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Abstract

Flexibility measures the ability to adapt to change and often has multiple dimensions that impact value jointly yet differently. We assess this joint impact in a theoretical model of a two-product firm that makes capacity, output and pricing decisions at three points in time with an underlying continuous-time information evolution. The firm's ability to adapt is characterized by three types of flexibility. The cost of switching capacity between the two products measures the firm's mix flexibility. The fraction of product costs that are postponed until demand information is updated measures the firm's volume flexibility. Finally, the relative timing of the output decision measures the firm's time flexibility. We show that mix and volume flexibilities are substitutes in creating firm value but both are complementary to time flexibility. Furthermore, the marginal values of mix and time flexibility are decreasing in demand correlation whereas the marginal value of volume flexibility increases in demand correlation. We discuss the implications of these results to the trade-offs faced by managers when deciding how much to invest in different aspects of flexibility. We also relate these results to corporate strategy and show when different types of flexibility can justify a company to pursue market diversification.

(Key Words: flexibility, mix, volume, valuation, capacity, inventory, investment, real options, forecast updating.)

1 Introduction

Flexibility measures the ability to adapt to change and often has multiple dimensions that impact value jointly yet differently. This paper aims to contribute to the understanding of the value of flexibility by examining the relationships among mix, volume and time flexibility. Although significant research efforts have been extended towards classifying and measuring different types of flexibility and studying their values, relatively little progress has been made in understanding their relationships. According to Parker and Wirth (1999),

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"The real challenges for managers and researchers are not only to appreciate the existence of a variety of flexibility types but also the existence of relationships and trade-offs among them. ... Understanding the relationships among flexibility types is paramount for understanding the managerial task required to manage enterprise flexibility. Given this, it is perhaps surprising there has been so little research into these relationships or trade-offs." We attempt to bridge this gap in the literature by proposing continuous measures for three key dimensions of flexibility and investigating their relationships in a theoretical model.

We consider a two-product firm that makes capacity, output and pricing decisions at three decision epochs with an underlying continuous-time information evolution. First, the firm chooses product-specific capacity levels based on an imperfect forecast of the future demand curves. Second, as the selling season approaches, the firm updates its forecast and at a certain point of time, called the *update time*, locks in output levels of its two products. The output of each product is constrained by the existing product-specific capacity but the firm has the option to convert, at a cost, one product-specific capacity to another. Finally, when the selling season comes and the demand curve uncertainty is resolved, the firm sets prices and realizes profit. We characterize the firm's optimal strategy in terms of its capacity, output and pricing decisions assuming the firm is an expected value maximizer.

Flexibility measures adaptability to dynamically evolving conditions. In our model, flexibility to adapt to additional information stems from three abilities, each modeled by a continuous parameter:

1. *Mix flexibility*, also known as product flexibility, is the ability to switch production among different products. In our model it is measured by the cost of switching one unit of specialized capacity to the other product. The lower this cost, the higher mix flexibility.

2. *Volume flexibility* is the ability to change production volume. In our model, it is measured by the fraction of total unit costs incurred after the update time. Two arguments explain why volume flexibility is high if this fraction is large. If unit capacity costs are small relative to marginal output costs incurred after the update time, the firm will invest in a relatively high capacity level that is unlikely to constrain output volume. In addition, the bulk of unit cost is still variable at the update time so that the firm has the ability and economic incentive to adapt its volume to the updated forecast. Other factors that determine this measure of volume flexibility include capital intensity of the production process or the ability to change supply quantities and adjust capacity.

3. *Time flexibility*, also known as postponement flexibility, is the ability to delay output decisions thereby benefiting from information updating. In our model, it is measured by the update time relative to the timing of capacity selection and the selling season. Other factors that determine a firm's time flexibility include its supply and production lead times, machine setup and changeover times, and/or the point of product differentiation within the production process.

In real options terminology, mix and volume flexibilities correspond to two distinct real options – the option to switch one product-specific capacity to another and the option to choose capacity utilization, respectively. The firm's time flexibility represents the "duration" of these two options. We show that while the mix and volume flexibilities are strategic substitutes in creating firm value, they are both complementary with time flexibility. Furthermore, the marginal values of mix and time flexibility are decreasing in demand correlation but the marginal value of volume flexibility increases in demand correlation. As we will explain in sections 5 and 6, these results have important implications for the trade-offs faced by managers when deciding how much to invest in different aspects of flexibility and for the value of market diversification. The empirical economic literature has recognized that market diversification coupled with technological or organizational flexibility may create value by reducing uncertainty in capacity planning through demand pooling. In addition to showing that the value premium for diversification increases in mix and time flexibility and decreases in volume flexibility and demand correlation, we provide insights into the relationships among these parameters with important repercussions for the trade-offs between related and unrelated diversification. These findings provide several hypotheses that can be tested empirically.

Our model is a stylized representation of a multi-product firm that faces substantial demand uncertainty during its short product life cycles and whose output is constrained by its irreversible capacity investment decision. The firm can mitigate the ensuing risk exposure by delaying the capacity utilization and output mix decisions. Besides automobile manufacturing, examples include the computer, apparel, and ski manufacturing industries. Leading computer manufacturers such as Dell have highly flexible in-house production processes that assemble "vanilla-box" computers in anticipation of market demand, but only differentiate (customize) them to market demand at the point of sale. A major printer manufacturer HP allocates the printers produced in its Vancouver plant in anticipation of worldwide demand to specific markets only a few weeks before sales. Apparel producers such as Benetton produce uncolored ("in greggio") garments long ahead of the sales season and postpone their dyeing until the sales season approaches. The leading Norwegian ski manufacturer Madshus is about to start making skate boards using the same core production process that is used for ski manufacturing.

The remainder of this paper is structured as follows. After the relevant literature is reviewed in Section 2, Section 3 presents the model which is analyzed in Section 4. The flexibility premium and the interplay among different dimensions of flexibility and market parameters are examined in Section 5. Section 6 discusses the link between flexibility and market diversification. Section 7 concludes. All proofs are relegated to Appendix 2.

2 Relation to the Literature

This paper relates to three literatures: the operations management literature on flexibility, the real options literature and the corporate strategy and economics literature on market diversification.

The operations management literature on flexibility is extensive and has paid particular attention to the problem of optimal investment in product-flexible resources. The first article showing the benefits of demand pooling in reducing the cost of under/overinvestment is Eppen (1979). Fine and Freund (1990) study the optimal investment in a mix of product-flexible and product-dedicated resources. Van Mieghem (1998) identifies the revenue maximizing option embedded in flexibility which differs from risk pooling. It explains why flexibility remains valuable even under perfectly positively correlated demands provided that the product profit margins are not identical. Bish and Wang (2004) and Chod and Rudi (2005a) study the optimal investment in a product-flexible resource by a firm with contingent pricing power. Tomlin and Wang (2005) investigate how dual sourcing can alleviate the loss of redundancy in product-flexible resources in unreliable networks under loss aversion and conditional value-at-risk objectives. Van Mieghem (2006) shows how risk aversion and the position of a product-flexible resource affects network configuration. Closely related to the concept of mix flexibility is the postponement of product differentiation. A comprehensive review of the postponement literature can be found in Swaminathan and Lee (2003). The resource management literature that studies the benefits of demand pooling under information updating includes Eppen and Iyer (1997), Aviv and Federgruen (2001), Petruzzi and Dada (2001) and Chod and Rudi (2005b).

The time aspect of flexibility has been captured by several papers that study "quick response" in inventory management for short life-cycle products. Fisher and Raman (1996) and Fisher et al. (2001) study the option of a retailer to reorder inventory after observing initial demand, demonstrating a considerable reduction in uncertainty cost. Iyer and Bergen (1997) examine the option of a retailer to postpone the order decision until additional demand information becomes available in a supply chain context. Van Mieghem and Dada (1999) analyze capacity, output and pricing decisions of a price-setting firm that faces a stochastic demand curve, considering several strategies differentiated by the sequence in which these decisions are made and uncertainty is resolved.

Many other types of operational flexibility (e.g., machine, sequencing, volume flexibility) have been described in the operations management literature but relatively little attention has been devoted to analyzing and valuing their economic benefits. The relationships among different flexibility types have been studied even less with a few notable exceptions: Browne et al. (1984) established the basic taxonomy of various flexibility types indicating the hierarchical relationships among them (e.g., machine flexibility is necessary for product flexibility). A comprehensive simulation study that shed light on many relationships among flexibility types was undertaken by Gupta and Goyal (1992) who examined the effect of various manufacturing system configurations on operational performance measures such as machine idle time and job waiting time. These relationships are further discussed by Parker and Wirth (1999).

The relationship between two fundamental flexibility types has been recently addressed in the theoretical model of Goyal and Netessine (2005). They consider the optimal technology and capacity choice of a two-product price-setting firm that faces uncertain demand functions in a two-step recourse model in which the firm produces at capacity. Available technologies may involve product and/or volume flexibility. Their product flexibility is perfect in the sense that it allows costless capacity allocation to products once uncertainty is resolved. (Our model allows a switching cost at the update time and a remaining production cost and time before perfect resolution of uncertainty.) Their notion of volume flexibility is that capacity levels can be adjusted up or down once uncertainty is resolved, but at a quadratic cost. In addition to discussing how different demand and cost parameters affect the optimal technology choice, Netessine and Goyal show that adding product flexibility to volume flexibility does not necessarily benefit the firm, even if it is costless, because of possible diseconomies of scope.

A general framework for modeling and analysis of flexibility was proposed by De Groote (1994) who defines flexibility of technology and diversity of environment generally as two complementary factors affecting a firm's performance. The operations literature on manufacturing flexibility has been surveyed by Sethi and Sethi (1990), Gerwin (1993) and Beach et al. (2000). The empirical work in this area has been reviewed by Vokurka and O'Leary-Kelly (2000).

