Valuing Risky Projects using Mixed Asset Portfolio Selection Models

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Abstract

We examine the valuation of projects in a setting where an investor can invest in a portfolio of private projects as well as in securities in financial markets, but where exact replication of project cash flows in financial markets is not necessarily possible. We consider both single-period and multi-period models, and develop an inverse optimization procedure for valuing projects in this setting. We show that the valuation procedure exhibits several important analytical properties, for example, that project values for a mean-variance investor converge towards prices given by the capital asset pricing model. We also conduct several numerical experiments.

Project valuation and selection has attracted a substantial amount of attention among researchers and practitioners over the past few decades. Several methods has been suggested for this purpose, including discounted cash flow analysis (DCF, see e.g. Brealey and Myers 2000), project portfolio optimization (see Luenberger 1998), and options pricing analysis, where the focus has been on the recognition of the managerial flexibility embedded in the project (Dixit and Pindyck 1994, Trigeorgis 1996). In DCF analysis, it is proposed that a project's cash flows should be discounted at the rate of return of an asset that is equivalent in risk to the project, so that the opportunity costs of alternative investment opportunities are properly accounted for. However, while it may be straightforward to use DCF analysis when opportunity costs are solely imposed by securities, it is less obvious how the discount rate should be adjusted to account for further opportunity costs imposed by alternative project opportunities in the firm's portfolio. Options pricing analysis, on the other hand, requires replication of the project's cash flows with financial instruments, which may be difficult in practice.

In this paper, we approach the problem of project valuation through portfolio optimization models. We examine the setting where an investor can invest both in market-traded, infinitely divisible assets as well as lumpy, non-market-traded assets, so that the opportunity costs of both classes of investments can be accounted for. We call this setting a *mixed asset portfolio selection* (*MAPS*) setting. Examples of such lumpy assets include corporate projects, but in general the analysis extends to any non-traded, all-or-nothing-type investments. In particular, we demonstrate how portfolio optimization models can be used to determine the value of each non-traded lumpy asset within the portfolio, and show that the resulting project values are consistent with options pricing analysis and that, for a mean-variance investor, they converge towards the prices given by the Capital Asset Pricing Model (CAPM; Sharpe 1964, Lintner 1965).

Our valuation procedure is based on the concepts of *breakeven selling and buying prices* (Luenberger 1998, Smith and Nau 1995, Raiffa 1968), which are the investor's subjective selling and buying prices for a project. We formulate the procedure for both expected utility maximizers and mean-risk optimizers. Using the Contingent Portfolio Programming (CPP) framework (Gustafsson and Salo 2005), we develop a general multi-period MAPS model, which allows the modeling of real options embedded in projects and the use of a broad variety of risk-averse preference models. For example, our model allows incorporating the option to sell the project in the financial market in order to explore how market prices influence the value of the project to the investor. In terms of risk preferences, the model accommodates, for example, Markowitz's (1952) mean-variance (MV) model, Konno and Yamazaki's (1991) *mean-absolute deviation* (MAD) model, the *mean-lower semi-absolute deviation* (MLSAD) model (Ogryczak and Ruszczynski 1999), and Fishburn's (1977) mean-risk models where risk is associated with deviations below a fixed target value.

MAPS-based project valuation is relevant in practice as well as in theory. At the theoretical level, MAPS provides us with a framework for calculating a financial value for a project where the opportunity costs imposed by alternative investment opportunities are properly accounted for. At the practical level, MAPS-based project valuation is relevant when a corporation makes investments both in projects as well as in securities. This is typical, for example, in investment banks, which invest in a portfolio of publicly traded securities and undertake venture capital investments, which are not quoted on stock exchanges. A similar situation may also arise in industrial companies when the company's product pipeline does not provide enough opportunities to invest the complete investment budget. As an example of the magnitude of investments that companies may make outside their internal project portfolio, Microsoft recently reported that it had invested about $47 billion in financial instruments (Microsoft 2005). Also, companies often obtain minority stakes in other companies and raise capital through issuance of corporate bonds, which is the same as going short in their own bonds. In this regard, when it is possible to short relevant bonds and bank loans, a MAPS problem can also be regarded as a combined investment and project-financing problem.

This paper is structured as follows. Section I introduces single-period MAPS models and discusses different formulations of the portfolio selection problem. Section II presents the multi-period MAPS model. In Section III, we introduce the valuation concepts, and produce theoretical results on the valuation properties of different types of investors. Section IV demonstrates the pricing properties of mean-variance investors through a series of numerical experiments in a single-period setting. Section V gives an example of the framework in a multi-period setting, which allows us to compare the results with a similar example in Gustafsson and Salo (2005). In Section VI, we summarize our findings and discuss the managerial implications of our results.

I Mixed Asset Portfolio Selection

In a MAPS problem, available investment opportunities are divided into two categories: (1) *securities*, which can be bought and sold in any quantities, and (2) *projects*, lumpy all-or-nothing type investments. From a technical point of view, the main difference between these two types of investments is that the projects' decision variables are binary, while those of the securities are continuous. Another difference is that the cost, or price, of securities is determined by a market equilibrium model, such as the CAPM, while a project's investment cost is an endogenous property of the project.

Portfolio selection models can be formulated either in terms of rates of return and portfolio weights, like in Markowitz-type formulations, or by using a budget constraint, expressing the *initial wealth level*, and maximizing the investor's *terminal wealth level*. When properly applied, both approaches yield identical results. We use the second approach with MAPS, because it is more suitable to project portfolio selection. We first formulate single-period MAPS models, where the investments are made at time 0 and the objective at time 1 is optimized. These models will allow us to generate several insights and show how MAPS is related to Markowitz's (1952) model and the CAPM. We then develop the multi-period MAPS model based on Contingent Portfolio Programming (CPP, Gustafsson and Salo 2005).

Early portfolio selection formulations (Markowitz 1952) were bi-criteria decision problems minimizing risk while setting a target for expectation. Later, the mean-variance model was formulated in terms of *expected utility theory* (EUT) using a quadratic utility function. However, there are no similar utility functions for most other risk measures, including the widely used absolute deviation (Konno and Yamazaki 1991). Therefore, we distinguish between two classes of portfolio selection models: (1) *preference functional models*, such as the expected utility model, and (2) *bi-criteria optimization models* or *mean-risk models*.

A single-period MAPS model using a preference functional can be formulated as follows. Let there be n risky securities, a risk-free asset (labeled as the 0^{th} security), and m projects. Let the price of asset i at time 0 be S_i^0 and the corresponding (random) price at time 1 is $\tilde{S}_i^1$. The price of the risk-free asset at time 0 is 1 and $1+r_f$ at period 1, where r_f is the risk-free interest rate. The amounts of securities in the portfolio are denoted by $x_i, i=0,...,n$. The investment cost of project k in time 0 is C_k^0 and the (random) cash flow at time 1 is $\tilde{C}_k^1$. The binary variable z_k indicates whether project k is started or not. The investor's budget is b. We can then formulate a MAPS model using a preference functional U as follows:

(i) *maximize utility at time 1:* $$\max_{\mathbf{x},\mathbf{z}} U\left[\sum_{i=0}^{n} \tilde{S}_i^1 x_i + \sum_{k=1}^{m} \tilde{C}_k^1 z_k\right]$$

subject to

(ii) *budget constraint at time 0:* $$\sum_{i=0}^{n} S_i^0 x_i + \sum_{k=1}^{m} C_k^0 z_k = b$$

(iii) *binary variables for projects:* $$z_k \in \{0,1\} \quad k=1,...,m$$

(iv) *continuous variables for securities:* $$x_i \text{ free} \quad i=0,...,n.$$

The budget constraint is formulated as equality, because in the presence of a risk-free asset all of the budget will be expended at the optimum. In this model and throughout the paper, it is assumed that there are no transaction costs or capital gains tax, and that the investor is able to borrow and lend at the risk-free interest rate without limit. These assumptions can be relaxed without introducing prohibitive complexities. For expected utility theory, the preference functional is $U[X] = E[u(X)]$, where u is the investor's von Neumann-Morgenstern utility function. When the investor is able to determine a certainty equivalent for any random variable X, U can be expressed as a strictly increasing transformation of the investor's certainty equivalent operator CE. Hence, the objective can also be written as $\max CE[X]$, which gives the total value of the investor's portfolio.

In addition to preference functional models, mean-risk models have been widely used in the literature. We concentrate on these models, because much of the modern portfolio theory, including the CAPM, is based on a mean-risk model, namely the Markowitz (1952) mean-variance model. Table I describes three possible formulations for mean-risk models: *risk minimization*, where risk is minimized for a given level of expectation (Luenberger 1998), *expected value maximization*, where expectation is maximized for a given level of risk (Eppen et al. 1989), and the *additive formulation*, where the weighted sum of mean and risk is maximized (Yu 1985). In Table I, ρ is the investor's risk measure, μ is the minimum level for expectation, and R is the maximum level for risk (i.e. risk tolerance). The parameters λ are tradeoff coefficients. The Markowitz (1952) model can be understood as a special case of a mean-variance MAPS model where the number of projects is zero.

Table I. Formulations of the mean-risk optimization problem.

	Objective	Constraints
Risk minimization	$\min_{\mathbf{x},\mathbf{z}} \rho\left[\sum_{i=0}^{n}\tilde{S}_i^1 x_i + \sum_{k=1}^{m}\tilde{C}_k^1 z_k\right]$	$E\left[\sum_{i=0}^{n}\tilde{S}_i^1 x_i + \sum_{k=1}^{m}\tilde{C}_k^1 z_k\right] \geq \mu$ $\sum_{i=0}^{n} S_i^0 x_i + \sum_{k=1}^{m} C_k^0 z_k = b$
Expected value maximization	$\max_{\mathbf{x},\mathbf{z}} E\left[\sum_{i=0}^{n}\tilde{S}_i^1 x_i + \sum_{k=1}^{m}\tilde{C}_k^1 z_k\right]$	$\rho\left[\sum_{i=0}^{n}\tilde{S}_i^1 x_i + \sum_{k=1}^{m}\tilde{C}_k^1 z_k\right] \leq R$ $\sum_{i=0}^{n} S_i^0 x_i + \sum_{k=1}^{m} C_k^0 z_k = b$
General additive	$\max_{\mathbf{x},\mathbf{z}} \lambda_1 \cdot E\left[\sum_{i=0}^{n}\tilde{S}_i^1 x_i + \sum_{k=1}^{m}\tilde{C}_k^1 z_k\right] - \lambda_2 \cdot \rho\left[\sum_{i=0}^{n}\tilde{S}_i^1 x_i + \sum_{k=1}^{m}\tilde{C}_k^1 z_k\right]$	$\sum_{i=0}^{n} S_i^0 x_i + \sum_{k=1}^{m} C_k^0 z_k = b$
Sharpe (1970)	$\max_{\mathbf{x},\mathbf{z}} \lambda \cdot E\left[\sum_{i=0}^{n}\tilde{S}_i^1 x_i + \sum_{k=1}^{m}\tilde{C}_k^1 z_k\right] - \rho\left[\sum_{i=0}^{n}\tilde{S}_i^1 x_i + \sum_{k=1}^{m}\tilde{C}_k^1 z_k\right]$	$\sum_{i=0}^{n} S_i^0 x_i + \sum_{k=1}^{m} C_k^0 z_k = b$
Ogryczak and Ruszczynski (1999)	$\max_{\mathbf{x},\mathbf{z}} E\left[\sum_{i=0}^{n}\tilde{S}_i^1 x_i + \sum_{k=1}^{m}\tilde{C}_k^1 z_k\right] - \lambda \cdot \rho\left[\sum_{i=0}^{n}\tilde{S}_i^1 x_i + \sum_{k=1}^{m}\tilde{C}_k^1 z_k\right]$	$\sum_{i=0}^{n} S_i^0 x_i + \sum_{k=1}^{m} C_k^0 z_k = b$

The general additive form can be turned into the model employed by Sharpe (1970) by dividing the additive form by λ_2, provided it is nonzero, and into the form used by Ogryczak and Ruszczynski (1999) by dividing it by λ_1. Apart from expectation and risk constraints, the Karush-Kuhn-Tucker (KKT) conditions of all of the formulations are identical and therefore will yield the same efficient frontiers, as long as optimal solutions in the additive formulation remain bounded. However, if limitless borrowing and shorting are allowed, the additive formulation can give unbounded solutions unless a risk constraint is introduced. Yet, in this case the formulation essentially coincides with expected value maximization.

Both risk minimization and expected value maximization models have advantages and disadvantages. Risk minimization requires the investor to set a minimum level for expectation, a readily understandable quantity, while the interpretation of a maximum risk level in expected value maximization may not always be clear. However, expected value maximization allows us to include multiple risk constraints, e.g. one for variance, one for expected downside risk, and one for the probability of getting an outcome below a particular level, so that the risk profile of the portfolio can be matched to the risk preferences of the investor as closely as possible. Due to this flexible property of regarding risk aversion as a set of risk constraints, we focus on this formulation in the following.

II A Multi-Period MAPS Model

A Framework

We develop a multi-period MAPS model using the Contingent Portfolio Programming (CPP) framework developed by Gustafsson and Salo (2005). In this framework, uncertainties are modeled using a state tree, representing the structure of future states of nature, as depicted in the leftmost chart in Figure 1. The state tree need not be binomial or symmetric; it may also take the form of a multinomial tree with different probability distributions in its branches. In each non-terminal state, securities can be bought and sold in any, possibly fractional quantities.

Projects are modeled using decision trees that span over the state tree. The two right-most charts in Figure 1 describe how project decisions, when combined with the state tree, lead to project-specific decision trees. The specific feature of these decision trees is that the chance nodes are shared by all projects, since they are generated using the common state tree. Security trading is implemented through state-specific trading variables, which are similar to the ones used in financial models of stochastic programming (e.g. Mulvey et al. 2000) and in Smith and Nau's (1995) method. Similar to a single-period MAPS model, the investor seeks either to maximize the utility of the terminal wealth level, or the expectation of the terminal wealth level subject to a risk constraint.

