Information in the first price auction*

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Abstract

We study how the structure of information can affect welfare outcomes in the first price auction. For a fixed distribution of buyers' valuations, we consider all of the Bayesian equilibria in weakly undominated strategies that could arise under different specifications of the buyers' higher-order beliefs, as long as those beliefs are consistent with a common prior and the given prior distribution of values. Tight bounds on revenue and bidder surplus are provided for the models we consider.

When there is no lower bound on how uninformed the buyers might be (beyond knowing the prior distribution), we derive tight upper and lower bounds on revenue and bidder surplus. These bounds hold for symmetric and arbitrarily correlated distributions of values with any number of bidders, and in particular the bounds apply to specifications where values are interdependent.

When buyers are assumed to at least know their own values, we derive a tight upper bound on revenue and a tight lower bound on bidder surplus for arbitrary distributions of values and for any number of bidders. We also derive a tight lower bound on revenue and a tight upper bound on bidder surplus when values are binary (high or low) and the distribution of values is symmetric. We apply these results to study how entry fees and reserve prices impact the welfare bounds.

Keywords: First price auction, information structure, Bayes correlated equilibrium, private values, interdependent values, common values, revenue, surplus, welfare bounds, reserve price, entry fee.

JEL Classification: C72, D44, D82, D83.

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1 Introduction

The first price auction has long been an important institution in both the theory and practice of mechanism design. The theoretical interest in the auction stems from the fact that it provides a good approximation of the mechanisms that are used to allocate a wide range of goods, from government contracts to real estate to cut flowers. Early mechanism design was largely motivated by the puzzle that in benchmark settings with independent and private values, the first price auction generates the same revenue as other common auction formats, leading one to ask why any one of these mechanisms should be used rather than all the others. The popularity of the first price auction may be due to its relatively transparent strategic incentives that pressure prospective buyers to bid positive amounts, and it seems less susceptible to bidding rings or other forms of collusion that might arise in the second price auction. At the same time, first price auctions are challenging to characterize theoretically; the optimal bidding strategy depends on beliefs about the distribution of opponents' bids, to say nothing of the potentially complex inference about the surplus from winning when others' information is correlated with one's own valuation. Indeed, the auction has been analyzed primarily in settings where it is possible to construct a "nice" equilibrium in monotonic bidding strategies, which necessitates a one-dimensional family of private signals and some notion of positive correlation among signals and values. Outside of these particular settings, we have limited understanding of the behavior of the first price auction, aside knowing that simple equilibria may not exist and that standard results such as revenue equivalence can fail.

For this reason, one might prefer to use a mechanism that is less sensitive to beliefs such as a second price auction, which at least in the case of private values has a unique equilibrium in weakly undominated strategies. With such a mechanism and such a refinement, there is no ambiguity about the welfare outcomes that will result. And yet, the first price auction is still an important mechanism in practice. Thus, it is important to understand how it might behave even in settings where a traditional equilibrium analysis is intractable. If equilibrium and welfare outcomes in the first price auction can vary with the structure of bidders' beliefs, then what kinds of beliefs might lead revenue to be greater than with other, more "robust" mechanisms? When might revenue be smaller? Ideally, we would like to understand how welfare outcomes are affected by the structure of beliefs, and perhaps even understand the range of outcomes that might obtain under different assumptions. With such understanding, we would be better able to compare the first price auction with other auction formats, or even learn when it is optimal to use the first price auction rather than its alternatives.

These are the questions that we study in this paper. In particular, we investigate how welfare outcomes in the first price auction could vary with the structure of bidders' information. We analyze this question by fixing the payoff relevant primitive of the model, which is the underlying joint distribution of the bidders' true valuations for the good. For this fixed distribution, we consider all possible ways to specify bidders' beliefs using the standard terminology and modeling device of type spaces: each bidder has a set of possible types, and bidders' types are jointly distributed along with the true values. This joint distribution must have the given prior as its marginal over valuations, and the sets of types and the joint distribution are common knowledge among the bidders. For each type space, we look at Bayes Nash equilibria (BNE) of the game where bidders first draw types and then submit bids conditional on their type. The question we answer is how much could revenue and bidder surplus vary over all type spaces and BNE that are consistent with the fixed distribution of values.

This sounds like a challenging task, given the myriad of possible type spaces and the well-known intractability of equilibria for particular examples. Fortunately, we can make the problem more managable by using a solution concept for games of incomplete information that two of us recently proposed, which we term Bayes correlated equilibrium (BCE) (Bergemann and Morris, 2013). In the auction setting, a BCE is a joint distribution of equilibrium bids and ex-post values such that bids are optimal conditional on information contained in the equilibrium bid and any other information bidders are assumed to know. Bergemann and Morris show that the set of BCE outcomes coincide with the set of joint distributions of bids and values that could arise for some pair of (i) a type space and (ii) a BNE on that type space. Essentially, this definition takes the equilibrium bid to be a sufficient statistic for whatever information in the bidder's type indicated that the equilibrium bid was optimal. Moreover, like its complete information cousin, correlated equilibrium, BCE is characterized by a set of linear inequalities that make it tractable for analysis.

Nonetheless, there is a large number of possible BCE of the first price auction. To discpline the model further, we implicitly assume in our solution concept that beliefs are consistent with a common prior. This rules out, for example, equilibria in which everyone bids zero because they believe others are bidding zero. We further refine our predictions by assuming that bidders do not use weakly dominated strategies, which in this setting means that buyers never place bids above the support of the interim belief about their own valuation. This rules out well known equilibria with complete information where bidders follow dominated strategies and in which the seller extracts all of surplus (Blume, 2003, cf).

In all of our results, we consider two different lower bounds on how much information the bidders might have about valuations. In the first specification, which we call the *unknown* *values* model, we assume that the bidders need not know anything about values except for the prior distribution. However, it is possible that bidders could have more information, ranging from just knowing their own value to knowing all bidders' values. This model encompasses many well known specifications that have been explored in the literature, e.g., mineral rights and affiliated values. In particular, it is possible that the bidders learn about their values from the equilibrium outcome, which can give rise to a winner's curse phenomenon that depresses equilibrium bids. In the second specification, we assume that bidders at least know their own true value for the good, but we impose no restrictions on how much bidders know about others' values, beyond what they can deduce from the prior and their own value. We call this the *known values* model. Known values, more commonly referred to in the literature as private values, is a natural assumption in many settings, and it is commonly supposed throughout the auction theory and applied auction econometrics literatures.

Let us comment briefly on these assumptions. The existing literature on first price auctions typically assumes, for the purposes of tractability, that bidders' information arrives in the form of one-dimensional signals that satisfy some form of positive correlation or single crossing property (Milgrom and Weber, 1982). Indeed, others have shown that equilibria could be quite complicated or intractable if signals are inherently multidimensional (Jackson, 2009). Our approach, on the other hand, allows for all possible type spaces, including ones in which bidders receive multidimensional signals and could have quite complicated inference about values and beliefs. Moreover, we impose minimal restrictions on the joint distribution of values, beyond assuming that the prior is symmetric for most of our results. In particular, bidders' values are allowed to be correlated in an arbitrary fashion. By working with the weaker solution concept of incomplete information correlated equilibria, and by considering all type spaces at once, we can abstract away from many of these issues of existence and tractability. For example, it is always a possibility that bidders have complete information about values, in which case trivial Bayes Nash equilibria exist, so the set of BCE is always non-empty. Moreover, the set of BCE is convex, being the intersection of a family of linear inequalities. Although some issues of non-compactness arise in the analysis due to the discontinuity of payoffs in the auction, there is enough structure in the set of BCE to conclude that BCE attaining welfare bounds exist.

The main results of our analysis are as follows: For each of these models (unknown and known values) we study maximum revenue and minimum bidder surplus, as well as minimum revenue and maximum bidder surplus. We group analyses and objectives by these pairs because for many specifications, they are achieved by the same BCE. For the unknown values model, as long as the distribution of values is symmetric, there exist trivial equilibria that minimize bidder surplus: if bidders do not know anything about their values except for the ex-ante distribution, it is an equilibrium for them all to bid same amount, which is the ex-ante expected value. This equilibrium results in zero bidder surplus, although the allocation of the good is ex-post inefficient. In fact, we construct equilibria that yield arbitrarily small bidder surplus and also generate an efficient allocation, while still satisfying the weakly undominated refinement. This establishes that it is possible for the seller to achieve revenue equal to the efficient surplus. More than anything else, this reflects the weakness of the weakly undominated criteria in this setting, since it is easy to construct complicated beliefs about one's value that make essentially any strategy undominated.

For the known values model, the analysis of maximum revenue and minimum bidder surplus is more subtle. With known values, it is generally impossible for bidders to receive zero surplus in equilibrium. In particular, bidders know that their opponents will not use weakly dominated strategies and hence will never bid above their own values. This means that a bidder could always bid a best response to a "worst case scenario" in which others bid exactly their values, which would guarantee the bidder a positive surplus that is a lower bound on what they could achieve in any equilibrium under a known values type space. In fact, we are able to construct BCE in which a bidder's surplus is exactly this lower bound. Moreover, it is possible for all bidders to attain the bound simultaneously, and this is even possible in an equilibrium in which the allocation of the good is efficient! Such an equilibrium must also maximize the seller's revenue, since total surplus is maximized and the bidders' share is minimized.

At first glance, the result is somewhat paradoxical: the lower bound described in the previous paragraph assumes that other bidders are following a strategy of bidding their values, whereas the best reply that generates the bound would, except in extreme cases, involve bidding below one's value. However, this "paradox" can be resolved by the bidders having additional information that allows them to determine whether or not they have the high value, and by the high-value bidder having just enough information to outbid losing opponents who bid their values. Moreover, it is possible to structure this information so that bidders obtain no additional information rents beyond what they get from knowing their known values. This establishes tight a upper bound on revenue and a tight lower bound on bidder surplus.

With regard to the opposite welfare directions, we would also like to know how badly the auction might perform in terms of revenue and how well it might perform in terms of bidder surplus. A trivial lower bound on revenue is zero, if bids are restricted to be non-negative (and values are non-negative as well). However, even in the unknown values case, this bound is not tight and the seller must earn positive revenue. The logic is essentially that given above: If everyone were to bid zero, then anyone could win for sure by deviating up to a positive but arbitrarily amount. Nonetheless, it is possible to push revenue well below the level of complete information, in which the bidder with the highest value wins and pays the second-highest value.[1] Intuitively, in the complete information case, there is a great deal of slack in incentive constraints; bidders strictly prefer to bid the second-highest value over any other bid. Such a bid would continue to be a best response even if it entailed a small probability of losing some other player making a slightly higher bid. Thus, a small amount of incomplete information about values, properly structured, could support equilibria with lower winning bids, because *losing* buyers are not certain whether they are bidding exactly the second highest value, are losing to others making slightly higher bids, or are themselves bidding slightly less than the second highest value and losing.

For the no-information case, we take this logic to its limit to derive a tight lower bound for revenue. This involves both a derivation of a bound and a construction of BCE that attain it. Here we give some intuition for how the bound is generated. The logic in the previous paragraph indicated a kind of recursive structure to the incentive constraints: I, as a bidder, am willing to lose with a given bid that is less than my value, as long as the probability of winning with that bid is sufficiently large relative to the gains from deviating upward. Thus, winning with low bids allows others to win against me with relatively low bids as well. Yet, the equilibrium is symmetric, so if others win against me with a given bid, then sometimes I must win with that bid as well. Thus, the probability mass of winning bids below a given level recursively constrains the number of winning bids at higher levels.

Indeed, we show that the cumulative distribution of winning bids in equilibrium has to satisfy a certain average incentive constraint, in that probability of the winning bid being below a given level b cannot rise too quickly as a function of b without violating some bidder's incentive to increase their bid. This constraint suggests a relaxation of the original problem, in which the choice variable becomes just the cumulative distribution of winning bids, and the objective is to minimize revenue by making this distribution rise as quickly as possible, subject to the average incentive constraint. It is a relaxation because (i) we only optimize over the marginal distribution of winning bids conditional on the true values, rather than the entire joint distribution of winning and losing bids, and (ii) we collapse incentive constraints down to this single average constraint. We solve this relaxed problem to obtain a lower bound on revenue over all equilibria. At the lower bound, the winning bid distributions will rise as quickly as possible and the average incentive constraint will always bind. In addition, the winner is always a bidder with the highest value, and the winner's bid is drawn from a distribution that depends only the losing buyers' values (in particular, on the average

[1] This coincides with expected revenue in the Bayes Nash equilibrium in settings with independent private values.

value among the losing buyers according to the true profile of values). These winning bid distributions have ordered supports, so winning bids are higher when the average losing value is higher. A bidder who wins does not know anything about his value, conditional on winning, except for the fact that his value is distributed according to the prior distribution conditional on the average losing value.

In addition to generating a bound, we show that it is possible to extend the solution of the relaxed problem to a full equilibrium, in which both upward and downward incentive constraints are satisfied. The relaxed problem pins down the distribution of winning bids conditional on the bidders' ex-post values, and these objects motivate a natural specification for the distribution of losing bids conditional on values and the winner's bid. It is a consequence of having solved the relaxed problem that bidders will not want to deviate upwards, and we verify from this construction that bidders will not want to deviate to lower bids as well. Incidentally, since the winning bidder has the highest ex-post value, the equilibrium is efficient, and this equilibrium necessarily maximizes bidder surplus as well.

This structure indicates a principle at work in information structures that push down revenue as much as possible. It is well known that when values are not known, a "winner's curse" can lead to lower equilibrium bids, since buyers interpret winning as a negative signal about their surplus from the good. In contrast, the equilibrium described above implies an even more perverse "upward deviator's curse". Consider a bidder who bids b in equilibrium. This bid b sometimes wins and sometimes loses, but when it is a winning bid it is made when the losing bidders' average value is v. If the bid wins, it must be because the losing bidders' average value is v, and all the winning bidder knows is that his own value is larger than v and drawn from the ex-ante distribution. Thus, in general, the expected winning valuation is significantly larger than v. If, however, this same bidder were to deviate up to a bid b' slightly higher than b, then the marginal event on which that bidder wins is when b was a losing a bid, i.e., when the bidder's own value was on average v, or just slightly higher. Thus, it is winning after deviating upwards that is a strong negative signal about one's value, which is why these beliefs are so effective at depressing the distribution of winning bids.

The unknown values model allows a great deal of flexibility to structure bidders' inference about their values as a function of their equilibrium bids. We are also interested in minimum revenue in the more constrained model where bidders at least know their own values. In Section 5.2, we extend our program to the analysis of known value models. First, we consider symmetric models with any number of bidders and in which buyers assign one of two possible values to the good, high or low. For this setting, we obtain a tight characterization of minimum revenue and maximum bidder surplus. We show that the solution of the relaxed problem in this case can be decomposed into a distribution of winning bids for each possible *number* of bidders with high values. The greater is the number of bidders with high values, the higher is the support of the distribution of winning bids. As in the no-information case, this marginal distribution extends to a full equilibrium in which both upward and downward constraints are satisfied. All equilibria are efficient when there are only two possible valuations, so bidder surplus is maximized.

Beyond binary values, additional complexities arise in the application of our program. We can write down a relaxed problem for many known values which is the analogue of the relaxations for unknown values and binary known values. The optimal distribution of winning bids will however generally depend on both the winner's and losers' values. This multi-dimensional structure precludes a straightforward ordering of the supports of winning bid distributions, and the solution to the relaxed problem could involve a complicated pattern of which types win with what bids. In addition, the solution need not be efficient, so that minimum revenue and maximum bidder surplus need not be attained with the same equilibrium. With the restriction to efficient solutions, we characterize the relaxed problem when there are three possible valuations. In that case, a more limited "ordered support" property characterizes the solution for a range of parameter values. However, even with three values, the bound need no longer be tight, and we can find distributions of values for which there do not exist equilibria that attain the bound from the relaxed problem. The reason is that the solution to the relaxed problem can involve a pattern of supports of winning bids in which one bidder's winning bid may be strictly higher than the support of bids he or she wins against, so that the bidder would want to deviate downwards. In Section 6, we explore computationally how minimum revenue behaves in general known value models.

Ultimately, there are two lessons that we take from our exercise: First, we learn that the first price auction is indeed sensitive to the structure of beliefs. In order to get a sense of how sensitive, we compare the maximum and minimum revenue to the revenue that would be obtained in the complete information setting in which all of the bidders know everyones' ex-post values, and revenue is just the second-highest value. We show in the known value model that the ratio between actual expected revenue and the complete information expected revenue can be arbitrarily large and arbitrarily close to zero. A fortiori, this result also extends to the unknown values model. On the other hand, the information and beliefs that attain the bounds are highly stylized. As bounds on welfare outcomes, they are impeccable, though they seem unlikely to arise in practice. Nonetheless, these characterizations indicate what kinds of information and belief structures are likely to make revenue and bidder surplus higher or lower than that suggested by a classical analysis.

The bulk of the paper is focused on the analysis of revenue and bidder surplus in a plain first price auction. In the second-to-last section, we consider extensions and applications of our model. In Section 6.1, we use a combination of theoretical and computational tools to explore the set of possible welfare outcomes in directions other than maximizing and minimizing revenue and bidder surplus. In particular, we look at how social efficiency and the distribution of surplus can vary with the bidders' information. In Section 6.3, we extend the model to a more general class of mechanisms: first price auctions with reserve prices and entry fees. We explore the how maximum and minimum revenue vary with these parameters of the mechanism.

1.1 Related literature

A small number of papers have solved for equilibria of private value first price auctions where bidders know their own value but have partial information about other bidders' values. Kim and Che (2004) consider the case where bidders are partitioned into groups, and there is complete information of valuations within elements of the partition, but no information about the valuations of bidders not in the same element of the partition. Equilibria in this setting are inefficient and thus reduce seller revenue. The implications of specific information structures in auctions, and their implication for online advertising market design, are analyzed in recent work by Abraham et al. (2013) and Celis et al. (2011). Both papers are motivated by asymmetries in bidders' ability to access additional information about the object for sale. Consequently, they examine the role of the distributions of valuations resulting from the private acquisition of data by a single bidder. In particular, Abraham et al. (2013) focus on second price auctions in a common value environment, while Celis et al. (2011) propose an approximately optimal mechanism in a private values model. In a closely related contribution to these two papers, Kempe et al. (2012) study the first-price, common-value auction with asymmetrically informed bidders.

By contrast, we focus on extremal information structures which give rise to extremal values of bidder surplus and revenue and this involves efficient allocations at least in the symmetric cases which we can solve. Fang and Morris (2006) and Āzacis and Vida (2015) consider the two bidder, two value, two signal case. We review their results in detail in the next Section. We end up solving for Bayes correlated equilibria for first price auctions (because these characterize what can happen in equilibrium for different information structures). Other papers have examined outcome in private value first price auctions under solution concepts weaker than Bayes Nash equilibrium (for a fixed information structure). Battigalli and Siniscalchi (2003) and Dekel and Wolinsky (2003) examine rationalizable outcomes. The set of rationalizable outcomes they consider are neither a subset nor a superset of the BCE outcomes we consider: more restrictive than us, they maintain that all bidders' interim beliefs about opponents' values are the same as the prior distribution; however, our solution concept maintains the common prior assumption. Lopomo et al. (2011) examine bidder collusion in first price auctions. They model bidder collusion as a mechanism design problem, and so the set of attainable equilibria corresponds to the set of communication equilibria in the sense of, e.g., Forges (1993), and they give analytic and computational results showing the impossibility of collusion. Communication equilibrium is another version of incomplete information correlated equilibrium which imposes "truth-telling" constraints (players must have an incentive to truthfully report their types to a mediator) that do not arise in Bayes correlated equilibria.

The rest of this paper proceeds as follows. In Section 2, we describe our basic model of the first price auction, with a fixed distribution of ex-post values but many possible specifications of beliefs. In Section 3, we preview our later results with a series of examples that illustrate how welfare outcomes could vary depending on the specification of the bidders' beliefs, and to provide intuition for the kinds of information that will induce extreme outcomes. In Section 4, we characterize tight bounds on maximum revenue and minimum bidder surplus over all possible specifications of bidders' beliefs consistent with a fixed ex-post distribution of values, first for the no-information model and then for the known values model. In Section 5, we repeat the exercise for the objectives of minimum revenue and maximum bidder surplus. Section 6 presents extensions of the model, and Section 7 concludes. Omitted proofs are contained in the Appendix.

2 Model

There are n potential buyers of a single unit of a good that can be produced at zero marginal cost. The bidders are index by $i \in \{1, \ldots, n\} = N$. Bidders' values are drawn from a compact set $V \subset \mathbb{R}$. For all of our formal results, we will assume that V is finite and denumerated as

$$V = \{v^0, \ldots, v^K\},$$

where K is the number of possible valuations. The profile of valuations is $v = (v_1, \ldots, v_n) \in V^n$. We will denote by $\mathbf{1}$ the vector in $\mathbb{R}^n$ with all coordinates equal to 1. Thus, $\mathbf{1}x$ denotes a vector in which all of the elements are x. There is a fixed prior joint distribution over values which we denote by $p \in \Delta(V^n)$,[2] which is common knowledge among the bidders.

Let Ξ denote the set of n-permutations, i.e., bijective mappings from $\{1, \ldots, n\}$ into itself. We associate to a permutation $\xi \in \Xi$ a mapping of the same name $\xi : V \to V$ where

[2]For a metric space X, $\Delta(X)$ denotes the set of Borel measures on X.

