Dual moments,
volatility, dependence and the copula

Dieter Denneberg and Nikola Leufer
Universität Bremen

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Abstract

The elementary volatility parameters of first and second order in spirit of dual theory of choice under risk or rank dependent expected utility are average sbsolute deviation and Gini index. Analogous to classical covariance a corresponding dual dependence parameter will be introduced and investigated in connection with the copula of a multivariate distribution. It is argued that the dual volatility and dependence parameters are better suited than the classical parameters for applications in finance and insurance. From the technical point of view it might be fascinating for a Choquet integrator to look at copulas, since for the latter ranking on the margins and comonotonicity play important roles.

1 Introduction

Predominantly thanks to David Schmeidler's work [10], the dual or rank dependent expected utility theory is now well established and applied in decision under risk and many other fields. Here *dual* means that not, like in the classical models, the outcomes of the random variables are transformed, but their distributions. The present paper applies the dual view to elementary volatility and dependence parameters. We survey the known first and second order dual volatility parameters, average absolute deviation and Gini index, and propose a new rank based dependence parameter.

The most common volatility and dependence parameters, standard deviation and correlation coefficient, are of the L_2-type, i.e. related to the second moments. The product of the random variables entering the respective formulas can hardly be interpreted directly in applications. We propagate parameters of L_1-type where the product is replaced by the min or a more sophisticated order or lattice relation, which had been investigated by Grabisch (see [5]). Our results apply to the copula of two random variables and generate a new concordance parameter.

The copula of a random vector is the essence of the common distribution of its components, obtained by normalizing the margins to become uniformly distributed. It is invariant under monotone transformations on the margins, so on the margins only the rank matters. The theory of the copula is well elaborated for continuous random variables. We propose a generalization or slight modification, where it is technically convenient to perceive distribution functions as interval valued functions if they are not continuous, or to attribute the midpoint of the interval.

In insurance, independence of the different claims had been, for a long time, a general assumption in the mathematical models. Since in our complex social world the interrelation of different risks increased, there is a new interest in models coping with dependence of random variables. Similarly in finance those models gain importance. During the last decade research on the copula and on concordance and discordance parameters has been intensified. We hope to convince the reader that the dual moments and rank based dependence parameters are better suited for certain applications than the classical parameters.

The results of this paper are in the framework of σ-additive probability theory, but methods of non-additive measure and integration come in quite naturally. First of all the dual view to distort probabilities leads to the Choquet integral. Next, comonotonicity of random variables plays a central role in both theories, for the copula and for the Choquet integral. Finally, to use the techniques of the other area will be fruitful for both. In probability theory and for the copula increasing distribution functions dominate, whereas for the Choquet integral decreasing distribution functions are the natural tool. So we employ both and, for the sake of concise formulations, perceive them as interval valued functions if they are not continuous.

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2 Volatility parameters

Let X be a real random variable on a probability space $(\Omega, \mathcal{A}, P)$. In this article F_X denotes the increasing distribution correspondence[1]

$$F_X(x) := \Big[P(X < x),\, P(X \leq x)\Big] \tag{1}$$

and $G_X := 1 - F(X)$ the decreasing one. The usual right continuous increasing distribution function is the upper selection of the distribution correspondence, $\overline{F}_X(x) = P(X \leq x)$. Similarly the left continuous increasing

[1] A correspondence is a set valued function, here all correspondences happen to be interval valued. A selection h of a correspondence H is a function with $h(x) \in H(x)$ for all x. If the values of H are closed intervals, the **upper selection** of H is $\overline{H}(x) := \bigvee\{u \mid u \in H(x)\}$, similarly the **lower selection** $\underline{H}$ is defined. We use the lattice operators $\vee$ and $\wedge$ to denote max and min or sup and inf, respectively.

distribution function is the lower selection $\underline{F}_X(x) = P(X < x)$. Let $\check{H}$ denote the inverse correspondence of a correspondence H, i.e. the graph of $\check{H}$ is the graph of H after intechanging the coordinate axes (for the explicit formulas see e.g. [5] Section 3). Again the pseudo inverse function of a distribution function is a selection of the inverse of the corresponding distribution correspondence.

