VARIETY AND SUBSTITUTION

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Abstract

We study two major issues in retail inventory management: (1) The optimal set of variants (the assortment problem), that is, how many and which variants should be carried by retailers; and (2) the optimal inventory policy, given a set of variants. It is well known that it is difficult to reach analytically tractable solutions solving these problems because of the randomness in the total market demand, in consumer choices, and the resulting substitution behaviors. To the best of our knowledge, this is the first study that provides structural results to a large family of substitution problems. We start by considering a single product category and a single profit-maximizing retailer. Demands for each of the variants are uncertain and are generated from a consumer choice model. For each customer, a realization of the consumer choice model is a ranking of the variants, from the most desirable variant to the least desirable one. Also, every consumer has a critical utility (no-purchasing-utility) such that if the utility resulting from the purchase of a certain variant is below the critical utility level, then the consumer will not purchase this variant. As a result, a consumer may have a set of variants that he or she is unwilling to purchase. When a certain variant that consumers desire to purchase is not available, they purchase the highest-ranked available variant with a utility that is higher than the no-purchase utility. We assume that each customer purchases at most one unit of the product, and we are able to determine an inventory policy which results in the optimal expected cost with a very high probability. We are also able to determine the optimal number of variants with a very high probability.

1. Introduction

In past few years, the number of products that are offered by retailers to consumers has increased dramatically. This suggests that line extension has become a favored strategy of retailers. Retailers may extend their product lines and offer a large variety in order to capture a larger share of the market and extract more consumer surplus. In this study, we concentrate on the effects of variety on market share. We start by studying the retailer's optimal inventory policy. Then, we continue by determining the optimal assortment. We build up a very general model of consumer choices, in which no specific substitution pattern is prescribed, but in which we make some limiting assumption. Specifically, we assume that all variants have the same price and quality. In this case, variety may play a role in increasing market share, keeping consumers loyal by preventing them from switching to other companies. Clearly, competition may play a major role in a retailer's decisions on which and how many variants to keep. In this paper, we consider a monopoly, and thus competition does not play a role. Yet, variety plays a major role in increasing the market share of the monopoly.

Retailers often face the issues of choosing the number of different variants and deciding the stocking level of each variant in a category. On one hand, too little variety limits the options for consumers and may reduce the retailer's market size as some customers may not find a variant they desire and thus decide to not purchase any variant. On the other hand, maintaining a large variety is costly and may complicate managing inventory and the stocking decision. Usually it is difficult to select the appropriate assortment and stocking levels as customers' preferences and the firm's market share are difficult to predict and manage beforehand.

We start with a consumer choice model to generate consumer preferences and demands. The out-come of the consumer choice model is a ranking of the K variants; from the most desirable (highest utility) to the least desirable (lowest utility). Clearly, each consumer may have a different ranking. In other words, the demand distributions of different variants are not exogenous; they are a function of the number of variants and are determined by the choice model. Consumers' preferences are assumed to be random, and it is assumed that retailers can only predict customers' preferences and estimate the demand for variants through predicted preferences. Thus, retailers are only able to infer the demand distribution for each of the K variants. To make the model more realistic, we include a no-purchase option in the buyer's choice set. For a consumer, some variants may have such small utilities that they are inferior to the option of not purchasing anything at all. In this setting, when customers try to buy a variant and it is not available, they try to substitute the first choice with the next-preferred option according to the realized utility ranking. If it turns out that the next variant has a smaller utility than the no-purchase utility, the choice is to not purchase any variant, and the consumer will leave without buying any product.

There are two common ways to model substitution. The first assumes that customers arrive sequentially and that the seller satisfies their demands one by one (*sequential substitution*). The second assumes that all consumers arrive simultaneously, reveal their preference at the same time (*simultaneous substitution*), and the retailer allocates the available inventory to consumers so that expected profits are maximized and consumers purchase variants with utilities that are higher than their no-purchase utilities.

Sequential substitution captures reality well but is comparably more difficult to model and solve. We show that by using the simultaneous substitution model, we are able to identify the optimal allocation of products to customers subject to their individual ranked preferences. Clearly, the optimal profit achieved using the simultaneous substitution model is an upper bound of the sequential substitution model, assuming everything else equal. In this study, we use the simultaneous substitution model because it enables us to obtain structural results. Perhaps more importantly, we prove that when the market size is large enough, the simultaneous substitution model provides a very tight upper bound on the optimal profit of the sequential substitution model. That is, the gap between the resulting optimal profits of the two models is very small, a result that we are not the first to point out. For example, (Gaur & Honhon) (2005) show, in a numerical study, that the gap between the profits of the two models is small.

Finally, we start by assuming that there are N consumers and that N is known with certainty. The source of uncertainty is generated by the consumer's choice. Later, we relax this assumption and let the number of customers to be a random variable.

The main objective of this paper is to study the following classical problems:

1. *Inventory substitution problem*—optimal stocking policy for each of the variants in a product line.
2. *The assortment problem*—the optimal number of variants that the retailer should carry.

In solving the assortment and inventory-substitution problems, we make the following important assumptions.

- **Variants have identical quality, price, and popularity**. Like many other studies, we start by assuming that different variants may have different characteristics (e.g., flavor, color) but are conceived to be of identical quality, and that all variants have the same costs and retail prices. The retailer does not know the preferences of the consumer but assumes that all consumers have the same choice model. In other words, the distributions of consumers' preferences are identical, but not their realized preferences. This differs from some other substitution studies that assume that variants (products) have different preference rankings or a predefined substitution order. For example, van Ryzin and Mahajan (1999) analyze the optimal assortment assuming that the variety choice set consists of variants with different degrees of popularity. Gaur and Honhon (2005) build up a Lancaster choice model assuming predefined substitution order (that is, once a customer's first choice is realized, the seller knows the entire preference rankings of the customer). Later, we relax this assumption and assume that consumers may have different choice models and that some variants are more popular than others. One major limitation of this study is our inability to consider variants with different costs and retail prices. There is no doubt that this assumption is limiting. Yet, many products have the same cost and retail price. For example, the cost and retail price of T shirts is not a function of the color, and similarly the cost and the retail price of yogurt are identical and are not a function of the flavor.