$v' = \xi(v)$ iff $v'_i = v_{\xi(i)}$ for all i. The distribution p is *symmetric* if $p(v) = p(\xi(v))$ for all $\xi \in \Xi$. Throughout most of the paper, we will assume that p is symmetric. Some of our results extend to asymmetric distributions of values, which we will discuss further in Section 6.5.

The buyers are participating in a first price auction in which they submit real-valued bids b_i in a compact interval $B = [M, v^K] \subset R$, with $M \leq v^0$. For a profile of bids $b \in B^n$, write $W(b) = \{i | b_i \geq b_j \; \forall j \neq i\}$ denote the set of high bidders. Bidder i receives the good with probability

$$q_i(b) = \frac{\mathbb{I}_{i \in W(b)}}{|W(b)|},$$

where $\mathbb{I}_X$ is equal to 1 if the event X occurs and zero otherwise. In other words, the high bidder receives the good and ties are broken uniformly. (Note that W and q_i are measurable with respect to the Borel σ-algebra on B^n.)

We assume that bidders may receive additional information about the profile of values, beyond knowing the prior distribution. This additional information comes in the form of signals, which we will call types, that are correlated with the profile of valuations. A *type space* is a collection $\mathcal{T} = (\{T_i\}_{i=1}^n, \mu)$, where T_i are compact metric spaces and $\mu : V^n \to \Delta(T)$ maps profiles of values into Borel probability measures over $T = \times_{i=1}^n T_i$. The interpretation is that T_i is the set of bidder i's types and μ describes the conditional distribution of types given values. Write $\mu_i(\cdot|v_i) \in \Delta_i$ for a version of the conditional distribution of buyer i's type given buyer i's value, i.e., μ_i almost surely satisfies

$$\mu_i(X|v_i) = \frac{\sum_{v_{-i} \in V^{n-1}} p(v_i, v_{-i}) \mu (X \times T_{-i} | v_i, v_{-i})}{\sum_{v_{-i} \in V^{n-1}} p(v_i, v_{-i})}.$$

for all Borel subsets $X \subseteq T_i$. We will say that $\mathcal{T}$ is a *known values* type space if for all v_i and v'_i in V, $\mu_i(\operatorname{supp} \mu_i(\cdot|v'_i)|v_i) = 0$, where supp denotes the support of a distribution. This implies that bidders can almost surely back out their valuation from their type.

For a fixed type space $\mathcal{T}$, the auction is a game of incomplete information in which bidders' strategies are measurable mappings $\beta_i : T_i \to \Delta(B)$. Let $\mathcal{B}_i$ denote the set of strategies for buyer i. Fixing a profile of strategies $\beta \in \mathcal{B} = \times_{i=1}^n \mathcal{B}_i$, bidder i's surplus from the auction is

$$U_i(\beta) = \sum_{v \in V^n} p(v) \int_{t \in T} \mu(dt|v) \int_{b \in B} q_i(b) \, (v_i - b_i) \, \beta(db|t),$$

where β is the unique measure induced on the product σ-algebra by the product measure $\beta_1 \times \cdots \times \beta_n$ (Cohn, 1980, Theorem 5.1.4). A strategy $\beta_i \in \mathcal{B}_i$ is *weakly undominated* if there does not exist a $\beta_i' \in \mathcal{B}_i$ such that

$$U_i(\beta_i', \beta_{-i}) \geq U_i(\beta_i, \beta_{-i})$$

for all $\beta_{-i} \in \mathcal{B}_{-i}$, with the inequality strict for some β_{-i}. For the first price auction mechanism, weakly undominated simply means that

$$\operatorname{supp} \beta_i \subseteq [v^0, \max\{v_i | t_i \in \operatorname{supp} \mu_i(\cdot|v_i)\}].$$

The profile $\beta \in \mathcal{B}$ is a *Bayes Nash equilibrium* (BNE) if and only if, for all i,

$$\begin{aligned} &\sum_{v \in V^n} p(v) \int_{t \in T} \mu(dt|v) \int_{b \in B} q_i(b)\,(v_i - b_i)\,\beta(db|t) \\ &\qquad \geq \sum_{v \in V^n} p(v) \int_{t \in T} \mu(dt|v) \int_{b \in B} q_i(b)\,(v_i - b_i)\,(\beta_i', \beta_{-i})(db|t) \end{aligned} \tag{1}$$

for all $\beta_i' \in \mathcal{B}_i$. Naturally, the BNE β is also weakly undominated if β_i is weakly undominated for all i.

Let $F : V^n \to \Delta(B^n)$ denote a mapping from profiles of values to probability distributions over profiles of bids. F is a *Bayes correlated equilibrium* (BCE) if, for all Borel measurable $\sigma_i : B \to B$,[3]

$$\begin{aligned} &\sum_{v \in V^n} p(v) \int_{b \in B^n} q_i(b)(v_i - b_i) F(db|v) \\ &\qquad \geq \sum_{v \in V^n} p(v) \int_{b \in B^n} q_i(\sigma_i(b_i), b_{-i})(v_i - \sigma_i(b_i)) F(db|v). \end{aligned} \tag{2.1}$$

We refer to such a σ_i as a *deviation mapping*. In addition, F is a *known values BCE* if for all $v_i \in V$ and for all $\sigma_i : B \to B$ that are Borel measurable,

$$\begin{aligned} &\sum_{v_{-i} \in V^{n-1}} p(v_i, v_{-i}) \int_{b \in B^n} q_i(b)(v_i - b_i) F(db|v_i, v_{-i}) \\ &\qquad \geq \sum_{v_{-i} \in V^{n-1}} p(v_i, v_{-i}) \int_{b \in B^n} q_i(\sigma_i(b_i), b_{-i})(v_i - \sigma_i(b_i)) F(db|v_i, v_{-i}). \end{aligned} \tag{2.2}$$

[3]This definition follows that given by Hart and Schmeidler (1989) for correlated equilibrium with infinite action spaces.

A *BCE in weakly undominated strategies* is one for which $\mathbb{E}[v_i|b_i] \geq b_i$ with probability one. A BCE is *symmetric* if for all $\xi \in \Xi$, $F(\xi(b)|\xi(v)) = F(b|v)$.

We note that the set of BCE is the intersection of the set of mappings from V^n to Borel measures over bid profiles with an (infinite) family of linear inequalities (2). Thus, the set of BCE is convex and bounded. However, it is not compact in the weak-* topology, as the outcome function is discontinuous and therefore the set of distributions satisfying (2) for each possible deviation function σ_i need not be closed.

Let us interpret these conditions. A BCE is an extension of correlated equilibrium to games of incomplete information, where players' actions are allowed to be correlated, not just with one another, but also with the underlying payoff relevant states. The incentive compatibility condition is that conditional on the equilibrium action, that action must be optimal. The equilibrium can be conceptualized as being engineered by a disinterested mediator who knows the true profile of values and privately recommends bids to each buyer. A BCE is a recommendation rule for the mediator such that the buyers would want to follow the recommendation. The distinction between regular and known value BCE is that the known value definition also requires that the recommendation be followed, even when the buyer knows their own value.

BCE are useful for our analysis because of the epistemic relationship between the set of BCE and the set of outcomes that can be induced by a BNE for some type space $\mathcal{T}$. In particular, let us say that the conditional distributions $F : V^n \to \Delta(B^n)$ are *induced* by the type space $\mathcal{T}$ and the BNE β if for all v and Borel subsets $X \subseteq B^n$,

$$F(X|v) = \int_{t \in T} \beta(X|t)\mu(dt|v). \tag{3}$$

Theorem 1 (Bergemann and Morris)**.** *F is induced by some (known value) type space $\mathcal{T}$ and some BNE β if and only if F is an unknown value (known value) BCE.*

Proof. We prove the result for the unknown value case. With known values, the proof is virtually identical.

For the if direction, take $T_i = B_i$ and $\mu = F$. It is straightforward to verify that $\beta_i(\{t_i\}|t_i) = 1$ is a BNE. In particular, suppose β_i' yields a higher payoff for player i. Let $\sigma_i : B_i \to B_i$ such that $\sigma_i(b_i) \in \operatorname{supp} \beta_i'(\cdot|b_i)$. If for every such σ_i, (2) holds, then it cannot be that β_i' is a profitable deviation, since β_i' could be implemented as a probability measure over σ_i. Thus, there must be some σ_i for which (2) is violated.

For the only if direction, let β be a BNE for the type space $\mathcal{T}$. These together induce an F by (3). Suppose that (2) is violated for some measurable deviation function σ_i. Then consider the alternative strategy β_i' which is the pushforward measure induced by the mapping σ_i, i.e.,

$$\beta_i'(\sigma_i(X)|t_i) = \beta_i(X|t_i)$$

for all Borel measurable sets X. We claim that this strategy must be a profitable deviation:

$$\begin{aligned}
&\sum_{v\in V^n} p(v)\int_{t\in T}\int_{b\in B^n} q_i(b)(v_i - b_i)(\beta_i', \beta_{-i})(db|t)\mu(dt|v) \\
&\quad = \sum_{v\in V^n} p(v)\int_{t\in T}\int_{b\in B^n} q_i(\sigma_i(b_i), b_{-i})(v_i - \sigma_i(b_i))\beta(db|t)\mu(dt|v) \\
&\quad = \sum_{v\in V^n} p(v)\int_{b\in B^n} q_i(\sigma_i(b_i), b_{-i})(v_i - \sigma_i(b_i))F(db|v) \\
&\quad > \sum_{v\in V^n} p(v)\int_{b\in B^n} q_i(b)(v_i - b_i)F(db|v) \\
&\quad = \sum_{v\in V^n} p(v)\int_{t\in T}\int_{b\in B^n} q_i(b)(v_i - b_i)\beta(db|t)\mu(dt|v),
\end{aligned}$$

This contradicts the hypothesis that β is a BNE. □

Our goal is to analyze how welfare varies across type spaces and BNE for a fixed distribution over values. Because of Theorem 1, we can equivalently ask how welfare varies across BCE. The welfare outcomes that we will investigate are:

$$\begin{aligned}
\text{Bidder surplus: } U(F) &= \sum_{i=1}^{n}\sum_{v\in V^n} p(v)\int_{b\in B^n} q_i(b)(v_i - b_i)F(db|v); \\
\text{Revenue: } R(F) &= \sum_{i=1}^{n}\sum_{v\in V^n} p(v)\int_{b\in B} q_i(b)b_iF(db|t); \\
\text{Total surplus: } S(F) &= \sum_{i=1}^{n}\sum_{v\in V^n} p(v)\int_{b\in B} q_i(b)v_iF(db|t).
\end{aligned}$$

We will study these objectives for both known and unknown value BCE.

We adopt the notation that $v^{(k)}$ denotes the kth order statistic of values, so that $v^{(1)}$ is the highest value and $v^{(2)}$ is the second-highest value. Note that the maximum feasible surplus is the expected first order statistic of values:

$$\overline{S} = \sum_{v\in V^n} p(v)v^{(1)}.$$

3 Examples

Before proceeding to our general results, we will illustrate the main ideas of our analysis with three examples. Though our results will be stated for the case where there are finitely many values, for the purposes of this section (and for later examples) we will suppose that there are two potential buyers whose values are independently and uniformly distributed on the interval $[0, 1]$.

3.1 Benchmark

A useful reference point for bidder surplus and revenue comes from assuming that the buyers know their own values, but do not know anything except for the prior distribution about the other buyer's value. Formally, we can assume that there is a type space $T_i = V$ for all i, and $\mu(t|v) = 1$ if $t_i = v_i$ for all i, and $\mu(t|v) = 0$ otherwise.

This case was famously first studied by Vickrey (1961), who showed that there is a simple Bayesian equilibrium in which each buyer bids half their value. From the revenue equivalence theorem (Myerson, 1981), we conclude that revenue is the same as that in the second price auction, in which the high value bidder wins and pays the second highest value, so that revenue is $1/3$. The outcome is efficient and so total surplus is the expected highest value of $2/3$, and each bidder receives one half of the surplus net of revenue, which is $1/6$.

Incidentally, these are the same welfare outcomes as would obtain if there was complete information about values, in which case the high value buyer would win and pay the second highest value. The losing bidder can follow any strategy of randomizing over bids below his own value such that the high bidder prefers to win with a bid equal to the second highest value.

Thus, for these two extreme assumptions of independent private values and complete information, welfare outcomes are the same. Naive intuition might suggest that welfare outcomes would be the same for intermediate information structures, in which the buyers know their own values but receive partial information about the other buyer's value. However, this is not the case.

3.2 Higher revenue and lower bidder surplus

Let us first consider how information might induce an outcome in which revenue is higher than the benchmark and bidder surplus is lower. We start close to complete information, and suppose that the buyers receive signals that indicate which of them has the higher valuation. In particular, we assume that the buyers' signals contain the maximum value.

In addition, let us suppose that rather than learning the losing value, the winning bidder receives a noisy signal. Fix $\epsilon \in (0, 1/2)$, and let us suppose that the bidders observe signals $t_i = (v_i, v^{(1)}, \max\{\epsilon v^{(1)}, v^{(2)}\})$. Thus, if $v^{(2)} > \epsilon v^{(1)}$, the high value buyer learns the lowest value, but otherwise all he learns is that the lowest value is less than ϵ of his own value.

We can construct a simple equilibrium in this case where if $v^{(2)} > \epsilon v^{(1)}$, the high value buyer bids $v^{(2)}$, and the low value buyer randomizes over bids below his value, say according to the cumulative distribution function $F(b|v^{(1)}, v^{(2)}) = \frac{v^{(1)}-v^{(2)}}{v^{(1)}-b}$ on $[0, v^{(2)}]$, and the high value buyer wins with a bid of $v^{(2)}$. On the other hand, if $v^{(2)} < \epsilon v^{(1)}$, the low value buyer simply bids his value and the winner bids $v^{(2)}$. This is incentive compatible because the winner thinks that the loser's bid is uniformly distributed on $[0, \epsilon v^{(1)}]$, so the surplus from a bid b is $(v^{(1)} - b)\frac{b}{\epsilon v^{(1)}}$, which is increasing as long as $b < v^{(1)}/2$.

With this information and in this equilibrium, the cumulative distribution of winning bids first-order stochastically dominates that under complete information, and so revenue must be strictly higher and bidder surplus strictly lower. Intuitively, under the complete information outcome, when $v^{(2)} = \epsilon v^{(1)}$, the high bidder strictly prefers to bid $v^{(2)}$ over any smaller bid. This bid would still be optimal even if there was a modest probability of $v^{(2)}$ and the losing bid being less than $\epsilon v^{(2)}$. Thus, by creating partial information about the losing bid and losing bidder's value, the winner could be induced to bid more than in the benchmark.

While this example illustrates the mechanism through which information could raise revenue and lower bidder surplus, it should be clear that this mechanism has hardly been pushed to its fullest extent. This construction only utilizes partial information about outcomes in which $v^{(2)}$ is quite small, and moreover the construction is specific to the uniform distribution. In Section 4, we will push these principles as far as they can go in order to characterize a tight upper bound on revenue and a tight lower bound on bidder surplus.

3.3 Lower revenue and higher bidder surplus

Partial information could also induce lower bids than in the benchmark. For example, consider again a small variation on the complete information benchmark. Bidders receive signals $t_i = (v_i, x_i)$ where $x_i \in [0, 1]$ is determined according to the following procedure.

The equilibrium is normally as with the complete information, with $x_i = v_j$, and bidder i bids x_i if $x_i < v_i$ and randomizes over $[0, v_i]$ if $x_i > v_i$ in order to support bidder j bidding v_i.

However, there is a small probability that the following happens: If $v_i > v_j$, then with probability $\epsilon v_j(v_i - v_j)$, bidder i is given the signal $x_i = \alpha(v_i, v_j)$, which is strictly increasing in v_i and v_j and for fixed v_j, has range $[2v_j/3, v_j]$, and bidder j is given the signal $x_j = x_i/2$. In particular, we use the function:

$$\alpha(v_i, v_j) = \frac{v_j}{3}\left(2 + \frac{v_i - v_j}{1 - v_j}\right),$$

which linearly interpolates bids between $[2v_j/3, v_j]$ and assigns then to winning bidders v_i in order, for $v_i \in [v_j, 1]$.

In equilibrium, when $x_i > v_i$, bidder i randomizes over the interval $[\alpha(v_j, v_i), v_i]$ so as to make bidder j indifferent to bidding v_i, and otherwise bidders bid their signal: $b_i = x_i$.

Let us show that this is an equilibrium. If $x_i > v_i$, buyer i is indeed willing to randomize as intended, since the "small probability" event has not happened and $b_j = v_i$.

If $x_i < v_i$, then there are three possibilities:

(a) $x_i = v_j < v_i$.

(b) $x_i = \alpha(v_i, v_j)$ for some $v_j < v_i$.

(c) $x_i = \alpha(v_j, v_i)/2$ for some $v_j > v_i$.

Note that event (c) could only occur if $x_i \in [v_i/3, 2v_i/3]$. Let us assess the conditional probabilities of these three events. Since v_j is uniformly distributed, event (a) simply occurs with ex-ante probability $1 - \epsilon x_i(v_i - x_i)$. Event (b) occurs with ex-ante probability $\epsilon\gamma(v_i, x_i)(v_i - \gamma(v_i, x_i))$ where $\gamma(v_i, x_i)$ is the solution to $\alpha(\gamma, v_i) = x_i$. Event (c) occurs with ex-ante probability $\epsilon v_i(\xi(v_i, x_i) - v_i)$, where $\xi(v_i, x_i)$ solves $\alpha(v_i, \xi) = 2x_i$.

Now, to verify that incentive constraints are satisfied, first observe that the surplus from following the recommendation is

$$(v_i - x_i)(1 - \epsilon x_i(v_i - x_i) + \epsilon\gamma(v_i, x_i)(v_i - \gamma(v_i, x_i))).$$

By deviating to a bid $b = x_i/2$ (the most attractive downward deviation), the bidder's surplus will be approximately

$$(v_i - x_i/2)\epsilon\gamma(v_i, x_i)(v_i - \gamma(v_i, x_i)),$$

which is less than the surplus from $b = x_i$ as long as

$$\epsilon \leq \frac{v_i - x_i}{\gamma(v_i, x_i)(v_i - \gamma(v_i, x_i))/2 + v_i - x_i}\frac{1}{x_i}.$$

Observing that $x_i \leq \gamma(v_i, x_i) \leq v_i$, it must be that $\gamma(v_i - \gamma) \leq v_i(v_i - x_i)$, so that the right-hand side is at least

$$\frac{1}{v_i/2 + v_i - x_i}\frac{1}{x_i}$$

which is at least 2/3, so for any $\epsilon < 2/3$, a downward deviation is not attractive.

Similarly, deviating up to a bid slightly larger than $2x_i$ would allow the bidder to win on event (c), yielding a surplus of

$$(v_i - 2x_i)(1 - \epsilon x_i(v_i - x_i) + \epsilon\gamma(v_i, x_i)(v_i - \gamma(v_i, x_i)) + \epsilon v_i(\xi(v_i, x_i) - v_i)),$$

which is less than the surplus from bidding x_i if

$$\epsilon \leq \frac{x_i}{(v_i - 2x_i)v_i(\xi(v_i, x_i) - v_i) + x_i^2(v_i - x_i) - \gamma(v_i, x_i)(v_i - \gamma(v_i, x_i))}.$$

But since $x_i \geq v_i/3$, the right-hand side must be at least

$$\frac{1}{3}\frac{1}{(v_i - 2x_i)(\xi(v_i, x_i) - v_i) + x_i^2/3},$$

which is itself at least 1/2. Thus, choosing $\epsilon \leq 1/2$, both upward and downward incentive constraints will be satisfied.

In the end, it is incentive compatible to follow the "recommendation" x_i. Moreover, this equilibrium induces winning bids where $x_i < v^{(2)}$ with positive probability, so that revenue must be strictly lower. The outcome is, however, still efficient, so bidder surplus must have risen relative to the complete information case.

This example illustrates a mechanism by which information could lead to lower revenue and higher bidder surplus than under complete information. By creating uncertainty about whether a given bid will win or lose, the buyers could be induced to make lower winning and losing bids. Otherwise slack incentive constraints, both for deviating up and for deviating down, can be tightened to induce lower equilibrium bids. Again, it should be clear that this example is not optimizing the structure of these constraints to push down or raise bidder surplus as much as possible: the only incentive constraints that we have tightened are deviating up to $2x_i$ or down to $x_i/2$. In Section 5, we will investigate the limits of how information can affect these welfare outcomes.

4 Maximum revenue and minimum bidder surplus

In this section, we explore the limits of how large revenue could be and how low bidder surplus could be, over all possible structures for bidders' beliefs. We briefly consider the unknown values model. For symmetric distributions of values, there are trivial type spaces that yield zero bidder surplus. Moreover, it is not hard to construct beliefs such that the seller extracts all the revenue. We then consider the known value model, in which neither of these statements is true: generically, bidders can always guarantee themselves positive surplus. We will characterize the exact limits of how much surplus the bidders need to receive and how much revenue the seller could earn.

4.1 Unknown values

If the distribution p is symmetric, then each bidder has the same ex-ante valuation for the good, which is

$$\mathbb{E}[v_i] = \sum_{v \in V^n} p(v) v_i.$$

In the absence of any additional information, there is a straightforward equilibrium where every bidder bids $b_i = \mathbb{E}[v_i]$. All bidders tie and earn a surplus of 0 in equilibrium. This equilibrium is, however, inefficient, since the winner need not be the bidder with the maximum ex-post valuation.