Let H be a **strongly increasing correspondence** from an interval $I \subset \mathbb{R}$ to $\mathbb{R}$, i.e. $x_1 < x_2$ implies $y_1 \leq y_2$ for all $y_i \in H(x_i)$. Distribution correspondences and their inverses have this property. The integral of H is defined as

$$\int_I H(x)\,dx := \int_I h(x)\,dx, \quad h \text{ a selection of } H\,. \tag{2}$$

This definition is unambiguous since H is single-valued except an at most countable set in I. The latter derives from the fact that on the real axis any family of disjoint open intervals is finite or countable.

As usual $EX := \int X dP = \int_0^1 \check{F}(p)\,dp = \int_0^1 \check{G}(p)\,dp$ (see (2)) denotes the expectation of X and $MX := \check{F}(.5)$ the median. The median is an interval in general and (3) is valid for any point in this interval. Average absolute deviation of X from its median,

$$\tau(X) := \int |X - MX|\,dP = \|X - MX\|_1\,, \tag{3}$$

is the first absolute central moment of X, hence related to the Lebesgue space $L_1 = L_1(\Omega, \mathcal{A}, P)$. Similarly, standard deviation $\sigma(X) = \|X - EX\|_2$ is related to the L_2-norm and the variance $\mathrm{var}(X) = \sigma(X)^2$ is the second central moment of X.

It is well known ([14], [2], [4] Example 8.2) that average absolute deviation $\tau(X)$ and Gini index $\mathrm{gini}(X)$ are volatility parameters that can be expressed by means of the Choquet integral w.r.t. a piecewise linear or a quadratic distortion $\gamma_i \circ P$ of P, $\gamma_i : [0,1] \to [0,1]$, respectively,

$$\tau(X) = EX - \int X\,d(\gamma_1 \circ P)\,, \quad \text{with } \gamma_1(p) := 0 \vee (2p-1)\,, \tag{4}$$

$$\mathrm{gini}(X) = EX - \int X\,d(\gamma_2 \circ P)\,, \quad \text{with } \gamma_2(p) := p^2\,, \tag{5}$$

where $\mathrm{Gini}(X) := \mathrm{gini}(X)/EX$ is the usual Gini index for an income distribution $X \geq 0$ of a population Ω. One might regard (4) as first and (5) as second order **dual moments**. Here dual refers to the functional $\int X dP$ of the two variables X and P, where the piecewise linear or quadratic transforms are applied to the variable X or, dually, to P.

In fact the first absolute central moment coincides with the first dual moment and there is an L_2-formula for the second dual moment as we will see in Proposition 4.2. In the proof we need the fact that quantiles behave a.s. multiplicative for nonnegative comonotonic random variables, which is of interest for its own.

Proposition 2.1 *Suppose* X, Y *are nonnegative and comonotonic, then almost everywhere* $\check{G}_{XY} = \check{G}_X \check{G}_Y$ *and* $\check{F}_{XY} = \check{F}_X \check{F}_Y$.

Proof with [3] Proposition 4.1, Corollary 4.6 and the logarithm. □

This result sheds some more light on the analogy between dual moments and the classical ones. In the dual case comonotinicity plays the role, independence plays for the classical case. The location parameter median MX, which is related to $\tau(X)$ (see (3)), behaves essentially multiplicative for comonotonic random variables, whereas expected value EX, related to variance $\mathrm{var}(X)$, behaves multiplicative for independent random variables.

The proof of Proposition 2.1 relies on a.s. comonotonic additivity of the quantile function ([3] Corollary 4.6). This fact implies comonotonic additivity of the Choquet integral, hence by (4), (5) also comonotonic additivity of the dual parameters $\tau(X)$ and $\mathrm{gini}(X)$. These facts are again analogous to independence additivity of the classical parameter $\mathrm{var}(X)$. Finally we remark that $\tau(X)$ and $\mathrm{gini}(X)$ are subadditive since γ_1 and γ_2 are convex.