The real options literature focuses on the valuation of flexibility mainly in a continuous-time framework using contingent claims pricing. The closely related paper by Triantis and Hodder (1990) develops a model to value investments in product-flexible production systems assuming an exogenous capacity level and stochastic demand functions. Production decisions are made at pre-set points in time maximizing the value of the manufacturing program at the beginning of each period. Triantis and Hodder obtain a set of analytical expressions to determine firm value in the two-product case, and suggest a Monte Carlo simulation to find this value in the case with more than two products. They show that the value of a flexible production system decreases in demand correlation, increases in demand volatility and increases in the frequency at which production mix adjustments are made. The last effect is more pronounced when demand correlation is low and demand volatility is large. The frequency of production mix adjustments is somewhat related to our concept of time flexibility as it measures the degree of uncertainty at the time production mix is adjusted. Triantis and Hodder also consider a fixed cost of production mix adjustment which is incurred, unlike our switching cost, no matter whether switching takes place or not.

He and Pindyck (1992) examine the technology and capacity choice of a two-product firm facing stochastic demand functions as a stochastic control problem. The firm can install product-specific capacity, or, at greater cost, flexible capacity that can be used to produce both products. The product mix is varied continuously and also capacity may be increased continuously. The literature on financial valuation of flexibility and real options has been surveyed by Bengtsson (2001). As extensive review of the option pricing literature has been recently undertaken by Broadie and Detemple (2004).

The corporate strategy literature studies motivations for companies to diversify.[4] Diversification reduces firm-specific risk which may be desirable in the presence of capital market imperfections. Corporate headquarters may be in a better position than the external capital markets to identify and cross-subsidize promising business opportunities. Moreover, generating funds internally avoids the transaction and agency costs associated with using external capital markets.[5] Diversification may also enable a higher utilization of assets such as sales staff, R&D or brand name that cannot be sold off because of their indivisibility or transaction cost but can be shared across several business units. Recent surveys of the strategic management and financial economics literatures on corporate diversification are Palich et al. (2000) and Martin and Sayrak (2003), respectively.

Among the few articles that explain corporate diversification with resource flexibility is the influential qualitative article of Teece (1982). It studies a firm that chooses a product mix according to constantly changing market conditions which create opportunities in different markets at different times. Levy and Haber (1986) exemplify this idea in a simple model of a firm that faces uncertainty in terms of relative profitability of its two products and has a fixed capacity of a key production resource. Levy and Haber illustrate graphically when the multiproduct firm is more profitable than two single-product firms. Haber and Levy (1988) formulate a deterministic program of cost minimization for a multiproduct firm that uses several flexible resources. Deriving the first order conditions for this allocation problem, they argue that the information required for the optimal allocation is hardly exchangeable in the market, suggesting superiority of a multiproduct firm over a comparable group of single-product firms. In the model of Von Ungern-Sternberg (1989), firms use flexibility together with diversification to hedge against price uncertainty in two perfectly competitive markets. Smith and Triantis (2001) show, through a numerical example, that a merger can create value as it enables a more efficient use of flexible capacity provided that the firms' demands are not too (positively) correlated. Matsusaka (2001) relates the value of diversification to resource flexibility by considering a value maximizing firm that possesses organizational capabilities that are transferable across products and industries. The productivity in each industry is ex ante uncertain and can only be ascertained by experimentation. Diversification then is a search for businesses that are good matches for the firm's

[4] There is significant empirical evidence suggesting that diversification generally destroys the firm value (see e.g., Lang and Stulz 1994, Berger and Ofek 1995, Comment and Jarrell 1995, and Servaes 1996) primarily because internal capital markets encourage investing in value destroying projects. Nevertheless, firms do diversify and often with success, General Electric being a classical example.

[5] The ability to reallocate financial capital among corporate divisions based on their profitability potential can be viewed as a specific form of resource or "mix" flexibility with the resource being financial capital.

capabilities. We formalize the idea that in the presence of demand uncertainty and transaction costs in the factor market, resource flexibility may justify corporate diversification in a rigorous mathematical model that links the diversification premium to three dimensions of flexibility.

The merit of related versus unrelated diversification has attracted significant research attention. Related diverification is defined as one "involving businesses that share related production or marketing technologies" (Lubatkin and O'Neill 1987). While the resource sharing argument favors related diversification, the risk and internal capital market considerations support unrelated diversification. The extensive empirical evidence on the effect of diversification relatedness is also very fragmented (see e.g., Palich et al. 2000). We address this issue by proposing distinct measures for market and technological relatedness of diversification and discussing their relationship.

3 The Model and Research Question

Consider a two-product firm that must make three decisions at three different points in time. Information availability and uncertainty, which are crucial to any investment strategy, are formalized by a standard probabilistic framework with a probability space $(\Omega, \mathcal{F}, P)$ and filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ as primitives. (The filtration $\mathbb{F}$ is an increasing family of sub-σ-fields that specifies how information arrives and uncertainty is resolved as time passes, with $\mathcal{F}_t$ representing the information available at time t.) Expectation conditional on $\mathcal{F}_t$ is denoted as $\mathbb{E}_t$.

At time 0, the firm must choose a vector $\mathbf{K} \in \mathbb{R}_+^2$ of two product-specific capacity levels based on the information then available and on its assessment of the uncertain future. Capacity is defined as the maximum output that can be produced given the firm's choice of capital, workforce, inventory or production levels. Following tradition in the irreversible investment literature, we assume that the capacity investment incurs a constant net marginal capacity cost c_K. Thus, c_K is adjusted for any residual value of capacity at time T. We assume, for simplicity, that all costs, capacity consumption rates as well as demand parameters are identical for the two products.

At the update time $\tau \in [0, T]$, the firm must decide on the actual output vector $\mathbf{Q} \in \mathbb{R}_+^2$ that will be available for sales at time T. Transformation of one unit of capacity into a unit of output incurs a constant marginal output cost c_Q, which includes any production costs incurred after the update time. Although the aggregate output cannot exceed total capacity, i.e., $Q_1 + Q_2 \leq K_1 + K_2$, the firm has the option to convert, or "switch", capacity i to capacity $j \neq i$. Because capacity i may be less cost-efficient at producing product j, switching may increase the marginal output cost by $c_S \geq 0$, which we refer to as switching cost.

Finally, at time T, production is complete and uncertainty is resolved. The firm sets output prices $\mathbf{p} \in \mathbb{R}_+^2$ and earns a revenue $\pi(\mathbf{p}, \mathbf{Q}) = \mathbf{p}'\mathbf{q}(\mathbf{p}, \mathbf{Q})$, where $\mathbf{q}(\mathbf{p}, \mathbf{Q}) \leq \mathbf{Q}$ is the optimal output vector to bring to market, and the prime denotes transpose. Given that no model dynamics occur after the start of the sales season, we compress the latter into an instantaneous sales event at time T, after which the firm is liquidated. We suppress the time value of money so that the firm terminal value, denoted as $v(T)$, is equal to the sales revenue minus the capacity investment, capacity switching and output costs:

$$v\left(T;\mathbf{K},\mathbf{Q},\mathbf{p}\right)=\pi(\mathbf{p},\mathbf{Q})-\sum_{i=1}^{2}\left(c_K K_i+c_S\max\left(Q_i-K_i,0\right)+c_Q Q_i\right). \quad (1)$$

We assume that the firm is a rational agent that makes the three decisions, $(\mathbf{K},\mathbf{Q},\mathbf{p})$, with the objective to maximize its value. The firm value at time $t\in[0,T]$, denoted as $v\left(t\right)$, is the expectation of the firm terminal value conditional[6] on information available then, i.e., $v\left(t\right)=\mathbb{E}_t v\left(T\right)$. The optimal decisions are found working backward and the Bellman optimality equations simplify as follows:

$$\begin{aligned}
\mathbf{p}^*\left(\mathbf{Q}\right) &= \operatorname*{arg\,max}_{\mathbf{p}\in\mathbb{R}_+^2}\ v\left(T;\mathbf{K},\mathbf{Q},\mathbf{p}\right), && (2)\\
\mathbf{Q}^*\left(\mathbf{K}\right) &= \operatorname*{arg\,max}_{\mathbf{Q}\in\mathbb{R}_+^2,\mathbf{1'Q}\le\mathbf{1'K}}\ v\left(\tau;\mathbf{K},\mathbf{Q},\mathbf{p}^*\left(\mathbf{Q}\right)\right), && (3)\\
\mathbf{K}^* &= \operatorname*{arg\,max}_{\mathbf{K}\in\mathbb{R}_+^2}\ v\left(0;\mathbf{K},\mathbf{Q}^*\left(\mathbf{K}\right),\mathbf{p}^*\left(\mathbf{Q}^*\left(\mathbf{K}\right)\right)\right), && (4)
\end{aligned}$$

where $\mathbf{1'}=(1,1)$. An optimal strategy $(\mathbf{K}^*,\mathbf{Q}^*,\mathbf{p}^*)$ is a solution $(\mathbf{K}^*,\mathbf{Q}^*(\mathbf{K}^*),\mathbf{p}^*(\mathbf{Q}^*(\mathbf{K}^*)))$ to (2)–(4). Finally, we let $v^*\left(t\right)=v\left(t;\mathbf{K}^*,\mathbf{Q}^*,\mathbf{p}^*\right)$ be the firm value at time t, when the optimal strategy is followed.

For concreteness and tractability, we assume that in each of the two product-markets, the firm is a monopolist facing an iso-elastic demand curve that is subject to a multiplicative random shock $\epsilon_i\left(T\right)$, $i=1,2$. Thus, the inverse demand curve in market i at time T is

$$p_i=\epsilon_i\left(T\right)q_i^{1/b},$$

where $b\in(-\infty,-1)$ is the constant price elasticity of demand. Notice that willingness to pay increases in ϵ_i, to which we informally refer as "larger ϵ_i is better."

The random shock $\boldsymbol{\epsilon}=\{\boldsymbol{\epsilon}\left(t\right),t\geq 0\}$ is assumed to follow a geometric Brownian motion.[7] Thus, $\ln\boldsymbol{\epsilon}\left(t\right)$ is a bivariate normal random vector with mean $\ln\boldsymbol{\epsilon}\left(0\right)$ and covariance matrix $t\mathbf{\Sigma}$, where $\mathbf{\Sigma}_{ii}=\sigma^2$ and $\mathbf{\Sigma}_{ij}=\rho\sigma^2$ if $i\neq j$. The information available at time t includes the history of $\boldsymbol{\epsilon}\left(t\right)$ or, formally, the filtration $\mathbb{F}$ is generated by ϵ. We refer to $\boldsymbol{\epsilon}$ as the *demand shock forecast evolution*, and to $\boldsymbol{\epsilon}\left(T\right)$ as the actual demand shock.