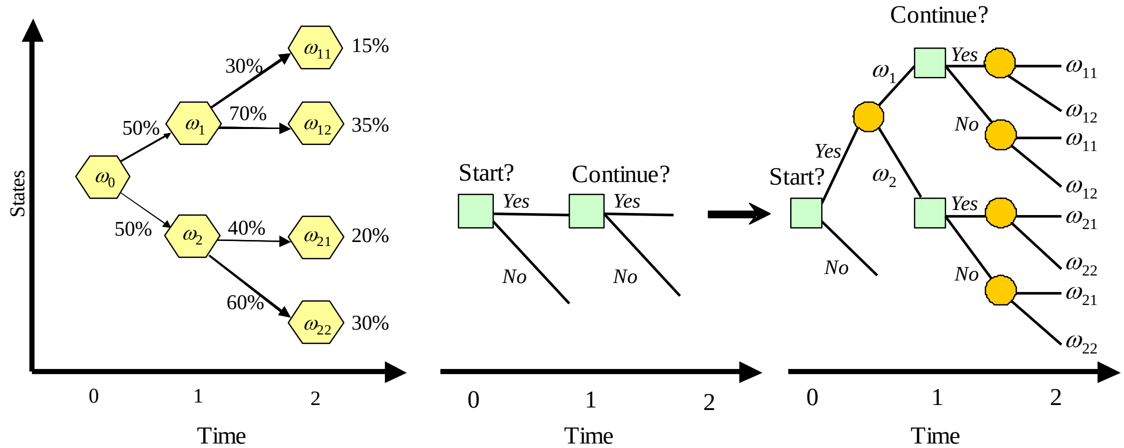

Figure 1. A state tree, decision sequence, and a decision tree for a project.

B Model Components

The two main components of the model are (i) states and (ii) the investor's investment decisions, which imply the cash flow structure of the model.

States

Let the planning horizon be $\{0,...,T\}$. The set of states in period t is denoted by Ω_t, and the set of all states is $\Omega = \bigcup_{t=0}^{T} \Omega_t$. The state tree starts with base state ω_0 in period 0. Each non-terminal state is followed by at least one state. This relationship is modeled by the function $B : \Omega \to \Omega$ which returns the immediate predecessor of each state, except for the base state, for which the function gives $B(\omega_0) = \omega_0$. The probability of state ω, when $B(\omega)$ has occurred, is given by $p_{B(\omega)}(\omega)$. Unconditional probabilities for each state, except for the base state, can be computed recursively from the equation $p(\omega) = p_{B(\omega)}(\omega) \cdot p(B(\omega))$. The probability of the base state is $p(\omega_0) = 1$.

Investment Decisions

Let there be *n* securities available in financial markets. The amount of security *i* bought in state ω is indicated by trading variable $x_{i,\omega}$, $i = 1,\dots,n$, $\omega \in \Omega$, and the price of security *i* in state ω is denoted by $S_i(\omega)$. Under the assumption that all securities are sold in the next period, the cash flow implied by security *i* in non-terminal state $\omega \neq \omega_0$ is $S_i(\omega) \cdot \left(x_{i,B(\omega)} - x_{i,\omega}\right)$. In base state ω_0, the cash flow is $-S_i(\omega_0) \cdot x_{i,\omega_0}$, and in a terminal state ω_T it is $S_i(\omega_T) \cdot x_{i,B(\omega_T)}$.

The investor can invest privately in *m* projects. The decision opportunities for each project, $k = 1,\dots,m$, are structured as a decision tree, where we have a set of decision points D_k and function $ap(d)$ that gives the action leading to decision point $d \in D_k \setminus \{d_k^0\}$, where d_k^0 is the first decision point of project *k*.

Let A_d be the set of actions that can be taken in decision point $d \in D_k$. For each action a in A_d, a binary action variable z_a indicates whether the action is selected or not. Action variables at each decision point *d* are bound by the restriction that only one z_a, $a \in A_d$, can be equal to one. The action in decision point *d* is chosen in state $\omega(d)$.

For a project *k*, the vector of all action variables z_a relating to the project, denoted by $\mathbf{z_k}$, is called the *project management strategy* of *k*. The vector of all action variables of all projects, denoted by **z**, is the *project portfolio management strategy*. We call the pair $(\mathbf{x}, \mathbf{z})$, composed of all trading and action variables, the *mixed asset portfolio management strategy*.

Cash Flows and Cash Surpluses

Let $CF_k^p(\mathbf{z_k}, \omega)$ be the cash flow of project *k* in state ω with project management strategy $\mathbf{z_k}$. When $C_a(\omega)$ is the cash flow in state ω implied by action a, this cash flow is given by

$$CF_k^p(\mathbf{z_k}, \omega) = \sum_{\substack{d \in D_k: \\ \omega(d) \in \Omega^B(\omega)}} \sum_{a \in A_d} C_a(\omega) \cdot z_a$$

where the restriction in the summation of the decision points guarantees that actions yield cash flows only in the prevailing state and in the future states that can be reached from the prevailing state. The set $\Omega^B(\omega)$ is defined as $\Omega^B(\omega) = \left\{\omega' \in \Omega \mid \exists\, k \geq 0 \text{ such that } B^k(\omega) = \omega'\right\}$, where $B^n(\omega) = B(B^{n-1}(\omega))$ is the n^{th} predecessor of ω ($B^0(\omega) = \omega$).

The cash flows from security *i* in state $\omega \in \Omega$ are given by

$$CF_i^s(\mathbf{x_i}, \omega) = \begin{cases} -S_i(\omega) \cdot x_{i,\omega} & \text{if } \omega = \omega_0 \\ S_i(\omega) \cdot \left(x_{i,B(\omega)} - x_{i,\omega}\right) & \text{if } \omega \neq \omega_0 \end{cases}$$

Thus, the aggregate cash flow $CF(\mathbf{x}, \mathbf{z}, \omega)$ in state $\omega \in \Omega$, obtained by summing up the cash flows for all projects and securities, is

$$\begin{aligned} CF(\mathbf{x}, \mathbf{z}, \omega) &= \sum_{i=1}^{n} CF_i^s(\mathbf{x_i}, \omega) + \sum_{k=1}^{m} CF_k^p(\mathbf{z_k}, \omega) \\ &= \begin{cases} \displaystyle\sum_{i=1}^{n} -S_i(\omega) \cdot x_{i,\omega} + \sum_{\substack{d \in D_k: \\ \omega(d) \in \Omega^B(\omega)}} \sum_{a \in A_d} C_a(\omega) \cdot z_a, & \text{if } \omega = \omega_0 \\ \displaystyle\sum_{i=1}^{n} S_i(\omega) \cdot \left(x_{i,B(\omega)} - x_{i,\omega}\right) + \sum_{\substack{d \in D_k: \\ \omega(d) \in \Omega^B(\omega)}} \sum_{a \in A_d} C_a(\omega) \cdot z_a, & \text{if } \omega \neq \omega_0 \end{cases} \end{aligned}$$

Together with the initial budget of each state, cash flows define *cash surpluses* that would result in state $\omega \in \Omega$ if the investor chose portfolio management strategy $(\mathbf{x}, \mathbf{z})$. Assuming that excess cash is invested in the risk-free asset, the cash surplus in state $\omega \in \Omega$ is given by

$$CS_\omega = \begin{cases} b(\omega) + CF(\mathbf{x},\mathbf{z},\omega) & \text{if } \omega = \omega_0 \\ b(\omega) + CF(\mathbf{x},\mathbf{z},\omega) + (1 + r_{B(\omega)\to\omega}) \cdot CS_{B(\omega)} & \text{if } \omega \neq \omega_0 \end{cases},$$

where $b(\omega)$ is the initial budget in state $\omega \in \Omega$ and $r_{B(\omega)\to\omega}$ is the short rate at which cash accrues interest from state $B(\omega)$ to ω. The cash surplus in a terminal state is the investor's terminal wealth level in that state.

C Optimization Model

When using a preference functional *U*, the objective function for the MAPS model can be written as a function of cash surplus variables in the last time period, i.e.

$$\max_{\mathbf{x},\mathbf{z},\mathbf{CS}} U(\mathbf{CS_T})$$

where $\mathbf{CS_T}$ denotes the vector of cash surplus variables related to period *T*. Under the risk-constrained mean-risk model, the objective is to maximize the expectation of the investor's terminal wealth level:

$$\max_{\mathbf{x},\mathbf{z},\mathbf{CS}} \sum_{\omega\in\Omega_T} p(\omega) \cdot CS_\omega .$$

Three types of constraints are imposed on the model: (i) *budget constraints*, (ii) *decision consistency constraints*, and (iii) *risk constraints*, which apply to risk-constrained models only. The multi-period MAPS model under both a preference functional and a mean-risk model is given in Table II. The constraints are explained in more detail in the following sections.

Table II. Multi-period MAPS models.

	Preference functional model	**Mean-risk model**
Objective function	$\max_{\mathbf{x},\mathbf{z},\mathbf{CS}} U\left(\mathbf{CS_T}\right)$	$\max_{\mathbf{x},\mathbf{y},\mathbf{CS}} \sum_{\omega\in\Omega_T} p(\omega) \cdot CS_\omega$
Budget constraints	$CF(\mathbf{x},\mathbf{z},\omega_0) - CS_{\omega_0} = -b(\omega_0)$ $CF(\mathbf{x},\mathbf{z},\omega) + (1 + r_{B(\omega)\to\omega}) \cdot CS_{B(\omega)} - CS_\omega = -b(\omega) \quad \forall \omega \in \Omega \backslash \{\omega_0\}$	
Decision consistency constraints	$\sum_{a\in A_{d_k^0}} z_a = 1 \quad k = 1,\ldots,m$ $\sum_{a\in A_d} z_a = z_{ap(d)} \quad \forall d \in D_k \backslash \left\{d_k^0\right\} \quad k = 1,\ldots,m$	
Risk constraints		$\rho\left(\Delta^-,\Delta^+\right) \le R$ $CS_\omega - \tau(\mathbf{CS_T}) - \Delta_\omega^+ + \Delta_\omega^- = 0 \quad \forall \omega \in \Omega_T$
Variables	$z_a \in \{0,1\} \quad \forall a \in A_d \ \forall d \in D_k \ k = 1,\ldots,m$ $x_{i,\omega}$ free $\quad \forall \omega \in \Omega \quad i = 1,\ldots,n$ CS_ω free $\quad \forall \omega \in \Omega$	$z_a \in \{0,1\} \quad \forall a \in A_d \ \forall d \in D_k \ k = 1,\ldots,m$ $x_{i,\omega}$ free $\quad \forall \omega \in \Omega \quad i = 1,\ldots,n$ CS_ω free $\quad \forall \omega \in \Omega$ $\Delta_\omega^- \ge 0 \quad \forall \omega \in \Omega_T$ $\Delta_\omega^+ \ge 0 \quad \forall \omega \in \Omega_T$

Budget Constraints

Budget constraints ensure that there is a nonnegative amount of cash in each state. They can be implemented using continuous *cash surplus variables* CS_ω, which measure the amount of cash in state ω. These variables lead to the budget constraints

$$CF(\mathbf{x},\mathbf{z},\omega_0) - CS_{\omega_0} = -b(\omega_0)$$
$$CF(\mathbf{x},\mathbf{z},\omega) + (1 + r_{B(\omega)\to\omega}) \cdot CS_{B(\omega)} - CS_\omega = -b(\omega) \quad \forall \omega \in \Omega \setminus \{\omega_0\}$$

Note that if CS_ω is negative, the investor borrows money at the risk-free interest rate to cover a funding shortage. Thus, CS_ω can also be regarded as a trading variable for the risk-free asset.

Decision Consistency Constraints

Decision consistency constraints implement the projects' decision trees. They require that (i) at each decision point reached, only one action is selected, and that (ii) at each decision point that is not reached, no action is taken. Decision consistency constraints can be written as

$$\sum_{a \in A_{d_k^0}} z_a = 1 \quad k = 1,\ldots,m$$
$$\sum_{a \in A_d} z_a = z_{ap(d)} \quad \forall d \in D_k \setminus \left\{d_k^0\right\} \quad k = 1,\ldots,m,$$

where the first constraint ensures that one action is selected in the first decision point, and the second implements the above requirements for other decision points.

Risk Constraints

A risk-constrained model includes one or more risk constraints. We focus on the single constraint case. When ρ denotes the risk measure and R the risk tolerance, a risk constraint can be expressed as

$$\rho(\mathbf{CS_T}) \le R .$$

In addition to variance (V), several other risk measures have been proposed in the literature on portfolio selection. These include *semivariance* (Markowitz 1959), *absolute deviation* (Konno and Yamazaki 1991), *lower semi-absolute deviation* (Ogryczak and Ruszczynski 1999), and their fixed target value counterparts (Fishburn 1977). Semivariance (SV), absolute deviation (AD) and lower semi-absolute deviation (LSAD) are defined as

$$\text{SV: } \bar{\sigma}_X = \int_{-\infty}^{\mu_X} \left(x - \mu_X\right)^2 dF_X(x), \text{ AD: } \delta_X = \int_{-\infty}^{\infty} \left|x - \mu_X\right| dF_X(x), \text{ and}$$
$$\text{LSAD: } \bar{\delta}_X = \int_{-\infty}^{\mu_X} \left|x - \mu_X\right| dF_X(x) = \int_{-\infty}^{\mu_X} (\mu_X - x) dF_X(x),$$

where μ_X is the mean of random variable X and F_X is the cumulative density function of X. The fixed target value statistics are obtained by replacing μ_X by some constant target value τ. All of these measures can be formulated in a MAPS program through *deviation constraints*. In general, deviation constraints are expressed as

$$CS_\omega - \tau(\mathbf{CS_T}) - \Delta_\omega^+ + \Delta_\omega^- = 0 \quad \forall \omega \in \Omega_T ,$$

where $\tau(\mathbf{CS_T})$ is a function defining the target value from which the deviations are calculated, and Δ_ω^+ and Δ_ω^- are nonnegative *deviation variables* which measure how much the cash surplus in state $\omega \in \Omega_T$ differs from the target value. For example, when the target value is the mean of the terminal wealth level, the deviation constraints are written as

$$CS_\omega - \sum_{\omega' \in \Omega_T} p(\omega') CS_{\omega'} - \Delta_\omega^+ + \Delta_\omega^- = 0 \qquad \forall \omega \in \Omega_T ,$$

Using these deviation variables, some common dispersion statistics can be written as follows:

AD: $\sum_{\omega \in \Omega_T} p(\omega) \cdot (\Delta_\omega^- + \Delta_\omega^+)$.