3 Dependence parameters

The most popular dependence parameter of a random 2-vector (X_1, X_2) is its covariance

$$\mathrm{cov}(X_1, X_2) := \int (X_1 - EX_1)(X_2 - EX_2)dP\,.$$

It is the inner product of the centralized random variables in the Hilbert space L_2 and

$$\mathrm{var}(X) = \|X - EX\|_2^2 = \mathrm{cov}(X, X)\,. \tag{6}$$

In (3) we have seen that average absolute deviation $\tau(X)$ is the L_1-norm of the centralized random variable. For defining the corresponding dependence parameter we have to find a substitute for the inner product, which does not exist in the Banach space L_1. For this purpose we have to replace the product in the definition of covariance with a new operation $\triangle$ having the property

$$|x| = x \triangle x\,. \tag{7}$$

The required operation had been introduced by Grabisch and is called **bipolar meet** in [5],

$$x \triangle y := \begin{cases} |x| \wedge |y| & \text{if } \operatorname{sign} x = \operatorname{sign} y \\ -(|x| \wedge |y|) & \text{else.} \end{cases}\,.$$

We emphasize that for positive x, y the bipolar meet is just the min $\wedge$ (or meet in lattice terminology). The bipolar meet is commutative and associative.

Now, in analogy with the covariance which could be called 2-covariance in our context, we define the **1-covariance**

$$\mathrm{cov}_1(X_1, X_2) := \int (X_1 - \dot{M}X_1) \triangle (X_2 - \dot{M}X_2) dP \,.$$

Here and in the sequel a dot on an interval denotes its barycenter,

$$\dot{I} := \frac{a+b}{2} \qquad \text{if } I = [a,b] \,.$$

Like in (6) we get, using (7),

$$\tau(X) = \|X - \dot{M}X\|_1 = \mathrm{cov}_1(X, X) \,.$$

Proposition 3.1 *Suppose, X_1 is symmetrically distributed on $\mathbb{R}$, i.e.*

$$P(X_1 - \dot{M}X_1 \leq -x_1) = P(X_1 - \dot{M}X_1 \geq x_1) \quad \textit{for all } x_1 \in \mathbb{R} \,.$$

If X_1, X_2 are independent then $\mathrm{cov}_1(X_1, X_2) = 0$.

For classical covariance this result holds without the symmetry assumption. The restricted validity of Proposition 3.1 does not harm us so much since our main concern is to apply it to copulas, where the assumption holds in the essential cases.

Proof We may suppose $\Omega = \mathbb{R}^2$, $X_i(x_1, x_2) = x_i$ and $\dot{M}X_i = 0$. By the independence assumption $P = P^{(X_1,X_2)}$ is the product of P^{X_1} and P^{X_2} so that we can apply Fubini's Theorem.

We decompose $\mathbb{R}^2$ in three parts: $A := \{x \in \mathbb{R}^2 \mid x_2 \geq |x_1|\}$, $B := \{x \in \mathbb{R}^2 \mid x_2 \leq -|x_1|\}$ and $C := \mathbb{R}^2 \setminus (A \cup B)$. We are done if we show that the integral of $X_1 \triangle X_2$ on each of these parts vanishes. First on A we get $x_1 \triangle x_2 = x_1$ so that

$$\begin{aligned}
\int_A X_1 \triangle X_2 \, dP &= \int_{-\infty}^{\infty} \int_0^{\infty} x_1 1_A(x_1, x_2) \, dP^{X_2}(x_2) \, dP^{X_1}(x_1) \\
&= \int_{-\infty}^{\infty} x_1 \int_0^{\infty} 1_A(x_1, x_2) \, dP^{X_2}(x_2) \, dP^{X_1}(x_1) \\
&= \int_{-\infty}^{\infty} x_1 P(X_2 \geq |x_1|) \, dP^{X_1}(x_1) \\
&= 0 \,.
\end{aligned}$$

For the last equation we needed that P^{X_1} is symmetrically distributed around 0 and that the function $x_1 P(X_2 \geq |x_1|)$ is an odd function of the variable x_1.