Perhaps not too surprisingly, there are also equilibria in weakly undominated strategies such that the outcome is very nearly efficient, and yet bidder surplus is arbitrarily close to zero. This result reflects, in some sense, the weakness of weakly undominated strategies as a refinement in our setting. It may be that beliefs are such that bidders are willing to bid a large amount because they think that the bid is less than their value conditional on winning, although conditional on losing their value might be quite a bit lower. Indeed, we construct such an equilibrium in the proof of the following theorem:

Theorem 2 (Maximum revenue and minimum bidder surplus with no-information)**.** *For all $\epsilon > 0$, there exists a weakly undominated BCE F such that $R(F) > \overline{S} - \epsilon$ and $U(F) < \epsilon$.*

Proof of Theorem 2. We construct the BCE as follows. If $v^{(1)} = v^{(2)}$, then all high bidders bid $v^{(1)}$, and losing bidders can bid anything less than $v^{(1)}$.

Otherwise, if $v^{(1)} > v^{(2)}$, the equilibrium first draws an x in $[0, 1]$ according to the distribution $F(x) = x^k$. The high value bidder is then told to bid $\overline{b} = xv^{(1)} + (1-x)v^{(2)}$, and losing bidders are told to bid $yv^{(1)} + (1-y)v^{(2)}$ with $y \in [0, x]$, where $y \sim \frac{G(y)}{G(x)} = \left(\frac{y}{1-y}\frac{1-x}{x}\right)^k$ and $G(x) = \left(\frac{x}{1-x}\right)^k$.

We claim that this is a BCE for all k. Clearly, there is always positive conditional probability that a given bid is both a winning bid and a losing bid. In particular, conditional on the profile of values, a recommendation in the range $[v^{(2)}, b]$ is a winning bid with probability

$$\frac{1}{n}G\left(\frac{b - v^{(2)}}{v^{(1)} - v^{(2)}}\right),$$

and a losing bid is recommended in the range $[v^{(2)}, b]$ with probability

$$\frac{n-1}{n}G\left(\frac{b - v^{(2)}}{v^{(1)} - v^{(2)}}\right)\left(1 + \int_{x=\frac{b-v^{(2)}}{v^{(1)}-v^{(2)}}}^{1} \frac{g(x)}{G(x)}dx\right).$$

Both of these expressions are differentiable and strictly increasing on the range $b \in [v^{(2)}, v^{(1)}]$, and are zero when $b = v^{(2)}$, and thus bids in any open subinterval of $[v^{(2)}, v^{(1)}]$ arise with positive probability conditional on both winning and losing. Thus, conditional on any bid, it is possible to have value $v^{(1)}$, so that the strategy of bidding b is undominated. However, if b is a losing recommendation, $b > v_i$, so that it is never profitable to deviate to a higher bid.

In addition, conditional on it being a winning bid $b = xv^{(1)} + (1-x)v^{(2)}$, and conditional on the highest and second highest values, the benefit of deviating downward to some $b' = yv^{(1)} + (1-y)v^{(2)}$ with $y \geq 0$ is proportional to

$$(v^{(1)} - v^{(2)})(1-y)\frac{G^{n-1}(y)}{G^{n-1}(x)} \leq (v^{(1)} - v^{(2)})(1-y)\frac{G(y)}{G(x)}$$

since $\frac{G(y)}{G(x)} < 1$, and moreover the left-hand side and right-hand side coincide at $y = x$. Thus, it is sufficient to show that $(1-y)G(y)$ is increasing in y. This function is $\frac{y^k}{(1-y)^{k-1}}$, which is obviously increasing in y since the numerator is increasing and the denominator is decreasing.

Finally, as k goes to ∞, the expected value of x converges to 1. For if we write $H(x)$ for any CDF that is strictly less than 1 with probability 1, then

$$\begin{aligned}\int_{x=0}^{1} x dH^k(x) &= xH^k(x)|_{x=0}^{1} - \int_{x=0}^{1} H^k(x)dx \\ &= 1 - \int_{x=0}^{1} H^k(x)dx \\ &\geq 1 - zH^k(z) - (1-z)\end{aligned}$$

for any $z \in (0,1)$ (this just comes from the observation that $H(x) \leq H(z)$ for $x \in [0,z]$ and $H(x) \leq 1$ for all $x \in [z,1]$). Since $H^k(z) < 1$, by choosing z sufficiently close to 1 and k sufficiently large, we can make this number arbitrarily close to 1. Thus, in the limit as k goes to ∞, it must be that $\mathbb{E}\left[\overline{b}\right] = \mathbb{E}[x]\sum_v p(v)v^{(1)} + (1-\mathbb{E}[x])\sum_v p(v)v^{(2)}$ goes to $\sum_v p(v)v^{(1)} = \overline{S}$. Since $S(F) = R(F) + U(F) \leq \overline{S}$, it must be that $U(F)$ goes to 0. $\square$

4.2 Known values

With known values, there is no scope for the extreme kind of updating that occurs in the equilibrium constructed above. The buyers always know their values, and weakly undominated strategies requires that bidders never bid above their values. This restriction guarantees that bidders receive a positive surplus in any BCE. In particular, if buyer i uses the simple strategy of always bidding b, there is a lower bound on the probability that the bidder will win the auction which is

$$g(b) = \sum_{\{v_{-i} \in V^{n-1} | \max_{j \neq i} v_j \leq b\}} p(v).$$

Let $b_i^*(v_i) = \max \arg\max \, (v_i - b)g(b)$, which must be an element of V. Then bidder i must receive in any equilibrium the surplus $\underline{U}_i(v_i) = (v_i - b_i^*(v_i))g(b_i^*(v_i))$. For if equilibrium surplus were lower, bidder i could deviate to $b_i = b_i^*(v_i) + \epsilon$ and guarantee himself surplus arbitrarily close to $\underline{U}_i(v_i)$. This implies that in ex-ante terms buyer i must receive

$$\underline{U}_i = \sum_v p(v)\underline{U}_i(v_i).$$

The main result of this section is to argue that this lower bound is in fact tight: there are BCE in which bidder i receives *exactly* $\underline{U}_i$ in surplus. Moreover, it is possible for all bidders to receive this surplus at the same time, and in an equilibrium in which the allocation is socially efficient. Thus, these equilibria simultaneously minimize bidder surplus and maximize the revenue of the seller, which is

$$\overline{R} = \overline{S} - \sum_{i=1}^{n} \underline{U}_i.$$

Theorem 3 (Maximum revenue and minimum bidder surplus with known values)**.** *For any BCE F, $U_i(F) \geq \underline{U}_i$ and $R(F) \leq \overline{R}$.*

For any distribution p, there exist weakly undominated BCE in which $U_i(F) = \underline{U}_i$ for all i and $R(F) = \overline{R}$.

Proof of Theorem 3. The BCE is constructed as follows. If $v_i < v^{(1)}$, we draw b_i independently from the cumulative distribution $\frac{v^{(1)}-v_i}{v^{(1)}-b}$ on a range $[v^{(1)}-\epsilon, v^{(1)}]$ for some ϵ sufficiently small.

If $v_i = v^{(1)}$, we will draw b_i according to the following procedure. Let $p_i^{(2)}(v^{(2)}|v^{(1)})$ denote the conditional distribution of the second-highest value given that bidder i has the highest value, which is $v^{(1)}$. In particular, this is:

$$p_i^{(2)}(w'|w) = \frac{\sum_{\{v \in V^n | v_i = w = v^{(1)}, v^{(2)} = w'\}} p(v)}{\sum_{\{v \in V^n | v_i = w = v^{(1)}\}} p(v)}.$$

Write $P_i^{(2)}(w'|w) = \sum_{w'' \leq w'} p_i^{(2)}(w''|w)$ for the corresponding cumulative distribution. (These objects are only defined when there exists a $v \in V^n$ such that $v_i = v^{(1)} = v$).

Let $k_i^*(v_i)$ such that $b_i^*(v) = v^{k_i^*(v_i)}$. Let α^l be defined by:

$$(v_i - v^l)\left(\alpha^l P_i^{(2)}(v^{k-1}|v_i) + p_i^{(2)}(v^l|v_i)\right) = (v_i - b_i^*(v_i))\alpha^l P_i^{(2)}(b_i^*(v_i)|v_i),$$

for $l > k_i^*(v_i)$, i.e.,

$$\alpha^l = \frac{(v_i - v^l)p_i^{(2)}(v^l|v_i)}{(v_i - b_i^*(v_i))P_i^{(2)}(b_i^*(v_i)|v_i) - (v_i - v^l)P_i^{(2)}(v^l|v_i)}.$$

(When $k_i^*(v_i) = l$, then α^l is defined to be 1, and if $p_i^{(2)}(v^l|v_i) = 0$, then it is defined to be 0.) Then bid $b_i = v^l$ is recommended to bidder i with probability

$$x^l = \prod_{l'=l+1}^{k} (1 - \alpha^{l'}).$$

whenever $v^{(2)} = v^l$ and with probability

$$y^l = \alpha^l x^l$$

when $v^{(2)} < v^l$.

We claim that these rules define a BCE. Note that the bidder i with $v_i = v^{(1)}$ only receives recommendation $b_i = v^l$ when $v^{(2)} \leq v^l$, so the only way to deviate upwards and win the auction when one would lose by following the recommendation is to bid more than $v^{(2)}$, which would result in negative surplus. Thus, it is sufficient to check that no bidder would like to deviate downwards. To assess the value of such deviations, first observe that the conditional distribution of $v^{(2)}$ conditional on $b_i = v^l$ and $v_i = v^{(1)}$ has a particular form: $v^{(2)} = v^l$ with probability

$$\frac{p_i^{(2)}(v^l|v_i)}{p_i^{(2)}(v^l|v_i) + \alpha^l P_i^{(2)}(v^{l-1}|v_i)},$$

and $v^{(2)} = v^{l'}$ for $l' < l$ with probability

$$\frac{\alpha^l p_i^{(2)}(v^{l'}|v_i)}{p_i^{(2)}(v^l|v_i) + \alpha^l P_i^{(2)}(v^{l-1}|v_i)}.$$

Thus, by construction bidder i is indifferent between a bid of v^l and $b_i^*(v_i)$, and moreover the latter bid is superior to any bid other bid $v^{l'}$ with $l' < l$ by the definition of $b_i^*(v_i)$ and the fact that probabilities of winning for $v^{l'}$ with $l' < l$ are all proportional to $P_i^{(2)}(v^{l'}|v_i)$. Moreover, by choosing ϵ sufficiently small, we can be sure that bids in V are superior to bids not in V.

Finally, this construction ensures that bidder i is always indifferent between following the recommendation b_i and bidding $b_i^*(v_i)$. Thus, the surplus in equilibrium must be equal to the surplus from following the latter strategy, which is precisely $\underline{U}_i(v_i)$. Moreover, a bidder with $v_i = v^{(1)}$ always wins the auction, so that the outcome is efficient. □

To illustrate, let us return to the example of two bidders with independent standard uniformly distributed values in $[0, 1]$. For a type v_i,

$$\underline{u}_i(v_i) = \max_{b \in [0,v]} (v - b)b = \frac{v^2}{4}.$$

Thus, the ex-ante lower bound on bidder surplus is

$$\underline{U}_i = \int_{v=0}^{1} \frac{v^2}{4} dv$$
$$= \frac{1}{12}.$$

Worst case bidder surplus is therefore 1/6, and since the efficient surplus is 2/3, maximum revenue is $2/3 - 1/6 = 1/2$. This is compared to revenue under the standard BNE, which is 1/3.

Let us explicitly construct a BCE that attains the outcome $(U, R) = (1/6, 1/2)$ when values are i.i.d. standard uniform. When values are (v_1, v_2), with $v_i > v_j$, bidders receive recommendations in $[v_i/2, v_i] \times \{v_j\}$. Consider the family of distributions, defined for $v/2 \leq b < v$:

$$G_{v,b}(v') = \begin{cases} 1 \text{ if } v' \geq b; \\ \frac{4(v-b)}{v^2} v' \text{ if } 0 \leq v' \leq b; \\ 0 \text{ otherwise.} \end{cases}$$

We also define the distribution $G_{v,v}$ as the pointwise limit of $G_{v,b}$ as $b \uparrow v$, which puts probability one on $v' = v$. We will construct the distribution of recommendations, given the loser's value v', so that the conditional distribution of the loser's value given a recommendation of b has the shape of $G_{v,b}$. As a result, it will always be optimal to bid b, and the bidder will be indifferent between bidding b and bidding $v/2$, which is the best response when others are bidding their values.

Let us write $F(b|v)$ for the marginal distribution of winning bid recommendations when the winner's value is v, which will have a continuously differentiable density. In order for probabilities to add up, it must be that the prior density of valuations is equal to the expected interim density. For $v' \in [v/2, v]$, this requirement can be written as

$$\int_{b=v'}^{v} \frac{4(v-b)}{v^2} f(b|v) db + f(v'|v) \left(1 - \frac{4(v-v')v'}{v^2}\right) = \frac{1}{v},$$

Note that there is a conditional mass point of size $1 - \frac{4(v-v')v'}{v^2}$ on $v' = b$, given the recommendation b. Differentiating this expression with respect to v', we conclude that

$$-\frac{4(v-v')}{v^2} f(v'|v) + \left(1 - \frac{4(v-v')v'}{v^2}\right) f'(v'|v) - f(v'|v) \frac{4(v-2v')}{v^2} = 0.$$

Finally, this rearranges to the differential equation

$$\frac{f'(v'|v)}{f(v'|v)} = \frac{4(v-2v')+4(v-v')}{(v-2v')^2},$$

which admits simple solutions of the form

$$f(v'|v) = \frac{C(v)}{(2v'-v)^3}\exp\left(-\frac{v}{2v'-v}\right).$$

The constant of integration has to be chosen so that the density of bid recommendations integrates to one, i.e.,

$$\begin{aligned} 1 &= \int_{b=v/2}^{v} f(b|v)db \\ &= \frac{C(v)}{v^2 e}. \end{aligned}$$

so that $C(v) = v^2 e$. It is straightforward to verify that the density adds to $1/v$ for $v' \in [0, v/2]$.

To complete the construction of the BCE, we simply have the losing buyer bid his or her value if the winning bid is strictly larger than the loser's value, and if the winner is told to bid the loser's value v', then the losing buyer randomizes over an interval, say $[v'/2, v']$ according to the cumulative distribution $G(b|v,v') = (v-v')/(v-b)$, where v is the winner's value.

The proof of Theorem 3 and the construction for the uniform distribution utilize ideas from our related paper on monopoly price discrimination (Bergemann et al., 2015). Essentially, a buyer who is facing a fixed distribution of opponents' bids is in an analogous situation as a monopolist who is facing a fixed distribution of consumers' valuations, each of whom demands a single unit of a good that can be produced at zero cost. In the buyer's case, submitting a bid b will result in a surplus of $v - b$ when the other buyers' bids are less than b. The monopolist, on the other hand, earns a revenue of p when consumers' valuations are greater than p. Thus, we have simply reversed the sign of how the agent's action enters the objective function, but it is still the case that surplus is a linear function of that action times the mass in a one-sided tail of a distribution that is, from the agent's perspective, exogenous.

In the monopoly case, partial information about consumers' values facilitates third degree price discrimination, whereby the monopolist offers different prices to different consumers. Bergemann et al. show that it is possible to structure information for the monopolist by creating pools of consumers so that there are enough low valuation consumers to justify dropping the price, but each pool also contains a fair number of high-valuation consumers who benefit when prices fall. In fact, regardless of the ex-ante distribution of consumers' valuations, it is always possible to construct these pools so that prices drop enough that all consumers purchase the good (as long as they value it above marginal cost), but the monopolist is just indifferent to dropping the price, so that monopoly profits do not increase relative to the no-discrimination outcome.

In the current setting, partial information about others' bids means that the buyer receives information that is correlated with the other buyers' information, and therefore with their equilibrium actions. Because of the similar features of the payoff function, the same logic that allows for efficient price discrimination with zero benefit to a monopolist means that it is possible to structure information for the high-valuation bidder so that he always outbids his opponents, but does not benefit from the additional information. Thus, the bidder is just as well off as if he knew nothing besides the prior distribution over others' values, and best responded to the conditional distribution of others' bids. Importantly, while in the monopoly setting the distribution of consumers' values is exogenous, the distribution of others' bids must be generated by best responses which have to be supported in equilibrium. However, this is easy to do when maximizing revenue and minimizing bidder surplus, because losing buyers know that they will lose to bids that are greater than their own known values.

5 Minimum revenue and maximum bidder surplus

We now explore the lower limit of revenue and the upper limit of bidder surplus over all possible structures for beliefs. We first consider minimum revenue in the unknown values model. There, we provide a tight lower bound on revenue and a tight upper bound on bidder surplus. The analysis proceeds by first solving a relaxed version of the problem of minimizing revenue over all BCE. We will solve this relaxed problem exactly, and then show that the solution to the relaxed problem can be extended to a full BCE. This provides a tight bound on revenue and bidder surplus for the unknown values model, as well as a loose bound for the known values model.

We then extend the analysis to the known values model. For a version of the model where values are binary (high or low), we are again able to obtain sharp bounds on these welfare objectives: there is an analogous and tractable relaxed problem, and its solution again extends. Beyond binary values, however, our methods are insufficient to characterize the exact solution. For three values, and restricting attention to efficient equilibria, we characterize the solution of the analogous relaxed problem. However, there are distributions of values for which the solution of the relaxed problem cannot extend to a full BCE, so the revenue and surplus bounds we obtain are not sharp. For the general many-value model, the solution to the relaxed problem may have a complicated structure. We report computational results that are illustrative of the structure of the minimum revenue BCE with many values.

5.1 Unknown values

5.1.1 Symmetric BCE

Before proceeding with the no-information case, we first state following result about symmetric BCE that will be useful throughout this section in characterizing minimum revenue.

Lemma 1. *If the distribution of values p is symmetric, then there exist symmetric BCE that minimize revenue and maximize total bidder surplus.*

This observation will greatly simplify the following analysis. The formal proof for this result appears in the Appendix, but here are the basic ideas. Essentially, BCE are a subset of the convex set of joint distributions over values and bids. For that reason, if one had an asymmetric BCE, it is always possible to "symmetrize" by (i) first drawing a permutation of the buyers' identities at random and (ii) drawing values and bids according to the original BCE and assigning them to the permuted identities. Note that this symmetrized BCE has the same distribution of values, since the original distribution was symmetric. Moreover, conditional on each permutation, the conditional distribution of bids and values is a BCE, so overall it must be a BCE as well. And finally, revenue and total bidder surplus for the symmetrized BCE are the expectation of those objectives conditional on the permutation, but conditional on the permutation, revenue and total bidder surplus are the same as under the original BCE. In light of Lemma 1, we will restrict attention to symmetric BCE for the remainder of this section.

5.1.2 The relaxed problem

The remainder of this subsection is devoted to solving for the minimum revenue BCE in the unknown values case. As mentioned earlier, our approach to characterizing minimum revenue over all BCE will involve first relaxing the problem by focusing on a particular subfamily of incentive constraints. At the minimum revenue equilibrium, the distribution of winning bids is being pushed down "as low as it could go." At this optimum, one intuits that the force that prevents the distribution from being pushed down further is the temptation of the bidders to increase their bids. Put differently, if the winning bid distribution were too low, then some bidder could deviate by increasing their bid a relatively small amount for a relatively large increase in the probability of winning the auction.

In principle there could be both upward and downward incentive constraints that bind at the minimum revenue BCE. However, this will turn out not to be the case for unknown values BCE. Thus, as a preliminary step in our characterization, we drop from the problem all downward incentive constraints, i.e., we only retain constraints with deviation mappings such that $\sigma_i(x) \geq x$ for all $x \in B$. Even so, there is still a large number of such deviation mappings. We will reduce the problem even further by only looking at deviation mappings of the particular form $\sigma_i^b(x) = \max\{b, x\}$. This deviation mapping says that whenever bidder i is told to bid some amount x which is less than b, that bidder deviates up to bidding b instead. It will turn out that this class of constraints is sufficient to characterize minimum revenue.

In addition, while the full BCE $F(db|v)$ describes the distribution of both winning and losing bids, revenue only directly depends on the distribution of winning bids. Let us write

$$H(x|v,k) = \int_{\{b \in B^n | b_i \leq x\}} q_i(b) F(db|v) \tag{4}$$

for the probability that a representative buyer i wins with a bid less than x when his value is v^k, and the profile of values is v. Since the BCE is symmetric, this function depends on the winner's value but is independent of the winner's identity i, since all bidders must be equally likely to win with a given bid when values are v and the bidder's value is v_i. Moreover, symmetry implies that this distribution should be invariant to permutations of the bidders' values. Thus, we can actually rewrite these functions in terms of (i) the winner's value and (ii) the distribution of values in v. Let

$$d^k(v) = |\{i | v_i = v^k\}|$$

count the number of times that the value v^k appears in the vector v, and let

$$d(v) = (d^0(v), \ldots, d^K(v))$$

denote a distribution of values. $D = N^n$ is the set of possible counting measures on values. The distribution p induces a distribution p^D on D according to

$$p^D(d) = \sum_{\{v \in V^n | d(v) = d\}} p(v).$$

Symmetry then implies that $H(b_i|v,k) = H(b_i|v',k)$ if $d(v) = d(v')$, so we can take d as a sufficient statistic and simply write $H(b|d,k)$ for the distribution of winning bids maked by a value v^k when the distribution of values is d, i.e.,

$$H(b|d,k) = \frac{1}{p^D(d)} \sum_{\{v \in V^n | d(v) = d\}} p(v) H(b|v,k).$$

Revenue can be calculated from the $H(b|d,k)$ distributions as:

$$\begin{aligned} R &= \sum_{d \in D} p^D(d) \sum_{k=0}^{K} d^k \int_{x \in B} x H(dx|d,k) \\ &= n v^K - \sum_{d \in D} p^D(d) \sum_{k=0}^{K} d^k \int_{x \in B} H(x|d,k) dx. \end{aligned}$$

where the second line comes from integration by parts (Cohn, 1980, Proposition 5.3.3). Note the coefficient of d^k in front of the integral, since there are d^k buyers whose winning bids are distributed according to $H(\cdot|d,k)$ when the distribution of values is d.