LSAD: $\sum_{\omega \in \Omega_T} p(\omega) \cdot \Delta_\omega^-$.

V: $\sum_{\omega \in \Omega_T} p(\omega) \cdot (\Delta_\omega^- + \Delta_\omega^+)^2$

SV: $\sum_{\omega \in \Omega_T} p(\omega) \cdot (\Delta_\omega^-)^2$.

The respective fixed-target value statistics can be obtained with the deviation constraints

$$CS_\omega - \tau - \Delta_\omega^+ + \Delta_\omega^- = 0 \quad \forall \omega \in \Omega_T ,$$

where τ is the fixed target level. EDR, for example, can then be obtained from the sum $\sum_{\omega \in \Omega_T} p(\omega) \cdot \Delta_\omega^-$.

Several different kinds of risk measures can also be modeled in the framework. For example, conditional value at risk (CVaR), or expected tail loss, can be implemented using the method explained in Uryasev (2000). Also, it is possible to set a limit for the (critical) probability of getting an outcome below some target level. Mathematically, the critical probability is defined as

$$P_X^{crit}(\tau) = F_X(X < \tau) ,$$

where F_X denotes the cumulative distribution function of random variable X and τ is the desired target level. This risk measure can be implemented by using the following linear constraints:

$$\tau - CS_\omega \le M \xi_\omega \quad \forall \omega \in \Omega_T \text{, and}$$

$$\xi_\omega \in \{0,1\} \quad \forall \omega \in \Omega_T ,$$

where τ is the target value from which critical probability is calculated and M is some very large number. Critical probability can then be constrained from above using the constraint $\sum_{\omega \in \Omega_T} p_\omega \xi_\omega \le R$.

Other Constraints

Other types of constraints can also be modeled, including short selling limitations, upper bounds for the number of shares bought, and credit limit constraints (Markowitz 1987). In the ensuing sections, we assume, for the sake of simplicity, that no such additional constraints have been imposed.

III Project Valuation

A Breakeven Buying and Selling Prices

Because we consider projects as non-tradable investment opportunities, there is no market price that can be used to value the project. In such a situation, it is reasonable to define the value of the project as the amount of money at present, time 0, that is equally desirable to the project. In a portfolio context, this can be interpreted so that the investor is indifferent between the following two portfolios: (A1) a portfolio with the project and (B1) a portfolio without the project and cash equal to the value of the project. However, we may alternatively define the value of a project as the indifference between the following two portfolios: (A2) a portfolio without the project and (B2) a portfolio with the project and a debt equal to the value of the project. The project values obtained in these two ways will not, in general, be the same. Analogously to Luenberger (1998), Smith and Nau (1995), and Raiffa (1968), we refer to the first value as the "*breakeven selling price*" (BSP), as the portfolio comparison can be understood as a selling process, and the second type of value as the "*breakeven buying price*" (BBP).

Table III. Definitions of the value of project *j*. Each setting is based on Table II.

	Breakeven selling price	**Breakeven buying price**
Definition	v_j^s such that $W_s^+ = W_s^-$	v_j^b such that $W_b^+ = W_b^-$
Status quo	Optimal objective function value: W_s^+ Additional constraint: $z_{a*} = 0$ (= invest in the project) Budget at time 0: $b(\omega_0)$	Optimal objective function value: W_b^- Additional constraint: $z_{a*} = 1$ (= do not invest in the project) Budget at time 0: $b(\omega_0)$
Second setting	Optimal objective function value: W_s^- Additional constraint: $z_{a*} = 1$ (= do not invest in the project) Budget at time 0: $b(\omega_0) + v_j^s$	Optimal objective function value: W_b^+ Additional constraint: $z_{a*} = 0$ (= invest in the project) Budget at time 0: $b(\omega_0) - v_j^b$

A crucial element in BSP and BBP is the definition of equal desirability of two different portfolios. In preference functional models, the investor is, by definition, indifferent between two portfolios whenever their utility scores are equal. In a mean-risk setting, an investor is indifferent between two portfolios if the means and risks of the two portfolios are identical. More generally, when the risks are considered as constraints, the investor is indifferent between two portfolios if their expectations are equal and they both satisfy the risk constraints. Hence, we obtain the pairs of optimization problems shown in Table III. Here, the variable z_{a*} is the action variable associated with *not starting* project *j*; thus, $z_{a*} = 0$ indicates that investor invests in the project, whereas $z_{a*} = 1$ means that the project is not started. In the case that the project starting decision is a binary "go / no go" decision, we can alternatively impose the restrictions on the variable indicating project starting instead. Finding a BSP and BBP is an *inverse optimization problem*: one has to find the values for the parameters v_j^s and v_j^b so that the optimal value of the second optimization problem matches the optimal value of the first problem.

B Inverse Optimization Procedure

Inverse optimization problems have recently attracted interest among researchers, such as Ahuja and Orlin (2001). In an inverse optimization problem, the challenge is to find the values for a set of parameters, typically a subset of all model parameters, that yield the desired optimal solution. Inverse optimization problems can broadly be classified into two groups: (i) finding an optimal value for the objective function, and (ii) finding a solution vector. Ahuja and Orlin (2001) discuss problems of the second kind, whereas the problem of finding a BSP or BBP falls within the first class.

In principle, the task of finding a BSP is equivalent to finding a root to the function $f^s(v_j^s) = W_s^-(v_j^s) - W_s^+$, where W_s^+ is the optimal value of the portfolio optimization problem in the status quo and $W_s^-(v_j^s)$ is the corresponding optimal value in the second setting as the function of parameter v_j^s. Similarly, the BBP can be obtained by finding the root to the function $f^b(v_j^b) = W_b^- - W_b^+(v_j^b)$. Note that these functions are increasing with respect to their parameters. To solve such root-finding problems, we can use any of the usual root-finding algorithms (see e.g. Belegundu and Chandrupatla 1999) such as the bisection method, the secant method, and the false position method, which have the advantage that they do not require the knowledge of the functions' derivatives, which are not typically known. If the first derivatives are known, or when approximated numerically, we can also use the Newton-Raphson method.

C General Analytical Properties

Sequential Consistency

Breakeven selling and buying prices are not, in general, equal to each other. While this discrepancy is accepted as a general property of risk preferences in expected utility theory (Raiffa 1968), it may also seem to contradict the rationality of these valuation concepts. It can be argued that if the investor were willing to sell a project at a lower price than at which he/she would be prepared to buy it, the investor would create an arbitrage opportunity and lose an infinite amount of money when another investor repeatedly bought the project at its selling price and sold it back at the buying price. In a reverse situation where the investor's selling price for a project is greater than the respective buying price, the investor would be irrational in the sense that he/she would not take advantage of an arbitrage opportunity – if such an opportunity existed – where it would be possible to repeatedly buy the project at the investor's buying price and sell it at a slightly higher price which is below the investor's breakeven selling price.

However, these arguments neglect the fact that the breakeven prices are affected by the budget and that therefore these prices may change after obtaining the project's selling price and after paying its buying price. Indeed, it can be shown that in a sequential setting where the investor first sells the project, adding the selling price to the budget, and then buys the project back, the investor's selling price and the respective (sequential) buying price are always equal to each other. This observation is formalized as the following proposition and it holds for any preference model accommodated by the model in Section II. The proof is given in the Appendix.

PROPOSITION 1. *A project's breakeven selling (buying) price and its sequential breakeven buying (selling) price are equal to each other.*

Consistency with Contingent Claims Analysis

Option pricing analysis, or *contingent claims analysis* (CCA; Luenberger 1998, Brealey and Myers 2000, Hull 1999), can be applied to value projects whenever the cash flows of a project can be replicated using financial instruments. According to CCA, the value of project *j* is given by the market price of the replicating portfolio (a portfolio required to initiate a replicating trading strategy) less the investment cost of the project:

$$v_j^{CCA} = -C_j^0 + \sum_{i=1}^{n} S_i\ (\omega_0) x_{i,\omega_0}^* \ ,$$

where x_{i,ω_0}^* is the amount of security *i* in the replicating portfolio and C_j^0 is the investment cost of the project in at time 0. A replicating trading strategy $\mathbf{x}^*$ for a project under a management strategy $\mathbf{z_k}$ is defined as follows: For each $\omega \setminus \omega_0 \in \Omega$, $\sum_{i=1}^{n} CF_i^s(\mathbf{x}_\mathbf{i}^*, \omega) - CF_k^p(\mathbf{z_k}, \omega) = 0$, i.e. the replicating trading strategy yields exactly the same cash flows as the project in each state of nature except in the base state.

It is straightforward to show that, when CCA is applicable, i.e. if there exists a replicating trading strategy, then the breakeven buying and selling prices are equal to each other and yield the same result as CCA (Smith and Nau 1995). For example, when v_j^{CCA} is positive, we know that *any* rational investor will invest in the project, since it is possible to make money for sure by investing in the project and shorting the replicating portfolio. Furthermore, any rational investor will start the project even when he/she is forced to pay a sum v_j^b less than v_j^{CCA} to gain a license to invest in the project, because it is now possible to gain $v_j^{CCA} - v_j^b$ for sure. On the other hand, if v_j^b is greater than v_j^{CCA}, the investor will be better off by investing the replicating portfolio instead and hence he/she will not start the project. A similar reasoning applies to breakeven selling prices. These observations are formalized in Proposition 2. The proof is in the Appendix. Due to the consistency with CCA, the breakeven prices can be regarded as a *generalization* of CCA to incomplete markets.

PROPOSITION 2. *If there is a replicating portfolio for a project, the breakeven selling price and breakeven buying price are equal to each other and yield the same result as CCA.*

Sequential Additivity

The BBP and BSP for a project depend on what other assets are in the portfolio. The value obtained from breakeven prices is, in general, an *added value*, which is determined relative to the situation without the project. When there are no other projects in the portfolio, or when we remove them from the model before determining the value of the project, we speak of the project's *isolated value*. We define the respective values for a set of projects as the *joint added value* and *joint value*. Figure 2 illustrates the relationship between these concepts.

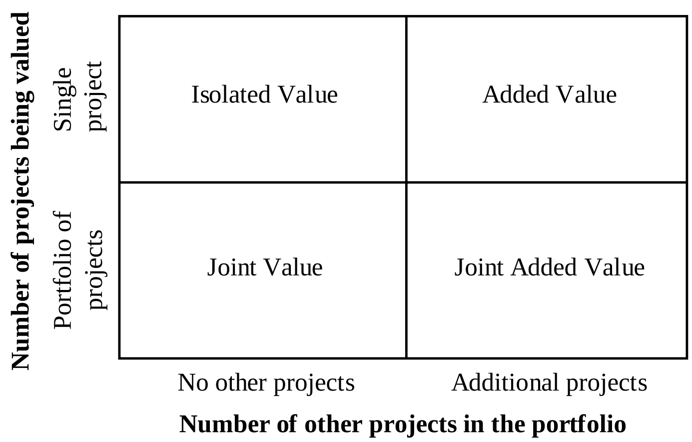

Figure 2. Different types of valuations for projects.

Isolated project values are, in general, *non-additive*; they do not sum up to the value of the project portfolio composed of the same projects. However, in a sequential setting where the investor buys the projects one after the other using the prevailing buying price at each time, the obtained project values do add up to the joint value of the project portfolio. These prices are the projects' *added* values in a sequential buying process, where the budget is reduced by the buying price after each step. We refer to these values as *sequential added values*. This *sequential additivity* property holds regardless of the order in which the projects are bought. Individual projects can, however, acquire different added values depending on the sequence in which they are bought. These observations are proven by the following proposition. The proof is in the Appendix.

PROPOSITION 3. *The breakeven buying (selling) prices of sequentially bought (sold) projects add up to the breakeven buying (selling) price of the portfolio of the projects regardless of the order in which the projects are bought (sold).*

D Analytical Properties of Mean-Risk Preference Models

Equality of Prices and an Alternative Valuation Formula

When the investor is a mean-risk optimizer and the risk measure is independent of an addition of a constant to the portfolio, like variance, breakeven selling and buying prices are identical, provided that unlimited borrowing and lending are allowed. In contrast, they are typically *different* under expected utility theory and under most non-expected utility models. Also, for mean-risk optimizers the breakeven prices can be computed directly by solving the expectations of terminal wealth levels when the investor invests and does not invest in the project and discounting their difference back to its present value at the risk-free interest rate. Therefore, with mean-risk models, there is no need to resort to possibly laborious inverse optimization. We formalize these claims in Proposition 4. The proof is in the Appendix.

PROPOSITION 4. *Let the investor be a mean-risk optimizer with risk measure ρ that satisfies $\rho[X+b]=\rho[X]$ for all random variables X and constants b. When there is a risk-free interest rate for each time period and limitless borrowing and lending are allowed, the breakeven selling price and the breakeven buying price of any given project are identical. Moreover, the prices are equal to*

$$v = v^b = v^s = \frac{W^+ - W^-}{\prod_{t=0}^{T-1}(1+r_t)},$$

*where W^+ is the expectation of the terminal wealth level when the investor invests in the project, W^- is the expectation of the terminal wealth level when the investor does not invest in the project, and r_t is the risk-free interest rate from time t to t+*1.

Proposition 4 is striking both in its generality and simplicity, since neither the given valuation formula, nor the equality of breakeven prices, generally holds under expected utility theory. The results are also intuitively appealing: the investor places only a single price on a given project, and this price is related to the investor's terminal wealth levels with and without the project. This indicates that mean-risk models exhibit a very reasonable type of pricing behavior.

Relationship to the Capital Asset Pricing Model

According to the CAPM, the market price of any asset is given by the certainty equivalent formula (see, e.g., Luenberger 1998):

$$v_j^{CAPM} = -C_j^0 + \frac{E\left[\tilde{C}_j^1\right]}{1+r_f} - \frac{\operatorname{cov}\left[\tilde{C}_j^1, \tilde{r}_M\right]}{\operatorname{var}\left[\tilde{r}_M\right]} \cdot \frac{E\left[\tilde{r}_M\right] - r_f}{1+r_f},$$

where C_j^0 is the investment cost of the asset, $\tilde{C}_j^1$ is the random value of the asset at time 1, and $\tilde{r}_M$ is the random return of the market portfolio. For non-market-traded assets like projects, the outcome of the formula can be interpreted as the price that the markets would give to the asset if it were traded.