On B we get $x_1 \triangle x_2 = -x_1$ and the proof runs like for A. Finally, on C we get $x_1 \triangle x_2 = x_2 \operatorname{sign}(x_1)$ and we proceed like for A but change the order of integration. Then the inner integral $\int_{-\infty}^{\infty} \operatorname{sign}(x_1) 1_C(x_1, x_2) P^{X_1}(x_1)$ vanishes by the symmetry assumption. □

By normalization the L_2- resp. L_1-**correlations** are defined,

$$\rho(X_1, X_2) := \frac{\operatorname{cov}(X_1, X_2)}{\sigma(X_1)\,\sigma(X_2)}, \qquad \rho_1(X_1, X_2) := \frac{\operatorname{cov}_1(X_1, X_2)}{\tau(X_1) \wedge \tau(X_2)}$$

if the denominator does not vanish, i.e. X_1 and X_2 are not constant a.e.. The well known fact $-1 \le \rho(X_1, X_2) \le 1$ holds for ρ_1, too.

Proposition 3.2 $-1 \le \rho_1(X_1, X_2) \le 1$ *if* $\rho_1(X_1, X_2)$ *is well defined.*

Proof Using $|x \triangle y| = |x| \wedge |y|$ one gets $|\operatorname{cov}_1(X_1, X_2)| \le \int |(X_1 - \dot{M}X_1) \triangle (X_2 - \dot{M}X_2)| dP = \int |X_1 - \dot{M}X_1| \wedge |X_2 - \dot{M}X_2| dP \le \int |X_1 - \dot{M}X_1| dP \wedge \int |X_2 - \dot{M}X_2| dP = \tau(X_1) \wedge \tau(X_2)$. □

Also, like for ρ we get

$$\begin{aligned} \rho_1(X_1, X_2) &= \rho_1(X_2, X_1), \\ \rho_1(X, X) = 1, &\quad \rho_1(X, -X) = -1, \\ \rho_1(X_1, -X_2) &= -\rho_1(X_1, X_2). \end{aligned}$$

4 The uniformisation of a random variable

In the context of the copula it is most convenient to use the barycenter

$$\dot{F}_X = \frac{1}{2}(\underline{F}_X(x) + \overline{F}_X(x)). \tag{8}$$

of the interval valued distribution correspondence $F_X(x) = [\underline{F}_X(x), \overline{F}_X(x)]$ of a random variable X. We refer to it as the **(midpoint) distribution function** of X. So we avoid the use of interval valued random objects which would blow up the technicalities.

Crucial for the copula is the **(up-)uniformisation** (or 'probability integral transform' in some literature) of X,

$$U_X := \dot{F}_X \circ X.$$

If we use the decreasing distribution correspondence $G_X = 1 - F_X$ of X, we get the **down-uniformisation** $V_X := \dot{G}_X \circ X = 1 - U_X$ of X. Our name 'uniformisation' anticipates the following well known result (i).

Proposition 4.1 *(i) If F_X is a continuous function, then U_X is uniformly distributed on* $[0, 1]$.

(ii) Let $\varphi : X(\Omega) \to \mathbb{R}$ be an injective and increasing function, i.e. $x_1 < x_2$ implies $\varphi(x_1) < \varphi(x_2)$. Then

$$U_{\varphi(X)} = U_X \,.$$

Proof (i) Applying [3] Proposition 4.1 with G_X as continuous transformation function we get $\check{G}_{V_X}(v) = \check{G}_{G_X \circ X}(v) = G_X \circ \check{G}_X(v) = v$ for almost all v. Hence V_X is uniformly distributed and so is U_X.

(ii) Go back to the definition of the uniformisation and use $\{\varphi(X) \leq \varphi(X(\omega))\} = \{X \leq X(\omega)\}$ and $\{\varphi(X) < \varphi(X(\omega))\} = \{X < X(\omega)\}$. □

The random variable U_X is not uniformly distributed in general. But in any case we get

$$\begin{aligned}
E(U_X) &= 1/2\,, && \\
\mathrm{var}(U_X) &= 1/12 && \text{if } F_X \text{ is continuous,} && (9)\\
\dot{M}(U_X) &= 1/2 && \text{if } X \text{ is symmetrically distributed,} && \\
\tau(U_X) &= 1/4 && \text{if } \dot{M}(U_X) = 1/2\,, && (10)\\
\mathrm{gini}(U_X) &= 1/6 && \text{if } F_X \text{ is continuous.} && (11)
\end{aligned}$$

Since the uniformisation of U_X is U_X itself, (11) computes easily with the following L_2-representation of the Gini index. It is known in case of the uniform distribution on a finite set [1] or a continuous random variable [8] (citations from [13]). The last paragraph of the subsequent proof will show us that defining the uniformisation with the midpoint distribution function (8) instead of another selection, say $\overline{F}_X$, of the distribution correspondence F_X is essential for the validity of Proposition 4.2 in the general case. So there is no arbitrariness in defining the uniformisation U_X of X as one might believe at first sight.