We can express the ex-ante value of the deviation mapping σ_i^b in terms of $H(b|d,k)$. In particular, if a bidder deviates from the recommended bid $x \leq b$ to b, he or she will continue to win on events in which the bid b would have won originally, but will obtain a surplus which is smaller by the difference in bids of $b - x$. The total loss, in ex-ante terms, is therefore

$$\sum_{d \in D} p^D(d) \sum_{k=0}^{K} \frac{d^k}{n} \int_{x=-\infty}^{b} (b-x) H(dx|d,k) = \sum_{d \in D} p^D(d) \sum_{k=0}^{K} \frac{d^k}{n} \int_{x=-\infty}^{b} H(x|d,k) dx, \quad (5)$$

where the equality is again from integration by parts.

On the other hand, if the bidder was supposed to lose to a bid less than b, meaning some other bidder $j \neq i$ was supposed to win with a bid less than b, then bidder i will now win and earn surplus $v_i - b$. Since the total probability that some buyer wins is

$$H(b|d) = \sum_{k=0}^{K} d^k H(b|d,k),$$

then

$$H(b|d) - H(b|d,k) = \sum_{k' \neq k} (d^{k'} - \mathbb{I}_{k=k'}) H(b|d,k')$$

is the probability that some other buyer wins with a bid less than b when buyer i's value is v^k and the distribution is d, where $\mathbb{I}_E$ is the indicator function for the event E. Thus, the total gain from deviating upwards must be

$$\sum_{d \in D} p^D(d) \sum_{k=0}^{K} \frac{d^k}{n}(v^k - b) \sum_{k' \neq k} (d^{k'} - \mathbb{I}_{k=k'}) H(b|d,k'). \tag{6}$$

Hence, the deviation σ^b is not attractive if (6) is less than (5).

In the end, instead of minimizing revenue over the entire set of BCE, we solve the following relaxed problem:

$$\max \sum_{d \in D} p^D(d) \sum_{k=0}^{K} d^k \int_{x \in B} H(x|d,k)dx \tag{R1.1}$$

$$\text{subject to } H(b|d,k) \in [0,1] \ \forall b \in B, d \in D, k \in N,$$

$$\sum_{k=0}^{K} d^k H(b|d,k) \leq 1 \ \forall b \in B, d \in D, \tag{R1.2}$$

$$\text{and } \sum_{d \in D} p^D(d) \sum_{k=0}^{K} \frac{d^k}{n}(v^k - b) \sum_{k' \neq k} (d^{k'} - \mathbb{I}_{k=k'}) H(b|d,k') \leq \sum_{d \in D} p^D(d) \sum_{k=0}^{K} \frac{d^k}{n} \int_{x=-\infty}^{b} H(x|d,k)dx \ \forall b \in B. \tag{R1.3}$$

(In particular, we are even dropping the requirement that the H functions be monotonically increasing and eventually reach one. As we shall see, this is not a problem, as the solutions to the relaxed problem end up being monotonic and rise to their upper limits.) Note that the objective, maximizing the area under the winning bid distributions, is in some sense aligned with relaxing the right-hand side of (R1.3) as much as possible.

This relaxed problem turns out to have an (almost) unique solution, which we will now explain. To start, note that increasing the $H(\cdot|d,k)$ functions will always make the objective (R1.1) larger. Thus, at the optimum, there must be some constraint which prevents us from raising some $H(\cdot|d,k)$. If (R1.2) is slack for some $b \in B$, then it must be that (R1.3) is binding at b. For if not, then it is clear that it is possible to increase some $H(b|d,k)$ at b without violating (R1.3). Moreover, increasing $H(b|d,k)$ simply increases the right-hand side of the constraint for all b', thus ensuring that an increased solution is still feasible.

In addition, it is also not hard to see that there must be an optimal solution to the relaxed problem with the feature that $H(b|d,k) = 0$ unless $k = k^*(d) = \max\{k' | d^{k'} > 0\}$. In other words, only bidders with the highest values make winning bids. The reason is that on the left-hand side of (R1.3), the distributions $H(b|d,k)$ are weighted according to the *loser's* valuation. By shifting probability mass from $H(b|d,k)$ to $H(b|d,k^*(d))$, we can rearrange the left-hand side so that the distribution is multiplied by smaller v^k and thereby relax the constraint. In particular, suppose that $H(b|d,k) > 0$ for some $k \neq k^*$. Then we can devise an alternative solution $\tilde{H}(b|d,k)$ where

$$\tilde{H}(b|d,k) = \begin{cases} \frac{1}{d^{k^*(d)}} H(b|d) \text{ if } k = k^*(d); \\ 0 \text{ otherwise.} \end{cases}$$

The change in the left-hand side is

$$\begin{aligned} &\sum_{d \in D} p^D(d) \sum_{k=0}^{K} \frac{d^k}{n}(v^k - b) \sum_{k' \neq k} (d^k - \mathbb{I}_{k=k'}) \left(\tilde{H}(b|d,k') - H(b|d,k') \right) \\ &= \sum_{d \in D} p^D(d) \sum_{k=0}^{k^*(d)-1} \frac{d^k}{n}(v^k - v^{k^*(d)}) H(b|d,k) \leq 0, \end{aligned}$$

so the left-hand side has decreased, whereas the right-hand side is clearly the same.

Thus, only bidders with the highest value win the auction. We can write

$$v^L(d) = \sum_{k=0}^{k^*(d)-1} \frac{d^k}{n-1} v^k + \frac{d^{k^*(d)} - 1}{n-1} v^{k^*(d)}$$

for the average losing value when the distribution of values is d. We also establish the notation that

$$\begin{aligned} V^L &= \{v^L(d) | d \in D, p^D(d) > 0\}; \\ \underline{v}^L &= \min V^L. \end{aligned}$$

Using this definition together with the fact that $d^{k^*(d)} H(b|d,k^*(d)) = H(b|d)$, (R1.3) can be rewritten as

$$\begin{aligned} &\sum_{d \in D} p^D(d) \frac{n-1}{n} (v^L(d) - b) H(b|d) \\ &= \sum_{d \in D} p^D(d) \frac{1}{n} \int_{x=-\infty}^{b} H(x|d) dx \text{ if } \exists d \text{ s.t. } H(b|d) < 1. \end{aligned} \tag{8}$$

At this point, we know that (8) should bind as long as $H(b|d) < 1$ for some d. Since the $H(b|d)$ are non-negative, the right-hand side of (8) has to increase, and eventually, at least for $b \geq v^K$, the equation can be trivially satisfied with $H(v^K|d) = 1$ for all d. Thus, there is some point less than v^K at which the distributions have all hit one. What remains to be determined is the order in which these distributions should increase. Should all of the $H(b|d)$ increase at the same rate? Should some increase faster than others?

The remaining property that pins down the solution to the relaxed problem specifies the correct order in which the winning bid distributions should rise, which in turn pins down who loses to what bids in the minimum revenue BCE. Notice that the distributions $p^D(d)H(b|d)$ appear symmetrically on the left and- right-hand sides of (8), except for the terms $v^L(d) - b$ on the left. Thus, all other things being equal, $H(x|d)$ with higher $v^L(d)$ contribute proportionally more to the left-hand side than do distributions with lower $v^L(d)$. This reflects the fact that losing buyers who have higher valuations have greater incentive to deviate upwards to win more often. (In fact, buyers do not know their exact values but are drawing inference based on the information in their equilibrium bids. A more proper way to make this point would be to say that the higher is the average losing value when b is bid, the more optimistic is the losing buyer's inference, and the tighter is the upward incentive constraint.)

Now, consider two candidate solutions to the relaxed problem, A and B. Let $d, d' \in D$ occur with positive probability and suppose that $v^L(d) < v^L(d')$. In solution A, we solve the differential equation (8) with equality for $H(b|d)$ starting from $b^0 = v^0$ up to some b^1 where $H(b^1|d) = 1$. Then, (8) is solved with equality for $H(b|d')$ from $b = b^1$ until b^2 where $H(b^2|d') = 1$. (Let us set aside for now how the solution proceeds beyond b^2.) In solution B, we reverse the order, first driving $H(b|d')$ to 1 and then $H(b|d)$. The question is, which of these candidates is more likely to be the solution to the relaxed problem? The answer must be solution A, and the reason is that $p^D(d)H(\cdot|d)$ contributes relatively less to the left-hand side than does $p^D(d')H(\cdot|d')$, for the same contribution to relaxing the right-hand side. Thus, $H(\cdot|d)$ can be driven up more quickly, which in turn increases the right hand side faster and leaves more room to subsequently increase $H(\cdot|d')$ faster. This discussion suggests that the remaining property that characterizes the solution is the following "ordering of supports":

$$v^L(d) < v^L(d') \text{ and } H(b|d) < 1 \implies H(b|d') = 0. \tag{9}$$

A subtlety here is that there might, and generically will, be two distributions d and d' such that $v^L(d) = v^L(d')$. In this case, the order in which these distributions increase is irrelevant. We can therefore write

$$p^L(v^L) = \sum_{\{d \in D | v^L(d) = v^L\}} p^D(d);$$
$$H(b|v^L) = \frac{1}{p^L(v^L)} \sum_{\{d \in D | v^L(d) = v^L\}} p^D H(b|d),$$

so that relaxed problem becomes

$$\max \sum_{v^L \in \mathbb{R}} p^L(v^L) \int_{x \in B} H(x|v^L) dx, \tag{R2.1}$$

$$\text{subject to } H(b|v^L) \in [0,1] \ \forall b \in B, v^L \in V^L \tag{R2.2}$$

$$\text{and } \sum_{v^L \in \mathbb{R}} p^L(v^L) \frac{n-1}{n} (v^L - b) H(b|v^L) \leq \sum_{v^L \in \mathbb{R}} p^L(v^L) \frac{1}{n} \int_{x=-\infty}^{b} H(x|v^L) dx \ \forall b \in B. \tag{R2.3}$$

As summarized in the following Lemma, (R2) is a relaxation of the problem of minimizing revenue over all BCE, and therefore must generate a lower bound on revenue and an upper bound on bidder surplus (assuming efficiency).

Lemma 2. *For any BCE $F(b|v)$, the $H(b|v^L)$ induced by F must satisfy* (R2.2) *and* (R2.3)*. Moreover, expected revenue under F must be equal to a constant minus* (R2.1)*.*

The ordered supports property (9) becomes

$$v^L < w^L \text{ and } H(b|v^L) < 1 \implies H(b|w^L) = 0. \tag{11}$$

The following proposition summarizes the solution to (R2):

Proposition 1. *The solution to the unknown values relaxed problem* (R2) *is the unique $H(\cdot|v^L)$ such that* (R2.3) *holds with equality for $b > v^0$ whenever $H(b|v^L) < 1$ for some $v^L \in V^L$ and that satisfies* (11)*.*

Let us recap and summarize the intuition behind this result. In the relaxed problem, we are choosing distributions of winning bids so as to make the cumulative distribution of winning bids rise as quickly as possible, thereby to push down revenue. We first concluded that the incentive to deviate up should be binding over the support of winning bids. A buyer who deviates upwards to a bid b draws inference about his or her valuation on the marginal events in which this deviation bid wins. We then concluded that it was optimal to have the bidder with the highest valuation win, thereby giving a *losing* buyer a more pessimistic inference about their own value and decreasing the incentives to deviate up. In general, a buyer who deviates upwards to a bid b draws inference about his or her valuation on the marginal events in which this bid wins. According to (11), deviating upwards to a slightly higher bid indicates that the marginal win will occur when the valuation is relatively low, generally much lower than the expected valuation conditional on winning if one follows the equilibrium strategy. In addition, by having lower valuation losing buyers lose to lower bids, we can "pack in" more low winning bids under the incentive constraints. Since every winning bid is also sometimes a losing bid, these low winning bids can be used to bootstrap even more low winning bids, through relaxing the constraint on the right-hand side of (R2.3).

Finally, we note that the allocation that is implied by Proposition 1 is efficient, since only high-valuation buyers make winning bids. Thus, the relaxed problem generates both a lower bound on revenue and an upper bound on total bidder surplus.

5.1.3 Construction of a BCE

The result of Proposition 1 is interesting in its own right as a bound on minimum revenue and an upper bound on bidder surplus. At this point, however, we do not know if these bounds are tight. There is good reason to think they might not be: we have only specified a particular marginal distribution of a BCE, namely the distribution of winning bids, and we have also not verified that the plethora of constraints can be satisfied, aside from the particular class of deviations σ_i^b.

It turns out that the solution to the relaxed problem can be extended to a BCE, thus verifying sharpness of the bounds derived above. Let $H(b|v^L)$ be the solution to the relaxed problem described in Proposition 1. Our procedure for extending this solution to a full BCE is as follows:

(i) Draw a profile of values v according to $p(v)$;

(ii) Draw a winning bid b from the distribution $H(b|v^L(v))$, where $v^L(v) = v^L(d(v))$, and assign it to one of the buyers with value $v^{(1)}$, breaking ties uniformly;

(iii) For the remaining bidders, draw bids independently from the distribution $L(b'|b, v^L)$ (to be specified shortly) conditional on the average losing value and the winning bid.

Since the $H(b|v^L)$ distributions satisfy the ordered supports property (11), we can write $B_v = \operatorname{supp} H(b|v)$ for the support sets, and we know that the B_v are ordered in the strong set order. The distributions L are defined inductively on v^L as follows. Assuming that

$L(x|y,w)$ has been defined for $w < v^L$ and for $x \leq y$, with $L(x|x,w) = 1$, we define

$$\begin{aligned} & p^L(v^L)L(b|b',v^L)H(db'|v^L) \\ & = \frac{1}{(v^L - b')^2}\left[\frac{1}{n-1}\sum_{w\leq\tilde{v}} p^L(w)\left(\int_{x=-\infty}^{b} H(x|w)dx + (v^L - b)H(b|w)\right)\right. \\ & \qquad \left. + \sum_{w<v^L}(v^L - w)p^L(w)\int_{x\in B_w} L(b|x,w)H(dx|w)\right]. \end{aligned} \tag{12}$$

where $b \in B_{\tilde{v}}$. The following Lemma confirms that the L's defined by (12) are well-defined distributions.

Lemma 3. *The functions $L(b|b',v^L)$ defined by* (12) *are monotonically increasing in b, and satisfy $L(b|b,v^L) = 1$.*

The equation (12) is derived from the requirement that buyers be indifferent to upward deviations. In particular, let $v^W(v^L)$ be the expected value of a buyer who wins when the average losing alue is v^L, i.e.,

$$v^W(v^L) = \frac{1}{p^L(v^L)} \sum_{\{v\in V^n|v^L(v)=v^L\}} p(v)v^{(1)}.$$

The change in the surplus of a bidder who is told to bid $b \in B_{\tilde{v}}$ and deviates up to $b' > b$ with $b' \in B_{v^L}$, in ex-ante terms, is

$$\sum_w (w - b')p^L(w)\frac{n-1}{n}\int_{x=-\infty}^{b'} L(db|x,w)H(dx|w) - (b'-b)\sum_w \frac{p^L(w)}{n}H(db|w). \tag{13}$$

If the bidder is indifferent to such a deviation, then this expression must be zero. Integrating with respect to db over the range $[-\infty, b]$, we conclude that

$$\begin{aligned} & \sum_w (w - b')p^L(w)\frac{n-1}{n}\int_{x=-\infty}^{b'} L(b|x,w)H(dx|w) \\ & = \sum_w \frac{p^L(w)}{n}\left(\int_{x=-\infty}^{b} H(x|w)dx + (b'-b)H(b|w)\right). \end{aligned}$$

Now, $H(dx|w)$ is zero for $x < b'$ if $w > v^L$, so this expression reduces to

$$
\begin{aligned}
& p^L(v^L) \int_{x=-\infty}^{b'} L(b|x, v^L) H(dx|v^L) \\
&= \frac{1}{v^L - b'} \sum_{w \leq \tilde{v}} \frac{p^L(w)}{n-1} \left(\int_{x=-\infty}^{b} H(x|w)dx + (b' - b)H(b|w) \right) \\
&\quad - \sum_{w < v^L} \frac{w - b'}{v^L - b'} p^L(w) \int_{x=-\infty}^{b'} L(b|x, w) H(dx|w).
\end{aligned}
$$

Differentiating this with respect to b' yields (12).

By construction, buyers will not want to deviate upwards. It only remains to show that no bidder has an incentive to deviate down to a lower bid. The payoff to a bidder who is told to bid $b' \in B_{v^L}$ and deviates down to $b < b'$ is

$$
(v^W(v^L) - b) L(b|b', v^L)^{n-1} H(db'|v^L),
$$

where the $n-1$ exponent comes from the fact that there are $n-1$ losing bidders who receive draws from this distribution. This expression is increasing in b if and only if

$$
(v^W(v^L) - b)(n-1) L(db|b', v^L) - L(b|b', v^L) \geq 0
$$

for all $b \in [-\infty, b']$. Using (12), a sufficient condition for this to be the case is that the following functions are weakly increasing:

$$
\begin{aligned}
& (v^W(v^L) - b) \left(\sum_{w \leq \tilde{v}} p^L(w) \int_{x=-\infty}^{b} H(x|w)dx \right)^{n-1}; \\
& (v^W(v^L) - b) L(b|x, w)^{n-1} \; \forall b \leq x \leq b', w < v^L.
\end{aligned}
$$

Again, if we inductively suppose that $(v^W(w) - b) L(b|x, w)^{n-1}$ is increasing for $w < v^L$, then since $v^W(v^L) > v^W(w)$, we have that $(v^W(v^L) - b) L(b|x, w)^{n-1}$ is increasing as well. Let us then argue that the first of these two functions is increasing. Again, a sufficient condition is that

$$
(v^W(v^L) - b)(n-1) \sum_{wleq\tilde{v}} p^L(w) H(b|w) - \sum_{w \leq \tilde{v}} p^L(w) \int_{x=-\infty}^{b} H(x|w)dx \geq 0.
$$

But since $v^W(v^L) \geq v^L$, this is implied by (R2.3). This verifies that downward deviations are not profitable, and therefore we have constructed a BCE. Thus, the bounds from Proposition 1 are sharp:

Theorem 4. *The solution to the unknown values relaxed problem, together with the losing bid distributions defined by* (12)*, constitute a BCE. This BCE attains a tight lower bound on revenue and a tight upper bound on total bidder surplus over all BCE.*

Why is it possible to construct the distributions $L(b|b', v^L)$ in a way that satisfies downward incentive constraints? The equilibrium is constructed so that losing buyers will be indifferent to increasing their bids to b when they are told to make lower bids $x \leq b$, and they know that the marginal event on which they win at the bid of b is some $v \leq v^L$. Because they are indifferent, it must be that the marginal surplus from winning with the bid of b, conditional on a recommendation $x < b$, is proportional to the surplus that the losing buyers receive in equilibrium from following the recommendation x. Thus, in some sense the suboptimality of downward deviations corresponds to the fact that equilibrium surplus from bids $x \leq b$ rises quickly enough as a function of b, from the perspective of the bidder who is told to win with a bid $b' \geq b$. According to (R2.3), the marginal surplus from winning increases just fast enough to keep the losing buyer of type v indifferent to deviating upwards. But a buyer who deviates down from a recommendation at which $v^L \geq v$ is the loser has a much higher expected value than v. Thus, this downward deviator stands to gain even more from increasing his bid than does the buyer that loses in equilibrium.