- Customers are willing to purchase any variant as long as it's utility is larger than the no-purchase utility. We assume that consumers who do not get their first priority variant do not perceive the service to be inferior (as long as they get an acceptable variant). Thus, the retailer does not try to maximize consumer's utility. Rather, the objective is to maximize profits.
- Like many other papers, we assume that a customer does not know the inventory level of each variant. In other words, remaining stocking level of all products will not interfere with the choice realization of each consumer.
- We start by assuming that the number of consumers (N) is known with certainty. We also assume that N is not too small. In section 6, we relax this assumption and assume that N is a random variable.
- In addition, we assume that each customer is willing to purchase at least two variants. That is each customer has a most desirable variant and at least one more variant with utility higher than the no-purchase utility

The main results of this study are:

- With a very large probability, it is possible to satisfy all consumers' demands with exactly N units of inventory and no safety stock. This result is quite remarkable and of practical value. It means that while the retailer faces uncertainty in consumer preferences and does not carry any safety stock, he is still able to satisfy the demands of all customers.
- The fact that no safety stock is needed depends heavily upon the assumption that the size of the population is known with certainty. When the size of the population is random, it is necessary to have safety stock. In other words, the safety stock is carried only because of the randomness in the market size but not due to the randomness in consumer preferences. We show that when population size is a random variable there is a simple way to calculate the optimal purchasing quantity of each of the variants. The optimal solution is obtained by solving a simple news vendor problem. The

above result allows us to solve for a complicated assortment of variants and substitution problems. We show that such problems can be simplified to a one-dimension profit maximization problem that solves for optimal variety. The solution shows that it might be optimal for a monopoly to have a large variety because increasing the available varieties decreases the probability of no purchasing, which in turn increases the market size and total profit. This eventually becomes a trade-off with the concurrently increasing variety cost. In other words, the choice of the length of the product line (number of variants) depends upon parameters that include no-purchasing probability, price, shortage cost, customer pool size, and variety cost.

- When variants are not equally popular, we can still achieve the optimal simultaneous substitution solution by appropriately stocking different variants. On the assortment issue, we show that the optimal assortment is composed of the most popular variants. Van Ryzin and Mahajan (1999) develop a similar result assuming no substitution. Gaur and Honhon (2005), using the Lancaster choice framework, show that the optimal assortment may not consist of the most popular variants. One important reason for the different results is their assumption of partially randomized customer preferences. In other words, assuming that the variants are numbered from 1,..K. They assume that if a customer is willing to purchase variants i and j then he is willing to purchase variants $i, i+1, ..., j-1, j$. Their analysis leads to the result that the optimal assortment may not consist of the most popular variants. This is different with the finding from Mahajan and van Ryzin (2000).
- Finally, we are able to show, analytically, that the gap between the simultaneous and sequential substitutions is small.

The structure of the paper is as follows. We review related literatures in section 2 and some consumer utility model in section 3. In Section 4, the general formulations of the problems regarding inventory and assortment decisions are developed through a large stochastic linear program, which turns out to be not practical. After analyzing the behavior of no-purchasing probability and consumer choice framework in Section 5, we continue to solve the general problem in two steps. As the first step, in Section 6, we assume given variety, and we address the inventory effect under simultaneous substitution. We develop a heuristic to calculate the optimal inventory level of each variant. We show that under quite general conditions, the demand of all consumers can be satisfied with very high probability and with no safety stock by using a simple stocking policy. Later, we relax certain assumptions to allow randomized market size and non-identical products. We show that using our approach inventory level is as close to optimal as desired. In the next stage, we apply the heuristic in Section 7 to solve the assortment problem assuming a single profit-maximizing retailer in the market. Using the heuristic, the general problem is simplified to one-dimensional profit maximization over variety and becomes analytically solvable. We find that, with high no-purchasing probability, a monopolist may offer more than one variant in order to increase market size. For unequally popular variants, the most preferred products will be chosen for the assortment.

2. Literature Review

Variety can be costly. Chamberlain (1933), in his seminal paper, assumed that production costs exhibit economies of scale. He showed that the higher the economies of scale the lower, in equilibrium, are the number of variants. This result is intuitive and very common in the economics literature. Stalk (1988) and Baumol (1956) claims that the cost of variety is almost proportional to the square root of the number of variants. We will use this result as an approximation to variety cost. In an empirical study, Kekre and Srinivasan (1990) examine over 1,400 business units and show that firms with higher variety have significantly higher relative product prices, which serves as an evidence that variety is costly. On the other hand, increasing variety gives the customers more options to choose when the desired item is not available, or out of stock. Grocery Manufacturers of America (2002) demonstrate the importance of substitution in retailing industry. The authors of this study argue that on average the percentage of shortages in the retail industry is 5%-10%. Also, on average, 45% of consumers who face shortages will substitute and purchase another product either from the same or a different retailer.

The operations literature that considers substitution aims to study the effects of substitution on inventory and procurement policies. In most of this literature, the demand distribution of each of the variants is exogenous and is independent of the number of variants. For examples, see Bassok et al. (1999), Ernst and Kouvelis (1999), and Netessine and Rudi (2003).

Pentico (1974) analyzes the deterministic multi-item inventory optimization problem with one-way substitution and shows with dynamic programming that there exists an optimal stocking policy. Pentico (1976) extends this formulation to concave production and substitution cost functions and presents algorithms to find optimal policies.

Bassok et al. (1999) build up a model analyzing the one-way substitution model for the manufacturers and characterize the structure of the optimal policy. They also use an example a problem involving only two products to illustrate the large effects of substitution and the significant gains that can be achieved when substitution is considered. Smith and Agrawal (2000) study the substitution effect on inventory assuming that a customer switches to the second most desired variant when his or her first choice is not available and each customer makes at most one such substitution attempt. They develop a method of determining the optimal level of inventory of each of the variants. Netessine and Rudi (2003) analyze the centralized and competitive models for substitution, and they show the concavity of the objective function in a non-competitive setting and establish the uniqueness of the equilibrium for the competitive n-product case. Jordan and Graves (1995) study manufacturing flexibility, which is analogous to the substitution problem. They show by simulation that limited flexibility, configured in a certain way, yields most of the benefits of total flexibility (substitution). Our findings are similar to theirs. Anupindi, et al. (1998) focus on only two different products and build up a model to estimate the demand for each of the products, assuming substitution and consumers arrive sequentially. They find that the model can be practically applied because parameters can be derived by using the MLEs. (Maximum Likelihood Estimates)

In this paper we take a different route, starting with a choice model to generate the demand for each of the products. The consumer choice model is one of the classic topics in marking. McFadden (1980) reviewed and discussed several probability models regarding consumer choice behavior. One of these is the MNL model, which is widely applied in many marketing and operation studies. Kim, et al. (2002) introduces a utility model that accounts for the effect of variety choice and substitution. In this model, a consumer can choose from a set of variants and decide not only which variants to buy but also the quantity of each of the variants to purchase. The model also introduces a budget constraint that limits the number of variants that consumers can purchase. In what follows, we use this model because of its flexibility and richness. In this study, we use this model in a limited way in that we assume that consumers will purchase only one item and that the costs of all items are identical. Thus, we assume that the budget of each consumer is equal to the cost of the product.