In general, breakeven selling and breakeven buying prices are inconsistent with CAPM valuations, because some of the CAPM assumptions do not hold in a MAPS setting. For example, private projects are not included in the derivation of the CAPM market equilibrium, yet they may be strongly correlated with the securities. Therefore, results that hold for the CAPM do not necessarily hold here. In particular, the optimal financial portfolio for the investor is not necessarily a combination of the risk-free asset and the market fund consisting of all securities in proportions according to their market capitalization.

However, there is a special case when the investor's optimal financial portfolio is what the CAPM predicts, namely when the project portfolio is uncorrelated with the market securities and the investor is a mean-variance optimizer. This is proven in Proposition 5.

PROPOSITION 5. *If the investor is a mean-variance optimizer and projects are uncorrelated with securities, the optimal financial portfolio is a combination of the market fund and the risk-free asset.*

Even though the optimal financial portfolio falls on the capital market line with and without the project being valued in this case, the CAPM valuation is incorrect, as an extra risk premium term appears in the breakeven selling and buying prices, as shown in Propositions 6 and 7. Proofs for the propositions can be found in the Appendix. Note that this is a particularly interesting special case, because project outcomes do not usually depend on the fluctuations of financial markets. Proposition 5 is to be understood as a mixed asset portfolio extension of the usual Separation Theorem (Tobin 1958), applicable when projects are uncorrelated with securities.

PROPOSITION 6. *If the conditions of Proposition 5 hold and the composition of the optimal project portfolio is the same with and without project j, the breakeven selling price of project j is*

$$v_j^s = -C_j^0 + \frac{E\left[\tilde{C}_j^1\right]}{1+r_f} - \left(\sqrt{1 + \frac{\operatorname{var}\left[\sum_{k=1}^{m} \tilde{C}_k^1 z_k\right] - \operatorname{var}\left[\sum_{\substack{k=1 \\ k \neq j}}^{m} \tilde{C}_k^1 z_k\right]}{\operatorname{var}\left[\sum_{i=1}^{n} \tilde{S}_i^1 x_i\right]}} - 1 \right) \frac{E\left[\tilde{r}_M\right] - r_f}{1+r_f} b_M ,$$

where $\tilde{r}_M$ *is the random rate of return of the market portfolio and* b_M *is the amount of money spent on securities in the status quo. That is,* $b_M = b - x_0 - \sum_{k=1}^{m} C_k^0 z_k$ *.*

PROPOSITION 7. *If the conditions of Proposition 5 hold and the composition of the optimal project portfolio is the same with and without project j, the breakeven buying price of project j is*

$$v_j^b = -C_j^0 + \frac{E\left[\tilde{C}_j^1\right]}{1+r_f} - \left(1 - \sqrt{1 - \frac{\operatorname{var}\left[\sum_{k=1}^{m} \tilde{C}_k^1 z_k\right] - \operatorname{var}\left[\sum_{\substack{k=1 \\ k \neq j}}^{m} \tilde{C}_k^1 z_k\right]}{\operatorname{var}\left[\sum_{i=1}^{n} \tilde{S}_i^1 x_i\right]}} \right) \frac{E\left[\tilde{r}_M\right] - r_f}{1+r_f} b_M ,$$

where $b_M = b - x_0 - \sum_{\substack{k=1 \\ k \neq j}}^{m} C_k^0 z_k$ *.*

Note that in Proposition 7 b_M is computed slightly differently than in Proposition 6, due to a different status quo situation. Despite the differences in the formulas in Propositions 6 and 7, we know from Proposition 4 that the formulas will give the same result. The following proposition generalizes Propositions 6 and 7 to the case where the optimal project portfolio changes with and without the examined project.

PROPOSITION 8. *If the conditions of Proposition 5 hold, the difference in budget required to make two portfolios with different projects (the decision variables of the first are denoted by* $z_k, k = 1,...,m$ *, and those of the second by* $z_k', k = 1,...,m$ *) equally desirable is*

$$\Delta = -\sum_{k=1}^{m} C_k^0 \left(z_k - z_k'\right) + \frac{\sum_{k=1}^{m} E\left[\tilde{C}_k^1\right]\left(z_k - z_k'\right)}{1+r_f} - \left(\sqrt{1 + \frac{\operatorname{var}\left[\sum_{k=1}^{m} \tilde{C}_k^1 z_k\right] - \operatorname{var}\left[\sum_{k=1}^{m} \tilde{C}_k^1 z_k'\right]}{\operatorname{var}\left[\sum_{i=1}^{n} \tilde{S}_i^1 x_i\right]}} - 1 \right) \frac{E\left[\tilde{r}_M\right] - r_f}{1+r_f} b_M ,$$

where $b_M = b - x_0 - \sum_{k=1}^{m} C_k^0 z_k$ *.*

As the CAPM predicts that the value of an uncorrelated project is equal to its NPV discounted at the risk-free interest rate, this suggests that breakeven buying and selling prices are, in general, different from CAPM values, even when the optimal financial portfolio falls on the capital market line. However, the prices approach CAPM values when the amount of securities in the portfolio increases (i.e. when the investor's risk tolerance increases). This can be verified by multiplying b_M inside the parentheses. The last term in Proposition 8 now becomes

$$\left(\sqrt{b_M^2+\frac{\operatorname{var}\left[\sum_{k=1}^{m}\tilde{C}_k^1 z_k\right]-\operatorname{var}\left[\sum_{k=1}^{m}\tilde{C}_k^1 z'_k\right]}{\operatorname{var}\left[\tilde{r}_M\right]}}-b_M\right)\frac{E\left[\tilde{r}_M\right]-r_f}{1+r_f},$$

which goes to 0 as b_M goes to infinity.

Convergence of the project values to CAPM prices as the investor's risk tolerance goes to infinity applies in general to all projects, regardless of their correlation with each other or with the market. As long as the optimal project portfolio is the same with and without the project being valued, every project's BSP and BBP will converge towards the CAPM price of the project, as shown in Proposition 9; otherwise, they will converge towards the CAPM value of a difference portfolio $\mathbf{z}-\mathbf{z}'$ specified in Proposition 10. The proofs of the propositions are in the Appendix.

PROPOSITION 9. *If the investor is a mean-variance optimizer and the composition of the optimal project portfolio is the same with and without project j, the project's breakeven buying and selling prices converge towards the project's CAPM price*

$$v_j^b=v_j^s=v_j^{CAPM}=-C_j^0+\frac{E\left[\tilde{C}_j^1\right]}{1+r_f}-\frac{\operatorname{cov}\left[\tilde{C}_j^1,\tilde{r}_M\right]}{\operatorname{var}\left[\tilde{r}_M\right]}\cdot\frac{E\left[\tilde{r}_M\right]-r_f}{1+r_f}$$

as the investor's risk tolerance goes to infinity.

PROPOSITION 10. *If the investor is a mean-variance optimizer, a project's breakeven buying and selling prices converge towards*

$$v^b=v^s=v^{CAPM}=-\sum_{k=1}^{m}C_k^0\left(z_k-z'_k\right)+\frac{\sum_{k=1}^{m}E\left[\tilde{C}_k^1\right]\left(z_k-z'_k\right)}{1+r_f}-\frac{\operatorname{cov}\left[\sum_{k=1}^{m}\tilde{C}_k^1\left(z_k-z'_k\right),\tilde{r}_M\right]}{\operatorname{var}\left[\tilde{r}_M\right]}\cdot\frac{E\left[\tilde{r}_M\right]-r_f}{1+r_f}$$

as the investor's risk tolerance goes to infinity. Here, $z_k, k=1,\ldots,m$*, denote the decision variables for the optimal project portfolio with the project and* $z'_k, k=1,\ldots,m$ *those without the project.*

Apart from this limit behavior, another case where breakeven buying and selling prices coincide with CAPM recommendations is when a replicating portfolio exists for the project. This is a direct result of the fact that each of the three valuation methods is consistent with CCA.

E Valuation of Opportunities and Real Options

When valuing a project, we can either value an already started project or an opportunity to start a project. The difference is that, although the value of a started project can be negative, that of an opportunity to start a project is always non-negative, because a rational investor does not start a project with a negative value. While BSP and BBP are appropriate for valuing started projects, new valuation concepts are needed for valuing opportunities.

Since an opportunity entails the right but not the obligation to take an action, we need selling and buying prices that rely on comparing the situations where the investor *can* and *cannot* invest in the project, instead of *does* and *does not*. The lowest price at which the investor would be willing to sell an opportunity to start a project can be obtained from the definition of the breakeven selling price by removing the requirement to start the project in the status quo, i.e. by removing the constraint $z_{a^*} = 0$ in the top-left quadrant of Table III. We define this price as the *opportunity selling price* (OSP) of the project. Likewise, the *opportunity buying price* (OBP) of a project can be obtained be removing the starting requirement in the second setting, i.e. the equation $z_{a^*} = 0$ in the bottom-right quadrant of Table III. It is the highest price that the investor is willing to pay for a license to start the project. Opportunity selling and buying prices have a lower bound of zero; it is also straightforward to show that the opportunity prices can be computed by taking a maximum of 0 and the respective breakeven price. Table IV gives a summary of breakeven and opportunity selling and buying prices.

Table IV. A summary of an investor's buying and selling prices.

Valuation concept	Idea	Difference from breakeven prices	Properties
Breakeven buying and selling prices	Comparison of settings where the investor *does* and *does not* invest in the project.	-	Breakeven buying price and selling price are in general different from each other.
Opportunity buying and selling prices	Comparison of settings where the investor *can* and *cannot* invest in the project.	The investor is not obliged to invest in the project.	Lower bound of 0. Equal to maximum of 0 and the respective breakeven price.

Opportunity buying and selling prices can also be used to value *real options* (Trigeorgis 1996) that may be contained within the project portfolio. These options result from management's flexibility to adapt later decisions to unexpected future developments. Typical examples include possibilities to expand production when markets are up, to abandon a project under bad market conditions, and to switch operations to alternative production facilities. Real options can be valued much in the same way as opportunities to start projects. However, instead of comparing portfolio selection problems with and without a possibility to start a project, we will compare portfolio selection problems with and without the real option. This can typically be implemented by disallowing the investor from taking a particular action (e.g. expanding production) in the setting where the real option is not present. Since breakeven prices are consistent with CCA, also opportunity prices have this property, and hence they can be regarded as a generalization of the standard CCA real option valuation procedure to incomplete markets.

From the perspective of finance theory, a particularly important application of the real options concept is the management's ability to sell the project to the market, e.g. through securitization. Indeed, the possibility to sell the project to the market is effectively an American put option embedded in the project. Where selling the project to the market is a relevant option to the management, the related selling decisions are to be implemented as a part of the project's decision tree. That is, at each state, in addition to any other options available to the firm, the firm can opt to sell the project to the market at the prevailing market price. The actual sale price can be computed by using standard techniques, such as the market-implied risk-neutral probability distribution, and by deducting from the result any applicable transaction expenses, such as legal fees arising from establishing a special purpose vehicle for the project. Like any other real option, the opportunity to sell the project to the market, potentially increases the firm's valuation of the project. Also, since the option is an American put, it also sets the minimum possible value for the project, which is the market price. A valuation where the option to sell the project to the market is accounted for can be regarded as a holistic valuation that accounts for both private and market factors influencing the value of the project. Note that opportunity buying and selling prices can also be used to calculate the value of the ability to sell the project to the market.

IV Single-Period Numerical Experiments

In this section, we demonstrate the use of a MAPS model in project valuation through a series of numerical experiments. We first construct a single-period model, because this will generate several insights, is easier to follow and replicate, and because we can then contrast the results with the CAPM, also a single-period model. An example of project valuation in a multi-period setting is given in Section V. The experiments in this section confirm some of the theoretical results obtained in the previous sections and also cast light on the following issues:

- How are project values affected by the presence of other projects in the portfolio?
- How are project values affected by the opportunity to invest in securities?
- How are project values affected by the investor's risk tolerance?
- How are project values related to the CAPM?
- How does the presence of twin securities affect the value of a project?

A Experimental Set-up

The experimental set-up includes 8 equally likely states of nature, four projects, A, B, C, and D (Table V), and two securities, 1 and 2, which together constitute the market portfolio (Table VI). The setting can be extended to include more securities, but for the sake of simplicity we limit our market portfolio to two assets only. This does not influence the generality of our results. Note that projects C and D and security 2 are the same that Smith and Nau (1995) use in their examples. Table VII shows the correlation between the assets' cash flows. Numbers in italic in Tables V and VI are computed values. The risk-free interest rate is 8%.

The market prices in Tables V and VI are obtained by using the CAPM, where the expected rate of return of the market portfolio has been chosen so that the price of security 2 is $20, the price used by Smith and Nau (1995). Technically, this can be accomplished by including all of the assets into the market portfolio, with projects having zero issued shares, and by finding the market prices, excluding projects' investment costs, that minimize the sum of squared errors (SSE) between the rate of return given by the CAPM formula and the real expected rate of return, as computed from the market price. The prices converge, resulting in SSE equal to zero. The desired expected rate of return is 15.33%. The standard deviation of the market portfolio then is 35.32%. In Table VI, the security capitalization weights represent the ratio between the market capitalization of the security and that of the entire market. These weights are of interest, because the CAPM predicts that an MV investor will always invest in a combination of the risk-free asset and the market fund, where securities are present according to their capitalization weights.

Table V. Projects.

	Project			
	A	**B**	**C**	**D**
Investment cost	$80	$100	$104	$0.00
State 1	$150	$140	$180	$67.68
State 2	$150	$140	$180	$67.68
State 3	$150	$150	$60	$0.00
State 4	$150	$110	$60	$0.00
State 5	$50	$170	$180	$67.68
State 6	$50	$100	$180	$67.68
State 7	$50	$90	$60	$0.00
State 8	$50	$90	$60	$0.00
Expected outcome	*$100*	*$123.75*	*$120*	*$33.84*
St. dev. of outcome	*$50*	*$28.26*	*$60*	*$33.84*
Beta	*0.000*	*0.431*	*1.637*	*3.683*
Market price (NPV)	*$12.59*	*$11.33*	*-$4.00*	*$25.07*

Table VI. Securities.