Our objective is to assess the firm's flexibility which stems from the following three dimensions:

[6] Thus P is an equivalent, risk-neutral probability measure.

[7] The choice of the geometric Brownian motion for $\boldsymbol{\epsilon}$ can be justified on the grounds of the Theory of Proportional Effect as formalized by the multiplicative Martingale Model of Forecast Evolution. See Chod and Rudi (2005b) and references therein for more discussion.

1. *Mix flexibility* φ measures the firm's ability to convert one type of capacity to another after updating the demand forecast. Since this ability depends on the switching cost c_S, we proxy mix flexibility by the continuous switching cost: $\varphi = 1/c_S$. Without mix flexibility ($\varphi = 0$), the firm does not have the option to switch capacity. With perfect mix flexibility ($\varphi \to \infty$), both products rely on the same capacity.

2. *Volume flexibility* γ measures the ability to adjust product volumes after the demand forecast is updated. We proxy volume flexibility by the (continuous) fraction of the total unit output cost that is incurred after the update time: $\gamma = c_Q / (c_K + c_Q)$. Without volume flexibility ($\gamma = 0$), the firm will always utilize full capacity, i.e., the aggregate output is effectively determined at time 0. With perfect volume flexibility ($\gamma = 1$), the firm's output is not constrained by capacity.

3. *Time flexibility* τ measures how long the firm can wait before finalizing the output decision. Without time flexibility ($\tau = 0$), the firm has to lock in the output quantities ex ante. With perfect time flexibility ($\tau = T$), the firm can postpone the output decision until all uncertainty is resolved.

4 The Optimal Strategy

We solve for the optimal strategy by backward induction starting with the pricing decision (2). It is a well-known property of the iso-elastic demand function that a monopoly always maximizes its revenue from a given output by selling *all* units at the market clearing price. In other words, $\mathbf{q}(\mathbf{p}^*(\mathbf{Q}), \mathbf{Q}) = \mathbf{Q}$ and

$$p_i^*(\mathbf{Q}) = \epsilon_i(T) Q_i^{1/b}, \qquad i = 1, 2. \tag{5}$$

Therefore, the firm's revenue under the optimal pricing policy is $\pi(\mathbf{p}^*(\mathbf{Q}), \mathbf{Q}) = \sum_{i=1}^{2} \epsilon_i(T) Q_i^{1+1/b}$.

The optimal output vector (3) is chosen to maximize the firm value at time τ, which can be written as

$$v(\tau; \mathbf{K}, \mathbf{Q}, \mathbf{p}^*(\mathbf{Q})) = \sum_{i=1}^{2} \left(\mathbb{E}_\tau \epsilon_i(T) Q_i^{1+1/b} - c_Q Q_i - c_S \max(Q_i - K_i, 0) - c_K K_i \right). \tag{6}$$

Iso-elastic demand functions have the property that marginal revenue tends to infinity as the product quantity approaches zero. It is therefore always optimal to produce a positive quantity of each product and, consequently, the non-negativity constraint in the capacity allocation problem (3) is non-binding and can be omitted. Since $b < -1$, i.e., $0 < 1 + 1/b < 1$, the objective function of the capacity allocation problem (6) is strictly concave in $\mathbf{Q}$, and the optimal output vector $\mathbf{Q}^*(\mathbf{K})$ is the unique solution to the Kuhn-Tucker

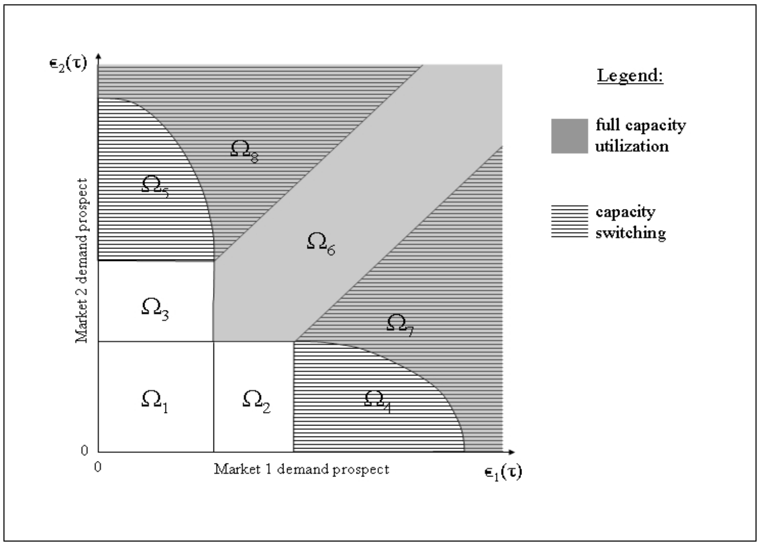

Figure 1: The state space of the demand prospects is partitioned into eight events that specify different optimal production vectors. If $\boldsymbol{\epsilon}(\tau) \in \Omega_{678}$, the total capacity is fully utilized. If $\boldsymbol{\epsilon}(\tau) \in \Omega_{4578}$, some capacity conversion or switching is optimal.

optimality conditions:

$$\begin{aligned} \nabla_{\mathbf{Q}} v(\tau; \mathbf{K}, \mathbf{Q}, \mathbf{p}^*(\mathbf{Q})) + \lambda \nabla_{\mathbf{Q}} (Q_1 + Q_2 - K_1 - K_2) &= \mathbf{0}, \\ \lambda (Q_1 + Q_2 - K_1 - K_2) &= 0, \\ Q_1 + Q_2 - K_1 - K_2 &\leq 0, \\ \text{and } \lambda &\leq 0, \end{aligned}$$

where λ is the Lagrangian multiplier for the capacity constraint. The optimal output vector $\mathbf{Q}^*(\mathbf{K})$ depends on the existing capacity $\mathbf{K}$, switching cost c_S, output cost c_Q, and the realization of $\boldsymbol{\epsilon}(\tau)$ that captures the demand prospects as they appear at time τ. To characterize the optimal output vector, we partition the state space of the market demand prospects $\boldsymbol{\epsilon}(\tau)$ into eight events $\Omega_1, ..., \Omega_8$ as illustrated in Figure 1. We also define $\Omega_{i_1 i_2 \cdots i_n} \equiv \Omega_{i_1} \cup \Omega_{i_2} \cup \cdots \cup \Omega_{i_n}$. (The formal definitions of $\Omega_1, ..., \Omega_8$ as well as the corresponding optimal output vectors are relegated to Appendix 1.) Recall that "larger ϵ_i is better" so that the intuitive interpretation of these eight events is as follows:

Event $\Omega_1(\mathbf{K})$: The prospects of both markets are poor and neither capacity is fully utilized: $Q_i^* < K_i$, $i = 1, 2$.

Event $\Omega_2(\mathbf{K})$: The prospects of market 1 are relatively good while those of market 2 are poor. Capacity 1 is fully utilized but capacity 2 is not and no capacity switching occurs: $Q_1^* = K_1$ and $Q_2^* < K_2$. (The event Ω_3 is symmetric: $Q_1^* < K_1$ and $Q_2^* = K_2$.)

Event $\Omega_4(\mathbf{K})$: The prospects of market 1 are very good so that not only full capacity 1 but also some of capacity 2 is used to make product 1: $Q_1^* > K_1$. The prospects of market 2, however, are poor so that only a fraction of the remaining capacity of resource 2 is used for product 2: $Q_2^* < K_2 - (Q_1^* - K_1)$. (The event Ω_5 is symmetric: $Q_2^* > K_2$ and $Q_1^* < K_1 - (Q_2^* - K_2)$.)

Event $\Omega_6(\mathbf{K})$: The prospects of both markets are good enough to justify full utilization of both resources and not sufficiently different to justify any capacity conversion: $Q_i^* = K_i$, $i = 1, 2$.

Event $\Omega_7(\mathbf{K})$: The overall market prospects are very good so that both resources are fully utilized. Furthermore, the prospects of market 1 are significantly better than those of market 2 warranting some capacity 2 conversion to product 1: $Q_1^* > K_1$ and $Q_2^* = K_2 - (Q_1^* - K_1)$. (The event Ω_8 is symmetric: $Q_2^* > K_2$ and $Q_1^* = K_1 - (Q_2^* - K_2)$.)

The capacity investment problem (4) is concave and so the optimal capacity vector is characterized by the first order condition. Furthermore, if the switching cost c_S is positive, the optimal capacity is unique.[8]

Proposition 1 *If $c_S > 0$, there exists a unique optimal capacity vector $\mathbf{K}^*$ which satisfies $K_1 = K_2$ and*

$$\begin{aligned} & \Pr(\Omega_{78}(\mathbf{K}))\,\mathbb{E}\left(\sum_{i=1}^{2} \frac{\partial Q_i^*}{\partial K_1} \frac{\partial (p_i^* Q_i^*)}{\partial Q_i^*} - c_Q - c_S \frac{\partial |Q_1^* - K_1|}{\partial K_1} \middle| \Omega_{78}(\mathbf{K}) \right) \\ + \quad & \Pr(\Omega_{26}(\mathbf{K}))\,\mathbb{E}\left(\frac{\partial (p_1^* K_1)}{\partial K_1} - c_Q \middle| \Omega_{26}(\mathbf{K}) \right) + \Pr(\Omega_4(\mathbf{K}))\, c_S = c_K, \end{aligned} \tag{7}$$

where $\mathbf{p}^* = \mathbf{p}^*(\mathbf{Q}^*(\mathbf{K}))$ is given by (5) and $\mathbf{Q}^* = \mathbf{Q}^*(\mathbf{K})$ is given in Appendix 1.