Van Ryzin and Mahajan (1999) set up an assortment model in which individual consumers make purchase decisions according to the multinomial logit (MNL) utility model. In their study, the role of variety is to minimize the probability of no purchasing. This is achieved by providing consumers with a large number of variants so that they are able to purchase their first choice variant. The authors assume that if the most preferred variant is not available then the consumer does not purchase any variant. In other words, they ignore the effects of substitution and study only the effects of the length of the product line on market share. They show the existence of optimal assortment, and it is composed of the most popular variants. Later, Mahajan and Van Ryzin (2000) present a retailing assortment model based on the MNL utility choice framework and sequential substitution, they use sample path algorithm to search for the optimal solution. They show that the algorithm is only able to find stationary points and that it is difficult to guarantee a global optimum for inventory stocking levels, though it appears robust in numerical testing. In addition, considering substitution, a retailer should stock more of the most popular variants than a traditional newsboy analysis would indicate.

Gaur and Honhon (2005) generalize the Lancaster attribute space model to analyze the assortment planning. They show that firms provide higher variety when considering the substitution effect and that the optimal assortment need not include the most popular product. Honhon, Gaur and Seshadri (2006) modeled consumer preference as consumer types to study the optimal assortment and inventory problem with different price and cost. They limit the preferences, or substitution behaviors to one-sided attempts. That is, only variant A being preferred to variant B, or B to A is allow, but not both of them. When the number of each type of customers is random, they show that a dynamic programming algorithm yields an upper bound of the optimal solution. Caro and Gallien (2006) build up a stylized multi-armed bandit model with Bayesian learning to study the multi-period optimal assortment problem in retailing industry. They assume a perfect replenishment system and thus inventory is not an issue in the model. Their analysis yields a closed-form dynamic index policy that is proved through numerical experiments to have solid performance. They later extend the model to capture substitution effect on variety and show that the index in the policy can be updated accordingly. A very good review of the substitution and assortment problem can be found in Kok et al (2006)

3. Utility Model

We start by presenting a utility model that was first presented by Kim, et al. (2002). An analogue to the MNL model, the nominal and random term in this model are assumed to be log linear. Also, the final choice can be constrained to the customer's budget limit. The i^{th} consumer's total utility is expressed in the following way:

$$u^i(x) = \sum_j u_j^i(x_j) = \sum_j \psi_j (x_j + \alpha_j)^{\gamma_j} \times e^{\varepsilon_j^i} \quad ,$$

where $\varepsilon \sim N(0,1)$.

In this model, ψ_j, γ_j, and α_j are parameters describing the j^{th} variant, and ε^i is a random variable representing the uncertainty that the retailer is facing. The realization of the random variable becomes known to the retailer only when the i^{th} customer comes to shop and reveal his or her preference. That is, the utility of consumer i is the sum of utilities resulting by consuming x_j units of variants j.

Every consumer solves the following problem:

$$\text{Maximize} \quad u^i(x)$$

$$s.t. RX \le E \ ,$$

where $R = (r_1,...,r_k)$ is the price of variant *1,…,j,…k,* and *E* is the total budget available for purchasing the variants.

We choose this model because of its richness—it has the capability to solve for multiple selections with a budget constraint for each customer. However, as a starting step, for simplicity we assume that each consumer buys only one product. This is equivalent to assuming that the consumer's budget is equal to the price of the product. We believe that limiting the model by ignoring the issue of consumers purchasing multiple units is justified by the simplicity of our model and the strong result we obtain.

4. Demand Approximation

4.1 The Demand Model

We assume that the market size is *N* (for the time being, *N* is known with certainty). Earlier, we mentioned that there might be consumers who will prefer to not purchase any of the variants, even though all variants are available in stock. We refer to such consumers as no-purchasing consumers, and we would like to calculate the probability that a consumer is a one of these. Let p_j ($\forall j \in \{1,2,...,K\}$) denote the probability of having variant j in the choice set—that is, $\Pr\{u_j > u_0\}$. We denote u_j as the utility of consuming one unit of variant j. A customer is called a "no-purchasing-customer" if the choice set of the customer is empty. That is, the probability that a customer is a no-purchasing-customer is:

$$P_0 = \prod_{j=1}^{K}(1-p_j) \quad .$$

We can interpret P_0 as the loss of market share. It is clear that this probability is decreasing with the number of variants, *K*.

For identical customers, we assume that the consumer uncertainty of choosing variant *j* is captured by ε_j, which is independent and identically distributed following standard normal distribution. In line with all of the previous assumptions, the utility for the i^{th} customer to choose item *j* is:

$$u_j = \psi_j (1+\alpha_j)^{\gamma_j} \times e^{\varepsilon_j} \quad .$$

The no-purchasing utility for this customer is given by:

$$u_0 = \psi_0 (1+\alpha_0)^{\gamma_0} \times e^{\varepsilon_0} \quad .$$

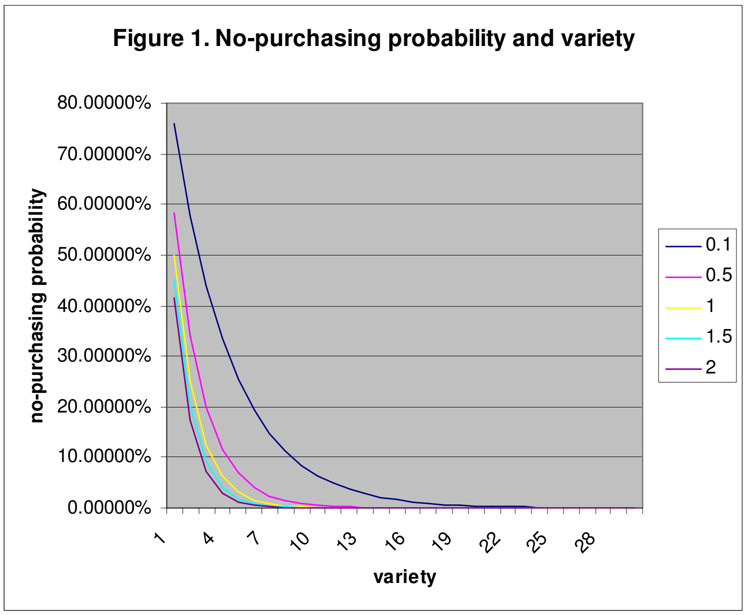
If we let $\delta_j = \dfrac{\psi_j (1+\alpha_j)^{\gamma_j}}{\psi_0 (1+\alpha_0)^{\gamma_0}}$, we have

$$p_j = P(u_j > u_0) = P(\varepsilon_j - \varepsilon_0 < \log \delta_j) = \Phi(\log \delta_j / \sqrt{2}),$$

where Φ is standard normal CDF (cumulative distribution function). As we assume identical products, $\delta_i = \delta_j = \delta (i \neq j)$.