	Security	
	1	**2**
Market price	*$39.56*	*$20.00*
Shares issued	15,000,000	10,000,000
Capitalization weight	*74.79%*	*25.21%*
State 1	$60	$36
State 2	$50	$36
State 3	$40	$12
State 4	$30	$12
State 5	$60	$36
State 6	$50	$36
State 7	$40	$12
State 8	$30	$12
Beta	*0.79*	*1.64*
Expected return	*13.76%*	*20.00%*
St. dev. of return	*28.26%*	*60.00%*

Table VII. Correlation matrix.

	A	**B**	**C**	**D**	**1**	**2**
A	1	0.398	0	0	0	0
B	0.398	1	0.487	0.487	0.653	0.487
C	0	0.487	1	1	0.894	1
D	0	0.487	1	1	0.894	1
1	0	0.653	0.894	0.894	1	0.894
2	0	0.487	1	1	0.894	1

The experiment comprises several steps. We start with a setting where the investor can invest only in the project being valued, which is then extended to cover a portfolio of all projects. We then add securities 1 and 2. Apart from the very first case, we assume that limitless borrowing at the risk-free rate and shorting of securities are allowed. As predicted by Proposition 4, the breakeven selling and buying prices are identical in this case and hence we display both of the prices in one entry. Unless otherwise noted, in each of the steps we use a budget of $500. The maximum risk level is defined in terms of the rate of return of the portfolio. For example, a 50% risk level implies that the maximum standard deviation for the terminal wealth level is $250.

B Project Values without Securities

In general, the value of a project depends on what would happen to the invested funds if the project were not started. In the absence of other projects and securities, three cases described in Table VIII are possible. These cases both provide a benchmark for project valuations obtained later, and show that even apparently small differences in available investment opportunities can make a significant difference in the values of projects. In the first case, where unused funds are lost, the net present value of the project is undefined, and hence we give the future value (FV) of the project instead.

Table VIII. Projects values when only the project being valued is available.

What happens to unused funds:	Project			
	A	**B**	**C**	**D**
Funds are lost (FV)	$100.00	$123.75	$120.00	$33.84
No interest gained	$20.00	$23.75	$16.00	$33.84
Deposited at risk-free interest rate	$12.59	$14.58	$7.11	$31.33

Let us next assume that the investor is able to invest in all of the projects and in the risk-free asset. The values of the projects are described in Table IX as a function of the investor's risk tolerance. In the optimal policy, the investor starts the projects with positive value.

Table IX. Projects values when all of the projects are available.

Risk level	Project			
	A	**B**	**C**	**D**
15%	-$1.99	$1.99	-$38.81	$18.74
20%	$12.59	$14.58	-$20.06	$24.22
25%	$5.48	$7.47	-$5.48	$24.22
30% and up	$12.59	$14.58	$7.11	$31.33

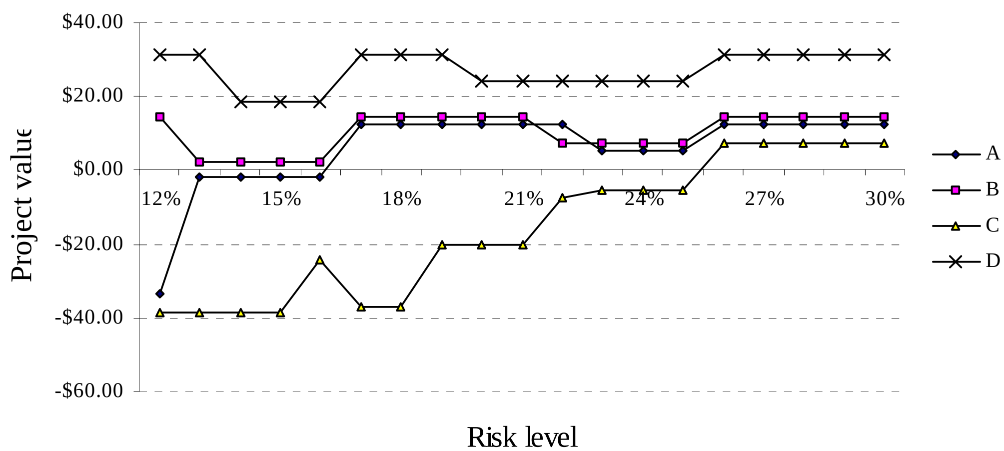

Figure 2. Project values for an MV investor.

Table IX and Figure 2 show that the project values behave rather erratically when the risk tolerance is varied. For example, at a 20% risk level, the MV values for projects A and B are identical to the values they have in isolation ($12.59 and $14.58), but then they drop to $5.48 and $7.47 at 25% and rise back to the values in isolation for higher risk tolerance levels. Intuitively, one might expect project values to rise as the risk constraint is relaxed, because more projects can be included into the portfolio so that fewer projects impose an opportunity cost. At the limit, this is correct: when the risk constraint is relaxed enough to allow the inclusion of all profitable projects, the project values coincide with the prices on the last row of Table VIII. For intermediate risk values, however, another effect confounds this result: the value of a project depends on the projects that fit into the portfolio when the project is started and when it is not. For example, the decrease of the price of projects A and B when the risk constraint is increased from 20% to 25% is due to the fact that, when the risk constraint is 25%, project C can be started if either of the projects is not included into the portfolio, but this is not the case at 20%. So at a 25% risk constraint, project C imposes an opportunity cost on these projects, but not at 20%, since the project does not fit into the portfolio regardless the decision on projects A and B.

C Project Values with Securities

We now examine what happens to the project values when securities are also available. In this case, the risk constraint will always be binding as long as the rate of return of the optimal security portfolio for the investor is higher than the risk-free interest rate: expectation of the whole portfolio will be maximized by purchasing as much of the optimal security portfolio as possible and by borrowing the necessary funds at the risk-free interest rate. When the investor does not invest in the projects, i.e. in a pure CAPM setting, the optimal security portfolio for a MV investor is to buy 75% of security 1 and 25% of security 2 at all risk levels. Note that these weights are the capitalization weights of the securities in Table VI.

Columns 2–5 of Table X describe the project values assuming that the investor can only invest in the project being valued and in securities. The presence of securities changes the project values for two reasons. On the one hand, securities impose an additional opportunity cost, which lowers project values. On the other hand, it is possible to hedge against project risks by buying negatively correlated or shorting positively correlated securities, which increases the project values.

Table X. Project values when securities 1 and 2 are available.

	Single project available				All projects available			
Risk level	**A**	**B**	**C**	**D**	**A**	**B**	**C**	**D**
15%	$8.92	$10.78	-$4.00	$25.07	$6.36	$8.22	-$4.00	$25.07
20%	$10.02	$10.92	-$4.00	$25.07	$8.61	$9.51	-$4.00	$25.07
30%	$10.94	$11.06	-$4.00	$25.07	$10.14	$10.26	-$4.00	$25.07
40%	$11.37	$11.13	-$4.00	$25.07	$10.80	$10.55	-$4.00	$25.07
50%	$11.62	$11.17	-$4.00	$25.07	$11.17	$10.72	-$4.00	$25.07
60%	$11.79	$11.19	-$4.00	$25.07	$11.42	$10.82	-$4.00	$25.07
70%	$11.90	$11.21	-$4.00	$25.07	$11.59	$10.90	-$4.00	$25.07
80%	$11.99	$11.23	-$4.00	$25.07	$11.72	$10.95	-$4.00	$25.07
90%	$12.06	$11.24	-$4.00	$25.07	$11.81	$11.00	-$4.00	$25.07
100%	$12.11	$11.25	-$4.00	$25.07	$11.89	$11.03	-$4.00	$25.07
200%	$12.35	$11.29	-$4.00	$25.07	$12.24	$11.18	-$4.00	$25.07
500%	$12.50	$11.31	-$4.00	$25.07	$12.45	$11.27	-$4.00	$25.07
2000%	$12.57	$11.32	-$4.00	$25.07	$12.56	$11.31	-$4.00	$25.07
10000%	$12.59	$11.33	-$4.00	$25.07	$12.59	$11.33	-$4.00	$25.07

Since project A is uncorrelated with the market portfolio, we can use its values to verify Propositions 6, 7, and 8. For example, let us calculate the breakeven buying price for project A at 50% risk level. The variance of the project portfolio with the project (including only project A) is $50^2 = 2500$ and 0 without the project. The variance of the optimal security portfolio in the status quo (no projects) is $250^2 = 62500$ and the amount of money spent on securities is \$708.3. As the market rate of return is 15.33%, we get from Proposition 7 that

$$v_A^b = -80 + \frac{100}{1.08} - \left(1 - \sqrt{1 - \frac{2500}{62500}}\right)\frac{0.0733}{1.08}708.3 = 12.59 - 0.0202 \cdot 48.07 = 11.62,$$

which coincides with the value in Table X. Also, as predicted by Proposition 9, when the risk tolerance approaches infinity, the project values converge to their CAPM prices, as shown in the row "Market price" in Table VI. Intuitively, one may expect that the values would approach risk-neutral values as the risk-constraint is being relaxed more and more, but this is not the case.

Notice that security 2 is a twin security for projects C and D. This explains why these projects are priced constantly to their CCA value across all risk levels. To replicate the cash flows of project C, one needs 5 shares of security 2, costing \$100. Since the investment cost of project C is \$104, we obtain the value of –\$4 for the project, regardless of the investor's risk tolerance. The cash flows of project D can be replicated by constructing a portfolio of 2.82 shares of security 2 and borrowing of \$31.33 at the risk-free interest rate. Therefore, the value of this project is $-\$0 + 2.82 \cdot \$20 - 31.33 \cdot \$1 = \$25.07$.

Let us next investigate how the presence of other projects, as well as that of securities, affects project values. Columns 6–9 of Table X give the results. First, we observe that the values of projects A and B still increase monotonically with increasing risk tolerance without any erratic behavior. We see also that the values still converge to CAPM prices. Additionally, the values of projects A and B decrease in comparison with the single-project case, because the optimal security portfolio with these projects is less profitable than what it was with a single project only.

D Summary of Experimental Results

Table XI summarizes our findings with respect to the research questions posed at the beginning of this section. Further, we have results on the valuation of real options in De Reyck et al. (2004) which indicate that the present approach can readily be extended to the valuation of real options in incomplete markets.

Table XI. Summary of research questions.

Research question	Answer
Influence of other projects on project values	Alternative projects can lower project values by imposing an opportunity cost.
Influence of securities on project values	Securities can both lower and raise project values. On the one hand, they impose an opportunity cost. On the other hand, they enable better diversification of risk.
Influence of risk tolerance on project values	When securities are not available, project values may rise non-monotonically as the risk tolerance rises. When securities are available, project values for MV investor may rise or decrease monotonically by risk tolerance depending on the correlation of the project with securities and other projects.
Relationship to CAPM prices	For an MV investor, project values converge towards CAPM prices as the risk tolerance goes to infinity.
Influence of twin securities on project values	A twin security makes the MAPS project value consistent with the project's CCA value at all risk levels, provided that limitless borrowing is allowed. As ordinary securities for other projects, twin securities also influence values of other projects.

V Multi-Period Example

In this section, we demonstrate project valuation in a multi-period MAPS setting through an example similar to the one in Gustafsson and Salo (2005). In this setting, the investor can invest in projects *A* and *B* in two stages as illustrated in Figure 1. At time 0, he/she can start either one or both of the projects. If a project is started, he/she can make an further investment at time 1. If the investment is made, the project generates a positive cash flow at time two; otherwise, the project is terminated with no further cash flows. In the spirit of the CAPM, it is assumed that the investor can also deposit and lend money at an 8% risk-free interest rate and invest in the equity market portfolio. The investor is able to buy and short the market portfolio and the risk-free asset in any quantities. The initial budget is \$9 million. The investor is a mean-LSAD optimizer with a risk (LSAD) tolerance of R = \$5 million. Note that we use a risk-constrained model instead of a preference functional model as in Gustafsson and Salo (2005), because otherwise the optimal strategy will be unbounded with the investor investing an infinite amount in the market portfolio and financing this by going short in the risk-free asset, or vice versa, depending on the value of the mean-LSAD model's risk aversion parameter.

Uncertainties are captured through a state tree where uncertainties are divided into market and private uncertainties (see Figures 3 and 4). The price of the market portfolio is entirely determined by the prevailing market state, while the projects' cash flows depend solely on the private states. At time 1, there are two possible private states, 1 and 2, and two possible market states *u* and *d*. These states imply four joint states, 1*u*, 1*d*, 2*u* and 2*d*, for time 1. At time 2, following the time-1 private state 1, private state may be 11 or 12; if the time-1 private state was 2, then the time-2 private state may be either 21 or 22. At time 2, the market state may be either *uu* or *ud*, if *u* obtained at time 1, or *du* or *dd* if the time-1 market state was *d*. These private and market states imply the following sixteen joint states for time 2 (the probability of the state is given in parentheses): 1*u*1*u* (3.75%), 1*u*1*d* (3.75%), 1*d*1*u* (3.75%), 1*d*1*d* (3.75%), 1*u*2*u* (8.75%), 1*u*2*d* (8.75%), 1*d*2*u* (8.75%), 1*d*2*d* (8.75%), 2*u*1*u* (5%), 2*u*1*d* (5%), 2*d*1*u* (5%), 2*d*1*d* (5%), 2*u*2*u* (7.5%), 2*u*2*d* (7.5%), 2*d*2*u* (7.5%), 2*d*2*d* (7.5%).

When project decisions are combined with the state tree in Figure 3, we obtain the simplified decision trees in Figures 5 and 6, where each action is associated with an indexed binary action variable z and the cash flows it generates. The market states are here indicated by "x" meaning that the value can be either one of u and d, because the project outcomes are independent of the market state and hence the project will generate the same cash flows regardless of which one of the market states obtains. The market portfolio is assumed to yield a return of 24% if u results and 0% if d results, with both having an equal chance of occurrence. This implies an expected excess rate of return of 4% for the market portfolio.

