \in V(x, r)$. Then, there exists $v, w \in V$ such that $\alpha = v(\theta)$ and $\beta = w(\theta)$. Let $\gamma \in]\alpha, \beta[$. There exists $\lambda \in]0, 1[$ such that $\gamma = \lambda\alpha + (1 - \lambda)\beta = \lambda v(\theta) + (1 - \lambda)w(\theta) = (\lambda v + (1 - \lambda)w)(\theta)$. Now, $\lambda v + (1 - \lambda)w \in V$, hence $\gamma \in V(\theta)$: $V(\theta)$ is an interval. This interval is bounded by theorem 3.4. This is in particular true of $\mathcal{V}(\theta)$.

Step 2: Let us define a family of functions over $2^{\mathcal{V}}$ by setting, for all $\theta \in \Theta$, for all $V \subseteq \mathcal{V}$ such that $V(\theta)$ is Lebesgue-measurable,

$$\mu_\theta(V) = \tfrac{\ell(V(\theta))}{\ell(\mathcal{V}(\theta))},$$

where ℓ is the Lebesgue measure over $\mathbb{R}$. If V is convex, $V(\theta)$ is an interval, so that:

$$\mu_\theta(V) = \tfrac{\sup_{v \in V} v(\theta) - \inf_{v \in V} v(\theta)}{\sup_{v \in \mathcal{V}} v(\theta) - \inf_{v \in \mathcal{V}} v(\theta)}.$$

μ_θ is a way of measuring V, but it is not a measure in the strict sense, nor a capacity, as its domain need not be a σ-algebra over $\mathcal{V}$.

Step 3: For all $\theta, \theta' \in \Theta \times \Theta$, let

$$\mathcal{V}_\theta := \{v \in \mathcal{V} \mid v(\theta) < u(\theta)\}$$

and

$$\mathcal{W}_{\theta,\theta'} := \{v \in \mathcal{V} \mid u(\theta') \leq v(\theta)\}.$$

Clearly que $\mathcal{V}_\theta$ and $\mathcal{W}_{\theta,\theta'}$ are convex, so that $\mathcal{V}_\theta(\theta)$ and $\mathcal{W}_{\theta,\theta'}(\theta)$ are intervals, and so is their intersection. By this fact, $u(\theta)$ always belong to the closure of $\mathcal{V}_\theta(\theta)$ by construction and to the closure of $\mathcal{W}_{\theta,\theta'}(\theta)$ whenever $u(\theta') \leq u(\theta)$ (as $u(\theta) \in \mathcal{V}(\theta)$ and, in this case, $\mathcal{W}_{\theta,\theta'}(\theta) =]\max\{u(\theta'), \inf \mathcal{V}(\theta)\}, \sup \mathcal{V}(\theta)[$). We therefore let

$$\overline{\mathcal{V}_\theta} := \mathcal{V}_\theta \cup \{u\}$$

and

$$\overline{\mathcal{W}_{\theta,\theta'}} := \{v \in \mathcal{V} \cup \{u\} \mid u(\theta') \leq v(\theta)\}.$$

Let us show the following fact:

$$\theta \succ \theta' \Leftrightarrow \mu_\theta(\mathcal{V}_\theta \cap \mathcal{W}_{\theta,\theta'}) > 0 \quad (*).$$

Suppose, indeed, that $\theta \succ \theta'$. Then $u(\theta) > u(\theta')$. Hence the interval $]u(\theta'), u(\theta)[$ is nonempty. But, as $a(\theta) \neq 1$ (assumption of the lemma), $u(\theta) \neq \inf \mathcal{V}(\theta)$. Hence the interval $]\inf \mathcal{V}(\theta), u(\theta)[$ is nonempty. As a consequence, the intersection of these two nested intervals , $]\max\{u(\theta'), \inf \mathcal{V}(\theta)\}, u(\theta)[$, is nonempty. But this set is exactly $\mathcal{W}_{\theta,\theta'}(\theta) \cap \mathcal{V}_\theta(\theta)$. Hence, there exists $v \in \mathcal{W}_{\theta,\theta'} \cap \mathcal{V}_\theta$ such that $v(\theta) \neq u(\theta)$, hence $v \neq u$. This implies the existence of $v_0 \in \overline{\mathcal{V}_\theta} \cap \overline{\mathcal{W}_{\theta,\theta'}}$ such that $v_0 \neq u$. But, because $u(\theta) > u(\theta')$, $u \in \overline{\mathcal{W}_{\theta,\theta'}}$ and, by definition, $u \in \overline{\mathcal{V}_\theta}$, therefore $u \in \overline{\mathcal{V}_\theta} \cap \overline{\mathcal{W}_{\theta,\theta'}}$. Thus, th interval $\overline{\mathcal{V}_\theta}(\theta) \cap \overline{\mathcal{W}_{\theta,\theta'}}(\theta)$ contains the open interval $]v_0(\theta), u(\theta)[$, with $v_0(\theta) \neq u(\theta)$. But $\mathcal{V}_\theta \cap \mathcal{W}_{\theta,\theta'}$ also contains this open interval, as $u(\theta)$ is the only point that can belong to $\overline{\mathcal{V}_\theta}(\theta) \cap \overline{\mathcal{W}_{\theta,\theta'}}(\theta)$ but not to $\mathcal{V}_\theta(\theta) \cap \mathcal{W}_{\theta,\theta'}(\theta)$. Therefore, $\mathcal{V}_\theta(\theta) \cap \mathcal{W}_{\theta,\theta'}(\theta)$ is an interval with nonempty interior, and this implies $\ell(\mathcal{V}_\theta(\theta) \cap \mathcal{W}_{\theta,\theta'}(\theta)) > 0$, so that $\mu_\theta(\mathcal{V}_\theta \cap \mathcal{W}_{\theta,\theta'}) > 0$. Conversely, suppose $\mu_\theta(\mathcal{V}_\theta \cap \mathcal{W}_{\theta,\theta'}) > 0$. Then, there exists $v \in \mathcal{V}_\theta \cap \mathcal{W}_{\theta,\theta'}$. By construction, this implies $u(\theta') \leq v(\theta) < u(\theta)$, hence $u(\theta) > u(\theta')$, *i.e.* $\theta \succ \theta'$.