$$P_0 = \prod_{i=1}^{K} (1 - p_i) = (1 - \Phi(\log \delta / \sqrt{2}))^K$$

It is easy to see that P_0 decreases as *K* increases. The relationship between P_0 and *K* is plotted in the Figure 1, assuming different δ.

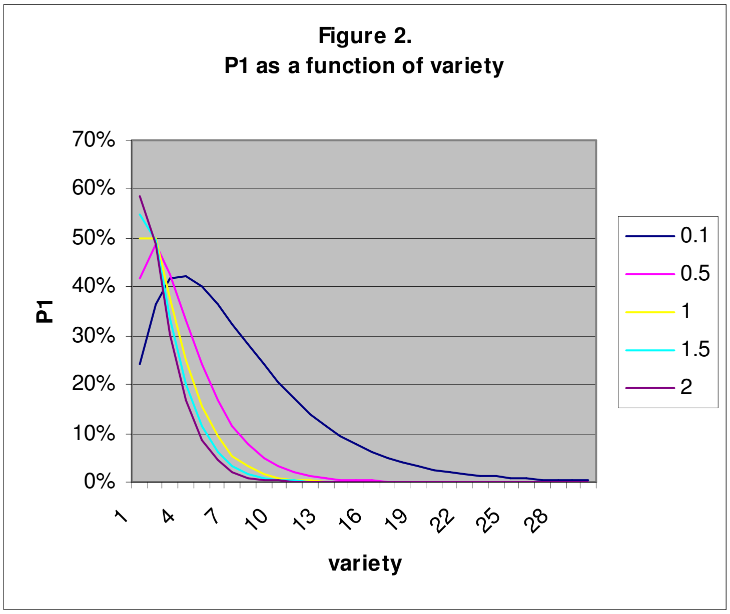


For example, $\delta = 1$ means that before customers' realization, the retailer predicts that each variant yields the same utility as the no-purchasing option. In other words, when considering one specific variant and the no-purchasing option, a customer is equally likely to purchase the single variant or decide not to purchase at all. In this case, if the seller increases variety from 9 to 10, P_0 will decrease by 0.1% (from 0.2% to 0.1%). In most cases, the utility of purchasing a variant will be larger than the no-purchasing utility ($\delta > 1$), and we can see that for a relatively small number of variants the no-purchasing probability is quite small.

In a similar fashion, we can calculate the probability that a customer is not willing to substitute. That is, if his or her first choice is not available, the customer will leave with not buying at all. Let us denote this probability as P_1.

$$P_1 = K * \Phi(\log\delta/\sqrt{2}) * (1-\Phi(\log\delta/\sqrt{2}))^{K-1}$$

Then we have the situation illustrated in Figure 2.



Notice that both P_0 and P_1 drop fast as variety increases. In other words, the probability $(1-P_0-P_1)$ of a customer being willing to substitute at least one time before selecting not to purchase and leave the system increases very quickly as variety increases. The fact that $1-P_0-P_1$ is large is important because we start by assuming that all consumers are willing to substitute at least one time. This is a valid assumption when the number of variants is relatively large.

4.2 An Approximation of Consumer Demands for Variants

In what follows, we develop an expression for the demand distribution for the different variants. These approximations will be used in determining the optimal number of products and the optimal inventory levels. We assume similar variants and identical customers, with each customer choosing variant $i, (i \in \{1,...,K\})$ as his or her first choice, with probability $\tilde{p}_i = \dfrac{1-P_0}{K}$. The initial demands (for the first-priority variant) of N customers are just samples of multinomial distribution with parameters $(N,(P_0,\tilde{p}_1,...,\tilde{p}_K))$. It is easy to calculate the correlation coefficient between any two variants' demands.

$$\rho_{ij}=\begin{cases} -\dfrac{1-P_0}{K-1+P_0} & i\neq j\neq 0 \\ -\sqrt{\dfrac{P_0}{K-1+P_0}} & ij=0, i+j\neq 0 \end{cases} \qquad i,j\in\{0,1,...,K\}\ .$$

Obviously, the correlation between any two variants' demands is negative (this is due to our assumption that the size of the population is fixed). The correlation approaches zero as variety increases because the no-purchasing probability decreases. Despite of this fact in all of our derivations, we treat the demands as if they are independent. In other words, treating the demands as independent to solve the problem gives us an upper bound solution. We will show shortly that despite simplifying the problem by assuming that the demands of the variants are independent we are still able to obtain the optimal solution and determine the optimal stocking levels

In the next section, we apply the assumption that demands are independent and approximate them as normal random variables. Let $D^i\{i=1,2,...,K\}$ denote the demand of initial first choices for variant i. D^i follows binomial distribution with parameters N and $\tilde{p}_i$. The expected number of customers having any one of the variants as a first choice is $N\tilde{p}_i = N(1-P_0)/k=\mu$, and the variance is $\sigma^2 = N\tilde{p}_i(1-\tilde{p}_i)=N(1-P_0)(K-1+P_0)/K^2$. We then approximate D^i by normal distribution with mean μ and variance σ^2, subject to the constraint that $\sum_{i=1}^{K} D^i = N$. In addition, as we approximate the distribution of the demand D^i $(i\neq j;\ \ i,j=1,2,...,K)$ with a normal distribution we assume that D^i is bounded by $[\mu-3\sigma,\mu+3\sigma]$, which happens with over 99% probability. In the same manner, for every *m* (1 < m < K), variants' total initial demand is bounded by $[\mu_m-3\sigma_m,\mu_m+3\sigma_m]$ with over 99% probability, where $\mu_m = nm(1-P_0)/K, \sigma_m^2 = nm(1-P_0)(K-m+mP_0)/K^2$. In Section 6, we will show that such an approximated demand framework is useful in proving the main theorem of the paper.