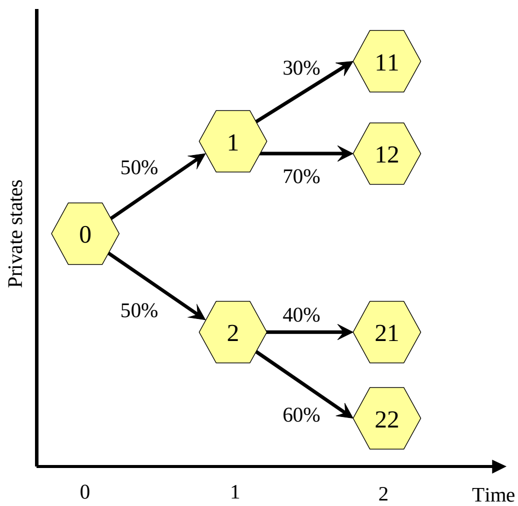

Figure 3 Private states

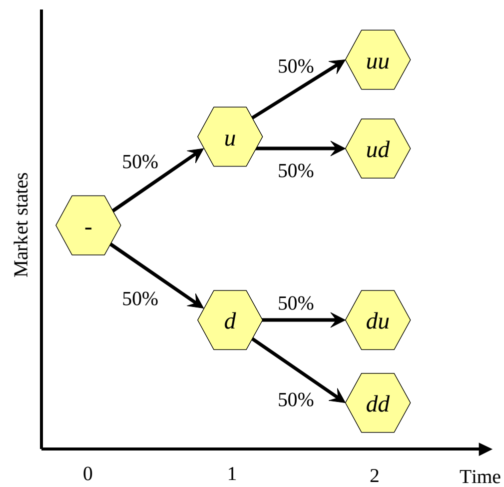

Figure 4 Market states

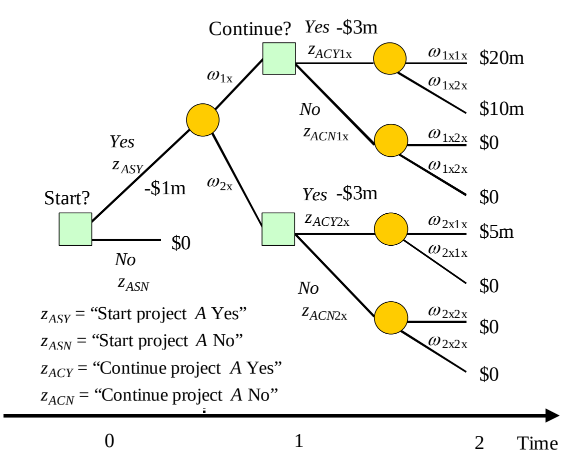

Figure 5 Decision tree of project *A*

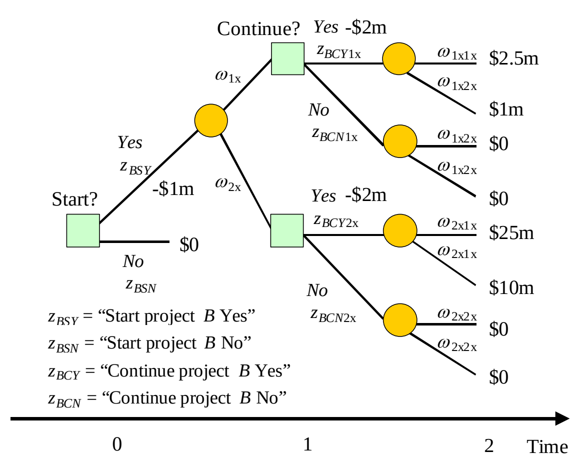

Figure 6 Decision tree of project *B*

Based on Figures 5 and 6, budget constraints can now be written as:

$$-1\cdot z_{ASY} - 2\cdot z_{BSY} - x_{\omega 0} + 9 - CS_{\omega 0} = 0$$

$$\begin{aligned}
&-3\cdot z_{ACY1u} - 2\cdot z_{BCY1u} + 1.24x_{\omega 0} - 1.24x_{\omega 1u} + 1.08\cdot CS_{\omega 0} - CS_{\omega 1u} = 0\\
&-3\cdot z_{ACY2u} - 2\cdot z_{BCY2u} + 1.24x_{\omega 0} - 1.24x_{\omega 2u} + 1.08\cdot CS_{\omega 0} - CS_{\omega 2u} = 0\\
&-3\cdot z_{ACY1d} - 2\cdot z_{BCY1d} + 1\cdot x_{\omega 0} - 1\cdot x_{\omega 1d} + 1.08\cdot CS_{\omega 0} - CS_{\omega 1d} = 0\\
&-3\cdot z_{ACY2d} - 2\cdot z_{BCY2d} + 1\cdot x_{\omega 0} - 1\cdot x_{\omega 2d} + 1.08\cdot CS_{\omega 0} - CS_{\omega 2d} = 0
\end{aligned}$$

$$\begin{aligned}
&20\cdot z_{ACY1u} + 2.5\cdot z_{BCY1u} + 1.5376x_{\omega 1u} + 1.08\cdot CS_{\omega 1u} - CS_{\omega 1u1u} = 0\\
&20\cdot z_{ACY1u} + 2.5\cdot z_{BCY1u} + 1.24x_{\omega 1u} + 1.08\cdot CS_{\omega 1u} - CS_{\omega 1u1d} = 0\\
&20\cdot z_{ACY1d} + 2.5\cdot z_{BCY1d} + 1.24x_{\omega 1d} + 1.08\cdot CS_{\omega 1d} - CS_{\omega 1d1u} = 0\\
&20\cdot z_{ACY1d} + 2.5\cdot z_{BCY1d} + 1\cdot x_{\omega 1d} + 1.08\cdot CS_{\omega 1d} - CS_{\omega 1d1d} = 0\\
&10\cdot z_{ACY1u} + 1\cdot z_{BCY1u} + 1.5376x_{\omega 1u} + 1.08\cdot CS_{\omega 1u} - CS_{\omega 1u2u} = 0\\
&10\cdot z_{ACY1u} + 1\cdot z_{BCY1u} + 1.24x_{\omega 1u} + 1.08\cdot CS_{\omega 1u} - CS_{\omega 1u2d} = 0\\
&10\cdot z_{ACY1d} + 1\cdot z_{BCY1d} + 1.24x_{\omega 1d} + 1.08\cdot CS_{\omega 1d} - CS_{\omega 1d2u} = 0\\
&10\cdot z_{ACY1d} + 1\cdot z_{BCY1d} + 1\cdot x_{\omega 1d} + 1.08\cdot CS_{\omega 1d} - CS_{\omega 1d2d} = 0
\end{aligned}$$

$$\begin{aligned}
&5\cdot z_{ACY2u} + 25\cdot z_{BCY2u} + 1.5376x_{\omega 2u} + 1.08\cdot CS_{\omega 2u} - CS_{\omega 2u1u} = 0\\
&5\cdot z_{ACY2u} + 25\cdot z_{BCY2u} + 1.24x_{\omega 2u} + 1.08\cdot CS_{\omega 2u} - CS_{\omega 2u1d} = 0\\
&5\cdot z_{ACY2d} + 25\cdot z_{BCY2d} + 1.24x_{\omega 2d} + 1.08\cdot CS_{\omega 2d} - CS_{\omega 2d1u} = 0\\
&5\cdot z_{ACY2d} + 25\cdot z_{BCY2d} + 1\cdot x_{\omega 2d} + 1.08\cdot CS_{\omega 2d} - CS_{\omega 2d1d} = 0
\end{aligned}$$

$$\begin{aligned}
&10\cdot z_{BCY2u} + 1.5376x_{\omega 2u} + 1.08\cdot CS_{\omega 2u} - CS_{\omega 2u2u} = 0\\
&10\cdot z_{BCY2u} + 1.24x_{\omega 2u} + 1.08\cdot CS_{\omega 2u} - CS_{\omega 2u2d} = 0\\
&10\cdot z_{BCY2d} + 1.24x_{\omega 2d} + 1.08\cdot CS_{\omega 2d} - CS_{\omega 2d2u} = 0\\
&10\cdot z_{BCY2d} + 1\cdot x_{\omega 2d} + 1.08\cdot CS_{\omega 2d} - CS_{\omega 2d2d} = 0\,.
\end{aligned}$$

For each terminal state $\omega_T \in \Omega_T$, there is a deviation constraint $CS_{\omega_T} - EV - \Delta^{+}_{\omega_T} + \Delta^{-}_{\omega_T} = 0$, where EV is the expected cash balance over all terminal states, viz.

$$EV = \sum_{\omega_T \in \Omega_T} p_{\omega_T} CS_{\omega_T}$$

In addition, the following decision consistency constraints apply:

$$\begin{aligned}
&z_{ASY} + z_{ASN} = 1 && z_{BSY} + z_{BSN} = 1\\
&z_{ACY1u} + z_{ACN1u} = z_{ASY} && z_{BCY1u} + z_{BCN1u} = z_{BSY}\\
&z_{ACY2u} + z_{ACN2u} = z_{ASY} && z_{BCY2u} + z_{BCN2u} = z_{BSY}\\
&z_{ACY1d} + z_{ACN1d} = z_{ASY} && z_{BCY1d} + z_{BCN1d} = z_{BSY}\\
&z_{ACY2d} + z_{ACN2d} = z_{ASY} && z_{BCY2d} + z_{BCN2d} = z_{BSY}
\end{aligned}$$

The risk constraint is now

$$\sum_{\omega_T \in \Omega_T} p_{\omega_T} \Delta^{-}_{\omega_T} \le 5$$

and the objective function is

$$Maximize\ EV = \sum_{\omega_T \in \Omega_T} p_{\omega_T} CS_{\omega_T}$$

For the total mixed asset portfolio, the optimal strategy is to start both projects; project *A* is terminated at time 1 if private state 2 occurs and project *B* if private state 1 occurs, i.e. variables $z_{ASY}, z_{ACY1u}, z_{ACY1d}, z_{BSY}, z_{BCY2u}, z_{BCY2d}$ are one and all other action variables are zero. The optimal amounts invested in the market portfolio and the risk-free asset are given in columns 2 and 3 of Table XII, respectively. This implies an expected cash balance of *EV* = \$25.63 million and LSAD of \$5.00 million at time 2. The portfolio has its least value, \$0.32 million, in state 1*d*2*d*. It is noteworthy that the value of the portfolio can be negative in some terminal states at higher risk levels, because the states' cash surplus variables are not restricted to non-negative values. Thus, the investor is able to borrow money at the risk-free rate and invest it in the market portfolio, and may hence default on his/her loan obligations if the market does not go up in either of the time periods and the project portfolio performs poorly.

Table XII. Investments in securities (\$ million)

State	x_M	CS
0	71.37	–64.37
1*u*	16.40	–4.35
1*d*	105.46	–106.61
2*u*	27.28	–16.85
2*d*	98.71	–98.86

Breakeven selling and buying prices for projects *A* and *B*, as well as for the entire project portfolio, are given on the last row in Table XIII. As predicted by Proposition 4, the prices are equal to each other, and we therefore record them in a single cell. For the purpose of using with the formula in Proposition 4, we also give the terminal wealth levels when the investor can and cannot invest in the project / portfolio being valued.

Table XIII. Project values (\$ million)

	Portfolio	A	B
W^+	\$25.63	\$25.63	\$25.63
W^-	\$17.16	\$22.17	\$20.54
$W^+ - W^-$	\$8.47	\$3.46	\$5.09
$v = v^b = v^s$	\$7.26	\$2.97	\$4.36

The portfolio value differs from the value of \$5.85 million obtained in Gustafsson and Salo (2005), where the investor did not have the possibility to invest in the market portfolio. The reason for the difference is twofold. First, due to the possibility to invest limitless amounts in the market portfolio, we use a risk-constrained preference model with R = \$5.00 million, whereas Gustafsson and Salo (2005) used a preference functional model with $\lambda = 0.5$. With different preference models we also have different risk-adjustment. Second, the ability to invest in the market portfolio influences the optimal mixed asset portfolio both when the investor is able and not able to invest in the project portfolio, whereby the project portfolio is likely to obtain a value different from the one obtained in Gustafsson and Salo (2005).

VI Summary and Conclusions

In this paper, we analyzed the valuation of private projects in a *mixed asset portfolio selection* (*MAPS*) setting, where an investor can invest in a portfolio of projects as well as securities traded in financial markets, but where the replication of project cash flows with financial securities is not necessarily possible. We developed a valuation procedure based on the concepts of *breakeven selling and buying prices*. This inverse optimization procedure requires the solution of portfolio selection problems with and without the project being valued and finding a lump sum that makes the investor indifferent between the two situations. To make the solution of these portfolio selection problems possible in a multi-period setting, we developed a multi-period MAPS model using the *Contingent Portfolio Programming* framework (Gustafsson and Salo 2005). We also produced several theoretical results relating to the analytical properties of breakeven prices, and studied the pricing behavior of mean-variance (MV) investors through a set of numerical experiments. Finally, we demonstrated the use of the multi-period framework through an illustrative example.

Our theoretical results indicate that the breakeven prices are, in general, consistent valuation measures, exhibiting sequential consistency, consistency with contingent claims analysis, and sequential additivity. The results also show that MV investors have several notable pricing properties. First, when limitless borrowing and lending are allowed, breakeven buying and selling prices are identical, which is not the case, in general, under expected utility theory (Smith and Nau 1995, Raiffa 1968). Also, the prices can be computed by solving the investor's terminal wealth level when he/she invests and he/she does not invest in the project and by discounting the difference back to its present value at the risk-free interest rate. In addition, we derived analytical formulas to calculate the breakeven prices for an MV investor when the optimal portfolio at present is known and projects are uncorrelated with securities. Finally, we showed that the project values given by an MV investor converge towards the projects' CAPM prices as the investor's risk tolerance goes to infinity.

Overall, our results suggest that alternative investment opportunities have a significant impact on the value of a project. Therefore, valuation of projects in isolation may potentially lead to biased estimates of the values of projects. This emphasizes the argument, which appears for example in the real options literature, that it is crucial to also consider financial instruments in project valuation. However, it is also important to recognize other projects in the portfolio.

Managers can draw several conclusions from our analysis. First, it is of central importance to clearly recognize the real investment alternatives to the projects. The presence of the possibility to borrow or short, or in general to invest in financial markets, may significantly influence the value of a project. Second, project values are non-additive. To obtain the value of a portfolio of two or more projects, it is necessary to calculate the terminal wealth levels when the investor starts and does not start the projects. This can be accomplished by solving the appropriate portfolio selection problems. Third, projects influence, in general, the optimal financial portfolio for the investor, so that when the projects are started, the optimal financial portfolio does not typically fall on the capital market line. Therefore, the allocation of funds to projects and securities separately may potentially lead to suboptimally diversified portfolios.