Step 4: Now define, for all $\theta, \theta' \in \Theta \times \Theta$,

$$b(\theta,\theta') = \begin{cases} 1 \quad \text{if} \quad u(\theta') \leq \inf \mathcal{V}(\theta) \\ \frac{u(\theta')-\sup \mathcal{V}(\theta)}{\inf \mathcal{V}(\theta)-\sup \mathcal{V}(\theta)} \quad \text{if} \quad \inf \mathcal{V}(\theta) < u(\theta') < u(\theta) \\ a(\theta) \quad \text{if} \quad u(\theta) \leq u(\theta'). \end{cases}$$

Let us show that

$$\mu_\theta(\mathcal{V}_\theta \cap \mathcal{W}_{\theta,\theta'}) = b(\theta,\theta') - a(\theta).$$

If $u(\theta') \leq \inf \mathcal{V}(\theta)$, then

$$\mathcal{V}_\theta(\theta) \cap \mathcal{W}_{\theta,\theta'}(\theta) = \mathcal{V}_\theta(\theta).$$

Hence, because $u(\theta)$ belongs to the closure of $\mathcal{V}_\theta(\theta)$, it is the upper bound of this set:

$$\begin{aligned} \sup_{v \in \mathcal{V}_\theta} v(\theta) &= u(\theta) \\ &= a(\theta) \inf_{v \in \mathcal{V}} v(\theta) + (1-a(\theta)) \sup_{v \in \mathcal{V}} v(\theta) \end{aligned}.$$

Moreover, by construction,

$$\inf_{v \in \mathcal{V}_\theta} v(\theta) = \inf_{v \in \mathcal{V}} v(\theta).$$

Therfore,

$$\begin{aligned} \mu_\theta(\mathcal{V}_\theta) &= \frac{a(\theta)\inf_{v\in\mathcal{V}} v(\theta)+(1-a(\theta))\sup_{v\in\mathcal{V}} v(\theta)-\inf_{v\in\mathcal{V}} v(\theta)}{\sup_{v\in\mathcal{V}} v(\theta)-\inf_{v\in\mathcal{V}} v(\theta)} \\ &= \frac{(1-a(\theta))(\sup_{v\in\mathcal{V}} v(\theta)-\inf_{v\in\mathcal{V}} v(\theta))}{\sup_{v\in\mathcal{V}} v(\theta)-\inf_{v\in\mathcal{V}} v(\theta)} \\ &= 1-a(\theta) \end{aligned}.$$

But,by dŐfinition of $b(\theta,\theta')$ in this case, *i.e.* $b(\theta,\theta') = 1$, we have:

$$\mu_\theta(\mathcal{V}_\theta \cap \mathcal{W}_{\theta,\theta'}) = b(\theta,\theta') - a(\theta).$$

Now, if

$$\inf \mathcal{V}(\theta) < u(\theta') < u(\theta),$$

then, as $u(\theta)$ belongs to the closure of $\mathcal{V}_\theta(\theta)$,

$$\sup \mathcal{V}_\theta(\theta) \cap \mathcal{W}_{\theta,\theta'}(\theta) = u(\theta)$$

and, as $u(\theta')$ belongs to the closure of $\mathcal{W}_{\theta,\theta'}(\theta)$,

$$\begin{aligned} \inf \mathcal{V}_\theta(\theta) \cap \mathcal{W}_{\theta,\theta'}(\theta) &= u(\theta') \\ &= b(\theta,\theta') \inf_{v\in\mathcal{V}} v(\theta) + (1-b(\theta,\theta')) \sup_{v\in\mathcal{V}} v(\theta). \end{aligned}$$