Optimality equation (7) sets marginal capacity value equal to its marginal cost c_K. A marginal increase in capacity 1 affects firm value only when resource 1 is fully utilized, which is in events $\Omega_2, \Omega_4, \Omega_6, \Omega_7$ and Ω_8. There are three cases to distinguish:

1. If all capacity is used and capacity switching takes place ($\boldsymbol{\epsilon}(\tau) \in \Omega_{78}$), an additional unit of capacity 1 will be split between *both* products in such ratio that the difference between their expected marginal revenues is equal to the switching cost. The sum of expected marginal revenues minus a unit output cost c_Q and the appropriate switching cost gives the marginal value of capacity 1. The appropriate switching cost depends on which capacity is being converted: If capacity 2 is being converted to capacity 1 ($\boldsymbol{\epsilon}(\tau) \in \Omega_7$), an additional unit of capacity 1 will decrease the amount of switching ($\partial |Q_1^* - K_1| / \partial K_1 < 0$) and hence save some switching cost. In contrast, if capacity 1 is being converted to capacity 2 ($\boldsymbol{\epsilon}(\tau) \in \Omega_8$),

[8] Note that if capacity can be switched between the products costlessly ($c_S = 0$), only the *total* capacity, $K_1 + K_2$, matters. In that case, there is a continuum of optimal capacity vectors characterized by the first order necessary and sufficient conditions, all of which represent the same total capacity.

an additional unit of capacity 1 will further increase the amount of switching ($\partial \left|Q_1^* - K_1\right| / \partial K_1 > 0$) and thus add to the switching cost.

2. If capacity 1 is fully utilized and no switching occurs ($\boldsymbol{\epsilon}(\tau) \in \Omega_{26}$), an additional unit of this capacity will increase the revenue from product 1 after incurring a unit output cost c_Q.

3. If capacity 2 is not fully utilized but some of it is converted to capacity 1 ($\boldsymbol{\epsilon}(\tau) \in \Omega_4$), an additional unit of capacity 1 reduces capacity conversion and saves a unit switching cost c_S.

In the boundary case that both the marginal output cost and the switching cost are zero ($c_Q = c_S = 0$), the optimal capacity and firm value can be characterized in closed form:

Corollary 1 *If $c_Q = c_S = 0$, the optimal total capacity and firm value are, respectively,*

$$\begin{aligned} K_1^* + K_2^* &= \left[\frac{1+1/b}{c_K} \mathbb{E}_0 \left(\left(\mathbb{E}_\tau^{-b} \epsilon_1(T) + \mathbb{E}_\tau^{-b} \epsilon_2(T) \right)^{-1/b} \right) \right]^{-b}, \\ \textit{and } v^*(0) &= \frac{c_K}{|1+b|} \left(K_1^* + K_2^*\right). \end{aligned}$$

Finally, to evaluate the benefits of flexibility, we consider the no-flexibility case as a reference case. A firm that has no flexibility must choose its output vector $\tilde{\mathbf{Q}}$ ex ante, together with the capacity vector $\tilde{\mathbf{K}}$. (Obviously, both are equal then.) Let $\tilde{\mathbf{p}}$ be the price vector set by the non-flexible firm. Flexibility typically comes at a cost which is reflected by lower capacity and production costs: $\tilde{c}_K \leq c_K$ and $\tilde{c}_Q \leq c_Q$. Except for possibly lower capacity and output costs, the non-flexible firm is a limiting case of the flexible firm with no time flexibility ($\tau = 0$) or, alternatively, with no mix and volume flexibilities ($\varphi = 0$ and $\gamma = 0$). Let $\tilde{v}\left(t; \tilde{\mathbf{K}}, \tilde{\mathbf{Q}}, \tilde{\mathbf{p}}\right)$, or $\tilde{v}(t)$ for short, denote the value of the non-flexible firm at time t, and let $\tilde{v}^*(t) \equiv \tilde{v}\left(t; \tilde{\mathbf{K}}^*, \tilde{\mathbf{Q}}^*, \tilde{\mathbf{p}}^*\right)$ be its value if the optimal strategy is followed. The optimal capacity vector and the corresponding value of a non-flexible firm can be both obtained in closed form:

Corollary 2 *The optimal capacity vector and value of the non-flexible firm are, respectively,*

$$\tilde{K}_1^* = \tilde{K}_2^* = \left(\frac{1+1/b}{\tilde{c}_Q + \tilde{c}_K} \mathbb{E}_0 \epsilon_i(T)\right)^{-b} \qquad \textit{and} \qquad \tilde{v}^*(0) = \frac{\tilde{c}_Q + \tilde{c}_K}{|1+b|} \left(\tilde{K}_1^* + \tilde{K}_2^*\right).$$

In the next section, we use these results to analyze the value premium for flexibility.

5 The Value of Flexibility and its Drivers

We define the value premium for flexibility as the relative difference between the value of a flexible and a non-flexible firm:

$$\Delta_F \equiv \frac{v^*(0) - \tilde{v}^*(0)}{\tilde{v}^*(0)}.$$

The value premium for flexibility is difficult to investigate analytically so that we must resort to a numerical analysis except for the boundary case of negligible switching and output costs.

Focusing on Mix Flexibility. In this part, we focus on the special case when $c_Q = c_S = 0$. The value of mix flexibility typically stems from (i) diversification or risk pooling, the benefit of which decreases in correlation and as product variances are less equal; and from (ii) revenue (profit) maximization imbedded in the switching option, the benefit of which decreases as product unit revenues (margins) are more equal. Both effects are captured in our model. (Products have different realized demand curves with probability one if demand correlation $\rho < 1$. Our variance symmetry assumption, however, does undervalue the revenue maximization benefit and nullifies it at $\rho = 1$.) The value that stems from risk pooling and the revenue maximizing option can be expressed in closed form:

Lemma 1 *If $c_Q = c_S = 0$, the value premium for flexibility simplifies into*

$$\Delta_F = \left(\frac{\tilde{c}_K}{c_K}\right)^{-b-1}\left(\frac{\|\boldsymbol{\epsilon}(T)\|_\tau}{\mathbb{E}_0\epsilon_i(T)}\right)^{-b} - 1, \text{ where } \|\boldsymbol{\epsilon}(T)\|_\tau = \mathbb{E}_0\left\{\left(\frac{\mathbb{E}_\tau^{-b}\epsilon_1(T) + \mathbb{E}_\tau^{-b}\epsilon_2(T)}{2}\right)^{-1/b}\right\}.$$

It follows from the Minkowski inequality that $\|\boldsymbol{\epsilon}(T)\|_\tau \geq \mathbb{E}_0\epsilon_i(T)$. The inequality holds as equality only if the firm has no time flexibility ($\tau = 0$), the demand curves are deterministic ($\sigma^2 = 0$), or the demand shocks are perfectly positively correlated ($\rho = 1$ and recall that we assumed $\sigma_1 = \sigma_2$). Except for these limiting cases, flexibility reduces the expected cost of over/underinvestment. Whether it justifies the higher cost of flexible technology depends on model parameters. In particular,

$$\Delta_F > 0 \qquad \Leftrightarrow \qquad \frac{c_K}{\tilde{c}_K} < \left(\frac{\|\boldsymbol{\epsilon}(T)\|_\tau}{\mathbb{E}_0\epsilon_i(T)}\right)^{\frac{b}{1+b}} \equiv \delta.$$

The next three lemmas confirm our intuition about the effects of key parameters on the optimal capacity vector $\mathbf{K}^*$, firm value $v^*(0)$, flexibility premium Δ_F and the maximum sustainable cost of flexibility δ.

The longer the firm can wait before exercising the option to switch capacity, the higher the value of this option and the higher the optimal capacity investment:

Lemma 2 *If $c_Q = c_S = 0$, the optimal capacity, firm value, flexibility premium and the maximal sustainable cost of flexibility increase in time flexibility τ: $\frac{\partial}{\partial\tau}(K_1^* + K_2^*) \geq 0$, $\frac{\partial}{\partial\tau}v^*(0) \geq 0$, $\frac{\partial}{\partial\tau}\Delta_F \geq 0$, and $\frac{\partial}{\partial\tau}\delta \geq 0$.*

Similar to financial options, price volatility increases the value of the option to switch and, hence, the value of a flexible firm. It also increases the optimal capacity level:

Lemma 3 *If $c_Q = c_S = 0$, the optimal capacity, firm value, flexibility premium and the maximal sustainable cost of flexibility increase in demand volatility σ: $\frac{\partial}{\partial\sigma}(K_1^* + K_2^*) \geq 0$, $\frac{\partial}{\partial\sigma}v^*(0) \geq 0$, $\frac{\partial}{\partial\sigma}\Delta_F \geq 0$, and $\frac{\partial}{\partial\sigma}\delta \geq 0$.*

Higher market correlation reduces the efficacy of statistical aggregation and, hence, the value of the switching option. It also leads to a lower capacity investment:

Lemma 4 *If $c_Q = c_S = 0$, the optimal capacity, firm value, flexibility premium and the maximal sustainable cost of flexibility decrease in demand correlation ρ: $\frac{\partial}{\partial \rho}(K_1^* + K_2^*) \leq 0$, $\frac{\partial}{\partial \rho} v^*(0) \leq 0$, $\frac{\partial}{\partial \rho} \Delta_F \leq 0$ and $\frac{\partial}{\partial \rho} \delta \leq 0$.*

While it is intuitive that higher market volatility and lower market correlation both increase the value of flexibility, it is less obvious that they also result in a higher optimal capacity level $(K_1^* + K_2^*)$. In the newsvendor model of Eppen (1979), the optimal flexible capacity increases in demand volatility and correlation if, and only if, the capacity exceeds the expected demand. In that model, the effect of demand volatility and correlation on the optimal capacity depends on whether it increases or decreases the probability that all capacity will be used. With zero marginal output cost ($c_Q = 0$), that probability is always one. But with price-setting, higher market volatility and lower market correlation increase the expected output prices and, hence, the marginal value of capacity. For the remainder, we return to the general case with nonnegative output and switching costs ($c_Q \geq 0$ and $c_S \geq 0$).