4. General Formulation

We start by solving the retailer profit-maximization problem. We will solve this multi-stage problem backward in the following way. First, we solve the allocation problem: the retailer knows not only the demand of each variant but also the individual consumer's preference. In addition, the inventory levels of all variants are known. Thus, the retailer in the first stage solves the allocation problem. That is, the retailer should allocate available inventory to consumers in such a way that sales are maximized. At the second stage (stage 2), after solving the allocation problem, we continue to develop a general expected profit maximization model, solving for optimal stocking levels, assuming randomized consumer utilities and a fixed number of variants. Finally, at the last stage, we identify the optimal number of variants (the assortment problem).

For a moment, let us assume that the set of variants the retailer has is $\overline{K}$:

$$\overline{K} = \{1,...,K\} \ .$$

Let *0* stands for the no-purchasing option. The size of the potential customer pool is *N*. We count consumers as potential customers if they are interested in the product (category). In other words, there is a variant of the product with utility larger than the no-purchase utility.

U denotes the utility matrix

$$U = [u_j^i] \qquad i \in \{1,...,N\}, j \in \{0,1,...,K\} \ ,$$

where u_j^i is the utility of the i^{th} customer consuming variant *j*.

We also assume that the retailer places an order quantity of $Q = (Q_{1,}Q_{2,} \dots, Q_K)$ for products, where $Q_i (i = 1,...,K)$ is the order for variant i. With realized utilities (U) for *N* customers, the following integer program can be used to determine the optimal allocation. The maximization solves the best sales a retailer can get with a specific realization of the utility:

$$f(Q,K,U) = \max \sum_{i=1}^{N} \sum_{j=1}^{K} x_j^i$$

$$\text{S.T}\quad \begin{aligned} &\sum_{j=0}^{k} x_j^i = 1, (i = 1,\dots, N) \\ &\sum_{j=1}^{k} x_j^i u_j^i > u_0^i, (i = 1,\dots, N) \\ &\sum_{i=1}^{n} x_j^i \le Q_j, (j = 1,\dots, K) \\ &x_j^i \in \{0,1\} \end{aligned} .$$

The first constraint follows the assumption that one customer buys at most one product. The second constraint prescribes that the product purchased has to have realized utility larger than the realized utility of the no-purchasing option. The last constraint is simply the inventory constraint, assuming that the seller orders $Q = (Q_{1,} Q_{2,} \dots, Q_K)$.

It is well known that such an integer program is equivalent to a transportation problem and that it can be solved as linear program. The above formulation assumes a specific realization of $N \times (K+1)$ random variables (utilities). Thus, given the utility preferences, variety, and purchasing quantities, it is easy to allocate the available inventory to consumers so that sales are maximized and consumers obtain a variant that has a utility larger than the no-purchase utility. The challenge is to find the optimal variety and purchasing quantity of each of the variants so that expected sales are maximized. This brings us to the development of the general profit maximization model that solves for optimal variety and stocking level.

$$\underset{k,Q}{\text{ax}} \quad -\tilde{c}\sqrt{K} + \Pi(K,Q)$$

$$\Pi(K,Q) = rE_U[f(Q,K,U)] - E_U\{h(Kq - f(Q,K,U)) - p(S_0 + N(1-P_0) - f(Q,K,U))\}$$

,

where

r :	Product unit price	p :	Penalty/unit shortage
c :	Purchasing cost/unit	$\tilde{c}$:	Constant factor of variety cost
h :	Holding cost/unit	P_0 :	Probability of no-purchasing

Kq is the total amount of inventory and $q = \frac{\sum_{i=1}^{k} Q_i}{K}$, where K is the number of variants.

$S_0 = nP_0$ denotes the penalty caused by the no-purchasing customers. Salvage value is assumed to be zero. $\tilde{c}$ is the cost incurred by the variety.

Such a problem is difficult to solve analytically. Intuitively, the two decision variables "variety" and "purchasing quantity" are not independent. That is, given a fixed population size, the increase in variety should increase the market size and is likely to decrease the order quantity for each variant. If we can establish their relationship, the problem can then be reduced to a one-dimension maximization problem that is easier to solve. To show that the general problem can indeed be simplified by approximation, we will quantify the approximated demand in the next section before we develop the main theorem in the paper.

5. The Inventory Problem

In this section, we will apply the approximated demand structure discussed in Section 4 to solve the inventory problem—i.e., we determine the optimal order quantity for each variant. At this stage, we assume that the number of variants is fixed. The objective of the retailer is to maximize expected profit. We have developed the expected profit function in the previous section. As mentioned above, it seems impractical to obtain closed-form expressions for the optimal values of the purchasing quantities and the number of variants. Since the retailer's objective is to maximize expected profits, and since all variants possess the same per-unit cost and revenue, it is sufficient to show that when the retailer purchases a total of *N* units (*N/K* units of each variant), then with a very high probability, the retailer satisfies the demands of all consumers. Thus, the retailer does not carry any inventory and does not face any shortages. Clearly, by doing so, the retailer maximizes his own expected profit.

In what follows, we will prove that purchasing *N/K* units of each of the variants is optimal (with a very high probability). In this case, no inventory is carried, and the demand of all *N* consumers is satisfied. We will first discuss some assumptions that are essential for our proofs. We will then present a lemma that is necessary for the main proof. Finally, we will present the main theorem, which shows that a very simple stocking policy is optimal almost always.

We consider the following assumptions that are fairly mild and practical.

I. *We consider K identical variants.*

II. *There are N customers where* $N > 18(K-2)$. Notice that if, for example, K = 10, then N > 144, which is satisfied in most markets.

III. *For customer* $i(i = 1,2,..,N)$*, there are at least two variants, j and l, such that:* $u_j^i > u_0^i, (i = 1,...,N)$ and $u_l^i > u_0^i, (i = 1,...,N)$. That is, for each customer, there are at least two variants with realized utility higher than the realized no-purchase utility.

We focus on the group of customers who are willing to substitute when their first-choice variant is out of stock. The reason is that the group of customers who have only one choice and the group who are no-purchasing customers are very small both decrease very fast as variety increases. This issue is discussed in Section 4.1.

For one specific utility realization of all customers, we have the following notations and definitions.

Definitions

$d_{ij} (i \neq j, \quad i, j \in \{1,...,K\})$ — **Pair demand**, which represents the group of customers having a first choice for product i followed by a second choice of product j.