Proposition 4.2 *A random variable X and its uniformisation U_X are comonotonic and*

$$\mathrm{gini}(X) = 2\,\mathrm{cov}(X, U_X) \quad \textit{for } X \geq 0.$$

Proof Comonotonicity of X and U_X is plain since U_X is an increasing transform of X (see e.g. [3] Proposition 4.5).

We first reformulate (5) with the increasing distribution function. Since (x, p) belongs to the graph of F_X iff $(x, 1-p)$ belongs to the graph of G_X one gets $\check{G}_X(p) = \check{F}_X(1-p)$. For evaluating the Choquet integral we need the decreasing distribution correspondence $G_{\gamma_2 \circ P, X} = \gamma_2 \circ G_{P,X}$ of X w.r.t. the distorted probability $\gamma_2 \circ P$. Notice that we write also like above G_X for $G_{P,X}$. Since $\gamma_2(t) = t^2$ is invertible on $[0,1]$ we get $\check{G}_{\gamma_2 \circ P, X} = \check{G}_{P,X} \circ \gamma_2^{-1}$.

These formulas together with the substitution $t = p^2$ imply

$$\begin{aligned}
\text{gini}(X) &= EX - \int_0^1 \check{G}_{\gamma_2 \circ P, X}(t)\, dt \;=\; EX - \int_0^1 \check{G}_X \circ \gamma_2^{-1}(t)\, dt \\
&= EX - \int_0^1 \check{G}_X(p)\, dp^2 \;=\; EX - 2\int_0^1 \check{G}_X(p)\, p\, dp \\
&= EX - 2\int_0^1 \check{F}_X(1-p)\, p\, dp \;=\; EX - 2\int_0^1 \check{F}_X(q)\,(1-q)\, dq \\
&= EX - 2\left(EX - \int_0^1 \check{F}_X(q)\, q\, dq\right) \\
&= 2\int_0^1 \check{F}_X(q)\, q\, dq - EX\,.
\end{aligned}$$

On the other hand we get

$$2\,\mathrm{cov}(X, U_X) = 2E(XU_X) - 2EXEU_X\,.$$

Since $EU_X = 1/2$, it is sufficient to show

$$\int_0^1 \check{F}_{XU_X}(q)\, dq = E(XU_X) = \int_0^1 \check{F}_X(q) q\, dq\,. \tag{12}$$

If F_X is a continuous function we know from Proposition 4.1 $\check{F}_{U_X}(q) = q$. Then, by Proposition 2.1, both integrands in (12) coincide a.e. and the proof is complete.

But also in the general case, in the right hand side integral, we can replace the factor q with the quantile correspondence $\check{F}_{U_X}(q)$ of U_X without changing the value of the integral.

If q is an inner point of the range of U_X in $[0,1]$, clearly $q = \check{F}_{U_X}(q)$. If not, regard the interval $[a,b] := F_X(x_0) \subset [0,1]$ with $q \in [a,b]$, i.e. $\check{F}_X(q) = x_0$. It is sufficient to show

$$\int_a^b \check{F}_X(t) t\, dt \;=\; \int_a^b \check{F}_X(t) \check{F}_{U_X}(t)\, dt\,.$$

This is an easy computation since for all $t \in [a,b]$ we get $\check{F}_X(t) = x_0$ and $\check{F}_{U_X}(t) = (a+b)/2$ so that $\int_a^b \check{F}_X(t) t\, dt = x_0 \int_a^b t\, dt = x_0(b^2 - a^2)/2 = x_0(b+a)(b-a)/2 = \int_a^b x_0(b+a)/2\, dt = \int_a^b \check{F}_X(t)\check{F}_{U_X}(t)\, dt$. □

5 The copula of a random vector

Let $X = (X_1, \cdots, X_n) \in \mathbb{R}^n$ be a random vector. Like in (1) the **increasing distribution correspondence of a random vector** is defined as

$$F_X(x) := [P(X_1 < x_1, \cdots, X_n < x_n), P(X_1 \le x_1, \cdots, X_n \le x_n)]\,, \quad x \in \mathbb{R}^n.$$

It is well known that the lower and upper selections $\underline{F}_X$ and $\overline{F}_X$ coincide on a dense subset of $\mathbb{R}^n$ and are continuous there. The distribution P^X of X is uniquely determined by F_X.