The three dimensions of flexibility. To understand the value of different types of flexibility, it is helpful to consider their impact on the expected demand-supply mismatch cost. This "mismatch cost" for short (Cachon and Terwiesch 2005) results from capacity investment and allocation decisions made under uncertainty. Mix flexibility mitigates the mismatch cost by giving the firm the opportunity to reallocate its capacity based on the additional information revealed by the update time. Volume flexibility reduces the mismatch cost by enabling the firm to postpone some of the irreversible expenditures until the update time when more information is available. Mix and volume flexibilities thus provide two distinct ways to reduce the initial capacity commitment. As a result, one would expect mix and volume flexibilities to be strategic substitutes. At the same time, the value of both mix and volume flexibility increases as more information is revealed. One would therefore expect both mix and volume flexibility to be complementary with time flexibility. We can formalize these notions as follows:

Conjecture 1 *Mix and time flexibilities are strategic complements: $\frac{\partial^2}{\partial \varphi \partial \tau} \Delta_F \geq 0$ for any given volume flexibility γ.*

Conjecture 2 *Time and volume flexibilities are strategic complements: $\frac{\partial^2}{\partial \tau \partial \gamma} \Delta_F \geq 0$ for any given mix flexibility φ.*

Conjecture 3 *Volume and mix flexibilities are strategic substitutes: $\frac{\partial^2}{\partial \varphi \partial \gamma} \Delta_F \leq 0$ for any given time flexibility τ.*

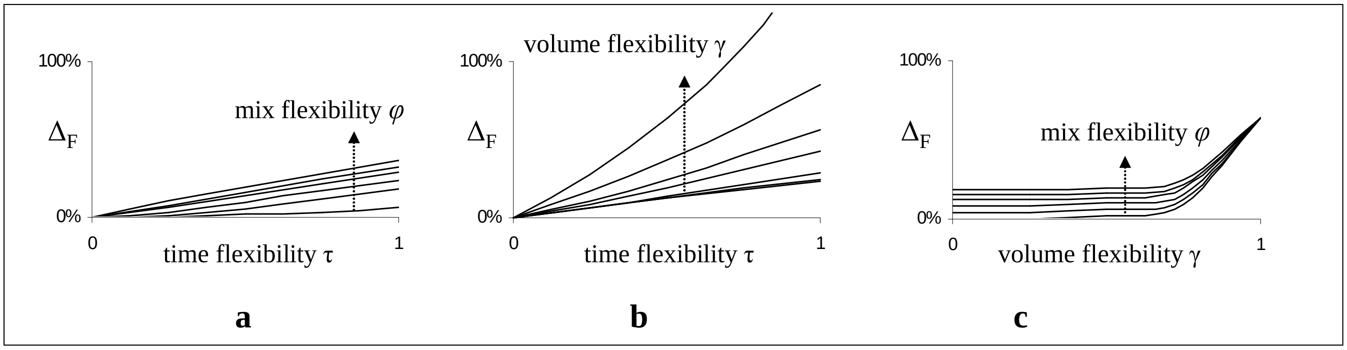

Figure 2: The effects of mix flexibility φ, cost flexibility γ and time flexibility τ on the flexibility premium Δ_F for independent demands. ($T = 1$, $b = -2$, $c_K + c_Q = 0.2$, $\epsilon(0) = 1$, $\sigma = 1$ and $\rho = 0$.)

Our numerical investigation supports all three conjectures, as illustrated by Figure 2 for a typical set of parameter values.[9]

Figure 2a shows the value premium for flexibility Δ_F for volume flexibility $\gamma = 0.5$, time flexibility $\tau \in [0, 1]$ and mix flexibility $\varphi \in \{0, ..., \infty\}$ (i.e., switching cost $c_S \in \{0, ..., \infty\}$). We observe that the marginal value of time flexibility $\partial\Delta_F/\partial\tau$ increases in mix flexibility φ, which confirms Conjecture 1 that mix and time flexibilities are strategic complements. Note that even in the absence of mix flexibility ($\varphi = 0$), time flexibility is valuable ($\partial\Delta_F/\partial\tau > 0$) because the firm has some volume flexibility ($\gamma > 0$). However, without time flexibility ($\tau = 0$), mix flexibility has no value ($\partial\Delta_F/\partial\varphi = 0$). This complementarity result has two managerial implications:

1. The closer to the selling season a firm chooses its output mix, or the more demand information it has at that time, the more it should invest in product-flexible technology, workforce cross-training and other enablers of mix flexibility.

2. The easier (cheaper) it is for a firm to convert one type of capacity to another, the more the firm should invest in obtaining accurate and timely market information and/or in postponing the point of product differentiation.

Figure 2b illustrates the complementarity of time and volume flexibilities for time flexibility $\tau \in [0, 1]$, volume flexibility $\gamma \in [0, 1]$ and mix flexibility $\varphi = 20$ (i.e., switching cost $c_S = 0.05$). We notice that the marginal value of time flexibility $\partial\Delta_F/\partial\tau$ increases in volume flexibility γ, confirming Conjecture 2. We also note that volume flexibility has no value ($\partial\Delta_F/\partial\gamma = 0$) without time flexibility ($\tau = 0$), while time flexibility has value ($\partial\Delta_F/\partial\tau > 0$) even without volume flexibility ($\gamma = 0$) due to the existing mix flexibility

[9] All numerical results used simulation to estimate probabilities and conditional expectations using 50,000 demand scenarios. All simulation errors were below 1% in the boundary cases that were also solved analytically.

($\varphi > 0$). The managerial implication is again twofold:

1. The more information a firm can gain by waiting, the more it should strive to postpone purchasing, hiring or production decisions.

2. And vice versa, the more of its quantity commitments a firm can postpone, the more it should invest in gathering information that will be available prior to making the postponed commitments.

Finally, **Figure 2c** shows how the flexibility premium Δ_F depends on volume flexibility $\gamma \in [0,1]$ and mix flexibility $\varphi \in \{0, ..., \infty\}$ for time flexibility $\tau = 0.5$. The marginal value of volume flexibility $\partial\Delta_F/\partial\gamma$ decreases in mix flexibility φ indicating that mix and volume flexibilities are strategic substitutes as conjectured. The figure also indicates that mix flexibility has no value ($\partial\Delta_F/\partial\varphi = 0$) under perfect volume flexibility ($\gamma = 1$), whereas volume flexibility is valuable ($\partial\Delta_F/\partial\gamma > 0$) even under perfect mix flexibility ($\varphi \to \infty$). In other words, volume flexibility can deliver all benefits of mix flexibility but not vice versa. The following two managerial implications ensue.

1. The higher the relative cost of assets (capacity), the more the firm should invest in its ability to switch between the production of different products. Thus, mix flexibility is particularly important in expensive, highly utilized capital equipment.

2. The higher the mix flexibility of a resource, the less can be gained from postponing its acquisition. In other words, it is more valuable to postpone the acquisition of product-specific resources than that of a product-flexible resource.

Flexibility premium and demand correlation. Demand correlation is an important driver of the value of flexibility. Positive demand correlation reduces the efficacy of risk pooling and, thus, diminishes the value of mix flexibility. This insight, which we stated in Lemma 4 for the boundary case of $c_Q = c_S = 0$ and which can be verified numerically for any value of c_Q and c_S, is intuitive and well-known (see e.g., Fine and Freund 1990).

It is less obvious how demand correlation affects the *marginal* value of different types of flexibility, which is important because it determines the optimal investment in flexibility. Given that diversification or risk pooling is a main value driver of mix flexibility we expect the marginal value of mix flexibility to decrease in demand correlation. Similarly for the marginal value of time flexibility (which stems from increasing the benefits of risk pooling as well as the benefits of volume flexibility). Since volume and mix flexibilities are strategic substitutes, less risk pooling increases the marginal value of volume flexibility so we expect increasing demand correlation to increase the marginal value of volume flexibility as well. We formalize our intuition in the following three conjectures:

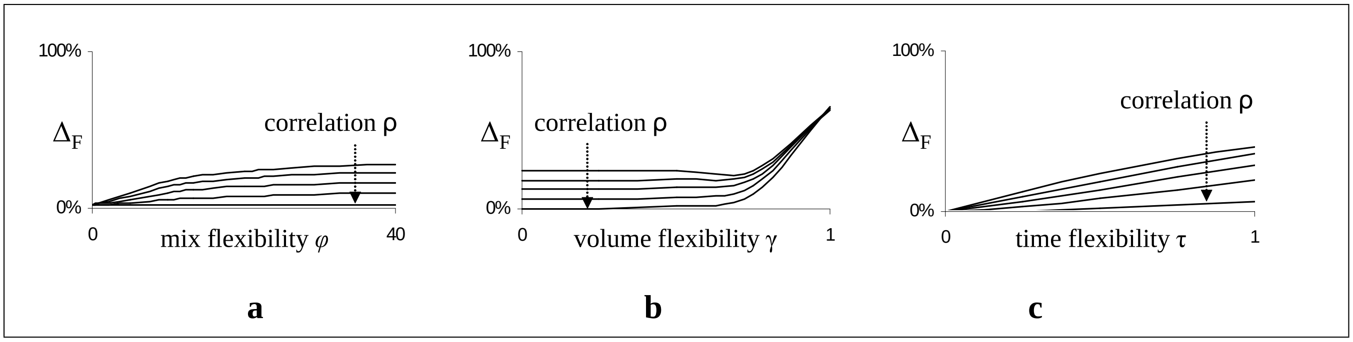

Figure 3: The effects of mix flexibility φ, cost flexibility γ and time flexibility τ on the flexibility premium Δ_F under different demand correlations.

Conjecture 4 *The value premium for flexibility is submodular in mix flexibility and demand correlation:* $\frac{\partial^2 \Delta_F}{\partial \varphi \partial \rho} \leq 0$ *for any given volume and time flexibility.*

Conjecture 5 *The value premium for flexibility is supermodular in volume flexibility and demand correlation:* $\frac{\partial^2 \Delta_F}{\partial \gamma \partial \rho} \geq 0$ *for any given mix and time flexibility.*

Conjecture 6 *The value premium for flexibility is submodular in time flexibility and demand correlation:* $\frac{\partial^2 \Delta_F}{\partial \tau \partial \rho} \leq 0$ *for any given mix and volume flexibility.*

Our numerical investigation confirms the three conjectures, as illustrated in Figure 3. (This figure is based on the same parameter values as Figure 2 except that the demand correlation coefficient ρ is varied between -1 and 1.) We note that the value premium for flexibility Δ_F is strictly decreasing in demand correlation except for the following three cases in which demand correlation has no effect on this premium: (i) the firm has no mix flexibility (switching cost is infinite); (ii) the firm has perfect volume flexibility (no capacity commitment has to be made under demand uncertainty); and (iii) firm's time flexibility is zero (output decision is made ex ante).