Direct allocation — An allocation of inventory in which the following steps are applied:

Step 1: Variant $i(i = 1,...,K)$ is allocated only to consumers whose first preference is variant i.

Step 2*:* If, at the end of step 1, there are consumers whose first choices can't be satisfied and there is excess inventory of variant $i(i = 1,...,K)$, then excess inventory of variant i is allocated to consumers whose first choices were not satisfied (in step 1) and their second choices were variant i.

At the end of step 2, we say that ***incomplete status*** (IS) is reached.

Assuming that the retailer purchases exactly *N* units of inventory, an ***IS*** might be a state in which there is excess inventory but it cannot be allocated to consumers because it consists of variants that are not among the first and second choices of the consumers. In such a case, there are consumers that cannot satisfy their demand and purchase a variant of their first or second choice. It can also be a state in which the demand of all consumers is satisfied either by their first or second choices; in this case no inventory is carried.

Observe that direct allocation is the most natural way to allocate available inventory to consumers. It is remarkable that there is an allocation such that all customers get a product of either their first or second choice without carrying any safety stock.

Communicate — In any IS, we say that variant i and variant j communicate in this IS if the demand of d_{ij} and d_{ji} ($j \neq i$) are not satisfied by a single variant. That is, at least one customer gets variant i, and the same is true for variant j. The importance of the communicate property is as follows: Suppose we have reached the *IS* and variant *i* and variant *j* communicate. Also suppose that there is an excess of variant *i* and a shortage of variant j. Because the two demand sets communicate, it is clear that at least one consumer purchases variant *j* but that this demand can also be satisfied by variant *i*. By satisfying this demand with variant *i*, it is possible to reduce the shortage for variant *j* and at the same time reduce the inventory of variant *i*.

D_M — Sum of all pair demands with at least one choice that belongs to *M*, a set of variants. For example,

$$\text{if } M = \{i, j\}\text{, then } D_M = \sum_{\substack{l=1\\ l\neq i}}^{K} (d_{il} + d_{li}) + \sum_{\substack{l=1\\ l\neq j, l\neq i}}^{K} (d_{jl} + d_{lj})\,.$$

Lemma 1: *Assuming the conditions I-III hold, and the following stocking policy is applied: Total purchasing quantity is exactly N, and the stocking quantity of each variant is N/k.*

Then for any set M containing m $(1 \leq m \leq K-2)$ *variants and for any demand realization,* D_M *is almost always (over 99% probability) larger than the total inventory allocated to the m variants. That is,* $D_M > m\frac{N}{K}$ *with probability over 99%.*

Proof:

For a set *M* containing $m\,(1 \leq m \leq K-2)$ variants, the total number of pair-demands in D_M is $\frac{2(K-1)+2(K-m)}{2}m = 2mK - m - m^2$. The probability of selecting any of these pair-demands is $\frac{(2mK - m - m^2)}{K(K-1)}$. Using normal approximation, we find that the mean and variance of the demand in D_M are $\mu_m = \frac{N(2mk - m - m^2)}{K(K-1)}$ and $\sigma_m^2 = \mu_m (1 - \frac{2mk - m - m^2}{K(K-1)})$, respectively.

The minimum D_M (over all demand realizations), with over 99% probability, is $\mu_m - 3\sigma_m$. We are now ready to find the condition for *N* such that this minimum demand is always larger than the total inventory allocated to these *m* variants. That is, $\mu_m - 3\sigma_m > m\mu, \quad where \quad \mu = N / K$.

Inserting μ_m, σ_m, and μ, after simplifying the inequality, we have $N > 9f(m)$, where $f(m) = \frac{(2mK - m - m^2)(K^2 - K - (2mK - m - m^2))}{(mK - m^2)^2}$. Taking the derivative with respect to *m*, we find that for $1 \le m \le K-2$, $\frac{\partial f}{\partial m} < 0$. Thus, $f(m)$ is a decreasing function of *m*, and $N > 9 \max f(m) = 9f(1) = 18(K-2)$ ensures that $\mu_m - 3\sigma_m > m\mu$.

Lemma 1 ensures that the demand D_M cannot be satisfied with only the inventory allocated to the variants in *M*. Thus, a situation in which, after the direct allocation, the demand D_M is satisfied and an excess inventory of variant *i* $(i \in M)$ is impossible. Notice that if such a situation were possible, it is clear that some inventory of variant *i* is never allocated. Since we assume that the retailer stocks exactly *N* units of inventory, this will also mean that some demands are not satisfied.

Theorem 1 Assuming the existence of conditions I-III and the above stocking policy, then for any demand realization there exists an allocation (assuming simultaneous substitution) such that demand of all customers is satisfied, with high probability.

Proof: We provide an algorithm so that for every demand realization the *N* units of inventory are allocated to the *N* customers. We start with direct allocation. After step 2, it is impossible to allocate any inventory to consumers. By then, we reach an IS, which we will call IS_0. In IS_0, let E be the set of variants that have excess inventory, and let S be the set of variants that are short. Because the retailer stocks exactly N units of inventory, it is clear that having one of these sets empty while the other is not is impossible. If both sets are empty, the theorem is proved. Thus, we assume that the two sets are not empty.

1. Suppose $i \in E$. If there does not exist $j \in S$ such that i communicates with j, go to step 2 below. Otherwise select $j \in S$ such that i communicates with j, and use one unit of i to fulfill one unit shortage of j. Repeat, using variant i to satisfy the demand for variant j until it is impossible to continue. Go to step 3.
2. By Lemma 1, there exists $l^1 \notin S, (l^1 \neq i, j)$ such that i and l^1 communicate. Otherwise it means that variant i satisfies all of the demands for pair demands in which variant i is the first choice, which is in contradiction to Lemma 1. If there is no j such that $j \in S$ and l^1 communicates with j, then select l^2 such that l^2 and l^1 communicate. Repeat this search until l^a is chosen such that l^a communicates with some $j \in S$ and l^{a-1}. Since the "communicate" relationship is transitive (see Chen and Bassok, 2005, Lemma 2), i and j communicate. Transfer excess of i to j until this process can no longer continue. Go to step 3.
3. If there is no more excess, the theorem is proved. If there is excess, we reach a new IS. Update set E and S in the new IS. Select $i \in E$, and return to step 1.