This implies,

$$\begin{aligned} \mu_\theta(\mathcal{V}_\theta \cap \mathcal{W}_{\theta,\theta'}) &= \tfrac{a(\theta)\inf_{v\in\mathcal{V}} v(\theta) + (1-a(\theta))\sup_{v\in\mathcal{V}} v(\theta) - b(\theta,\theta')\inf_{v\in\mathcal{V}} v(\theta) - (1-b(\theta,\theta'))\sup_{v\in\mathcal{V}} v(\theta)}{\sup_{v\in\mathcal{V}} v(\theta) - \inf_{v\in\mathcal{V}} v(\theta)} \\ &= \tfrac{(a(\theta)-b(\theta,\theta'))\inf_{v\in\mathcal{V}} v(\theta) + (b(\theta,\theta')-a(\theta))\sup_{v\in\mathcal{V}} v(\theta)}{\sup_{v\in\mathcal{V}} v(\theta) - \inf_{v\in\mathcal{V}} v(\theta)} \\ &= \tfrac{(b(\theta,\theta')-a(\theta))(\sup_{v\in\mathcal{V}} v(\theta) - \inf_{v\in\mathcal{V}} v(\theta))}{\sup_{v\in\mathcal{V}} v(\theta) - \inf_{v\in\mathcal{V}} v(\theta)} \qquad . \\ &= b(\theta,\theta') - a(\theta) \end{aligned}$$

Finally, if $u(\theta) \leq u(\theta')$, then

$$\mathcal{V}_\theta(\theta) \cap \mathcal{W}_{\theta,\theta'}(\theta) = \emptyset,$$

hence

$$\mu_\theta(\mathcal{V}_\theta \cap \mathcal{W}_{\theta,\theta'}) = 0.$$

But, by definition of $b(\theta,\theta')$ in this case, *i.e.* $b(\theta,\theta') = a(\theta)$, this implies

$$\mu_\theta(\mathcal{V}_\theta \cap \mathcal{W}_{\theta,\theta'}) = b(\theta,\theta') - a(\theta).$$

□

Now, if $\theta = (x,r)$ and $\theta' = (y,r)$, it suffices to let

$$m_r(x,y) := \min\{b(\theta,\theta'), b(\theta',\theta)\},$$

and to apply the lemma. □

Theorem 2.

By lemma 4, setting: $b(x,r) := b((x,r),(r,r))$ it easy to see that:

$$(\forall x \in X, b(x,r) - a_r(x) \leq b(x,r') - a_{r'}(x)) \Rightarrow r \gtrsim_{SQ} r'.$$

Now this is equivalent to:

$$(\forall x \in X, a_r(x) - a_{r'}(x) \geq b(x,r) - b(x,r')) \Rightarrow r \gtrsim_{SQ} r'$$

and implies

$$(\forall x \in X, a_r(x) - a_{r'}(x) \geq |b(x,r) - b(x,r')|) \Rightarrow r \succsim_{SQ} r'.$$

Now, by construction, $b(x,r) \in [a_r(x), 1]$. Hence, $b(x,r) - b(x,r') \in [a_r(x) - 1, 1 - a_{r'}(x)] \subseteq [-1,1]$ and $|b(x,r) - b(x,r')| \in [0,1]$. Hence,

$$\sup_{x \in X} |b(x,r) - b(x,r')| < +\infty.$$

Set

$$d(r,r') = \sup_{x \in X} |b(x,r) - b(x,r')|.$$

Then, we have:

$$(\forall x \in X, a_r(x) - a_{r'}(x) \geq d(r,r')) \Rightarrow r \succsim_{SQ} r'.$$

d is symmetric and it is easy to show, as for the distance of uniform convergence, that d satisfies the triangle inequality. Therefore, it is a pseudo-distance.

Now, suppose

$$a_r(r) - a_{r'}(r) \geq d(r,r').$$

Then,

$$a_r(r) - a_{r'}(r) \geq b(r,r) - b(r,r'),$$

hence

$$a_r(r) - b(r,r) \geq a_{r'}(r) - b(r,r').$$

But $a_r(r) = b(r,r)$, hence $a_{r'}(r) - b(r,r') \leq 0$. But this possible only if $a_{r'}(r) = b(r,r')$, the negation of $r \succ_{r'} r'$. The theorem is thus proved. $\square$

Theorem 3.

For $\theta = (x,r)$ and $\theta' = (y,r)$, let

$$b_r(x,y) := b(\theta, \theta')$$

where $b(\theta,\theta')$ is the object found in lemma 4. Let $\gamma(x) = \sup_{y \in X} |b_y(x,y) - b_x(x,y)|$. It easy to see, applying lemma 4 twice, that

$$a_y(x) - a_x(x) \geq \gamma(x) \Rightarrow xGy.$$

$\square$

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