For the purposes of illustration, let us consider what this equilibrium would look like for two players with a continuous distribution of values on $[\underline{v}, \overline{v}]$. The following discussion is heuristic, and is not meant to be a formal derivation. If $n = 2$, then $v^L(v)$ is always just the second-highest value, $v^{(2)}$, so that $p^L(v^{(2)}) = 2p(v^{(2)})(1 - P(v^{(2)}))$. In the continuum limit, (11) implies that the support of $H(\cdot|v^{(2)})$ collapses to a singleton and there is a deterministic winning bid conditional on the loser's value, which we write as $\beta(v^{(2)})$. Assuming for now that $\beta(v)$ is strictly increasing and differentiable, the continuum analogue of (R2.3) is

$$\int_{v=\underline{v}}^{\beta^{-1}(b)} (v - b)p(v)(1 - P(v))dv = \int_{v=\underline{v}}^{\beta^{-1}(b)} (b - \beta(v))p(v)(1 - P(v))dv.$$

This follows from the fact that $H(b|v) = 1$ if and only if $b \geq \beta(v)$. Let us substitute $\widehat{v} = \beta^{-1}(b)$ to obtain

$$\begin{aligned} \int_{v=\underline{v}}^{\widehat{v}} (v - \beta(\widehat{v}))p(v)(1 - P(v))dv &= \int_{v=\underline{v}}^{\widehat{v}} (\beta(\widehat{v}) - \beta(v))p(v)(1 - P(v))dv \\ \iff \int_{v=\underline{v}}^{\widehat{v}} (v + \beta(v))p(v)(1 - P(v)) &= 2\beta(\widehat{v}) \int_{v=\underline{v}}^{\widehat{v}} p(v)(1 - P(v))dv. \end{aligned}$$

Differentiating both sides with respect to $\widehat{v}$ gives

$$\beta'(\widehat{v}) = (\widehat{v} - \beta(\widehat{v}))\frac{p(\widehat{v})(1 - P(\widehat{v}))}{1 - (1 - P(\widehat{v}))^2}. \tag{14}$$

There is a solution to this differential equation that is given by

$$\beta(\widehat{v}) = \frac{\int_{v=\underline{v}}^{\widehat{v}} \left(\frac{vp(v)(1-P(v))}{\sqrt{1-(1-P(v))^2}} \right) dv}{\sqrt{1 - (1 - P(\widehat{v}))^2}}.$$

Since the winner's bid is deterministic given the loser's value, all that remains to be specified for the equilibrium is the joint distribution of the loser's bid and the loser's true value. One can show that the density of the losing bid $\beta(v)$ when the lowest value is v' is

$$f(v, v') = \begin{cases} g(v)g(v') \text{ if } v \leq v'; \\ 0 \text{ otherwise,} \end{cases}$$

where

$$g(x) = \frac{\sqrt{2}p(x)(1 - P(x))}{\sqrt{1 - (1 - P(x))^2}}.$$

This is the distribution of losing bids that makes bidders indifferent to deviating upwards, and it is derived analogously to the derivation of $L(b|b', v^L)$ above.

We briefly verify that it is not attractive to deviate downwards. Suppose that the winner is told to bid $\beta(v')$, which is a winning bid only if the losing buyer's value is v'. Then the surplus from deviating downwards to a bid $\beta(v)$ is proportional to

$$\left(\int_{x=v'}^{\overline{v}} x\frac{2p(x)(1 - P(x))}{1 - (1 - P(x))^2}dx - \beta(v) \right) \int_{x=\underline{v}}^{v} g(x)dx.$$

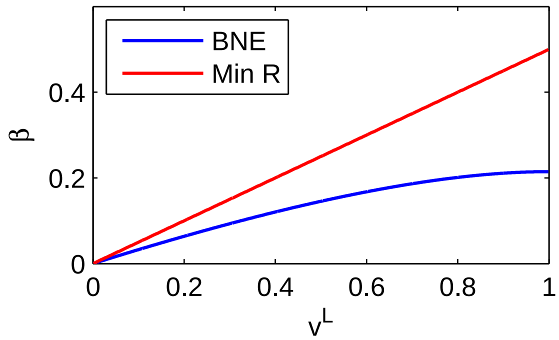

Figure 1: Winning bid as a function of the loser's value in the no information minimum revenue BCE, with two bidders whose values are i.i.d. standard uniform. Also depicted in red is the known values Bayes Nash equilibrium bid of half of one's own value. The unknown values ex-ante bid is 0.5.

The first term is the expected winner's valuation given the loser's valuation is v', and the last term is the probability that the losing buyer bids less than $\beta(v)$. Differentiating this object with respect to v, we conclude that surplus is increasing in v if

$$\left(\int_{x=v'}^{\overline{v}} x\frac{2p(x)(1-P(x))}{1-(1-P(x))^2}dx - \beta(v)\right) g(v) \geq \beta'(v)\int_{x=\underline{v}}^{v} g(x)dx. \tag{15}$$

Substituting in (14), conclude that the right-hand side is

$$(v-\beta(v))\frac{p(v)(1-P(v))}{1-(1-P(v))^2}\int_{x=\underline{v}}^{v} g(x)dx.$$

But the integral in this expression is simply $\sqrt{2}\sqrt{1-(1-P(v))^2}$, so that the right-hand side of (15) is just $(v-\beta(v))g(v)$, which is necessarily smaller than the left-hand side (since v is less than the expectation of $v^{(1)}$ given that $v^{(2)} = v' > v$).

For the standard uniform distribution, in which $P(v) = v$ on $[0,1]$, these expressions simplify to

$$\beta(\widehat{v}) = \frac{\int_{v=0}^{\widehat{v}} \frac{v(1-v)}{1-(1-v)^2}dv}{\sqrt{1-(1-v)^2}},$$

which is depicted in Figure 1. The winning bid ranges from 0 to 0.2146, expected revenue is 0.133, and bidder surplus is 0.537. In comparison, expected revenue under the BNE with no information is 0.5, and expected revenue and bidder surplus with independent known values are each 0.33.

5.2 Known values

The unknown values minimum revenue BCE is a useful benchmark, and it also provides a bound for models in which buyers are assumed to have more information than just knowing the common prior distribution of ex-post values. Even so, we are interested in how the bounds would change under more restrictive assumptions about the buyers' information. In this section, we investigate minimum revenue and maximum bidder surplus when buyers at least know their own ex-post valuations for the good. This seems like quite a lot of information, but as we illustrated in Section 3, there is still ample room for particular belief structures to lead to BNE with lower revenue.

Our approach is much the same as with the unknown values model. First, formulate a relaxed problem in which we optimize over the distribution of winning bids and only observe a subset of the incentive constraints. Then, solve the relaxed problem to obtain bounds on revenue and surplus. And finally, construct an equilibrium that attains the bounds from the relaxed problem.

First, we will derive a generalized relaxed problem for the case of known values. The solution of the relaxed problem with known values turns out to be more complicated than in the unknown values model, and we will solve it completely only for particular cases. A feature of the unknown values relaxed problem that made it especially tractable is that the relaxed incentive constraints reduce to a one-dimensional family of integral inequalities in (R2.3), from which we can guess the order of the supports of winning bids as in (11). With general known values, there is such a family of one-dimensional integral inequalities *for each possible value*. This adds a second-dimension to our problem, and in general the correct ordering of the supports of the winning bid distributions could be quite complex.

When buyers' valuations can take on only one of two values, high or low, we are able to solve the relaxed problem and construct a BCE that attains the bounds. In this case, the bidder with the low valuation has to bid his value with probability one in any BCE. After solving out the low valuation buyer's bids, there remains only a single family of incentive constraints for the high valuation buyer, and we derive an ordering of supports that again characterizes the solution.

Beyond binary values, we demonstrate the following features:

(i) For three values, we show that there is a generalization of the ordered supports property that characterizes the solution to the relaxed problem for a range of parameters, when we restrict to efficient solutions. We conjecture that this result generalizes, and that for an open set of distributions, the solution to the relaxed problem will similarly be characterized by an ordered supports property;

(ii) We also show that in general, the bounds from the solution to the relaxed problem *cannot* be sharp, and downward constraints will frequently be binding;

(iii) Finally, we report simulation results that give a sense of how minimum revenue and maximum bidder surplus BCE behave for general known values models. In particular, the minimum revenue or maximum bidder surplus BCE need not be efficient and need not coincide.

5.2.1 A known values relaxed problem

As before, we will optimize over the winning bid distributions $H(b|d,k)$. In addition, we consider the same class of deviation mappings σ_i^b. The difference is that now we calculate the ex-ante value of these deviations from the perspective of a bidder who knows his own value is v^k, and therefore knows that the true distribution of values must satisfy $d^k \geq 1$. Thus, a known values relaxed problem for minimizing revenue is

$$\max \sum_{d\in D} p^D(d) \int_{x\in B} H(x|d)dx \tag{R3.1}$$

$$\text{subject to } H(b|d,k) \geq 0 \; \forall b \in B, d\in D, k \in N, \tag{R3.2}$$

$$\sum_{k=0}^{K} H(b|d,k) \leq 1 \; \forall b\in B, d\in D, \tag{R3.3}$$

$$\begin{aligned} \text{and } (v^k - b) &\sum_{\{d\in D|d^k\geq 1\}} p^D(d) \sum_{k'} (d^{k'} - \mathbb{I}_{k'=k}) H(b|d,k') \\ &\leq \sum_{\{d\in D|d^k\geq 1\}} p^D(d) \int_{x=-\infty}^{b} H(x|d,k)dx \; \forall k = 0,\ldots,K, b\in B, \end{aligned} \tag{R3.4}$$

For the remainder of this section, we will explore solutions to the relaxed problem of minimizing revenue or maximizing total bidder surplus, and their connection to extremal BCE.

5.2.2 Binary values

We start with a simple model with binary valuations. In particular, let $V = \{v^0, v^1\}$ be the set of possible valuations. To keep our notation concise, we will write $m = d^1$ for the number of bidders with high values, and let

$$p^M(m) = \sum_{\{d\in D|d^1=m\}} p^D(d)$$

denote the distribution of the number of bidders with high values.

It turns out that with known values, bids less than v^0 can never win in equilibrium. This is a general result that extends beyond the binary values case. The loose intuition is that there is common knowledge that the good is worth at least v^0, and Bertrand competition bids up the price to dissipate all of the surplus that the low type might receive from winning the good. Thus, whenever the low type wins, he wins with a bid of v^0, and he only wins against the other type v^0. This also implies that any equilibrium with binary values is efficient, since the high type would always prefer to win against the low type by bidding an amount slightly more than v^0, which is the minimum equilibrium bid. This reduces the relaxed problem to simply finding the distribution of winning bids for high types, conditional on the number of high types:

$$H(b|m) = \sum_{\{d \in D | d^1 = m\}} H(b|d,1),$$

for $m \geq 1$. Using this transformation, we can rewrite (R3) as

$$\max \sum_{m=1}^{n} p^M(m) m \int_{x=-\infty}^{v^1} H(x|m) dx \tag{R4.1}$$

$$\text{subject to } m\, H(b|m) \in [0,1] \ \forall b \in B, m \in N \tag{R4.2}$$

$$\begin{aligned} \text{and } (v^1 - b) \sum_{m=1}^{n} p^M(m)(m-1) H(b|m) \\ \leq \sum_{m=1}^{n} p^M(m) \int_{x=-\infty}^{b} H(x|m) dx \ \forall b \in B. \end{aligned} \tag{R4.3}$$

Again, the objective (R4.1) is the area under the winning bid distributions, the maximization of which is equivalent to the minimization of the expected winning bid. The constraints (R4.3) are the binary value analogue of (R3.4). These constraints express the equilibrium requirement that the gains from deviating up and winning on events when one was supposed to lose (i.e., one of the $m-1$ other high-value buyers wins the auction) must not exceed the loss of surplus when one was supposed to win.

Since the outcome is necessarily efficient, the objectives of minimizing revenue and maximizing total bidder surplus must coincide. By similar principles as before, we can guess the solution of the relaxed problem to be the following. If $H(b|m) < 1$ for some m, then (R4.3) must bind at b. Otherwise, it is possible to increase some $H(b|m)$, shift to the left the distribution of winning bids, and relax the right-hand side of the constraint. Thus, at an optimal solution, (R4.3) is binding whenever some $H(b|m)$ is below its upper bound.

Furthermore, notice that $H(b|m)$ enters the left-hand side of (R4.3) with weight $m-1$, but enters the left-hand side with weight 1. This implies that if $m < m'$, $H(b|m)$ is a more efficient choice for relaxing the right-hand side of (R4.3) than $H(b|m')$. Thus, the optimal solution is characterized by (R4.3) binding and the following ordered supports property:

$$H(b|m) < \frac{1}{m} \implies H(b|m') = 0 \ \forall m' > m. \tag{18}$$

Proposition 2. *The solution to the binary known values relaxed problem is the unique $H(b|m)$ that satisfies* (R4.3) *with equality whenever $H(b|m) < 1/m$ for some m and also satisfies* (18).

In addition, similar techniques as with the unknown values case can be used to construct an equilibrium that attains the bounds from the relaxed problem. In particular, we can construct a BCE where (i) we draw the number of high-valuation bidders according to p^M, (ii) we pick one of the high-valuation bidders uniformly to be the winning bidder and assign him or her a winning bid b from $H(b|m)$, and (iii) we independently draw losing bids b' for the other high-valuation bidders from a distribution $L(b'|b, m)$. The particular L we use is defined by

$$p^M(m')(m'-1)L(b|b', m')H(db'|m') = \frac{1}{(v^1 - b')^2} \sum_{l=1}^{m} p^M(l) \int_{x=-\infty}^{b} (v^1 - x)H(dx|l), \tag{19}$$

where $b \in B_m = \operatorname{supp} H(\cdot|m)$ and $b' \in B_{m'} = \operatorname{supp} H(\cdot|m')$ with $b' > b$.

This losing bid distribution is derived analogously as in the unknown values case. In particular, it should be the case that a high-valuation buyer who is told to bid b is indifferent to deviating up to some $b' > b$. The condition for this to be the case is that

$$\begin{aligned}(v^1 - b) &\sum_{m=1}^{n} p^M(m)H(db|m) \\ &= (v^1 - b') \sum_{m=1}^{n} p^M(m) \left(H(db|m) + (m-1) \int_{x=b}^{b'} L(db|x, m)H(dx|m) \right),\end{aligned} \tag{20}$$

which implies that

$$\sum_{m=1}^{n} p^M(m)(m-1) \int_{x=b}^{b'} L(db|x, m)H(dx|m) = \frac{b' - b}{v^1 - b'} \sum_{m=1}^{n} p^M(m)H(db|m).$$

Differentiating both sides of this equation with respect to b', we conclude that if $b \in B_m$ and $b' \in B_{m'}$, then

$$p^M(m')(m'-1)L(db|b',m')H(db'|m') = \frac{v^1-b}{(v^1-b')^2}p^M(m)H(db|m).$$

Integrating with respect to db over the range $[0,b]$, we obtain (19).

As before, upward incentive constraints will necessarily be satisfied because of (20). In order to verify that we have constructed a BCE, we therefore only have to check that downward incentive constraints are satisfied, i.e., after being told to bid b', it is not profitable to deviate to some $b' < b$. This is equivalent to verifying that

$$(v^1-b)(L(b|b',m')^{m'-1}$$

is increasing for $b \in [v^0, b']$, when $b' \in B_{m'}$. Recall that the exponent of $m'-1$ comes from the fact that the $m'-1$ losing high-valuation buyers are bidding independent draws from the distribution $L(db|b',m')$. This expression is increasing if

$$(v^1-b)(m'-1)L(db|b',m') - L(b|b',m') \geq 0.$$

Suppose $b \in B_m$. Taking out terms which depend on b' and m' but not on b and m, this is equivalent to

$$(v^1-b)^2 p^M(m)(m'-1)H(db|m) - \sum_{l=1}^{m} p^M(l) \int_{x=-\infty}^{b} (v^1-x)H(dx|l) \geq 0.$$

The second term integrates by parts to

$$\begin{aligned}&\sum_{l=1}^{m} p^M(l)\left((v^1-b)H(b|l) + \int_{x=-\infty}^{b} H(x|l)dx\right)\\ &= \sum_{l=1}^{m} p^M(l)l(v^1-b)H(b|l),\end{aligned}$$

where the substitution comes from (R4.3). Thus, we have reduced the sufficient condition to

$$(v^1-b)p^M(m)(m'-1)H(db|m) - \sum_{l=1}^{m} p^M(l)lH(b|l) \geq 0.$$

But this follows from the differential form of (R4.3):

$$(v^1 - b)\sum_{m=1}^{n} p^M(m)(m-1)H(db|m) = \sum_{m=1}^{n} p^M(m)H(b|m),$$

together with the fact that $m' \geq m$. This proves that downward incentive constraints are satisfied, and we have the following result:

Theorem 5. *The solution to the binary known value relaxed problem, together with the losing bid distributions defined by* (19)*, constitute a BCE. This BCE attains a tight lower bound on revenue and a tight upper bound on total bidder surplus over all binary known value BCE.*

Let us illustrate the binary known value result in the case where there are just two buyers, $v^0 = 0$ and $v^1 = 1$. Let us suppose that values are independently and identically distributed, with $p \in [0,1]$ denoting the probability that one of the buyer's has a valuation 1. In this case, $H(b|0)$ and $H(b|1)$ put probability 1 on $b = 0$, and $H(b|1)$ solves the following differential equation

$$(1-b)pH(b|1) = (1-p)b + p\int_{x=0}^{b} H(x|1)dx.$$

The solution is

$$H(b|1) = \frac{1-p}{p}\frac{b(2-b)}{(1-b)^2},$$

which hits 1 at

$$\overline{b} = 1 - \sqrt{1-p}.$$

In fact, the v^1 type is indifferent to a strategy of always bidding $\overline{b}$, so the surplus of the high type is easily calculated as $1 - \overline{b} = \sqrt{1-p}$. The allocation is efficient, so total surplus is $1-(1-p)^2$, and therefore revenue is

$$1-(1-p)^2 - 2\sqrt{1-p}.$$

Note that in the complete information benchmark, each bidder earns a surplus of 1 when facing the v^0 type and a surplus of 0 when facing the v^1 type, so that surplus is only $1-p < \sqrt{1-p}$. Revenue under complete information is p^2.

In this case, there is an extremely simple information structure and BNE that attain the bound. The high-valuation buyers receive signals, which are either high (H) or low (L), according to the following distribution:

$$\begin{array}{c|c|c|c|} & 0 & (1,L) & (1,H) \\ \hline 0 & (1-p)^2 & p(1-p) & 0 \\ \hline (1,L) & p(1-p) & 0 & \alpha p^2 \\ \hline (1,H) & 0 & \alpha p^2 & (1-2\alpha)p^2 \\ \hline \end{array} \tag{21}$$

where $\alpha \in (0,1)$ and the column/row headings denote valuation as well as signal if the value is high. In equilibrium, the 0 type randomizes over $(-\epsilon, 0)$ and the $(1, L)$ types bid 0. The high type randomizes over the interval $[0, \bar{b}]$ according to the cumulative distribution function $F(b) = \frac{\alpha}{1-2\alpha}\frac{1}{1-b}$. This distribution ensures that the $(1, H)$ type is indifferent between bidding any amount in $[0, \bar{b}]$. We will choose the α so that the $(1, L)$ type is also indifferent to randomizing over the same range, thereby ensuring that all upward incentive constraints are binding. A necessary and sufficient condition for this to be the case is that the relative probabilities of the *other* player being type $(1, H)$ and being type 0 conditional on the signal $(1, L)$ are the same as the conditional probabilities of $(1, H)$ and $(1, L)$ conditional on the signal $(1, H)$, i.e.,

$$\frac{1-p}{\alpha p} = \frac{\alpha}{1-2\alpha},$$

which is satisfied with $\alpha = 1 - \frac{1-\sqrt{1-p}}{p}$. In a previous version of this paper (Bergemann et al., 2013), we additionally solve for the entire set of bidder surplus pairs that can attain in a BCE. Extreme surplus pairs are achieved using a generalization of this construction.

5.2.3 Beyond binary values

Beyond binary values, the solution to the relaxed problem is more complicated, and bounds derived from the relaxed problem are generally not tight. This is evident even with just three possible valuations, which we will now demonstrate. Suppose that $V = \{v^0, v^1, v^2\}$ and that $n = 2$, i.e., there are only two potential buyers. To condense notation, we will write

$$H(b|k, k')$$

for the cumulative distribution of winning bids when v^k wins against $v^{k'}$, and $p_{kk'}$ will denote $p(v^k, v^{k'})$. For example, $H(b|1, 2)$ denotes the cumulative distribution of winning bids when v^1 wins against v^2. We will restrict attention to efficient solutions to this relaxed problem, i.e., ones in which $H(b|1, 2) = 0$ for all b. As with binary values, the buyer with valuation v^0 always bids v^0, and there are incentive constraints for values v^1 and v^2. The full relaxed problem in this case is:

$$\max \sum_k p_{kk'} \int_{x=-\infty}^{v^k} H(x|k,k')dx. \tag{R5.1}$$

$$\text{subject to } H(b|k,k') + H(b|k',k) \in [0,1] \; \forall b \in B, \; k,k' \in N \tag{R5.2}$$

$$(v^1 - b)\left(p_{11}H(b|1,1) + p_{21}H(b|2,1)\right) \leq \int_{x=v^0}^{b} \left(p_{10} + p_{11}H(x|1,1)\right) dx; \tag{R5.3.1}$$

$$(v^2 - b)p_{22}H(b|2,2) \leq \int_{x=v^0}^{b} \left(p_{20} + p_{21}H(x|2,1) + p_{22}H(x|2,2)\right) dx, \tag{R5.3.2}$$

We have built into this formulation that both types v^1 and v^2 win against the v^0 type with bids of v^0.

We can consider the relaxed problem where we maximize (R5.1) subject to (R5.2) and (R5.3). Based on the earlier analyses, we can guess some features of the optimal solution. First, nothing is constraining $H(b|2,2)$ except for (R5.3.2), so that constraint should bind whenever $H(b|2,2) < 1/2$. This pins down the path for $H(b|2,2)$ given the path of $H(b|2,1)$. In addition, $H(b|1,1)$ and $H(b|2,1)$ are being constrained by (R5.3.1), and it is clear, as with the relaxations we considered previously, that increasing either of these objects only relaxes the right-hand sides of (R5.3). It is therefore apparent that if either $H(b|1,1) < 1/2$ or $H(b|2,1) < 1$, then one of these objects can be increased while maintaining feasibility, and decreasing revenue. Thus, the equations (R5.3) should be binding whenever one of the distributions on the respective left-hand sides is below its upper bound.