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Appendix

PROOF OF PROPOSITION 1: Let us prove the proposition first for the breakeven selling price and the sequential buying price. Let the breakeven selling price for the project be v_j^s. Then, based on Table III, v_j^s will be defined by the portfolio setting in the second column of Table A1:

Table A1.

	Step 1: Breakeven selling price	**Step 2: Breakeven buying price**
Status quo	Optimal objective function value: W_s^+ Additional constraint: $z_{a*} = 0$ (= invest in the project) Budget at time 0: $b(\omega_0)$	Optimal objective function value: $W_b^-(= W_s^-)$ Additional constraint: $z_{a*} = 1$ (= do not invest in the project) Budget at time 0: $b(\omega_0) + v_j^s$
Second setting	Optimal objective function value: W_s^- Additional constraint: $z_{a*} = 1$ (= do not invest in the project) Budget at time 0: $b(\omega_0) + v_j^s$	Optimal objective function value: W_b^+ Additional constraint: $z_{a*} = 0$ (= invest in the project) Budget at time 0: $b(\omega_0) + v_j^s - v_j^b$

Next, we can observe that the status quo in determining the sequential buying price is the same as the second setting for the breakeven selling price, wherefore also the optimal objective function values will be the same, i.e. $W_b^- = W_s^-$. Since by definition of breakeven prices we have $W_s^+ = W_s^-$ and $W_s^+ = W_s^-$, it follows that we also have $W_s^+ = W_s^-$. Since the portfolio optimization models in BSP's status quo and BBP's second setting are otherwise the same, except that the first has the budget of $b(\omega_0)$ and the second $b(\omega_0) + v_j^s - v_j^b$, and because the optimal objective function value is strictly increasing with respect to the budget, it follows that $b(\omega_0) + v_j^s - v_j^b = b(\omega_0)$, or $v_j^b = v_j^s$. The proposition for the breakeven buying price and the respective sequential selling price is proven similarly. Q.E.D.

PROOF OF PROPOSITION 2: Recall that the definition of a replicating trading strategy $\mathbf{x}^*$ for a project under project management strategy $\mathbf{z_k}$ is: For each state $\omega \setminus \omega_0 \in \Omega$, $\sum_{i=1}^{n} CF_i^s(\mathbf{x_i^*}, \omega) - CF_k^p(\mathbf{z_k}, \omega) = 0$. The value of the (replicating) security portfolio needed initiate the replicating trading strategy is $S^*(\omega_0) = \sum_{i=1}^{n} S(\omega_0) \cdot x_{i,\omega_0}^*$. Let us also denote the time-0 cash flow of project *k* by $CF_k^p(\mathbf{z_k}, \omega_0) = C_k^0$. Thus we have $CF_k^p(\mathbf{z_k}, \omega_0) - CF^s(\mathbf{x}^*, \omega_0) = C_k^0 - S^*(\omega_0)$. Observe now that starting the project using strategy $\mathbf{z_k}$ and shorting the replicating trading strategy will lead to a situation where cash flows net each other out in each state except in the base state, where the cash flow will be $C_k^0 - S^*(\omega_0)$. Therefore, a setting where the investor starts the project using a fixed strategy $\mathbf{z_k}$ and shorts the replicating trading strategy (the total trading strategy is $\mathbf{x} = \mathbf{x}' - \mathbf{x}^*$), will be exactly the same as a case where the project is not included in the portfolio, the trading strategy is $\mathbf{x}'$, and the time-0 budget is increased by $C_k^0 - S^*(\omega_0)$ (and hence the same objective function values). Therefore, the investor's breakeven selling price for the project is, by definition, $C_k^0 - S^*(\omega_0)$. The proposition for the breakeven buying price can be proven similarly. Q.E.D.

PROOF OF PROPOSITION 3: Let us begin with the breakeven buying price and a setting where the portfolio does not include projects and the budget is $b(\omega_0)$. Suppose that there are *n* projects that the investor buys sequentially. Let us denote the optimal value for this MAPS problem by $W_{b,1}^-$. Suppose then that the investor buys a project, indexed by 1, at his or her breakeven buying price, v_1^b. Let us denote the resulting optimal value for the MAPS problem by $W_{b,1}^+$. By definition of the breakeven buying price, $W_{b,1}^- = W_{b,1}^+$. Suppose then that the investor buys another project, indexed by 2, at his or her breakeven buying price, v_2^b. The initial budget is now $b - v_1^b$, and after the second project is bought, it is $b - v_1^b - v_2^b$. Since the optimization problems in the second project's status quo and the first project's second setting are the same, the optimal value for the resulting MAPS problem $W_{b,2}^-$ is equal to $W_{b,1}^+$. Add then the rest of the projects in the same manner, as illustrated in the following table:

Table A2.

	1st project	2nd project	3rd project	*n*th project
SQ	Optimal value: $W_{b,1}^-$ Budget at time 0: $b(\omega_0)$	Optimal value: $W_{b,2}^-(= W_{b,1}^+)$ Budget at time 0: $b(\omega_0) - v_1^b$	Optimal value: $W_{b,3}^-(= W_{b,2}^+)$ Budget at time 0: $b(\omega_0) - v_1^b - v_2^b$	Optimal value: $W_{b,n}^-(= W_{b,n-1}^+)$ Budget at time 0: $b(\omega_0) - v_1^b - v_2^b - \ldots - v_{n-1}^b$
SS	Optimal value: $W_{b,1}^+$ Budget at time 0: $b(\omega_0) - v_1^b$	Optimal value: $W_{b,2}^+$ Budget at time 0: $b(\omega_0) - v_1^b - v_2^b$	Optimal value: $W_{b,3}^+$ Budget at time 0: $b(\omega_0) - v_1^b - v_2^b - v_3^b$	Optimal value: $W_{b,n}^+$ Budget at time 0: $b(\omega_0) - v_1^b - v_2^b - \ldots - v_n^b$

The resulting budget in the last optimization problem, which includes all the projects, is $b(\omega_0) - v_1^b - v_2^b - \ldots - v_n^b$. Because the second setting of a project in the sequence is always the status quo of the next project and because by definition of the breakeven buying price, for each project, the objective function values in the status quo and the second setting are equal, we have $W_{b,n}^+ = W_{b,n}^- = W_{b,n-1}^+ = W_{b,n-1}^- = W_{b,n-2}^+ = \ldots = W_{b,1}^-$. Therefore, by definition of the breakeven buying price, the breakeven buying price for the portfolio including all the project must be $v_{ptf}^b = v_1^b + v_2^b + \ldots + v_n^b$. By re-indexing the projects and using the above procedure, we can change the order in which the projects are added to the portfolio. In doing so, the projects can obtain different values, but they still sum up to the same joint value of the portfolio. Similar logic proves the proposition for breakeven selling prices. Q.E.D.

PROOF OF PROPOSITION 4: Let us first examine the breakeven selling price. Let us denote the optimal objective function value for the given risk constraint in the status quo with W_s^+ and the one in the second setting when $v^s = 0$ with $W_s^- = W_s^+ + \Delta^s$. Since there is a risk-free interest rate, for each state at time *t*, we have $r_{B(\omega_{t+1})\to\omega_{t+1}} = r_t$, where $B(\omega_{t+1})$ is the time-*t* state and r_t is the risk-free interest rate from time *t* to *t*+1. Thus, $1 invested at time 0 will grow to $\prod_{t=0}^{T-1}(1+r_t)$ by time *T*.

Let us examine a MAPS problem in Table II, and add Δ to the budget at time 0, i.e. $b^\Delta(\omega_0) = b(\omega_0) + \Delta$. We can next study how the money would flow forwards without immediately re-optimizing the decisions. Denoting the cash flows in state ω_0 by $CF(\omega_0)$, the budget constraint would now become $CF(\omega_0) - CS_{\omega_0}^\Delta = -b^\Delta(\omega_0)$, where $CS_{\omega_0}^\Delta$ indicates cash surplus under the modified budget. Since in the original optimization problem we have $CF(\omega_0) - CS_{\omega_0} = -b\ (\omega_0)$, we get $CS_{\omega_0}^\Delta = CS_{\omega_0} + \Delta$. A time-1 budget constraint would now be $CF(\omega_1) + (1+r_0)CS_{\omega_0}^\Delta - CS_{\omega_1}^\Delta = -b\ (\omega_1)$, which can be modified into $CF(\omega_1) + (1+r_0)CS_{\omega_0} - CS_{\omega_1}^\Delta + (1+r_0)\Delta = -b\ (\omega_1)$, wherefrom it follows that $CS_{\omega_1}^\Delta = CS_{\omega_1} + \Delta(1+r_0)$. Taking the analysis forward, we get that for a time-*T* state ω_T, $CS_{\omega_T}^\Delta = CS_{\omega_T} + \Delta\prod_{t=0}^{T-1}(1+r_t)$.

Now, observe that, if we would change $CS_{\omega_0}^\Delta$ s to be decision variables instead of CS_{ω_0} s, the optimization problem would remain from the mathematical perspective effectively unchanged, because

(i) by changing $CS_{\omega_0} \to CS_{\omega_0}^\Delta$ we just change one unconstrained decision variable to another;

(ii) $\rho[X+b] = \rho[X]$, also $\rho\left[X + \Delta\prod_{t=0}^{T-1}(1+r_t)\right] = \rho[X]$; if the risk measure is implemented through deviation constraints, this can be also understood so that the deviation constraints $CS_{\omega_T}^\Delta - \sum_{\omega_T' \in \Omega_T} p(\omega_T')CS_{\omega_T'}^\Delta - \Delta_{\omega_T}^+ + \Delta_{\omega_T}^- = 0\ \forall \omega_T \in \Omega_T$ immediately reduce to $CS_{\omega_T} - \sum_{\omega_T' \in \Omega_T} p(\omega_T')CS_{\omega_T'} - \Delta_{\omega_T}^+ + \Delta_{\omega_T}^- = 0\ \forall \omega_T \in \Omega_T$;

(iii) the objective is $\sum_{\omega_T \in \Omega_T} p(\omega_T)\cdot CS_{\omega_T}^\Delta = \sum_{\omega_T \in \Omega_T} p(\omega_T)\cdot CS_{\omega_T} + \Delta\prod_{t=0}^{T-1}(1+r_t)$, and we know that adding a constant to the objective function does not influence the decisions that yield the optimum to the problem.

Therefore, the optimal portfolio decisions **x** and **z** must be the same as in the original optimization problem, as otherwise the solution to the original portfolio optimization problem would not be optimal. Thus, we can deduce from point (iii) that increasing the budget by Δ will increase the optimal objective function value by

$\Delta\prod_{t=0}^{T-1}(1+r_t)$.

Since we desire to make the expectations in the status quo and in the second setting under a modified budget (denoted by W_s^{-*}) equal, we have the equation $W_s^{+} = W_s^{-*} \Leftrightarrow W_s^{+} = W_s^{+} + \Delta^s + v^s \prod_{t=0}^{T-1}\left(1+r_t\right)$, from which it follows that $v^s = -\Delta^s / \prod_{t=0}^{T-1}\left(1+r_t\right)$, which is the breakeven selling price of the project.

Let us then examine the breakeven buying price and denote the optimal expectation in the status quo by W_b^- and the one in the second setting when $v_b = 0$ by $W_b^+ = W_b^- + \Delta^b$. As before, the expectations can be matched by lowering the budget by δ^b. The expectation in the second setting is now $W_b^{+*} = W_b^- + \Delta^b - v^b \prod_{t=0}^{T-1}\left(1+r_t\right)$. By requiring that $W_b^- = W_b^{+*} \Leftrightarrow W_b^- = W_b^- + \Delta^b - v^b \prod_{t=0}^{T-1}\left(1+r_t\right)$, we get $v^b = \Delta^b / \prod_{t=0}^{T-1}\left(1+r_t\right)$, which is the breakeven buying price of the project.

Finally, we observe that, as the respective optimization problems are identical (remember that budgets are not modified in any of these settings), $W_s^- = W_b^-$ and $W_b^+ = W_s^+$, from which it follows that $W_s^- = W_b^- \Leftrightarrow W_s^+ + \Delta^s = W_b^+ - \Delta^b \Leftrightarrow \Delta^s = -\Delta^b$. Hence, $v^s = v^b$. To obtain the valuation formula, we can just drop the subscript from W_s^+ and W_s^-. Q.E.D.

PROOF OF PROPOSITION 5: Since projects are uncorrelated with securities, we know that $\mathrm{var}\left[\sum_{i=0}^{n}\tilde{S}_i^1 x_i + \sum_{k=1}^{m}\tilde{C}_k^1 z_k\right] = \mathrm{var}\left[\sum_{i=0}^{n}\tilde{S}_i^1 x_i\right] + \mathrm{var}\left[\sum_{k=1}^{m}\tilde{C}_k^1 z_k\right]$ and hence the risk constraint can be rewritten as $\mathrm{var}\left[\sum_{i=0}^{n}\tilde{S}_i^1 x_i\right] \le R - \mathrm{var}\left[\sum_{k=1}^{m}\tilde{C}_k^1 z_k\right]$. Similarly, the objective can rewritten as $\max E\left[\sum_{i=0}^{n}\tilde{S}_i^1 x_i + \sum_{k=1}^{m}\tilde{C}_k^1 z_k\right] \Leftrightarrow \max E\left[\sum_{i=0}^{n}\tilde{S}_i^1 x_i\right] + E\left[\sum_{k=1}^{m}\tilde{C}_k^1 z_k\right]$. For any project portfolio with fixed $z_k^*, k = 1,...,m$, the optimal financial portfolio will be acquired by solving an optimization problem $\max E\left[\sum_{i=0}^{n}\tilde{S}_i^1 x_i\right] + a$ subject to $\sum_{i=0}^{n} S_i^0 x_i + \sum_{k=1}^{m} C_k^0 z_k^* = b$ and $\mathrm{var}\left[\sum_{i=0}^{n}\tilde{S}_i^1 x_i\right] \le R^*$, where $a = E\left[\sum_{k=1}^{m}\tilde{C}_k^1 z_k^*\right]$ is a constant and $R^* = R - \mathrm{var}\left[\sum_{k=1}^{m}\tilde{C}_k^1 z_k^*\right]$ is a constant. The parameter b does not influence the x_i's that yield the maximum to the problem, and R^* is just an adjusted maximum risk level for the investor. Thus, for any given project portfolio, the optimal financial portfolio is obtained by the solution of a usual mean-variance problem. We know from the Separation Theorem (Tobin 1958) that a mean-variance investor will invest in the combination of the market fund and the risk-free asset. Q.E.D.