Figure 3a plots the flexibility premium Δ_F as a function of mix flexibility φ (switching cost c_S) for given volume flexibility $\gamma = 0.5$ and time flexibility $\tau = 0.5$. It confirms our conjecture that the marginal value of mix flexibility $\partial \Delta_F / \partial \varphi$ decreases in demand correlation. The higher the demand correlation, the less capacity is likely to be converted and, therefore, the less value results from reducing the switching cost. As a result, the higher the demand correlation, the lower the optimal investment in mix flexibility (switching cost reduction).

Figure 3b graphs the flexibility premium Δ_F as a function of volume flexibility γ for given mix flexibility $\varphi = 20$ (switching cost $c_S = 0.05$) and time flexibility $\tau = 0.5$. As conjectured, the marginal value of volume flexibility $\partial \Delta_F / \partial \gamma$ increases in demand correlation. As demand correlation increases, mix flexibility becomes less effective in reducing the mismatch between supply and demand, which makes volume flexibility relatively more important. (Recall that mix and volume flexibilities are substitutes.) Therefore, the higher the demand correlation, the more a firm should invest in volume flexibility. At the same time, as volume flexibility increases, capacity becomes less constraining and its conversion as well as demand correlation become less important.

Finally, **Figure 3c** shows how the flexibility premium Δ_F depends on time flexibility τ for given mix flexibility $\varphi = 20$ (switching cost $c_S = 0.05$) and volume flexibility $\gamma = 0.5$. As expected, the marginal value of time flexibility $\partial\Delta_F/\partial\tau$ decreases in demand correlation. More time flexibility means that more information is available before capacity conversion has to be made. As demand correlation increases, additional demand information is less likely to result in capacity conversion and, hence, is less valuable. Therefore, when demand correlation is high, the firm should invest less in time flexibility (forecasting, shorter lead times, etc.) than when demand correlation is low.

The next section links the value of mix flexibility to the value of market diversification with flexible assets.

6 Flexibility and Market Diversification

Economic theory recognizes that diversification can create shareholder value in the presence of market imperfections such as transaction costs in the factor market. In particular, if capacity investment is irreversible, market diversification coupled with technological or organizational flexibility may create value by reducing the aggregate uncertainty in capacity planning through statistical aggregation or "pooling" of demands. Although this argument has been formulated in the economic literature (e.g., Teece 1982, Smith and Triantis 2001), the value of diversification with flexible assets has not been studied in a rigorous mathematical model.

We quantify the benefits of diversification by comparing the value of a two-product firm to the sum of the values of two separate single-product firms under the assumption of sufficiently high transaction costs in the factor market that make firms' capacity investments irreversible. Thus, once acquired by either type of firm, capacity cannot be traded before the selling season is over. The optimal value of a two-product, or "diversified" firm is $v^*(0)$. The sum of the values of two single-product firms that cannot trade capacity is equal to $v^*(0)$ with $\varphi = 0$. We define the relative diversification premium as

$$\Delta_D \equiv \frac{v^*(0)|_{\varphi>0} - v^*(0)|_{\varphi=0}}{v^*(0)|_{\varphi=0}}. \tag{8}$$

Notice that in the boundary case of negligible output and switching costs characterized in Lemma 1, the value of diversification equals the value of flexibility. Given that the diversification premium decreases in volume flexibility and increases in mix flexibility, this value of flexibility is also an upper bound for the diversification premium in general:

$$\Delta_D \leq \left(\frac{\tilde{c}_K}{c_K}\right)^{-b-1}\left(\frac{\|\boldsymbol{\epsilon}(T)\|_\tau}{\mathbb{E}_0\epsilon_i(T)}\right)^{-b} - 1, \tag{9}$$

where equality holds in the boundary case when $c_Q = c_S = 0$ and where c_K and $\tilde{c}_K$ are the unit capacity costs of the diversified and single-product firms, respectively. It directly follows from Lemmas 2-4 that in the boundary case the diversification premium Δ_D increases in time flexibility τ and demand volatility σ and decreases in demand correlation ρ.

Diversification premium, flexibility and market correlation. The key drivers of the diversification premium are demand volatility, demand correlation, time and volume flexibility (which are both assumed to be the same for the diversified firm and the single-product firms) and mix flexibility of the diversified firm. Note that even though we are assuming the same time and volume flexibility for the diversified firm as for the two single-product firms, both of these parameters affect the diversification premium due to their complementarity/substitutability with mix flexibility. These effects are illustrated in Figure 4, which is based on the same parameter values as Figures 2 and 3.

The value of diversification stems from the diversified firm's ability to switch capacity between the products based on the additional information revealed up to time τ. As a consequence, the diversification premium increases in time flexibility and the diversified firm's mix flexibility and decreases in demand correlation. At the same time, the diversification premium decreases in the firms' volume flexibility. This is because volume and mix flexibilities are substitutes, i.e., as volume flexibility increases, additional mix flexibility is less valuable. In the extreme case of perfect volume flexibility ($\gamma = 1$), the firms do not have to make any capacity investment ex ante and, therefore, the ability to switch capacity within a diversified firm is worthless. In conclusion, diversification creates more value if demand correlation is not high, the diversified firm can switch capacity relatively close to the selling season (when accurate demand information is available) and at a relatively low cost, and the irreversible capacity investment represents a considerable part of the firm's total cost.

Figure 4 further indicates the following (admittedly intuitive) complementarity and substitutability results. More time flexibility magnifies both the positive effect of mix flexibility ($\partial^2\Delta_D/\partial\varphi\partial\tau \geq 0$) and the negative effect of volume flexibility ($\partial^2\Delta_D/\partial\gamma\partial\tau \leq 0$). Higher volume flexibility reduces the positive effect of mix flexibility ($\partial^2\Delta_D/\partial\varphi\partial\gamma \leq 0$). Higher demand correlation reduces the positive effects of mix flexibility ($\partial^2\Delta_D/\partial\varphi\partial\rho \leq 0$) and time flexibility ($\partial^2\Delta_D/\partial\tau\partial\rho \leq 0$) and the negative effect of volume flexibility ($\partial^2\Delta_D/\partial\gamma\partial\rho \geq 0$). Thus, the lower the correlation among the different business segments of a diversified firm, the more this firm should invest in technological and organizational flexibility that allows it employing its resources across different business segments based on existing market conditions.

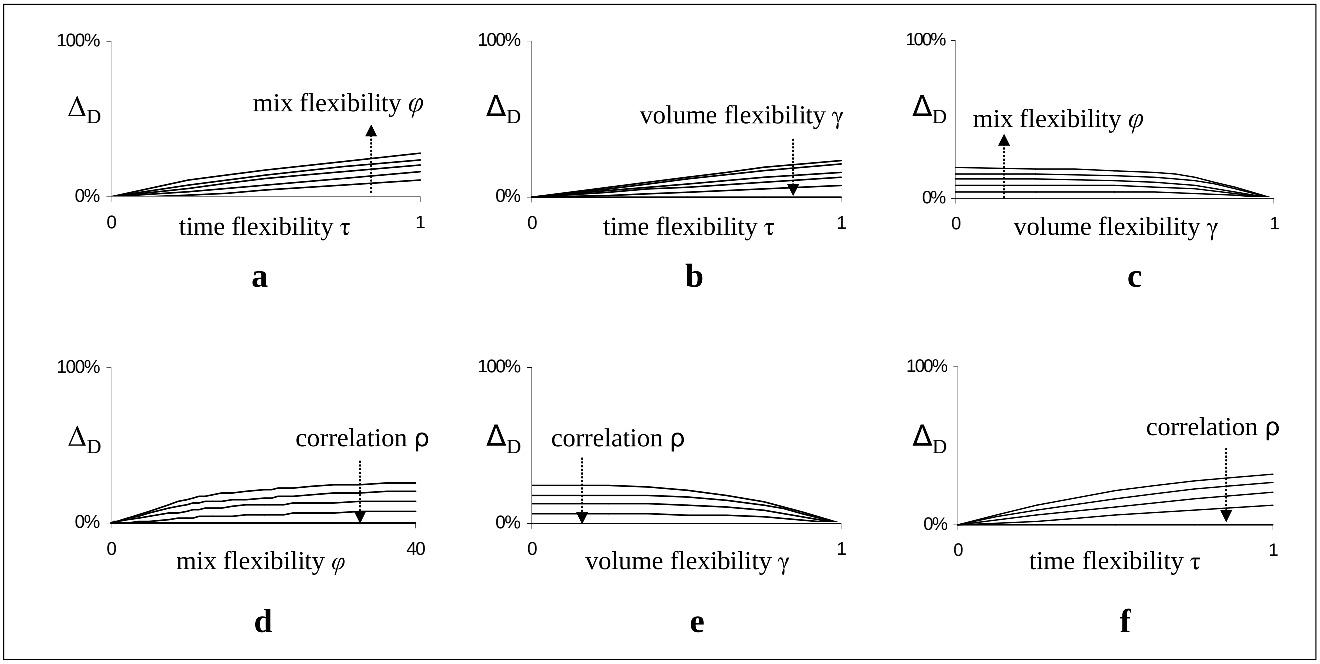

Figure 4: The effects of mix flexibility φ, cost flexibility γ, time flexibility τ and demand correlation ρ on the diversification premium Δ_D.

Diversification relatedness. One aspect of diversification that has attracted significant attention in the empirical finance and strategy literatures is the relative benefit of diversification into related versus unrelated industries. The notion of relatedness in the existing literature is, however, rather ambiguous as it involves relatedness of markets as well as relatedness of technology. These two aspects of relatedness have different implications for the value of diversification. Our model offers distinct measures for both types of relatedness. Because more similar markets for the two products are likely to be more (positively) correlated, market correlation ρ is a reasonable proxy for (at least some parts of) *market relatedness* of diversification.