What remains uncertain is the order in which $H(b|1,1)$ and $H(b|2,1)$ should rise. Let us consider the tradeoffs. At a given b, there is a certain amount of slack in the right-hand side of (R5.3.1) which could be allocated either to $H(b|1,1)$ or to $H(b|2,1)$. If that space is allocated to $H(b|1,1)$, then the slack on the right-hand side of (R5.3.1) increases more quickly, thereby allowing both $H(b|1,1)$ and $H(b|2,1)$ to increase faster in b. On the other hand, if the slack in the constraint is allocated to $H(b|2,1)$, then the right-hand side of (R5.3.2) increases faster, and $H(b|2,2)$ can increase more quickly. Thus, there is a genuine tradeoff between whether we want to drive $p_{11}H(b|1,1) + p_{21}H(b|2,1)$ up faster or drive $p_{22}H(b|2,2)$ up faster. In economic terms, the question is whether or not the buyer of type v^1 should lose to the other player's v^1 type or lose to the other player's v^2 type at the minimum revenue BCE. If v^1 loses to v^1 at b, then that effectively means that v^1 is winning with bid b as well and will therefore be willing to lose more often to slightly higher bids. If v^2 wins against v^1 with the bid b, then v^2 is more willing to lose to the other v^2 at bids slightly above b.

The resolution of this tradeoff turns out to depend very much on the parameters of the model, i.e., the prior distribution of values. The reason is that these "returns" to relaxing one of the incentive constraints in (R5.3) are only generated while the incentive constraint binds. Thus, if one of the two constraints is binding for a wider range of bids, then there is more benefit to relaxing that inequality. For example, if $\frac{p_{11}+p_{21}}{p_{10}}$ is much smaller than $\frac{p_{22}}{p_{20}+p_{21}}$, it is more valuable to relax the constraint on $H(b|2,2)$ than to increase $p_{11}H(b|1,1)+p_{21}H(b|2,1)$ more quickly. For in this situation, $p_{11}H(b|1,1) + p_{2,1}H(b|2,1)$ will hit its upper bound much sooner than $H(b|2,2)$, so that there is only a small scope for increasing $p_{11}H(b|1,1) + p_{21}H(b|2,1)$ more quickly. On the other hand, if the parameters are such that $H(b|2,2)$ hits $1/2$ much sooner than $p_{11}H(b|1,1)+p_{21}H(b|2,1)$ hits $p_{11}/2+p_{21}$, then the opposite intuition holds, that it is better to let v^1 win against the other v^1 with low bids.

For this latter case, we can provide an exact characterization of the solution to the relaxed problem. In particular, if $\frac{p_{11}}{p_{10}}$ is sufficiently smaller than $\frac{p_{22}}{p_{20}}$, then the optimal solution should satisfy the following ordered supports property:

$$H(b|1,1) < 1/2 \implies H(b|2,1) = 0. \tag{22}$$

We formalize this in the following proposition:

Proposition 3. *If* $\frac{p_{11}}{p_{10}}$ *is sufficiently large relative to* $\frac{p_{22}}{p_{20}}$*, the solution to the efficiency-constrained trinary known value relaxed problem is the unique* $H(b|k,k')$ *that satisfies* (R5.3) *with equality whenever* (R5.2) *is slack and also satisfies* (22)*.*

Thus, for these cases, we can find the revenue minimizing distributions by first solving (R5.3.1) with equality for $H(b|1,1)$ with $H(b|2,1) = 0$ until $H(b|1,1)$ hits $1/2$, and then solving (R5.3.1) with equality for $H(b|2,1)$ with $H(b|1,1) = 1/2$. Given these paths, we define $H(b|2,2)$ by solving (R5.3.2) with equality until $H(b|2,2) = 1/2$.

Alas, while we can characterize the solution to the relaxed problem for certain parameters, we have little hope of constructing BCE that will attain the bounds. The reason is that there is no constraint on the supports of the winning bid distributions. This is at issue particularly for the cases where we can pin down the solution for the relaxed problem, which is for situations where $H(b|2,2)$ hits $1/2$ before $H(b|1,1)$ and $H(b|2,1)$ hit their upper bounds. Consider a bid recommendation b in the support of $H(b|2,1)$ that is above both the supports of $H(b|1,1)$ and $H(b|2,2)$, which must exist because of (22). Such a bid is a winning bid for the v^2 type, but b is almost surely strictly above the support of the v^1 type's winning bids. This leads to the following quandary: if v^1 were told to *lose* with bids close to b, those recommendations would imply a loss for sure, which would give v^1 an incentive to increase his bid and win with positive probability. If v^1 *never* loses with bids close to b, then the v^2 type with a recommendation of b would know that the bid is strictly above the support of bids that he is winning against, and there would be an incentive to shade further. Thus, downward incentive constraints cannot be satisfied.

Corollary 1. *There exist distributions of values such that the welfare bounds from the the trinary known value relaxed problem are not tight.*

In the end, we conclude that the approach that worked in the unknown values and binary known value cases will not extend to a characterization of minimum revenue and maximum bidder surplus in general known values models. Nonetheless, the BCE solution concept remains highly tractable from a computational perspective. We will revisit the behavior of the known values model in the next section.

6 Further topics

6.1 The entire surplus set

We have thus far focused on characterizing bounds on the welfare outcomes of revenue and bidder surplus. Maximizing and minimizing these outcomes correspond to characterizing the BCE that are maximal in four directions in welfare space. Of course, there are many other welfare objectives we might have considered. For example, what do BCE look like that maximize weighted sums of revenue and bidder surplus, or even sums of bidders' surpluses with unequal weights? In this section, we will broaden our perspective and consider the shape of the entire set of welfare outcomes that might obtain in a first price auction under some information structure.

BCE is a tractable solution concept because of its simple structure. In particular, BCE are joint distributions over values and bids that satisfy a family of linear incentive constraints, and welfare outcomes such as revenue and surplus are linear functions of that distribution. This linearity is essential to our theoretical results, but it also facilitates a computational analysis: the problem of maximizing a weighted sum of expected welfare outcomes over all BCE is an instance of a linear program, for which there exist many fast and robust computational algorithms. We applied large-scale linear programming software to compute the BCE of discretized versions of the auction model that maximize various welfare objectives. These simulation results give us a broader sense of the range of possible welfare outcomes.

In particular, we studied a discretized auction model in which there are two potential buyers who have 10 valuations and 50 possible bids. Values and bids are respectively evenly spaced between 0 and 1, and the distribution of values is uniform. Figure 2 depicts the pairs of revenue and total bidder surplus that can arise in a BCE. The boundary of the set of such pairs for the unknown values model is colored in blue, and the boundary for the known values model is in red.

These sets have a number of prominent features that align with our earlier theoretical results. First, the red set is contained within the blue set, which is consistent with the observation that any known value BCE is also an unknown values BCE: once can always construct an unknown values BCE from a known value BCE by integrating out values. Second, a number of the equilibria we have constructed or that we know exist can be seen on the boundaries of these sets. For example, minimum total bidder surplus is zero and is attained at points A, B, and C, and the unknown values BNE outcome of $(0, 0.5)$ (point B) is on the boundary of the unknown values surplus set. We also see the known value BNE at point E on the boundary of both the known value and unknown values surplus sets. In this outcome, revenue is the expected second-highest value and the outcome is socially efficient. With the uniform grid of 10 valuations, the expected highest valuation is $0.68\overline{3}$, and the expected lowest vauation is $0.31\overline{6}$. Also as predicted, there are BCE that attain point A in which the outcome is socially efficient, but buyers are held down to their lower bound surplus of zero and revenue is $0.68\overline{3}$. Finally, we can also see that minimum unknown values revenue is attained in an efficient equilibrium at point D, which is consistent with the prediction of Theorem 4.

An interesting feature of the unknown values surplus set is the inefficient southwestern frontier, on which our theorems have hitherto been silent. Social efficiency is the expected valuation of the buyer who wins the auction. At best, the good is allocated to the buyer who values the good the most, and indeed there are many BCE (including the known values BNE) at which this outcome is attained and social efficiency is maximized. A simple lower bound on social efficiency is generated by always giving the good to the buyer who values it the *least*. Thus, as long as the good is always allocated, it is impossible for social efficiency to fall below the expected lowest valuation, which in this case is $0.31\overline{6}$. A striking implication of Figure 2 is that social efficiency is minimized at point C, at which the surplus from the auction is precisely attains this bound. Moreover, the bound is attained while both bidders receive zero surplus!

While we have not explored the fullest generality of this result, we can provide the intuition and a simple construction for why it should be true with two bidders and independent values. Consider a case where buyers' values are independently and identically distributed according to a cumulative distribution $P(v)$, with the maxima and minima of the support being $\underline{v}$ and $\overline{v}$. In the following BCE, each buyer observes the *other* buyer's valuation, and

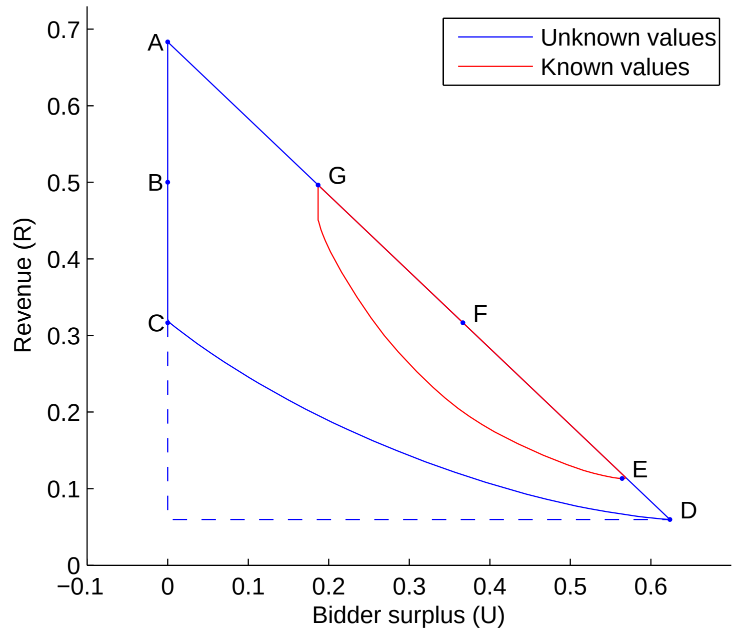

Figure 2: The set of revenue/total bidder surplus pairs that can arise in a BCE. Computed for uniform distribution with grids of 10 valuations and 50 bids between 0 and 1.

bids

$$\beta(v_j) = \frac{\int_{x=\underline{v}}^{v_i} x\, dP(x)dx}{P(v)}, \tag{23}$$

where $\beta(\underline{v}) = \underline{v}$. In other words, each buyer bids the other buyer's expected value, conditional on it being below their true valuation. As long as the density is almost everywhere positive, this bidding function will be strictly increasing on the support of valuations. We claim that this constitutes a BCE: for each buyer knows nothing about their own value, except for its prior distribution. Conditional on a bid of b, the buyer will win whenever $v_i \leq \beta^{-1}(b_i)$, so the expected valuation conditional on winning with a bid of b is the expectation of their value given $v_i \leq \beta^{-1}(b_i)$, which is precisely $\beta(v_i)$! Thus, all bids in the support of P result in an expected payoff of zero. The equilibrium strategy is therefore incentive compatible. Moreover, it is clear that since equilibrium bids are increasing in the *other* buyer's value, the winner of the auction will be the bidder with the lowest valuation, thus attaining point C. Similar arguments and constructions can be provided when values are discrete, while maintaining the assumptions of independence and two bidders.

The argument clearly relies on independence, and would fail in cases of extreme correlation. For example, if one buyer's value is always 1 while the other's is 0, then it is impossible to sustain all buyers bidding zero all the time (which would be required for bidder surplus to be zero while maintaining a minimally efficient allocation). Nonetheless, this example provides additional evidence, if more was needed, that an analysis of a mechanism for all information structures simultaneously can lead to striking and novel results.

The known values surplus set is depicted in red in Figure 2, and it is significantly smaller than the unknown values surplus set. At point G, revenue is maximized and bidders are held down to the lower bound from Section 5. In this example, minimum revenue for the known values model is approximately 0.11, while minimum revenue for the unknown values model is approximately 0.06. This difference corresponds to roughly 8 percent of the efficient surplus.

The extreme points of the surplus sets of Figure 2 are maximal for welfare objectives that are symmetric with respect to the bidders. For example, when we maximize revenue or total bidder surplus, the objective does not depend on which bidder bids which amount, but only on the average maximum bid. For symmetric objectives, it is without loss of generality to consider symmetric BCE (Lemma 1). However, even when the distribution of values is symmetric, asymmetries in information about values or in behavior could induce differences in welfare outcomes across buyers.

Figure 3 displays the sets of bidder surplus pairs (U_1, U_2) that can arise in our discretized uniform example. Again, blue denotes the unknown values model and red denotes the known values model. Points are labeled in an order that highlights the correspondence between Figures 2 and 3. For example, point F in Figure 3 is the BNE outcome under known values, and corresponds to point F in Figure 2. Point ABC, at which bidder surpluses are minimized, corresponds to points A, B, and C in Figure 2, etc. Point G corresponds to the minimum bidder surplus/maximum revenue point for the known values model. Point D is where bidder surplus is maximized and revenue is minimized under the unknown values model. The point marked E' is the only labeled point on Figure 3 that does not have a direct counterpart in Figure 2: although the two are close together and are visually difficult to distinguish, minimum revenue and maximum bidder surplus need not coincide for the known values model, so that point E is distinct from E'.

Let us highlight two additional features of the bidder surplus sets. The first is the large flats on the southwestern frontier for both the unknown values and known value sets. We already know that it is possible to simultaneously hold both bidders to lower bound surpluses simultaneously. These flats indicate that it is also possible to hold one bidder down to minimum surplus over a range of surpluses for the other bidder. This is fairly easy to see in the unknown values case. Let us construct an equilibrium that attains a surplus for buyer 1 equal to that of the Bayes Nash equilibrium point F, while maintaining zero surplus for the other buyer. Starting from point A, in which both buyers essentially bid

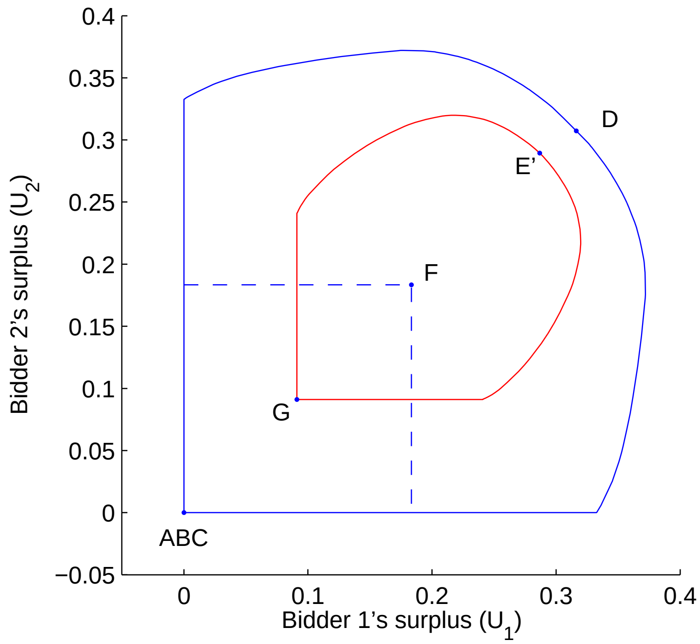

Figure 3: The set of bidder surplus pairs that can arise in a BCE. Computed for uniform distribution with grids of 10 valuations and 50 bids between 0 and 1.

the highest value, we could switch to an equilibrium where both buyers are told to bid the highest value when $v_1 < v_2$ and bid the second-highest value when $v_2 > v_1$. By perturbing this equilibrium using similar methods as we applied in the proof of Theorem 2, it is possible to adjust tie breaks so that the high valuation buyer always wins and strategies are weakly undominated, at arbitrarily small cost to revenue. Thus, buyer 1 will always win and nearly pay the second-highest value, whereas buyer 2 wins and nearly pays his own value. Similarly, for the known values model, one can construct the equilibrium as in Theorem 3 when $v_1 > v_2$, and use complete information when $v_2 > v_1$, which again would induce an outcome where buyer 2's surplus is the same as that at point G, but buyer 1's surplus is at the level of F. Of course, these arguments merely show that the flat should extend *at least* as far as the known values BNE. In Figure 3, they extend much further. A tight characterization of the extent of these flats remains an open question.

Finally, we note the smooth Pareto frontier for bidder surplus pairs. In the earlier version of this paper (Bergemann et al., 2013), we characterized the frontier for the version of the model with two players and binary values. In that case, it is possible to generalize the relaxed problem (R3) to asymmetric winning bid distributions. There is a class of undominated solutions to (R3) that extend to BCE that attain the frontier. Again, we consider a more general characterization of the efficient frontier to be an intriguing direction for future research.

6.2 Max/min ratios

Ultimately, we have been able to characterize maximum and minimum revenue and total bidder surplus for a variety of models. But what is the broader message? Are these bounds wide or narrow? Are welfare outcomes of the first price auction sensitive to information, or not so sensitive? At the end of the day, such a judgement is inherently subjective, but we can clarify the debate by condensing our bounds down to summary statistics that describe the degree to which surplus varies across information structures.

We have argued that a useful benchmark for revenue and bidder surplus comes from the complete information outcome, which coincides with the known values Bayes Nash equilibrium in the case of independent private values. Thus, we can ask how much revenue and bidder surplus can vary over all BCE relative to the complete information benchmark. Let $\widehat{R} = \mathbb{E}[v^{(2)}]$ denote the expected second-highest value, and $\widehat{U} = \mathbb{E}[v^{(1)} - v^{(2)}]$ denote the residual surplus for bidders, which is the difference between the highest and second-highest values. Analogously, let $\overline{R}$, $\underline{R}$, $\overline{U}$, and $\underline{U}$ denote the highest and lowest revenues and highest and lowest total bidder surpluses, respectively, that occur in BCE. To give our analysis the best possible shot of obtaining relatively informative bounds, we will restrict attention to known value BCE. All of these objects are implicitly being defined as functions of the underlying distribution of values, p.

We will consider the ratios:

$$\underline{R}/\widehat{R},\ \overline{R}/\widehat{R},\ \underline{U}/\widehat{U},\ \text{and } \overline{U}/\widehat{U}.$$

A bounds on how far these ratios can differ from 1 would mean that these welfare objectives cannot be either too much larger or too much smaller than the complete information benchmark, regardless of the distribution p.

In fact, there exist distributions of values such that these welfare ratios can be arbitrarily far from one. In particular, $\underline{R}/\widehat{R}$ and $\underline{U}/\widehat{U}$ can be arbitrarily close to zero, and $\overline{R}/\widehat{R}$ and $\overline{U}/\widehat{U}$ can be arbitrarily large. Thus, in a proportional sense, our welfare bounds can be arbitrarily wide. We illustrate this with three examples. We note that since these examples all assume known values, the conclusion about max/min ratios extends a fortiori to the unknown values case.

The first example shows that $\underline{R}/\widehat{R}$ and $\overline{U}/\widehat{U}$ can be arbitrarily small and arbitrarily large, respectively. Consider a model with binary values in $\{0, 1\}$, and let p_1 and p_2 denote the probabilities of one or two values being 1, respectively. Revenue under complete information is p_2, and minimum revenue is $(1-2\alpha)p_2$, where α satisfies the ratio condition

$$\frac{\alpha}{1-2\alpha} = \frac{p_1}{\alpha p_2}.$$

(We refer the reader to the construction in Section 5.2.2 for details.) The solution is

$$\alpha = \frac{\sqrt{1+p_2/p_1}-1}{p_2/p_1}.$$

Thus, the ratio of minimum revenue to benchmark revenue is $1-2\alpha$. But as p_2 goes to 0, α goes to 1/2, as can be verified from L'Hôspital's rule, so that the ratio $\underline{R}/\widehat{R}$ goes to zero. With regard to $\widehat{U}/\overline{U}$, bidder surplus is $2p_1$ under the benchmark but is $2(p_1+\alpha p_2)$ under the BCE described above. Thus, the ratio $\overline{U}/\widehat{U}$ is $1+\alpha p_2/p_1$. As p_1 goes to 0, α goes to ∞, so that $\overline{U}/\widehat{U}$ goes to infinity.

Our second example illustrates that it is possible for $\overline{R}/\widehat{R}$ to be arbitrarily large. Consider the case of two bidders and independent private values distributed according to the cumulative distribution $P(v) = v^\beta$. The lower bound on bidder surplus is

$$\underline{U} = \int_{v=0}^{1} \max_b (v-b)P(b)p(v)dv.$$

With the distribution we have chosen, the optimal bid is $b^*(v) = \frac{\beta}{1+\beta}v$, so that lower-bound bidder surplus is

$$\frac{1}{1+2\beta}\left(\frac{\beta}{1+\beta}\right)^{\beta+1}.$$

The expected highest value is

$$\int_{v=0}^{1} vP(v)p(v)dv = \frac{\beta}{1+2\beta},$$

so that maximum revenue is

$$\int_{v=0}^{1} vP(v)p(v)dv = \frac{\beta}{1+2\beta} - \frac{1}{1+2\beta}\left(\frac{\beta}{1+\beta}\right)^{\beta+1}.$$

The expected second-highest value is therefore

$$\int_{v=0}^{1} v(1-P(v))p(v)dv = \frac{\beta}{1+\beta}\frac{\beta}{1+2\beta}.$$

Thus, the ratio of maximum to benchmark revenue is

$$\frac{1+\beta-\left(\frac{\beta}{1+\beta}\right)^{\beta}}{\beta},$$

which clearly blows up as $\beta \to 0$.