We now show by contradiction that the above process must end with all inventory allocated and all demand satisfied. Suppose the process cannot continue. In this case, it must stop in step 2, where we cannot use the excess inventory to satisfy any shortage. It is easy to see by Lemma 1 that all variants can be separated into two groups, $\tilde{E}$ and $\tilde{S}$, such that $E \subseteq \tilde{E}$ and $S \subseteq \tilde{S}$, and that no variant in $\tilde{E}$ communicates with any variant in $\tilde{S}$. However, this is a violation of Lemma 1 because $\tilde{E}$ has excess inventory while all demands in $\tilde{E}$ are fulfilled by variants in $\tilde{E}$. In other words, $D_{\tilde{E}} < m\mu$ serves as a contradiction to Lemma 1, where m is the number of variants in $\tilde{E}$, and $m\mu$ is their total inventory. □

Also notice that with each step the total excess and shortage decrease. Thus, it is impossible to have infinite cycles, and so after a finite number of iterations all inventory is allocated and all demand is satisfied.

Recall that we have assumed, wrongly, that the demands are independent when they are actually negatively correlated. When demands are negatively correlated, the total inventory needed to satisfy the demand is decreasing. Yet, we just proved that all demand can be satisfied with the minimum possible inventory, and thus assuming independence of the demands has no effect on our main result.

6.1 Uncertain demand

As we assumed that *N* is known with certainty in the previous discussion, we will relax this assumption by the following corollary.

Corollary 1.1 *Suppose that the number of consumers, N, is random with CDF* F_N. *Then the optimal stocking level is* $\tilde{Q} = F_N^{-1}(\alpha)$, *where* $\alpha = \dfrac{r-c}{r+h}$. *The stocking level for each variant is then* $\dfrac{\tilde{Q}}{K}$.

Proof: For every realization n of N where $\tilde{Q} \geq n$ it is clear that all of the demand is satisfied and the excess inventory is equal to $\tilde{Q} - n$. Also, for $\tilde{Q} < n$ it is clear that the demand of $\tilde{Q}$ customers is satisfied and the shortage is equal to $n - \tilde{Q}$. Thus, it is easy to see that the expected profit function is identical to the profit function of a news vendor problem. □

Theorem 1, together with corollary 1.1, illustrates that safety stock is kept because of the uncertainty in the market size but not because of the uncertainty in consumer choice. It also demonstrates that when the market size is random it is possible to solve the problem by solving a simple news vendor problem.

6.2 Gap between simultaneous and sequential substitution problems

Theorem 2: Assume a ***sequential substitution*** problem in which *N* is known with certainty. Also, assume that *N/K* units of each variant are stocked. Let Λ be the number of customers whose demand is not satisfied (the number of units in inventory), and let $\tilde{\Lambda}$ be the percentage of customers whose demand is not satisfied. Then $\Lambda < \frac{3}{2}\sqrt{N}$ and $\tilde{\Lambda} \leq \frac{3}{2\sqrt{N}}$.

Proof: Notice that the direct allocation provides a feasible allocation to the sequential substitution problem. First, customers satisfy their demand with their first priority variant. When the first priority variant is not available, customers chose their second priority. If the second priority variant is not available, they do not purchase any product and leave the system.

Therefore, the maximum unsatisfied demand, after direct allocation, indeed serves as a gap between the sequential substitution and simultaneous substitution. Let us denote this gap by Λ.

We will now estimate the magnitude of Λ. Suppose that after direct allocation we have $m (1 \le m \le K-1)$ variants that are in shortage. By the demand approximation in Section 4.2 (and assuming that $P_0 = 0$), we know that with high probability (over 99%) the maximum shortage for these m variants is $3\sigma_m$. Therefore, we have

$$\Lambda = \max_{\substack{m \\ 1\le m\le K-1}} 3\sigma_m \ ,$$

where

$$\sigma_m^2 = N\frac{m(K-m)}{K^2} \ .$$

Obviously, the variance is maximized when $m = \frac{K}{2}$ and $\Lambda = \frac{3}{2}\sqrt{N}$.

Assuming our simple ordering policy and sequential substitution, we then know that the maximum shortage, or excess, as a fraction of the total customers, is $\frac{3}{2\sqrt{N}}$. For example, when $N = 10000$, then in the worse case only 1.5% of the customers are facing shortages, and at the same time only 1.5% of the goods are in excess.

Theorem 2 tells us that when the size of the market is large and known with certainty the expected shortage and excess inventory is relatively small. The above numbers present the worst possible case. We conjecture that in average expected shortages and excess inventory will be lower.

6.3 Unequal popularity

In the above discussion we assumed that all variants are "equally popular". By "equally popular" we mean that all variants have the same probability of being at a certain position in the utility ranking of each of the consumers. We will now show that this assumption can be relaxed. We say that variant i is more popular than variant j if the probability of i being on the top of the utility list is higher that of j. That is $\hat{p}_i > \hat{p}_j$.

With out loss of generality assume that $\hat{p}_1 > \hat{p}_2 ... > \hat{p}_K$ and let $\hat{p}$ be a base probability such that $\hat{p}_i = e_i \hat{p}$ and for i, j , $e_i > e_j$. Notice that there are many choices of base probability, $\hat{p}$, depends on the users' preference. For example, one can set $\hat{p} = \hat{p}_K$. However, the resulting stocking level for each variant will not be altered by different $\hat{p}$. From now on we will say that there are $\sum_{i=1}^{K} e_i$ "variants" (and not *K*). Clearly, some of these "variants" are identical to each other but we will consider them as different. Notice that now we have a problem with $\sum_{i=1}^{K} e_i$ "variants" and these variants are identical in the sense that they have the same probability to have the highest utility in the utility ranking list. Stocking $N / \sum_{i=1}^{K} e_i$ units of each of the "variants" and applying Theorem 1 ensures that we can satisfy all the demand without keeping any safety stock.