PROOF OF PROPOSITIONS 6, 7, AND 8: Let us prove first Proposition 8. Let the project portfolio in the status quo be $P = \sum_{k=1}^{m} \tilde{C}_k^1 z_k$ and the security portfolio $M = \sum_{i=1}^{n} \tilde{S}_i^1 x_i$. The respective portfolios in the second setting are $P' = \sum_{k=1}^{m} \tilde{C}_k^1 z_k'$ and $M' = \sum_{i=1}^{n} \tilde{S}_i^1 x_i'$. The amount of money lent in the status quo is denoted by x_0 and the respective amount in the second setting by x_0'. By Proposition 4 we know that the portfolio weights in the security portfolios *M* and *M'* are identical, i.e., that $x_i' / x_i = a, i = 1, ..., n$, where *a* is the positive ratio of funds invested in risky securities in the second setting and in the status quo. That is,

$$a = \frac{b' - x_0' - \sum_{k=1}^{m} C_k^0 z_k'}{b - x_0 - \sum_{k=1}^{m} C_k^0 z_k}, \tag{A1}$$

where b' is the budget in the second setting. This implies that $M' = aM$. Let us require next that

$$E\left[(1+r_f)x_0 + M + P\right] = E\left[(1+r_f)x_0' + M' + P'\right]$$
$$\mathrm{var}\left[(1+r_f)x_0 + M + P\right] = \mathrm{var}\left[(1+r_f)x_0' + M' + P'\right]$$

By using the relationship $M' = aM$, the equivalence of expectations yields us

$$(1+r_f)x_0 + E[M] + E[P] = (1+r_f)x_0' + aE[M] + E[P']$$
$$\Rightarrow x_0' = x_0 + \frac{(1-a)E[M] + E[P] - E[P']}{1+r_f} \tag{A2}$$

Similarly, from the equivalence of variances we obtain

$$\mathrm{var}[M] + \mathrm{var}[P] = a^2\, \mathrm{var}[M] + \mathrm{var}[P']$$
$$\Rightarrow a = \pm\sqrt{1 + \frac{\mathrm{var}[P] - \mathrm{var}[P']}{\mathrm{var}[M]}} \tag{A3}$$

The minus sign alternative can be dropped, because we know that *a* has to be positive. Let us define $\Delta = b' - b$. By the definition of *a* in (A1), we get

$$\Delta = b' - b = a\left(b - x_0 - \sum_{k=1}^{m} C_k^0 z_k\right) + x_0' + \sum_{k=1}^{m} C_k^0 z_k' - b\,.$$

By substituting (A2) into this, we obtain

$$\Delta = b' - b = a\left(b - x_0 - \sum_{k=1}^{m} C_k^0 z_k\right) + x_0 + \frac{(1-a)E[M] + E[P] - E[P']}{1+r_f} + \sum_{k=1}^{m} C_k^0 z_k' - b\,.$$

By denoting $b_M = b - x_0 - \sum_{k=1}^{m} C_k^0 z_k$ and $M = (1+\tilde{r}_M)b_M$, we get

$$\Delta = ab_M + x_0 + \frac{(1-a)\left(1+E\left[\tilde{r}_M\right]\right)B_M + E\left[P\right] - E\left[P'\right]}{1+r_f} + \sum_{k=1}^{m} C_k^0 z_k' - b\,.$$

Let us then add and subtract $\sum_{k=1}^{m} C_k^0 z_k$ from the right-hand side of the equation to obtain $b_M = b - x_0 - \sum_{k=1}^{m} C_k^0 z_k$ on the right-hand side. Thus we get

$$\Delta = (a-1)b_M + \frac{(1-a)\left(1+E\left[\tilde{r}_M\right]\right)b_M + E\left[P\right] - E\left[P'\right]}{1+r_f} + \sum_{k=1}^{m} C_k^0 z_k' - \sum_{k=1}^{m} C_k^0 z_k\,.$$

By adding and subtracting r_f inside $\left(1+E\left[\tilde{r}_M\right]\right)$ we are able to eliminate the term (a – 1)b_M from the beginning and obtain $\Delta = \frac{(1-a)\left(E\left[\tilde{r}_M\right]-r_f\right)b_M + E\left[P\right] - E\left[P'\right]}{1+r_f} + \sum_{k=1}^{m} C_k^0 z_k' - \sum_{k=1}^{m} C_k^0 z_k\,.$

By denoting $E\left[P\right] = \sum_{k=1}^{m} E\left[\tilde{C}_k^1\right]\cdot z_k$ and $E\left[P'\right] = \sum_{k=1}^{m} E\left[\tilde{C}_k^1\right]\cdot z_k'$ and by rearranging the terms, we get

$$\Delta = -\sum_{k=1}^{m} C_k^0 (z_k - z_k') + \frac{\sum_{k=1}^{m} E\left[\tilde{C}_k^1\right]\cdot (z_k - z_k')}{1+r_f} - (a-1)\frac{E\left[\tilde{r}_M\right]-r_f}{1+r_f} b_M\,. \tag{A4}$$

Finally, by substituting (A3) into (A4), we obtain the desired formula

$$\Delta = -\sum_{k=1}^{m} C_k^0 (z_k - z_k') + \frac{\sum_{k=1}^{m} E\left[\tilde{C}_k^1\right]\cdot (z_k - z_k')}{1+r_f} - \left(\sqrt{1 + \frac{\mathrm{var}\left[\sum_{k=1}^{m} \tilde{C}_k^1 z_k\right] - \mathrm{var}\left[\sum_{k=1}^{m} \tilde{C}_k^1 z_k'\right]}{\mathrm{var}\left[\sum_{i=1}^{n} \tilde{S}_i^1 x_i\right]}} - 1\right)\frac{E\left[\tilde{r}_M\right]-r_f}{1+r_f} b_M\,.$$

Proposition 6 is obtained by assuming that $z_k = z_k' \quad \forall k \neq j$ and $z_j = 1$ and $z_j' = 0$. We get now

$$v_s^j = \Delta = -C_j^0 + \frac{E\left[\tilde{C}_j^1\right]}{1+r_f} - \left(\sqrt{1 + \frac{\mathrm{var}\left[\sum_{k=1}^{m} \tilde{C}_k^1 z_k\right] - \mathrm{var}\left[\sum_{\substack{k=1 \\ k\neq j}}^{m} \tilde{C}_k^1 z_k\right]}{\mathrm{var}\left[\sum_{i=1}^{n} \tilde{S}_i^1 x_i\right]}} - 1\right)\frac{E\left[\tilde{r}_M\right]-r_f}{1+r_f} b_M\,.$$

Similarly, Proposition 7 is obtained by assuming that $z_k = z_k' \quad \forall k \neq j$ and $z_j = 0$ and $z_j' = 1$. Then,

$$v_b^j = -\Delta = -C_j^0 + \frac{E\left[\tilde{C}_j^1\right]}{1+r_f} - \left(1 - \sqrt{1 - \frac{\mathrm{var}\left[\sum_{k=1}^{m} \tilde{C}_k^1 z_k\right] - \mathrm{var}\left[\sum_{\substack{k=1 \\ k\neq j}}^{m} \tilde{C}_k^1 z_k\right]}{\mathrm{var}\left[\sum_{i=1}^{n} \tilde{S}_i^1 x_i\right]}}\right)\frac{E\left[\tilde{r}_M\right]-r_f}{1+r_f} b_M\,. \text{ Q.E.D.}$$

PROOF OF PROPOSITIONS 9 AND 10: Let us prove first Proposition 10. The notation and the proof logic are as in the proof of Propositions 6–8 with the main differences being (i) in that the covariance terms do not disappear in the equality of variances, and (ii) the underlying reason for equality of security portfolio weights is different.

Observe first that when the investor's risk tolerance increases, he/she invests more and more in securities and the effect of projects on the total mixed asset portfolio diminishes. At the limit, the investor invests in an infinitely large security portfolio whose composition matches that of the market portfolio regardless of what projects there are in the portfolio. Therefore, the portfolio weights in the security portfolios *M* and *M'* are identical, so that $M' = aM$, where *a* is some constant. Now, we have again

$$\operatorname{var}\left[(1+r_f)x_0 + M + P\right] = \operatorname{var}\left[(1+r_f)x_0' + M' + P'\right],$$

from which we obtain

$$\begin{aligned}
&\operatorname{var}[M] + \operatorname{var}[P] + 2\operatorname{cov}[M,P] = a^2\operatorname{var}[M] + \operatorname{var}[P'] + 2a\operatorname{cov}[M,P'] \\
&\Rightarrow a^2\operatorname{var}[M] + 2a\operatorname{cov}[M,P'] - \operatorname{var}[M] + \operatorname{var}[P'] - \operatorname{var}[P] - 2\operatorname{cov}[M,P] = 0.
\end{aligned}$$

Solving this for *a* gives

$$\begin{aligned}
&\Rightarrow a = \frac{-2\operatorname{cov}[M,P'] \pm \sqrt{4\operatorname{cov}[M,P']^2 + 4\operatorname{var}[M]\left(\operatorname{var}[M] - \operatorname{var}[P'] + \operatorname{var}[P] + 2\operatorname{cov}[M,P]\right)}}{2\operatorname{var}[M]} \\
&= \frac{-\operatorname{cov}[M,P']}{\operatorname{var}[M]} \pm \sqrt{\frac{\operatorname{cov}[M,P']^2}{\operatorname{var}[M]^2} + \frac{\operatorname{var}[M]^2 + \operatorname{var}[M]\left(\operatorname{var}[P] - \operatorname{var}[P']\right) + 2\operatorname{var}[M]\operatorname{cov}[M,P]}{\operatorname{var}[M]^2}} \\
&= \frac{-\operatorname{cov}[M,P']}{\operatorname{var}[M]} \pm \sqrt{\frac{\rho_{M,P'}^2\operatorname{var}[P']}{\operatorname{var}[M]} + 1 + \frac{\operatorname{var}[P] - \operatorname{var}[P']}{\operatorname{var}[M]} + 2\frac{\rho_{M,P}stdev[P]}{stdev[M]}} \\
&= \frac{-\operatorname{cov}[M,P']}{\operatorname{var}[M]} \pm \sqrt{\frac{\rho_{M,P}^2\operatorname{var}[P]}{\operatorname{var}[M]} + 2\frac{\rho_{M,P}stdev[P]}{stdev[M]} + 1 - \frac{\rho_{M,P}^2\operatorname{var}[P]}{\operatorname{var}[M]} + \frac{\rho_{M,P'}^2\operatorname{var}[P']}{\operatorname{var}[M]} + \frac{\operatorname{var}[P] - \operatorname{var}[P']}{\operatorname{var}[M]}} \\
&= \frac{-\operatorname{cov}[M,P']}{\operatorname{var}[M]} \pm \sqrt{\left(\frac{\rho_{M,P}stdev[P]}{stdev[M]} + 1\right)^2 + \frac{(\rho_{M,P'}^2 - 1)\operatorname{var}[P']}{\operatorname{var}[M]} - \frac{(\rho_{M,P}^2 - 1)\operatorname{var}[P]}{\operatorname{var}[M]}} \\
&\underset{\operatorname{var}[M]\to\infty}{\longrightarrow} \frac{-\operatorname{cov}[M,P']}{\operatorname{var}[M]} + \frac{\operatorname{cov}[M,P]}{\operatorname{var}[M]} + 1 = \frac{\operatorname{cov}[M,P-P']}{\operatorname{var}[M]} + 1
\end{aligned}$$

The minus sign alternative can be dropped, because we know that *a* has to be positive. Now let us substitute this into Equation (A4) in the proof of Propositions 6–8 to obtain

$$\Delta = -\sum_{k=1}^{m} C_k^0 (z_k - z_k') + \frac{\sum_{k=1}^{m} E\left[\tilde{C}_k^1\right]\cdot(z_k - z_k')}{1+r_f} - \frac{\operatorname{cov}\left[M, \sum_{k=1}^{m}\tilde{C}_k^1\cdot(z_k - z_k')\right]}{\operatorname{var}[M]}\frac{E[\tilde{r}_M] - r_f}{1+r_f} b_M. \qquad \text{(A5)}$$

For the BSP of project *j*, we have $z_j = 1$ and $z_j' = 0$. Since Δ is the budget increment required to make two portfolios equally desirable, Δ is, by definition, the BSP of the project. Formula for the BBP, which is a budget reduction, is obtained by multiplying both sides of Equation (A5) by –1. Finally, since

$M = (1+\tilde{r}_M)b_M$, we obtain the CAPM formula

$$v^{CAPM} = \Delta = -\sum_{k=1}^{m} C_k^0 (z_k - z'_k) + \frac{\sum_{k=1}^{m} E\left[\tilde{C}_k^1\right]\cdot(z_k - z'_k)}{1+r_f} - \frac{\operatorname{cov}\left[\tilde{r}_M, \sum_{k=1}^{m} \tilde{C}_k^1 \cdot (z_k - z'_k)\right]}{\operatorname{var}\left[\tilde{r}_M\right]} \frac{E\left[\tilde{r}_M\right] - r_f}{1+r_f}. \quad \text{(A6)}$$

Note that the definition of z_i's and z'_i's in Proposition 10 is different from the definition in Proposition 8. Here, Proposition 10 for the BSP case can be obtained directly using Equation (A6). The BBP case is obtained by changing the notation so that z'_i indicates the status quo (no project) and z_i indicates the second setting (project present). Proposition 9 is obtained for the BSP case from (A6) by assuming that $z_k = z'_k \quad \forall k \neq j$ and $z_j = 1$ and $z'_j = 0$. Proposition 9 for the BBP case is again obtained by changing the meaning of z'_i (to no project) and z_i (to project present) and assuming that $z_k = z'_k \quad \forall k \neq j$ and $z_j = 1$ and $z'_j = 0$. Q.E.D.