The final example shows that $\underline{U}/\widehat{U}$ can be arbitrarily small. We will suppose that $v^{(1)} \sim U[0,1]$, and that conditional on the highest value, the second-highest value follows a truncated Pareto distribution on $[0, \gamma v^{(1)}]$, i.e.,

$$P(v^{(2)}|v^{(1)}) = \begin{cases} 1 \text{ if } v^{(2)} > \gamma v^{(1)}; \\ (1-\gamma)\frac{v^{(1)}}{v^{(1)}-v^{(2)}} \text{ if } 0 \leq v^{(2)} \leq \gamma v^{(1)}; \\ 0 \text{ otherwise.} \end{cases}$$

It should be clear that the expected highest value is $1/2$, and the optimal bid is always $\gamma v^{(1)}$, so that the lower bound on bidder surplus is just $(1-\gamma)/2$. Benchmark revenue is a slightly more complicated calculation. The expected second-highest value, conditional on $v^{(1)}$, is

$$(1-\gamma)v^{(1)} \int_{x=0}^{\gamma v^{(1)}} \frac{x}{(v^{(1)}-x)^2} dx = (\gamma + (1-\gamma)\log(1-\gamma))v^{(1)}.$$

Hence, the ratio of benchmark to lower bound bidder surplus is

$$1 - \log(1-\gamma),$$

which goes to infinity as γ goes to one.

Thus, the interpretation of our bounds in terms of max/min ratios depends very much on the distribution of values. The range of welfare outcomes can be very wide, depending on the specification of P. The extreme ratios are however obtained in extreme cases where either benchmark revenue or benchmark bidder surplus is going to zero. It remains an open question whether or not there are intuitive restrictions on the distribution that lead to uniformly narrower bounds.

We note that a weakness of these ratios as measures of the variability of welfare is that while they are invariant to scaling of values, they are not invariant translation. Thus, by adding a constant to all values, it is possible to make any of these ratios arbitrarily close to 1. In that sense, our bounds can also be arbitrarily narrow. It remains an open question whether there are other intuitive ratio measures that are invariant to affine transformations and with respect to which welfare ratios are uniformly bounded.

6.3 Reserve prices and entry fees

The first price auction is an important mechanism to study for many reasons, but at the end of the day it is just one of many possible mechanisms. Indeed, even when there is no ambiguity about the structure of buyers' information, the plain first price auction need not be an optimal mechanism for many objectives of interest, e.g., maximizing revenue. For that particular objective and in the benchmark setting when values are known and the distribution is symmetric, independent, and regular, the first price auction is generally only optimal if low valuation buyers excluded from the auction using a minimum bid or participation fee (Myerson, 1981).

Since our primary purpose is to understand the sensitivity of mechanisms to different informational assumptions, it is natural to try to extend our analysis to other mechanisms as much as possible. A general treatment of welfare bounds in mechanism design goes well beyond the scope of this paper. We can, however, make some progress along this dimension by considering the consequences of information in variations of the first price auction, namely those that involve reserve prices and entry fees. In this section, we explore how the introduction of such features into the auction affect maximum and minimum revenue over all BCE. These revenue bounds would be useful for an auction designer who knows the distribution of values, but is ambiguity averse with respect to the information structure and the equilibrium, and who is restricted to only using mechanisms within this class. In particular, is it possible to introduce a reserve price or entry fee that raises minimum revenue, regardless of the true information structure? If so, which of these devices induces greater worst-case revenue?

We will explore this issue via a numerical example. Suppose that there are two buyers whose values are uniformly distributed on a grid between 0 and 1. Reserve prices and entry fees are modeled as follows: With a reservation price $r \geq 0$, buyers can submit any bid they want, but if the high bid is less than the reservation price, the good will not be allocated and no transfers take place. Otherwise, the good is allocated to the buyer who bids the most, and the winning buyer pays their bid to the seller. With an entry fee, buyers first choose whether or not to enter the auction. Only entrants can bid, but an entrant must pay

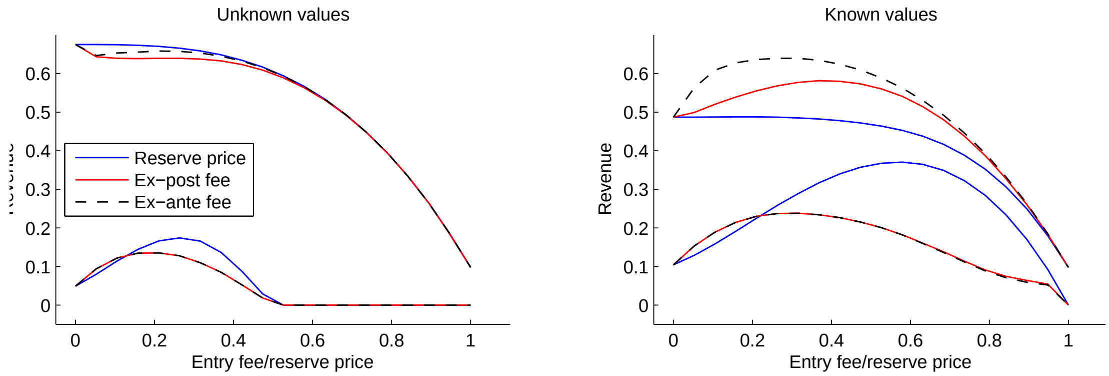

Figure 4: Comparison of reserve prices and entry fees. Computed for uniformly distribution with grids of 20 valuations and 20 bids between 0 and 1.

a fee of $e \geq 0$ in addition to the normal payments that result from winning the auction, and regardless of whether or not the entrant wins.

Note that in the absense of an entry fee, we can interpret a bid of 0 (or any bid below the reserve price) as "not entering," so that the reserve price model effectively captures a model with a zero entry fee. With a positive entry fee, however, some modeling choices arise with regard to the timing of information. In particular, do buyers make their entry decisions before or after acquiring whatever information will inform the equilibrium bid? Similarly, in the known values model, we may wish to assume that buyers learn their values before or after the entry decision. We think it is reasonable to suppose that information could be revealed both before and after the entry decision, although the two assumptions could have different implications for revenue.

Again, we used linear programming software to numerically calculate maximum and minimum revenue for a range of reserve prices and entry fees. Figure 4 displays the output of these computations for a specification with 20 values and 20 bids, evenly spaced between 0 and 1. The left and right panels illustrate the revenue bounds for the unknown and known value models, respectively. The blue lines indicate bounds for reserve prices, while the black and red lines cover entry fees. In particular, the black lines bound revenue over all BCE with an "ex-ante" fee, which must be paid *before* the buyers acquire whatever information informs their equilibrium bid, while the red lines bound revenue with an "ex-post" fee, which is paid *after* learning one's equilibrium bid. Note that the ex-post fee entails more incentive constraints than the ex-ante fee: if it is not optimal to deviate to not entering conditional on each possible equilibrium bid, then clearly it will not be optimal if you only learn that you should enter the auction and without learning which bid you will make. Indeed, the red lines in Figure 4 are bounded above and below by the black lines, which is consistent with the ex-post fee being more restrictive. In addition, the bounds on the left panel are all wider than the corresponding bounds on the right panel, which is consistent with the set of unknown values BCE being larger than the set of known value BCE for each possible reserve price or fee. We should also point out that for the ex-ante fee, the entry decision is assumed to be after one learns one's value but before one learns the equilibrium bid. We could have assumed alternatively that buyers first make their entry decisions and then learn their values and equilibrium bids, which would result in a bound that is intermediate between the black lines on the left and right panels.

The simulation indicates some intriguing features of how reserve prices and entry fees could affect the auction. First, it is very much possible for both reserve prices and entry fees to raise minimum revenue over all information structures, which is consistent with intuition from the classical model. Moreover, for this example, reserve prices are much more effective at boosting minimum revenue than are entry fees. For example, in the known values model, max min revenue over all reserve prices is approximately 0.3705, whereas max min revenue over entry fees is 0.2379. We note that in the limit when values are drawn from the standard uniform distribution, the optimal reserve price is 0.5, which results in revenue of $5/12 \approx 0.4167$. Entry fees and reserve prices are equally effective exclusion devices under the Bayes Nash equilibrium with independent private values. With regard to entry fees, a surprising result is that ex-ante and ex-post entry fees result in virtually identical minimum revenue curves. We do not know of any theoretical justification for this phenomenon.

Figure 4 also depicts the effects of reserve prices and entry fees on maximum revenue. Under the unknown values model, maximum revenue without a minimum bid or an entry fee is already the efficient surplus, so excluding low valuation buyers can only depress revenue. With known values, however, introducing a minimum bid or a fee can increase revenue. In addition, there is a stark ordering over maximum revenues. For each $x \in [0, 1]$, maximum revenue with an ex-ante fee of x is strictly higher than maximum revenue with an ex-post fee of x, which is strictly higher than maximum revenue with a reserve price of x. The weak ordering of ex-ante and ex-post fees is fairly transparent, but maximum revenue ranking of entry fees versus reserve prices is more subtle.

In fact, our characterization from Theorem 3 can explain why ex-ante entry fees will dominate reserve prices in terms of maximum revenue when values are known. Suppose that there are n buyers whose values are i.i.d. draws from a cumulative distribution $P(v)$ on $[\underline{v}, \overline{v}]$. An entry fee e and reserve price r together induce an *exclusion level* $\underline{x}(r, e)$, which is the lowest-value that would enter the auction in a revenue maximizing equilibrium. It is still the case that, conditional on entering, buyers will not bid more than their values, and indeed, it is possible to construct an equilibrium via similar methods in which bidder surplus is exactly what bidders could attain by best responding when others bid their values. The difference is that when a buyer does not enter, that effectively imputes a bid for that buyer which is equal to the reserve price. Moreover, the allocation is efficient conditional on the set of types that enter, so that the lowest entering type will only win when no other buyer enters, and in ex-ante terms, the cutoff type must receive a surplus of zero. This means that the exclusion level $\underline{x}(r, e)$ is implicitly defined by

$$e = (\underline{x} - r)P^{n-1}(\underline{x}). \tag{24}$$

It is straightforward to verify that $\underline{x}$ is increasing in both of its arguments and if P is strictly increasing, then $\underline{x}$ will be strictly increasing in each argument. (We will assume P is strictly increasing for the remainder of the heuristic argument.)

Now, let us consider two distinct pairs (r, e) and (r', e') that induce the same exclusion level x. Then it is without loss of generality to assume that $r < r'$ and $e > e'$. We will argue that revenue is higher under (r, e) by arguing that bidder surplus is uniformly lower for all types that enter under (r, e) than under (r', e'). Let $b^*(v)$ be the solution to

$$\max_b (v - b)P^{n-1}(b) = \underline{u}(v).$$

Thus, $\underline{u}(v)$ is the interim lower bound bidder surplus when others bid their values and all types enter. It is straightforward to show that $b^*(v)$ is (weakly) increasing and $\underline{u}(v)$ is strictly increasing. Thus, among buyers who enter, the optimal lower bound bidding strategy will involve another cutoff $\widehat{x}(r, e) \geq \underline{x}(r, e)$, where buyers with values between $\underline{x}(r, e)$ and $\widehat{x}(r, e)$ will bid r and buyers with values above $\widehat{x}(r, e)$ will bid $b^*(v)$, and at the cutoff,

$$(\widehat{x} - r)P^{n-1}(\underline{x}(r, e)) = (\widehat{x} - b^*(\widehat{x}))P^{n-1}(b^*(\widehat{x})).$$

The cutoff $\widehat{x}$ must also be increasing in (r, e). Thus, inducing a given exclusion level with a lower reserve price and a higher fee tends to increase bidding at the reserve price. Specifically, if $r < r'$ and $\underline{x}(r, e) = \underline{x}(r', e') = \underline{x}$, then $\widehat{x}(r, e) \geq \widehat{x}(r', e')$. This is intuitive, because the probability of winning at the reserve price is fixed but the cost of winning at the reserve price goes down, then clearly bidding at the reserve price must become more attractive to all types.

We can now compare bidder surplus across the two exclusion mechanisms. If v is between $\underline{x}$ and $\widehat{x}(r', e')$, then interim bidder surplus must be the same using either reserve and fee pair, because these types bid the reserve price and obtain surplus

$$(v - r)P^{n-1}(\underline{x}) - e = (v - v')P^{n-1}(\underline{x}) = (v - r')P^{n-1}(\underline{x}) - e'.$$

This follows from the entry condition (24). If v is greater than $\widehat{x}(r, e)$, lower bound surplus is necessarily lower with the lower reserve price, since the reserve price does not distort bidding behavior (leaving interim surplus the same) but the entry fee is higher. Finally, for valuations that are between $\widehat{x}(r', e')$ and $\widehat{x}(r, e)$, the difference in surplus is

$$u(v) - (v - r)P^{n-1}(\underline{x}) - e \geq (v - r)P^{n-1}(\underline{x}) - e - (v - r')P^{n-1}(\underline{x}) + e' = 0,$$

so that these buyers attain higher surplus under (r', e') than under (r, e).

Intuitively, conditional on the exclusion level, a lower reserve price induces greater distortion in bidding behavior upon entry away from the unconditional optimum b^*. Both distortion and higher entry fees tend to decrease bidder surplus, relative to the no exclusion case. Thus, by setting a lower reserve and a higher fee, the seller simultaneously induces more distortion and extracts more rents from the buyers' whose behavior is not distorted, which must be decreasing the lower bound bidder surplus.

We note that this conclusion corresponds with a result of Milgrom and Weber (1982) that entry fees induce greater revenue than reserve prices when signals and values are affiliated. Their argument also involves comparison of two different pairs of reserve prices and fees that induce the same exclusion level, and to some extent, a similar logic may be underlying the two results.

In sum, this example demonstrates that reserve prices and entry fees can have a large impact on revenue bounds. In future research, we hope to extend the analysis of welfare bounds to more general classes of mechanisms.

6.4 Continuum of values

Though our formal results have been stated for the case of discrete values, we have frequently illustrated these results with examples in which there is a continuum of valuations. While discreteness has been useful for our inductive contructions, we do not regard it as essential, and we expect that our results would generalize to distributions of values characterized by Borel measures. However, the extension of the results to the continuum case seems to be a non-trivial technical exercise, for the simple reason that the set of BCE is not weak-* compact, and so sequences of BCE need not converge to BCE in the limit. For example, it may be that all along a sequence of BCE, bids are not equal with probability 1, but in the limit, bids are equal with probability 1, thus invoking the tie breaking rule. This is a well-studied phenomenon, and we refer the reader to Jackson and Swinkels (2005) and Jackson et al. (2002) for a more detailed discussion of the issue. While it is beyond the scope of this paper, we believe that our constructions will extend to continuum models in a manner that avoids the tie break entirely, as has been the case with all of our discrete value results.

6.5 Extension to asymmetric p

Throughout our analysis, we have maintained the assumption that the distribution p is symmetric. This has greatly simplified our arguments, especially for deriving solutions to the minimum revenue and maximum bidder surplus optimal control problems. However, we wish to point out that some of our results extend readily to models with asymmetric value distributions. Nothing in the proof of Theorem 3 relied on symmetry, and indeed, that result extends unchanged to the case where p is asymmetric. In particular, it will generically be the case that the lower bounds $\underline{U}_i$ depend on i, but the same construction defines a BCE in which all bidders simultaneously attain surpluses of $\underline{U}_i$ respectively and the allocation is efficient.

For our other results, symmetry is more important. The maximum revenue and minimum bidder surplus result can be extended for the unknown values model to a more general class of distributions, although obviously there are asymmetric distributions in which it is impossible to hold bidders down to zero surplus in an efficient equilibrium, e.g., when the support of one buyer's valuation is strictly above the support of the other buyers' valuations. The constructions for minimum revenue and maximum bidder surplus also need not generate BCE with asymmetric p. However, our results do generate bounds for asymmetric models. For suppose that p is asymmetric and minimum expected revenue over all BCE is R. Then R would also be minimum revenue over all BCE for the permuted distribution of values given by $p \circ \xi$ for some permutation $\xi \in \Xi$. Now consider the "symmetrized" distribution

$$\tilde{p} = \frac{1}{n!} \sum_{\xi \in \Xi} p \circ \xi.$$

Then clearly, there is a BCE for $\tilde{p}$ which is just the symmetrization of the BCE that yielded revenue of R for the original p, and this BCE for $\tilde{p}$ will also yield revenue of R. As a result, R must be weakly larger than minimum revenue for $\tilde{p}$, which we have characterized above for various cases. Thus, one can generate bounds for asymmetric distributions by symmetrizing and then calculating bounds for the symmetrized distribution.

7 Conclusion

The purpose of this paper has been to study the range of welfare outcomes that might obtain in a first price auction. In this exercise, we have sought to relax classical assumptions on the nature of the bidders' information that were made primarily for the purpose of tractability. For general specifications of information, in which values can be arbitrarily correlated and signals can be multidimensional, the structure of Bayes Nash equilibrium could be quite complicated. By focusing on those information structures that generate extreme welfare outcomes, we have maintained tractability while generating new insights into the range of welfare outcomes that might occur. In particular, we have shown that while there is a wide range of behavior that could occur, there are non-trivial limits on what can happen to revenue, bidder surplus, and total surplus. The information structures that generate extreme outcomes give us further insight into what kinds of information might be good or bad for the seller and buyers. In particular, revenue is lower when buyers receive partial information about whether or not they have the high value that induces them to bid less, opening the door for other partially informed buyers to win with lower bids as well. Revenue is higher when buyers receive precise information about whether or not they have the high value, but partial information about the losing buyer's value. Social surplus can be harmed when there is precise information about others' values and in the absense of precise information about the buyer's own value.

Many important questions remain. In a computational exercise, we showed that the introduction of devices for excluding low valuation buyers can raise both minimum and maximum revenue over all information structures. While these results are suggestive, clearly they represent a very limited exploration of welfare bounds across the universe of mechanisms. A rich and open question is how to design mechanisms in the face of large uncertainty about the beliefs held by agents. A broad takeaway from our work is that the design of equilibria can be complemented by the design of information. By simultaneously constructing knowledge and behavior, we obtain bounds on the outcomes of interest as well as gain insight into how information interacts with the given game form. It is our hope that this methodology will find use for a wide range of problems within mechanism design.

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A Proofs

Lemma 1. *If the distribution of values p is symmetric, then there exist symmetric BCE that minimize revenue and maximize total bidder surplus.*

Proof of Lemma 1. □

Proposition 1. *The solution to the unknown values relaxed problem* (R2) *is the unique* $H(\cdot|v^L)$ *such that* (R2.3) *holds with equality for* $b > v^0$ *whenever* $H(b|v^L) < 1$ *for some* $v^L \in V^L$ *and that satisfies* (11).

Proof of Proposition 1. First, we observe that a solution to (R2) must exist. The set of bounded and measurable functions from B to $[0,1]$ is compact. Moreover, for each $b \in B$, (R2.3) is a linear restriction that is continuous in H in the weak-* topology, so that the set of distributions satisfying (R2.3) is closed. Therefore, the feasible set for (R2) is compact, and the objective is weak-* continuous, so a solution exists.

Next, let us show that $H(b|v^L) = 0$ almost surely for all $b < \underline{v}^L$. Let us define the function

$$H(b) = \sum_{v^L \in V^L} p^L(v^L) H(b|v^L).$$

Then (R2.3) implies that

$$(\underline{v}^L - b)\frac{n-1}{n} H(b) \leq \frac{1}{n} \int_{x=-\infty}^{b} H(x) dx.$$

Clearly, $H(b|v^L) > 0$ for some v^L if and only if $H(b) > 0$ as well, so it is sufficient to prove that $H(b) = 0$ almost surely for $b < \underline{v}^L$. If $H(b) > 0$ for some non-null set of $b < \underline{v}^L$, then there exists a $b < \underline{v}^L$ such that $\int_{x=-\infty}^{b} H(x)dx > 0$. Let $\underline{b}$ be the infimum of such b (here we are using the assumption that B is bounded below). Then it must be that $H(x) = 0$ for $x < b$. Let $z(\epsilon) > 0$ be the supremum of $H(b)$ over the range $[\underline{b}, \underline{b}+\epsilon]$ for some small ϵ. Then it must be that

$$(\underline{v}^L - \underline{b} - \epsilon)\frac{n-1}{n} z(\epsilon) \leq \frac{1}{n} \epsilon z(\epsilon),$$

which implies that $(\underline{v}^L - \underline{b})\frac{n-1}{n} \leq \epsilon$. But this has to hold for arbitrarily small ϵ. Thus, $\underline{b} \geq \underline{v}^L$.

As a result, it must be that

$$H(b|\underline{v}^L) = \begin{cases} 1 \text{ if } b \geq \underline{v}^L; \\ 0 \text{ otherwise.} \end{cases}$$

The reason is that the weight on $H(b|\underline{v}^L)$ in (R2.3) is non-positive, so that increasing $H(b|\underline{v}^L)$ always weakly decreases the left-hand side and increases the right-hand side. Incidentally, this argument also implies that $H(b|v^L) = 1$ for $b \geq v^L$, for all $v^L \in V^L$.