Denote with $U := (U_1, \cdots, U_n) \in [0,1]^n$ the vector of the uniformisations $U_i := U_{X_i}$ of the X_i. If the X_i are continuously distributed the U_i are uniformly distributed on $[0,1]$ (Proposition 4.1). This is often supposed in the literature on the copula.

The **copula** C_X of the random vector X is the increasing distribution correspondence of the random vector U,

$$C_X := F_U \,.$$

C_X is just another way to prescribe the distribution P^U of U, which might be called the **copula distribution** of X. The usual definition of the copula for continuously distributed X_i (see e.g. [9]) is the upper selection $\overline{C}_X = \overline{F}_U$ of our one. Since our definition differs slightly from the usual one, we reprove some basic facts about the copula.

The copula distribution P^U can also be characterized as the image measure

$$P^U = (P^X)^\Gamma \tag{13}$$

of P^X under the measurable application

$$\Gamma : \mathbb{R}^n \to [0,1]^n, \quad \Gamma(x) := (\dot{F}_1(x_1), \cdots, \dot{F}_n(x_n)) \,.$$

As usual $\dot{F}_i := \dot{F}_{X_i}$ denotes the midpoint distribution function of X_i. Observe that $U = \Gamma \circ X$.

The image $A := \{\Gamma(x) \mid x \in \mathbb{R}^n\}$ of Γ is the cartesian product $\prod_{i=1}^n A_i$ of sets A_i which consist of at most countable many intervals on the u_i-axis. Hence A is Borel measurable. P^U lives on A, i.e. $P^U(A) = 1$. In the other direction we introduce the application

$$\Xi : [0,1]^n \to \mathbb{R}^n, \quad \Xi(u) := (\dot{\check{F}}_1(u_1), \cdots, \dot{\check{F}}_n(u_n)) \,.$$

Recall that $\check{F}_i$ is the inverse correspondence of F_i and $\dot{\check{F}}_i(u_i)$ is the midpoint of the interval $\check{F}_i(u_i)$.[2] Again Ξ is measurable and its image $B := \{\Xi(u) \mid u \in [0,1]^n\}$ is a Borel measurable set. The functions $\dot{F}_i$ and $\dot{\check{F}}_i$ are not surjective, but if we restrict each of both to the image of the other, we get bijections,

$$\dot{\check{F}}_i \circ \dot{F}_i(x_i) = x \text{ for } x_i \in \operatorname{im}\dot{\check{F}}_i \,, \quad \dot{F}_i \circ \dot{\check{F}}_i(u_i) = u_i \text{ for } u_i \in \operatorname{im}\dot{F}_i \,.$$

[2]If X_i is essentially bounded below, $\check{F}_i(0)$ is an unbounded interval $]-\infty, b]$. Then we set $\dot{\check{F}}_i(0) := b - 1$ in order to remain real valued. Similarly proceed with $\dot{\check{F}}_i(1)$.

Hence

$$\Xi \circ \Gamma(x) = x\,,\ x \in B \subset \mathbb{R}^n, \qquad \Gamma \circ \Xi(u) = u\,,\ u \in A \subset [0,1]^n$$

and the applications $\Xi|A : A \to B$ and $\Gamma|B : B \to A$ are inverse to each other. Then the measures $(P^U)^\Xi$ and P^X coincide on measurable subsets of B, especially $P^X(B) = (P^U)^\Xi(B) = P^U(\Xi^{-1}(B)) = P^U(A) = 1$, so they are identical on the Borel σ-algebra of $\mathbb{R}^n$,

$$P^X = (P^U)^\Xi\,. \tag{14}$$

Hence in addition to (13), we can also recover P^X from the copula distribution P^U by means of Ξ or Γ, which are determined through the marginal distributions of X alone.