Because the switching cost is likely to be lower for technologically similar products, a natural proxy for *technological relatedness* of diversification is mix flexibility $\varphi = 1/c_S$. An alternative proxy for technological relatedness would be time flexibility τ because it is typically easier to postpone product differentiation until a later stage of the production process for technologically close products. No matter which of the two possible surrogates for technological relatedness we consider, Figure 4 shows that the diversification premium Δ_D increases in technological relatedness and market unrelatedness and, furthermore, technological relatedness and market unrelatedness are complementary in increasing this premium. These findings are supported by the empirical evidence of Amit and Livnat (1989) who demonstrate that efficient diversifiers operate in related business segments that have differential responses to business cycle changes and thereby enjoy the benefits from (technologically) related diversification as well as from the portfolio effect (market unrelatedness).

7 Conclusion

This paper develops a stylized model of a two-product firm that chooses capacities, outputs, and prices at three different points in time while continuously updating its demand forecast. We identify three aspects of the firm's ability to respond to the dynamically evolving market conditions which we refer to as mix, volume and time flexibility. We suggest a continuous measure for each of these flexibilities and discuss their effects on the firm value as well as their relationships. While mix and volume flexibilities are shown to be strategic substitutes, they are both complementary with time flexibility. We also discuss the relationships between each flexibility type and demand correlation, showing that while demand correlation decreases the optimal investment in mix and time flexibility, it increases the optimal investment in volume flexibility. Finally, we link the value of flexibility to the value of market diversification discussing the main drivers of the diversification premium such as the three types of flexibility and demand correlation and their relationships.

Our stylized model has several limitations. The symmetry assumptions undervalue the revenue maximization option of flexibility and thus undervalue flexibility. The assumption that the firm is a monopoly in both product markets may need to be relaxed because it is has been shown that competition may significantly impact the value of flexibility. For example, Anand and Girotra (2004) show that in the presence of Cournot-like competition, a firm's inability to adjust output has a "commitment" value which can sometimes dominate the risk pooling benefits of flexibility. While competitive considerations are beyond the scope of our paper, they would be a natural yet nontrivial extension.

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Appendix 1

The partitioning of the state space of $\boldsymbol{\epsilon}(\tau)$:

$$
\begin{aligned}
\Omega_1(\mathbf{K}) &\equiv \left\{\mathbf{x}\in\mathbb{R}^2_+ : x_1 < \frac{c_Q K_1^{-1/b}}{(1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T)},\ x_2 < \frac{c_Q K_2^{-1/b}}{(1+1/b)\,\mathbb{E}_0\epsilon_2(\tau,T)}\right\},\\
\Omega_2(\mathbf{K}) &\equiv \left\{\mathbf{x}\in\mathbb{R}^2_+ : \frac{c_Q K_1^{-1/b}}{(1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T)} < x_1 < \frac{(c_Q+c_S)\,K_1^{-1/b}}{(1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T)},\ x_2 < \frac{c_Q K_2^{-1/b}}{(1+1/b)\,\mathbb{E}_0\epsilon_2(\tau,T)}\right\},\\
\Omega_3(\mathbf{K}) &\equiv \left\{\mathbf{x}\in\mathbb{R}^2_+ : x_1 < \frac{c_Q K_1^{-1/b}}{(1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T)},\ \frac{c_Q K_2^{-1/b}}{(1+1/b)\,\mathbb{E}_0\epsilon_2(\tau,T)} < x_2 < \frac{(c_Q+c_S)\,K_2^{-1/b}}{(1+1/b)\,\mathbb{E}_0\epsilon_2(\tau,T)}\right\},\\
\Omega_4(\mathbf{K}) &\equiv \left\{\mathbf{x}\in\mathbb{R}^2_+ : x_1 > \frac{(c_Q+c_S)\,K_1^{-1/b}}{(1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T)},\ \left(\frac{x_1}{c_Q+c_S}\right)^{-b} + \left(\frac{x_2}{c_Q}\right)^{-b} < \frac{K_1+K_2}{\left((1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T)\right)^{-b}}\right\},\\
\Omega_5(\mathbf{K}) &\equiv \left\{\mathbf{x}\in\mathbb{R}^2_+ : x_2 > \frac{(c_Q+c_S)\,K_2^{-1/b}}{(1+1/b)\,\mathbb{E}_0\epsilon_2(\tau,T)},\ \left(\frac{x_1}{c_Q}\right)^{-b} + \left(\frac{x_2}{c_Q+c_S}\right)^{-b} < \frac{K_1+K_2}{\left((1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T)\right)^{-b}}\right\},\\
\Omega_6(\mathbf{K}) &\equiv \left\{\mathbf{x}\in\mathbb{R}^2_+ : x_1 > \frac{c_Q K_1^{-1/b}}{(1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T)},\ x_2 > \frac{c_Q K_2^{-1/b}}{(1+1/b)\,\mathbb{E}_0\epsilon_2(\tau,T)},\ \left|x_1K_1^{1/b} - x_2K_2^{1/b}\right| < \frac{c_S}{(1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T)}\right\},\\
\Omega_7(\mathbf{K}) &\equiv \left\{\mathbf{x}\in\mathbb{R}^2_+ : \left(\frac{x_1}{c_Q+c_S}\right)^{-b} + \left(\frac{x_2}{c_Q}\right)^{-b} > \frac{K_1+K_2}{\left((1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T)\right)^{-b}},\ x_1K_1^{1/b} - x_2K_2^{1/b} > \frac{c_S}{(1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T)}\right\},\\
\Omega_8(\mathbf{K}) &\equiv \left\{\mathbf{x}\in\mathbb{R}^2_+ : \left(\frac{x_1}{c_Q}\right)^{-b} + \left(\frac{x_2}{c_Q+c_S}\right)^{-b} > \frac{K_1+K_2}{\left((1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T)\right)^{-b}},\ x_2K_2^{1/b} - x_1K_1^{1/b} > \frac{c_S}{(1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T)}\right\}.
\end{aligned}
$$

The corresponding optimal output vector $\mathbf{Q}^*(\mathbf{K},\boldsymbol{\epsilon}(\tau))$:

$$
\begin{aligned}
\text{If } \boldsymbol{\epsilon}(\tau) &\in \Omega_1(\mathbf{K}),\ Q_i^* = \left(\frac{(1+1/b)\,\mathbb{E}_\tau\epsilon_i(T)}{c_Q}\right)^{-b}, i=1,2.\\
\text{If } \boldsymbol{\epsilon}(\tau) &\in \Omega_2(\mathbf{K}),\ Q_1^* = K_1, Q_2^* = \left(\frac{(1+1/b)\,\mathbb{E}_\tau\epsilon_2(T)}{c_Q}\right)^{-b}.\\
\text{If } \boldsymbol{\epsilon}(\tau) &\in \Omega_3(\mathbf{K}),\ Q_1^* = \left(\frac{(1+1/b)\,\mathbb{E}_\tau\epsilon_1(T)}{c_Q}\right)^{-b}, Q_2^* = K_2.\\
\text{If } \boldsymbol{\epsilon}(\tau) &\in \Omega_4(\mathbf{K}),\ Q_1^* = \left(\frac{(1+1/b)\,\mathbb{E}_\tau\epsilon_1(T)}{c_Q+c_S}\right)^{-b}, Q_2^* = \left(\frac{(1+1/b)\,\mathbb{E}_\tau\epsilon_2(T)}{c_Q}\right)^{-b}.\\
\text{If } \boldsymbol{\epsilon}(\tau) &\in \Omega_5(\mathbf{K}),\ Q_1^* = \left(\frac{(1+1/b)\,\mathbb{E}_\tau\epsilon_1(T)}{c_Q}\right)^{-b}, Q_2^* = \left(\frac{(1+1/b)\,\mathbb{E}_\tau\epsilon_2(T)}{c_Q+c_S}\right)^{-b}.\\
\text{If } \boldsymbol{\epsilon}(\tau) &\in \Omega_6(\mathbf{K}),\ \mathbf{Q}^* = \mathbf{K}.\\
\text{If } \boldsymbol{\epsilon}(\tau) &\in \Omega_7(\mathbf{K}),\ \mathbf{Q}^* \text{ is the unique solution to } Q_1+Q_2 = K_1+K_2 \text{ and } \mathbb{E}_\tau\epsilon_1(T)\,Q_1^{1/b} - \mathbb{E}_\tau\epsilon_2(T)\,Q_2^{1/b} = \frac{c_S}{(1+1/b)}.\\
\text{If } \boldsymbol{\epsilon}(\tau) &\in \Omega_8(\mathbf{K}),\ \mathbf{Q}^* \text{ is the unique solution to } Q_1+Q_2 = K_1+K_2 \text{ and } \mathbb{E}_\tau\epsilon_2(T)\,Q_2^{1/b} - \mathbb{E}_\tau\epsilon_1(T)\,Q_1^{1/b} = \frac{c_S}{(1+1/b)}.
\end{aligned}
$$

Appendix 2

Proof of Proposition 1: For simplicity, we use $\mathbf{p}^*$ and $\mathbf{Q}^*$ as shorthand for $\mathbf{p}^*(\mathbf{Q}^*(\mathbf{K}))$ and $\mathbf{Q}^*(\mathbf{K})$, respectively, where $\mathbf{p}^*$ is given by (5) and $\mathbf{Q}^*$ is characterized in Appendix 1. Given the optimal pricing and output decisions, the firm value at time τ is

$$v(\tau;\mathbf{K}) = \sum_{i=1}^{2}\left(\mathbb{E}_\tau \epsilon_i(T)(Q_i^*)^{1+1/b} - c_Q Q_i^* - c_S \max(Q_i^* - K_i, 0) - c_K K_i\right). \tag{10}$$