6. The Assortment Problem

As was mentioned earlier, variety is costly. On the other hand, higher variety has advantages, mainly decreasing the no-purchasing probability and increasing the market size. In what follows we take a simplified approach — we assume that the market size is known with certainty and is equal to NP_0 (which is, actually, the mean market size), where $P_0 = q^K$ and $q = 1 - \Phi(\log \delta / \sqrt{2})$. Since all the demand is satisfied without keeping any inventory, we need to consider the trade off between cost of variety and the revenue. We get the following profit function: $-\tilde{c}\sqrt{K} + (r-c)N(1-P_0) - pNP_0$. The first term represents the cost of variety. The second term represents the revenue less the purchasing cost and the third term represents the penalty due to not carrying enough variants and the inability to satisfy some of the consumers' demand. Substituting q^K for P_0, we get the following objective function: $Max_K \; -\tilde{c}\sqrt{K} + (r-c)N(1-q^K) - pNq^K$. In general, this objective function is not concave. Taking derivative with respect to *K* we get: $-\frac{\tilde{c}}{2\sqrt{K}} - (r-c)Nq^K Log(q) - pNq^K Log(q)$. A sufficient condition for optimality is: $-\frac{\tilde{c}}{2(r-c+p)NLog(q)} = q^K \sqrt{K}$. It is not difficult to see that for some values of the parameters there is no *K* that satisfies this condition. For example, when $\tilde{c}$ is very large the first derivative is always negative, meaning that it is not worth at all to produce any of the variants. On the other hand, for most reasonable values of the parameters it is possible to easily calculate the values of the optimal number of variants.

7. Special substitution structure --- Chain

Jordan and Graves (1995) study manufacturing flexibility analogous to the substitution problem. They configured limited flexibility in what we called the chain substitution structure. By simulation, they show that the chain substitution structure yields most of the benefits of total flexibility (substitution). In this section we will show similar result by applying theorem 1 to such structure.

We hold the following assumptions for chain substitution structure

Ia. *Considering K identical variants, the seller order the variety from 0 through K-1*

IIa. *There are N customers, where* $N > 9K^2/4$. Notice that if, for example, K = 10, then N > 225 which is satisfied in most markets.

IIIa. *For customer* $i(i = 1,2,.., N)$, *if his first choice is variant l and it is not available, he is willing to substitute with a second choice, variant l+1. If the fist choice is variant k-1, the second choice is variant 0.*

Notice that assumption IIIa specifies the rule of substitution, which is exactly the same as the limited flexibility defined by Jordan and Graves. The identical variants in the first assumption can be relaxed later just as what's discussed in section 6. Assumption IIa gives a mild condition for the number of customers. This is higher than that in section 6, which is intuitive because we have rules for substitution.

We follow the same definition for pair demand, direct allocation and incomplete status. Notice that communicate and D_M in this case become the following:

Communicate — In any IS, we say that variant i and variant j ($i, j \in \{0,..., K-1\}$) are communicate if pair demand $d_{i,i+1 \bmod K}$ is not satisfied by single variant in this IS.

D_M — Sum of demands in partial chain $M = \{i \bmod K,..., i+m-1 \bmod K\}$ $(K-1 \geq i \geq 0, m < K)$. For example, if $M = \{i \bmod K, i+1 \bmod K,..., j \bmod K\}$ $(i \leq K-1, i < j, j-i < K-1)$, then $D_M = \sum_{l=i}^{j+1 \bmod K} d_{l-1,l}$.

We also need to update Lemma 1 accordingly.

Lemma 1a: *Assume the conditions Ia to IIIa hold and the following stocking policy is applied: Total purchasing quantity is exactly N, and the stocking quantity of each variant is N/K. For any variant set* $M = \{i \bmod K, ..., i+m-1 \bmod K\}$ $(i \geq 0, m < k)$ *and for any demand realization,* D_M *is almost always (over 99% probability) larger than the total inventory allocated to variants* $M = \{i \bmod K, ..., i+m-1 \bmod K\}$ *. That is,* $D_M > m\frac{N}{k}$ *with probability over 99%.*

Proof:

For a set $M = \{i \bmod K, ..., i+m-1 \bmod K\}$ containing m $(1 \leq m \leq K-2)$ variants, the total number of pair-demands in D_M is $m+1$. The probability of selecting any of these pair-demands is $\frac{(m+1)}{K}$. Using normal approximation, we find that the mean and variance of the demand in D_M is: $\mu_m = \frac{N(m+1)}{K}$ and $\sigma_m^2 = \mu_m(1 - \frac{m+1}{K})$.

The minimum D_M (over all demand realizations), with over 99% probability, is $\mu_m - 3\sigma_m$. We are now ready to find the condition for *N* such that this minimum demand is always larger than the total inventory allocated to these *m* variants. That is, $\mu_m - 3\sigma_m > m\mu$, *where* $\mu = N/K$.

Insert μ_m, σ_m, μ and simplify the inequality, we have $N > 9f(m)$, where $f(m) = 9(m+1)(K-m-1)$. It is easy to show that $N > 9\max f(m) = 9K^2/4$ ensures that $\mu_m - 3\sigma_m > m\mu$.

By theorem 1 with Lemma 1a in place of Lemma 1, assuming the existence of conditions of Ia to IIIa and the above stocking policy, for any demand realization, there exists an allocation (assuming simultaneous substitution) such that demand of all customers is satisfied, with high probability.

8. Managerial Insight and Conclusion

This study is extremely helpful for managers in making assortment and inventory decisions for the following reasons: Typically, managers must make the inventory and assortment decisions assuming demand uncertainty. We show that there are two sources of demand uncertainty: 1. Uncertainty that is related to taste and choice of consumers (Consumer choice uncertainty), and 2. Uncertainty that is related to the size of the market (market size uncertainty). Different sources of uncertainty must be dealt differently. Consumer choice uncertainty is dealt by choosing the right assortment of variants while market size uncertainty is dealt by choosing the right inventory levels and safety stocks. For example, if the market size uncertainty is negligible then despite of the choice uncertainty it is possible to satisfy the demand of all consumer, with high probability, with no safety stock. In this case the right assortment is crucial to satisfy consumer demands and maximize profit. In addition, to the best of our knowledge we are the first to offer a simple closed form solution to the inventory problem that ensures that the optimal solution is achieved with a very high probability.

There are several possible venues for extending this work: a) bring into account the different product cost issue and develop appropriate inventory policy addressing the substitution effect; b) to investigate the role of variety in competition, in which profit margin may changed with the product-line length; c) to collect real demand and inventory data from some stores to verify the heuristic.

In conclusion, the major contribution of this paper has been to show that uncertainty in consumer preference can be eliminated by the substitution effect with a simple heuristic. Such approximation approach is fruitful in proving that when the market size is large the not keeping safety stock will result in a small percentage of consumers who are unable to obtain the variants of their desire. Inventory becomes much less of an issue than the right choice of product-line length. We are also able to extend this result to situations in which the market size is random. In this case we show that it is sufficient to solve a simple news-vendor problem to obtain the optimal stocking level. Finally we are able to develop a very simple model to calculate the optimal number of variants. This simple approach seems to provide a promising tool for the analysis of variety decisions, and can be extended to address future research issues in this area.

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