Less abstract this result reads as follows. The distribution of the random vector X, i.e. the common distribution of $X_1, \cdots, X_n$, can be reconstructed from its copula by means of the distribution functions $\dot{F}_i$ of its components.

The copula is invariant under increasing transformations on the margins. This fact follows immediately from Proposition 4.1 (ii).

Proposition 5.1 *Suppose $\varphi_i : X_i(\Omega) \to \mathbb{R}$ are injective and increasing functions and set $\varphi(x) := (\varphi_1(x_1), \cdots, \varphi_n(x_n)) \in \mathbb{R}^n$ for $x \in X_1(\Omega) \times \cdots \times X_n(\Omega) \subset \mathbb{R}^n$. Then $C_{\varphi(X)} = C_X$.*

The copula of a continuously distributed random vector X is well known if its components are independent or comonotonic. In the first case P^U is the uniform distribution on $[0,1]^n$. If the components of X are comonotonic, P^U is the uniform distribution on the main diagonal of $[0,1]^n$, the corresponding copula being called 'Fréchet-Höffding upper copula'. Here is another basic special case which is not of the continuous type.

Example 5.1 *Suppose, X_2 is constant and X_1 arbitrary with continuous distribution function. Then $U_2 \equiv 1/2$ and U_1 is uniformly distributed. Hence P^U is the uniform distribution on the horizontal line $\{(u_1, .5) \mid u_1 \in [0,1]\} \subset [0,1]^2$. Notice that X_1 and X_2 are independent and comonotonic simultaneously.*

6 Dependence parameters for the copula

Denoting again with U_i the uniformization $\dot{F}_{X_i} \circ X_i$ of X_i we define the **copula 1-correlation** of the random 2-vector (X_1, X_2) with X_1, X_2 not being constant a.e.

$$\mathrm{COR}_1(X_1, X_2) := \rho_1(U_1, U_2)\,.$$

This definition is the L_1 analogue to **Spearman's rho**

$$\mathrm{COR}_2(X_1, X_2) := \rho(U_1, U_2)\,.$$

Using (10), (9) we get

$$\begin{aligned} \mathrm{COR}_1(X_1, X_2) &= 4\,\mathrm{cov}_1(U_1, U_2) \qquad \text{if } \dot{M}U_i = \frac{1}{2}\,,\ i = 1, 2 & (15) \\ \mathrm{COR}_2(X_1, X_2) &= 12\,\mathrm{cov}(U_1, U_2) \qquad \text{if } F_X \text{ is continuous.} & (16) \end{aligned}$$

COR_2 is a measure of concordance as defined by Scarsini (cf. [9] Theorem 5.1.9). Also COR_1 has most, if not all, properties of a measure of concordance. Especially, by Proposition 3.1,

Proposition 6.1 *Suppose U_1 is symmetrically distributed and X_1, X_2 are not constant a.e., then*

$$\mathrm{COR}_1(X_1, X_2) = 0 \quad \textit{if } X_1, X_2 \textit{ are independent}.$$

The symmetry assumption holds if X_1 is continuously distributed. A similarly important result is

Proposition 6.2 *Suppose X_1, X_2 are continuously distributed then for $i =$ 1, 2*

$$\mathrm{COR}_i(X_1, X_2) = 1 \quad \textit{if } X_1, X_2 \textit{ are comonotonic}.$$

Of course for Spearman's rho, i.e. $i = 2$, this result is known already [6].

Proof Since X_1 and X_2 are comonotonic, the common distribution $P^{(U_1,U_2)}$ of U_1 and U_2 is the uniform distribution on the diagonal $D \subset [0, 1]$. Hence by the transformation rule for the second equation below and (7) for the third

$$\begin{aligned} \mathrm{cov}_1(U_1, U_2) &= \int_D (u_1 - 1/2) \bigtriangleup (u_2 - 1/2)\, dP^{(U_1,U_2)}(u_1, u_2) \\ &= \int_0^1 (u_1 - 1/2) \bigtriangleup (u_1 - 1/2)\, du_1 \\ &= \int_0^1 |u_1 - 1/2|\, du_1 \\ &= \tau(U_1)\,. \end{aligned}$$