The Hessian matrix of (10) with respect to $\mathbf{K}$ is

$$H_{\mathbf{K}} v(\tau;\mathbf{K}) = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } \boldsymbol{\epsilon}(\tau) \in \Omega_{145}, \\ \frac{1+b}{b^2}\mathbb{E}_\tau \epsilon_1(T) K_1^{1/b-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } \boldsymbol{\epsilon}(\tau) \in \Omega_2, \\ \frac{1+b}{b^2}\mathbb{E}_\tau \epsilon_2(T) K_2^{1/b-1}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } \boldsymbol{\epsilon}(\tau) \in \Omega_3, \\ \begin{pmatrix} \frac{1+b}{b^2}\mathbb{E}_\tau \epsilon_1(T) K_1^{1/b-1} & 0 \\ 0 & \frac{1+b}{b^2}\mathbb{E}_\tau \epsilon_2(T) K_2^{1/b-1} \end{pmatrix} & \text{if } \boldsymbol{\epsilon}(\tau) \in \Omega_6, \\ \frac{1+b}{b^2}\frac{\mathbb{E}_\tau \epsilon_1(T)\mathbb{E}_\tau \epsilon_2(T)(Q_1^*)^{1/b-1}(Q_2^*)^{1/b-1}}{\mathbb{E}_\tau \epsilon_1(T)\left(Q_1^*\right)^{1/b-1} + \mathbb{E}_\tau \epsilon_2(T)\left(Q_2^*\right)^{1/b-1}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \text{if } \boldsymbol{\epsilon}(\tau) \in \Omega_{78}. \end{cases}$$

Thus, $H_{\mathbf{K}} v(\tau;\mathbf{K})$ is negative definite if $\boldsymbol{\epsilon}(\tau) \in \Omega_6$ and negative semidefinite otherwise. Therefore, $v(\tau;\mathbf{K})$ is concave in $\mathbf{K}$ for any $\boldsymbol{\epsilon}(\tau)$ and the concavity is strict if $\boldsymbol{\epsilon}(\tau) \in \Omega_6$. This means that $v(0;\mathbf{K}) = \mathbb{E}_0 v(\tau;\mathbf{K})$ is concave in $\mathbf{K}$ and the first-order optimality condition $\nabla_{\mathbf{K}} v(0;\mathbf{K}) = \mathbf{0}$ is sufficient. Furthermore, if $c_S > 0$, then $\Pr(\Omega_6(\mathbf{K})) > 0$ and the concavity is strict, implying that $\mathbf{K}^*$ is unique. The uniqueness of $\mathbf{K}^*$ together with the symmetry of all parameters implies that $K_1^* = K_2^*$. Taking the derivatrive of $v(0;\mathbf{K})$ with respect to K_1 yields

$$\frac{\partial v(0;\mathbf{K})}{\partial K_1} = \frac{\partial}{\partial K_1}\mathbb{E}_0 v(T;\mathbf{K}) = \frac{\partial}{\partial K_1}\sum_{i=1}^{8}\Pr(\Omega_i(\mathbf{K}))\,\mathbb{E}_0\left(v(T;\mathbf{K})\middle|\,\Omega_i(\mathbf{K})\right), \tag{11}$$

where the firm terminal value, given the optimal pricing and output decisions, is

$$v(T;\mathbf{K}) = \sum_{i=1}^{2}\left(Q_i^* p_i^* - c_P Q_i^* - c_S \max(Q_i^* - K_i, 0) - c_K K_i\right).$$

Note that $v(T;\mathbf{K})$ is continuous in $\boldsymbol{\epsilon}(\tau)$ and, therefore, the terms from differentiating the boundaries of $\Omega_1, ..., \Omega_8$ with respect to K_1 in (11) cancel out. This leaves us with

$$\frac{\partial v(0;\mathbf{K})}{\partial K_1} = \sum_{i=1}^{8}\Pr(\Omega_i(\mathbf{K}))\,\mathbb{E}_0\left(\frac{\partial v(T;\mathbf{K})}{\partial K_1}\middle|\,\Omega_i(\mathbf{K})\right).$$

Differentiating $v(T;\mathbf{K})$ with respect to K_1 and setting $\partial v(0;\mathbf{K})/\partial K_1 = 0$ results in (7).□

Proof of Corollary 1: The result follows from Proposition 1 with $c_S = c_Q = 0$.□

Proof of Corollary 2: The result follows from Proposition 1 with $\tau = 0$, $c_K = \tilde{c}_K$ and $c_Q = \tilde{c}_Q$.□

Proof of Lemma 1: The result follows from Corollaries 1 and 2.□

Proof of Lemma 2: To simplify the notation, we normalize $T = 1$ and $\boldsymbol{\epsilon}(0) = \mathbf{1}$. To prove the desired results, it is sufficient to show that $\frac{\partial}{\partial\tau}\|\boldsymbol{\epsilon}(1)\|_\tau \geq 0$. Recall that $\|\boldsymbol{\epsilon}(1)\|_\tau = \mathbb{E}_0\left\{\left(\frac{\mathbb{E}_\tau^{-b}\epsilon_1(1)+\mathbb{E}_\tau^{-b}\epsilon_2(1)}{2}\right)^{-1/b}\right\}$ and $\ln\boldsymbol{\epsilon}(t) \sim N(\ln\boldsymbol{\epsilon}(0), t\boldsymbol{\Sigma})$. Using the fact that $\mathbb{E}_\tau\epsilon_i(1) = \epsilon_i(\tau)\exp\left(\frac{1}{2}\sigma^2(1-\tau)\right)$, we can write

$$\begin{aligned}
\|\boldsymbol{\epsilon}(1)\|_\tau &= \mathbb{E}_0\left[\left(\frac{\left(\epsilon_1(\tau)\exp\left(\frac{1}{2}\sigma^2(1-\tau)\right)\right)^{-b}+\left(\epsilon_2(\tau)\exp\left(\frac{1}{2}\sigma^2(1-\tau)\right)\right)^{-b}}{2}\right)^{-1/b}\right] \\
&= 2^{1/b}\exp\left(\frac{1}{2}\sigma^2(1-\tau)\right)\mathbb{E}_0\left[\left(\epsilon_1^{-b}(\tau)+\epsilon_2^{-b}(\tau)\right)^{-1/b}\right].
\end{aligned}$$

The normal vector $\ln\boldsymbol{\epsilon}(\tau)$ can be rewritten in terms of two independent standard normal random variables as $\ln\boldsymbol{\epsilon}(\tau) = \sqrt{\tau\boldsymbol{\Sigma}}\mathbf{Z}$, where $\mathbf{Z} \sim N(\mathbf{0},\mathbf{I})$ and $\mathbf{I}$ is a 2×2 identity matrix. Since $\tau\boldsymbol{\Sigma}$ is positive definite, $\sqrt{\tau\boldsymbol{\Sigma}}$ exists and can be obtained using eigenvector decomposition, $\sqrt{\tau\boldsymbol{\Sigma}} = \sqrt{\tau}\sigma\begin{pmatrix}\sqrt{(1-\rho)/2} & \sqrt{(1+\rho)/2}\\ -\sqrt{(1-\rho)/2} & \sqrt{(1+\rho)/2}\end{pmatrix}$. Using this transformation, we obtain

$$\begin{aligned}
\|\epsilon(1)\|_\tau &= 2^{1/b}\exp\left(\frac{1}{2}\sigma^2(1-\tau)\right)\times \\
&\quad \mathbb{E}_0\left[\exp\left(\sqrt{\tau}\sigma\sqrt{(1+\rho)/2}Z_2\right)\left(\exp\left(-b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right)+\exp\left(b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right)\right)^{-1/b}\right].
\end{aligned}$$

Since Z_1 and Z_2 are independent, we can further simplify

$$\begin{aligned}
\|\epsilon(1)\|_\tau &= 2^{1/b}\exp\left(\frac{1}{2}\sigma^2-\frac{1}{4}\tau\sigma^2+\frac{1}{4}\tau\sigma^2\rho\right)\times \\
&\quad \mathbb{E}_0\left[\left(\exp\left(-b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right)+\exp\left(b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right)\right)^{-1/b}\right]. \qquad (12)
\end{aligned}$$

Next, we take the derivative of (12) with respect to τ. After some algebra, we obtain

$$\begin{aligned}
\frac{\partial}{\partial\tau}\|\epsilon(1)\|_\tau &= \frac{1}{4}\sigma^2(\rho-1)\|\epsilon(1)\|_\tau + 2^{1/b}\exp\left(\frac{1}{2}\sigma^2-\frac{1}{4}\tau\sigma^2+\frac{1}{4}\tau\sigma^2\rho\right)\frac{\sigma\sqrt{(1-\rho)/2}}{\sqrt{\tau}}\times \\
&\quad \mathbb{E}_0\left[\left(\exp\left(-b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right)+\exp\left(b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right)\right)^{-1/b-1}\exp\left(-b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right)Z_1\right]. \qquad (13)
\end{aligned}$$

To evaluate (13), we make use of the fact that for a differentiable function g and a standard normal random variable Z_1, $\mathbb{E}(g(Z_1)Z_1) = \mathbb{E}g'(Z_1)$ (Rubinstein 1976). Applying this result and some algebra to (13), we obtain

$$\begin{aligned}
\frac{\partial}{\partial \tau} \left\| \epsilon\left(1\right) \right\|_{\tau} &= -\sigma^2\left(1-\rho\right)\left(1+b\right)2^{1/b}\exp\left(\frac{1}{2}\sigma^2 - \frac{1}{4}\tau\sigma^2 + \frac{1}{4}\tau\sigma^2\rho\right) \times \\
&\mathbb{E}_0\left[\left(\exp\left(-b\sqrt{\tau}\sigma\sqrt{\left(1-\rho\right)/2}Z_1\right) + \exp\left(b\sqrt{\tau}\sigma\sqrt{\left(1-\rho\right)/2}Z_1\right)\right)^{-1/b-2}\right] \\
&= -\sigma^2\left(1-\rho\right)\left(1+b\right)2^{1/b}\mathbb{E}_0\left[\left(\prod_{i=1,2}\mathbb{E}_{\tau}^{-b}\epsilon_i\left(1\right)\right)\left(\sum_{i=1,2}\mathbb{E}_{\tau}^{-b}\epsilon_i\left(1\right)\right)^{-1/b-2}\right] \\
&\geq 0.\square
\end{aligned}$$

Proof of Lemma 3: The proof is similar to the proof of Lemma 2 and is omitted.□

Proof of Lemma 4: The proof is similar to the proof of Lemma 2 and is omitted.□