Now $\rho_1(U_1, U_2) = 1$ and case $i = 1$ is proved. Replacing the bipolar meet $\bigtriangleup$ with the product in the argument above we get $\mathrm{cov}_1(U_1, U_2) = \mathrm{var}(U_1)$ and $\rho(U_1, U_2) = 1$. □

Again the question arises, if the assumptions in propositions 6.1 and 6.2 can be relaxed. Example 5.1 shows that $\mathrm{cov}_i(X_1, X_2) = 0$ can happen for comonotonic random variables, but $\mathrm{COR}_1(X_1, X_2)$, $\mathrm{COR}_2(X_1, X_2)$ are not defined in this case.

7 Applications to insurance and finance

Premium functionals using average absolute deviation τ or, more general, distorted probabilities can be found in the literature, [2], [11] and [2] Section 8. [12] gives an axiomatization based on Schmeidler's work [10].

A survey on the use of the copula in insurance and finance is given in [7]. We regard a simple example to illustrate the usefulness of our new dependence parameters.

In insurance a claim $X \geq 0$ often has high probability of no payments being due, say $P(X = 0) > .5$. Then $MX = 0$ and $\tau(X) = E(X)$. Regard a portfolio of two random variables X_1, X_2 of this type. The 1-covariance is $\text{cov}_1(X_1, X_2) = \int X_1 \wedge X_2 dP$. High values of $\text{cov}_1(X_1, X_2)$ indicate that both claims in the portfolio assume large values simultaneously with high probability, unfavorable for the insurance company since this fact excludes diversification. If one relates 'large' and 'small' to $\tau(X_1) \wedge \tau(X_2) = EX_1 \wedge EX_2$ the same indication is done by 1-correlation $\rho_1(X_1, X_2)$. The classical covariance cov and correlation ρ are not useful here, since the product of the claim amounts of the insurance contracts has no immediate interpretation, but the minimum has.

The transformation $\dot{F}_{X_i}$ transforms a set $[x_0, \infty[$ of high claims of X_i to the set $[\dot{F}_{X_i}(x_0), 1]$ of high quantiles. So, like above, values of $\text{COR}_1(X_1, X_2)$ close to 1 indicate that both claims in the portfolio have high quantiles simultaneously with high probability.

In finance, the **value at risk** given a security level $\alpha \in [0, 1]$, e.g. $\alpha = 5\%$ or 1%,

$$\text{VaR}_\alpha(X) = \check{G}_X(\alpha)$$

is a popular indicator for high values of a random loss X.[3] Could the 1-covariance $\text{cov}_1(X_1, X_2)$ supplement the value at risk $\text{VaR}_\alpha(X_1 + X_2)$ of the portfolio in order to overcome the well known shortcomings of VaR_α w.r.t. diversification?

8 Conclusions and outlook

We have sketched a first approach to dual moments, especially to dependence parameters, and to their application in connection with the copula. Many details and more complex applications still have to be investigated. Our dependence parameter $\text{cov}_1(X_1, X_2)$ is a hybrid one as it combines order relations and algebraic operations. The generalization from 2-vectors to n-vectors seems to be promising and feasible since the bipolar meet is commutative and associative. We proposed the 1-covariance, but how should be

[3] In the literature, often the scale is in terms of gains, not losses, which changes the sign.

defined the dual copula 2-covariance corresponding to the Gini index, the dual second order volatility parameter?

We looked at first and second moments and their dual. What about null moments? The space $L_0(\Omega, \mathcal{A}, P)$ of measurable random variables is a metric space with the Ky Fan metric $\|X - Y\|_0$ and an ordinal analogue of an inner product can be defined by means of the Fan-Sugeno functionals in [5]. So 0-covariance cov_0 and copula 0-correlation COR_0 might be defined. But also the quantiles VaR_α might be perceived as *dual null moments*. They are the Choquet integral w.r.t. a distorted probability with $\{0, 1\}$-valued distortion function.[4] But this distortion function is not convex like the distortions in (4), (5). This fact is the source for the shortcomings of VaR_α in the applications to finance.

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[4] W.r.t. this distorted probability, since it is $\{0, 1\}$-valued, Choquet integral and Sugeno integral coincide for $[0, 1]$-valued random variables.

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