Now, consider a solution to (R2) such that there exists a non-null set $X \subseteq B$ such that for all $x \in B$, $H(b|v^L) < 1$ for some $v^L \in V^L$ and (R2.3) is slack. Let

$$G(b) = \frac{1}{n} \sum_{v^L \in V^L} p^L(v^L) \left(\int_{x=-\infty}^{b} H(x|v^L) dx - (n-1)(v^L - b) H(b|v^L) \right)$$

denote the slack in the constraint at b, so that $G(b) > 0$ on X. Then we can define an alternative solution:

$$\tilde{H}(b|v^L) = H(b|v^L) + \begin{cases} 0 \text{ if } b \notin X \text{ or if } b \geq v^L; \\ \min \left\{ 1 - H(b|v^L), \frac{G(b)}{(n-1)(v^L - b)} \right\} \text{ otherwise.} \end{cases}$$

Thus, it must be that $\tilde{H}(b|v^L) \geq H(b|v^L)$, and for a non-null set of $b \in X$, it must be that $\tilde{H}(b|v^L) > H(b|v^L)$. Hence, we have that

$$\begin{aligned}
\sum_{v^L \in V^L} p^L(v^L) \frac{n-1}{n}(v^L - b)\tilde{H}(b|v^L) &\leq \sum_{v^L \in V^L} p^L(v^L)\frac{1}{n}\left[(n-1)(v^L - b)H(b|v^L) + G(b)\right] \\
&= \sum_{v^L \in V^L} p^L(v^L)\frac{1}{n}H(x|v^L)dx \\
&\leq \sum_{v^L \in V^L} p^L(v^L)\frac{1}{n}\tilde{H}(x|v^L)dx,
\end{aligned}$$

so that $\tilde{H}(b|v^L)$ is feasible for (R2). However, since $\tilde{H}(b|v^L) > H(b|v^L)$ on a non-null set, we must have that (R2.1) is higher with $\tilde{H}$ than with H, so the objective has improved. Thus, if (R2.3) is slack on a non-null set such that $H(b|v^L) < 1$ for some $v^L \in V^L$, then (R2.1) must be strictly below its optimal value.

Now, let us argue that (11) must be satisfied at the optimum. Suppose it is not the case, and let X be a non-null set such that $H(b|w) > 0$ while $H(b|v) < 1$ for some $w > v$. Let

$$\phi(b) = \min\left\{p^L(v)\frac{1 - H(b|v)}{2}, p^L(w)H(b|w)\right\}.$$

Note that $\phi(b) > 0$ on x. We can then define an alternative solution

$$\tilde{H}(b|v^L) = \begin{cases} H(b|v^L) \text{ if } v^L \notin \{v, w\}; \\ H(b|v) + \frac{1}{p^L(v)}\phi(b) \text{ if } v^L = v; \\ H(b|w) - \frac{1}{p^L(w)}\phi(b) \text{ if } v^L = w. \end{cases}$$

From the definition of ϕ, we know that $\tilde{H}(b|v^L) \in [0,1]$ for all $b \in B$ and for all $v^L \in V^L$. In addition, whenever $\phi(b) > 0$,

$$\begin{aligned}\sum_{v^L \in V^L} p^L(v^L)\frac{n-1}{n}(v^L - b)\tilde{H}(b|v^L) &= \sum_{v^L \in V^L} p^L(v^L)\frac{n-1}{n}(v^L - b)H(b|v^L) \\ &\quad + \frac{n-1}{n}\phi(b)(v-w) \\ &< \sum_{v^L \in V^L} p^L(v^L)\frac{n-1}{n}(v^L - b)H(b|v^L) \\ &\leq \sum_{v^L \in V^L} p^L(v^L)\frac{1}{n}\int_{x=-\infty}^{b} H(x|v^L)dx \\ &= \sum_{v^L \in V^L} p^L(v^L)\frac{1}{n}\int_{x=-\infty}^{b} \tilde{H}(x|v^L)dx,\end{aligned}$$

$$\sum_{v^L \in V^L} p^L(v^L)H(b|v^L) = \sum_{v^L \in V^L} p^L(v^L)\tilde{H}(b|v^L)$$

for all b. Thus, we conclude that (R2.3) is slack on a non-null set even though $\tilde{H}(b|v^L) < 1$ for some v^L, and therefore (R2.1) must be strictly below the optimum at $\tilde{H}$. But (R2.1) is the same at $\tilde{H}$ and at H. Thus, H cannot be the optimal solution either. We conclude that (11) must hold almost surely.

We are essentially done. $H(b|v^L)$ is now inductively pinned down by (R2.3) and (11). In particular, suppose that we have defined $H(b|v^L)$ for all $v^L < v \in V^L$. Let $\widehat{b} = \sup\{b|H(b|v^L) < 1 \text{ for some } v^L < v\}$. Then it must be that $H(b|v^L) = 1$ for all $v^L < v$ and $x \leq \widehat{b}$, and thus

$$\begin{aligned}&(n-1)(v-b)H(b|v) - \int_{x=-\infty}^{b} H(x|v)dx \\ &= \frac{1}{p^L(v)}\left[\sum_{v^L < v} p^L(v^L)\left(\int_{x=-\infty}^{\widehat{b}} H(x|v^L)dx - (n-1)(v^L - b)\right) + (b - \widehat{b})\right] \\ &= C_1 + C_2 b.\end{aligned}$$

This is a first-order linear ordinary differential equation. The homogenous solutions would be of the form $(v-b)^{\frac{1}{n-1}}$, and using this, one can derive the non-homogenous solution

$$\int_{x=-\infty}^{b} H(x|v)dx = C_3(v-b)^{-\frac{1}{n-1}} - C_1 - C_2\frac{(n-1)v+b}{n},$$

and thus,

$$H(b|v) = \frac{C_3}{n-1}(v-b)^{-\frac{n-2}{n-1}} - \frac{C_2}{n}.$$

The constant C_3 has to be chosen so that $H(\widehat{b}|v) = 0$, so

$$C_3 = C_2\frac{n-1}{n}(v-\widehat{b})^{\frac{n-2}{n-1}}.$$

Our final form for $H(b|v)$ is

$$H(b|v) = \frac{1}{n}C_2\left(\left(\frac{v-\widehat{b}}{v-b}\right)^{1+\frac{1}{n-1}} - 1\right). \tag{25}$$

Strictly speaking, $H(b|v)$ is zero for $b < \widehat{b}$, given by (25) for $b \in [\widehat{b}, \overline{b}]$ where $\overline{b}$ is the point at which $H(b|v)$ hits 1, and then is 1 for $b > \overline{b}$. Clearly, as $b \uparrow v$, the right-hand side of (25) blows up, so that $H(b|v)$ must hit 1 before b hits v. Now we can inductively continue the solution for the next higher $v \in V^L$, and since there are only finitely many elements of v^L, this process eventually terminates and we have defined $H(b|v)$. We note that the functions $H(b|v)$ so defined are monotonically increasing in b, so that they are in fact CDFs. □

Lemma 2. *For any BCE $F(b|v)$, the $H(b|v^L)$ induced by F must satisfy* (R2.2) *and* (R2.3)*. Moreover, expected revenue under F must be equal to a constant minus* (R2.1).

Proof of Lemma 2. The proof essentially follows the derivation of the relaxed problem in Section 5.1.2. It is obvious that revenue is equivalent to (R2.1), where $H(b|v^L)$ is defined from a BCE F according to (4), suitably summing over v and k where $v^L(v) = v^L$. In any BCE, the buyers must weakly prefer to follow their recommendations rather than use any other deviation mapping. Let us consider the deviation mapping $\sigma_i^x(b) = \max\{x, b\}$. Then using (2.1), we can write the IC constraint for these deviations as

$$\sum_{v\in V^n} p(v)\int_{\{b\in B^n|b_i=b^{(1)},b_i\le x\}} \left[(v_i-b_i)q_i(b) - (v_i-x)q_i(x,b_{-i})\right] F(db|v) \ge 0.$$

Since $q_i(x, b_{-i}) \leq 1$, then clearly this constraint implies that

$$\sum_{v \in V^n} p(v) \left[\int_{\{b \in B^n | b_i = b^{(1)}, b_i \leq x\}} (v_i - b_i) q_i(b) F(db|v) - (v_i - x) F(x\mathbf{1}|v) \right] \geq 0.$$

Now, from (4) and the assumption of symmetric, it must be that

$$q_i(b) F(db|v) = H_i(db_i|v)$$

where $v_i = v^k$. Moreover, $F(x\mathbf{1}|v)$ is the total probability that *some* buyer wins with a bid less than x when values are v, which is $\sum_{i \in N} H_i(x|v)$. Thus, we can rewrite this as

$$\sum_{v \in V^n} p(v) \left[\int_{b_i = -\infty}^{x} (x - b_i) H_i(db_i|v) - (v_i - x) \sum_{j \neq i} H_j(x|v) \right] \geq 0.$$

So now, integrating by parts, we conclude that

$$\sum_{v \in V^n} p(v) \left[\int_{b_i = -\infty}^{x} H_i(b_i|v) db_i - (v_i - x) \sum_{j \neq i} H_j(x|v) \right] \geq 0.$$

Note that by symmetry, $\sum_{v \in V^n} H_i(x|v) = \sum_{v \in V^n} H_j(x|v)$ for all $i, j \in N$, so by exchanging the order of integration and summation, the constraint becomes

$$\left[\int_{b_i = -\infty}^{x} \sum_{v \in V^n} p(v) H_i(b_i|v) db_i - (v_i - x)(n-1) \sum_{v \in V^n} p(v) H_i(x|v) \right] \geq 0.$$

Finally, grouping terms, we can write $\sum_{v \in V^n} p(v) H_i(x|v) = \sum_{v^L \in V^L} p^L(v^L) H(x|v^L)$, which proves that (R2.3) must be satisfied. □

Lemma 3. *The functions $L(b|b', v^L)$ defined by* (12) *are monotonically increasing in b, and satisfy $L(b|b, v^L) = 1$.*

Proof of Lemma 3. If $v^L = \underline{v}^L = \min V^L$, then we define $L(b|b', v^L)$ to randomize over an interval$[\underline{v}^L - \epsilon, \underline{v}^L]$ so as to support bidding at $\underline{v}^L$ by a type with expected valuation $v^W(\underline{v}^L)$. In particular, we can have

$$L(b|\underline{v}^L, \underline{v}^L) = \begin{cases} 1 \text{ if } b \geq \underline{v}^L; \\ \frac{v^W(\underline{v}^L) - \underline{v}^L}{v^W(\underline{v}^L) - b} \text{ if } b \in [\underline{v}^L - \epsilon, \underline{v}^L]; \\ 0 \text{ otherwise.} \end{cases}$$

Now, inductively suppose that $L(b|b',w)$ has been defined for $w < v^L$, and satisfies the properties in the lemma. The derivative of the first term in (12) is

$$\sum_{w \le \tilde{v}} p^L(w)(v^L - b)H(db|w)$$

which is necessarily non-negative, since $b \le v^L$ and $H(db|w)$ is positive from Proposition 1. And since we have inductively assumed that $L(b|x,w)$ is increasing for all $w < v^L$, it must be that $L(b|b',v^L)$ must be increasing as well.

Now, let us argue that $L(b|b,v^L) = 1$. Since $L(x|x,w) = 1$ for all $w < v^L$ and since $b > x$ for all $x \in B_w$ with $w < v^L$, (12) can be rewritten as

$$\begin{aligned} &p^L(v^L)L(b|b,v^L)H(db|v^L) \\ &= \frac{1}{(v^L-b)^2}\Bigg[\frac{1}{n-1}\sum_{w \le v^L} p^L(w)\left(\int_{x=-\infty}^{b} H(x|w)dx + (v^L - b)H(b|w)\right) \\ &\qquad + \sum_{w < v^L}(v^L - w)p^L(w)\int_{x \in B_w} H(dx|w)\Bigg] \end{aligned}$$

which further rearranges to

$$\begin{aligned} &p^L(v^L)(v^L-b)^2 L(b|b,v^L)H(db|v^L) \\ &= \frac{1}{n-1}\sum_{w \le v^L} p^L(w)\left(\int_{x=-\infty}^{b} H(x|w)dx + (v^L - b)H(b|w)\right) \\ &\quad + \sum_{w < v^L}(v^L - w)p^L(w)\int_{x \in B_w} H(dx|w) \end{aligned}$$

Since (R2.3) holds with equality and the $H(b|v^L)$ are almost everywhere differentiable, it must be that

$$\sum_{w \le v^L} p^L(w)\frac{n-1}{n}(w-b)H(db|w) = \sum_{w \le v^L} p^L(w)H(b|w). \tag{26}$$

But $H(db|w) = 0$ for all $w \neq v^L$, so in fact

$$p^L(v^L)\frac{n-1}{n}(v^L - b)H(db|v^L) = \sum_{w \le v^L} p^L(w)H(b|w).$$

We can substitute this and (R2.3) into (12) to rewrite it as

$$
\begin{aligned}
&L(b|b,v^L)(v^L-b)\frac{n}{n-1}\sum_{w\le v^L}p^L(w)H(b|w)\\
&=\frac{1}{n-1}\sum_{w\le v^L}p^L(w)\big((n-1)(w-b)H(b|w)+(v^L-b)H(b|w)\big)\\
&\quad+\sum_{w<v^L}(v^L-w)p^L(w)H(b|w)
\end{aligned}
$$

or

$$
\begin{aligned}
&\left(L(b|b,v^L)\frac{n}{n-1}-\frac{1}{n-1}\right)(v^L-b)\sum_{w\le v^L}p^L(w)H(b|w)\\
&=\sum_{w\le v^L}p^L(w)(w-b)H(b|w)+\sum_{w<v^L}(v^L-w)p^L(w)H(b|w)\\
&=\sum_{w\le v^L}p^L(w)(v^L-b)H(b|w)
\end{aligned}
$$

which implies that $L(b|b,v^L)=1$.

We observe that L is continuously differentiable for $b>\underline{v}^L$ and is continuous from the right at $\underline{v}^L$. □

Theorem 4. *The solution to the unknown values relaxed problem, together with the losing bid distributions defined by* (12)*, constitute a BCE. This BCE attains a tight lower bound on revenue and a tight upper bound on total bidder surplus over all BCE.*

Proof of Theorem 4. As a preliminary result, we observe that any BCE $F(b|v)$ can be disintegrated into a marginal distribution $F_i(b_i|v)$ over buyer i's bid and a conditional marginal distribution $F_i(b_{-i}|b_i,v)$ over other buyers' bids, given the profile of values and buyer i's bid (Çinlar, 2011, Theorem 2.18). In order for incentive compatibility to be satisfied in the sense of (2), it is sufficient to verify that there exists a version of these objects such that it is almost surely not beneficial to deviate after any given recommendation, with respect to the conditional marginal distribution. In particular, the following condition most hold $F_i(\cdot|v)$ almost surely:

$$
\begin{aligned}
&\sum_{v\in V^n}p(v)(v_i-b_i)\int_{b_{-i}\in B^{n-1}}q_i(b_i,b_{-i})F_i(db_{-i}|b_i,v)F_i(db_i|v)\\
&\ge\sum_{v\in V^n}p(v)(v_i-x)\int_{b_{-i}\in B^{n-1}}q_i(x,b_{-i})F_i(db_{-i}|b_i,v)F_i(db_i|v)
\end{aligned}
$$

for all $x \in B$. If this condition were violated on a non-null subset of B, then we could choose σ_i to be the deviation mappinng that selects any x for which the inequality is strict, and clearly such a deviation would be profitable.

Let us prove that this is the case for the equilibrium constructed in Section 5.1.3. We first verify the constraints for $x \leq b_i$, following the steps we outlined previously. For the special case where $v^L = \underline{v}^L$, it is trivial to verify since there is only a single winning recommendation, $b = \underline{v}^L$, and the losing buyer is always told to bid less than $\underline{v}^L$. Thus, upon being told to bid $b < \underline{v}^L$, the buyer knows that his or her value is $\underline{v}^L$ and it is impossible to win with a bid less than $\underline{v}^L$.

On the other hand, with a recommendation b greater than $\underline{v}^L$, the conditional probability of winning with a bid $x < b$ is $L(x|b, v^L)^{n-1}$, where $b \in B_{v^L}$. Thus, it is sufficient to verify that

$$(v^W(v^L) - x)L(x|b, v^L)^{n-1} \tag{27}$$

is increasing in x. We will prove the even stronger result that

$$(v^L - x)L(x|b, v^L)^{n-1}$$

is increasing, which clearly implies the former result, since $v^W(v^L) \geq v^L$. Let us assume that this is true for recommendations in B_{v^L} for $v^L < v$ and prove that it is true for bids in B_v as well. Since L is continuously differentiable for $b > \underline{v}^L$, a sufficient condition for (29) to be increasing is that

$$(v - x)(n - 1)L(dx|b, v^L) \geq L(x|b, v).$$

Let us prove that this is the case. Using the recursive definition (12) (Cohn, 1980, Theorems 5.2.1 and 6.3.6), we can calculate $L(dx|b, v)$ as

$$L(dx|b, v) = \frac{1}{(v - b)^2} \sum_{v^L \leq v} p^L(v^L) \left(\frac{1}{n - 1} H(dx|v^L) + (v - v^L) \int_{y \in B_{v^L}} L(dx|y, v^L) H(dy|v^L) \right).$$

The inductive hypothesis implies that

$$(v^L - x)(n - 1)L(dx|y, v^L) \geq L(x|y, v^L)$$

for all $v^L < v$, $y \in B_{v^L}$. We will now argue that

$$\begin{aligned}(v-x)^2(n-1)\sum_{v^L \le v} p^L(v^L)H(dx|v^L) \\ \ge \sum_{v^L \le v} p^L(v^L)\left(\int_{y=-\infty}^{x} H(y|v^L)dy + (v-x)H(x|v^L)\right)\end{aligned} \tag{28}$$

for $v^L < v$. This follows from (R2.3), since

$$\begin{aligned}\sum_{v^L \le v} p^L(v^L)\left(\int_{y=-\infty}^{x} H(y|v^L)dy + (v-x)H(x|v^L)\right) \\ = \sum_{v^L \le v} p^L(v^L)\left((n-1)(v^L-x)+(v-x)\right)H(x|v^L)\end{aligned}$$

and the differential form of (R2.3), which is (26), implies that the left-hand side of (28) is

$$(v-x)n\sum_{v^L \le v} p^L(v^L)H(x|v^L)$$

which must be strictly larger. This concludes the proof that $(v-x)L(x|b,v)$ is increasing for all $x \le b$, and therefore no downward deviations are attractive.

The final step is to verify that there are no attractive upward deviations. This follows from reverse engineering the incentive constraints that we used to derive (12). In particular, (12) integrates to

$$\begin{aligned}\sum_{v^L \in V^L} p^L(v^L)(v^L-x)p^L(v^L)(n-1)\int_{y=-\infty}^{x} L(b|y,v^L)H(dy|v^L) \\ = \sum_{v^L \in V^L} p^L(v^L)\left(\int_{y=-\infty}^{b} H(y|v^L)dy + (x-b)H(b|v^L)\right)\end{aligned}$$

Differentiating this with respect to b, we conclude that almost surely

$$\begin{aligned}\sum_{v^L \in V^L} p^L(v^L)(v^L-x)p^L(v^L)(n-1)\int_{y=-\infty}^{x} L(db|y,v^L)H(dy|v^L) \\ = \sum_{v^L \in V^L} p^L(v^L)(x-b)H(db|v^L)\end{aligned}$$

Recall that the change in surplus from deviating upwards from $b \in B_v$ to $x > b$ is given by

$$\sum_{v^L \in V^L} (v^L - x) p^L(v^L) \frac{n-1}{n} \int_{y=-\infty}^{x} L(db|y, v^L) H(dy|v^L) - (x - b) \sum_{v^L \in V^L} \frac{p^L(v^L)}{n} H(db|v^L).$$

Thus, the net gains from upward deviations are zero, and the specified joint distribution is a BCE.

Since (R2) is a relaxation of the problem of minimizing expected revenue over all BCE and the winning bid distributions $H(b|v^L)$ attain that optimum, it is impossible for any other BCE to generate lower revenue. Moreover, the allocation is efficient, so no other BCE can generate higher revenue. □

Proposition 2. *The solution to the binary known values relaxed problem is the unique $H(b|m)$ that satisfies* (R4.3) *with equality whenever $H(b|m) < 1/m$ for some m and also satisfies* (18)*.*

Proof of Proposition 2. The proof is similar to that of Proposition 1. The objective is weak-* continuous and the feasible set is weak-* compact, by analogous arguments, so an optimum exists. We first argue that no buyer can bid below v^0. Then, we argue that (R4.3) must bind whenever $H(b|m) < 1/m$ for some m, or else there is another solution that strictly improves the objective and lowers revenue. Finally, we show that the ordered supports property (18) must be satisfied, or else it is possible to find an alternative solution for which revenue is the same but also for which (R4.3) is slack even though $H(b|m) < 1/m$ for some m.

We can write

$$H(b) = \sum_{m=0}^{n} p^M(m) H(b|m)$$

for the ex-ante cumulative distribution of the winning bid. Because $v^1 > v^0$, (R3.4) implies that $H(b)$ satisfies the following inequality:

$$(v^0 - b) H(b) \leq \int_{x=-\infty}^{b} H(x) dx.$$

Since the support of bids is bounded below, there is an infimum b such that $H(b) > 0$, which we denote by $\underline{b}$. For ϵ small, let

$$z(\epsilon) = \sup\{H(b) | b \in [\underline{b}, \underline{b} + \epsilon]\}.$$

Note that $z(\epsilon) > 0$ by definition of $\underline{b}$. Then we have that for all $b \in [\underline{b}, \underline{b} + \epsilon]$,

$$(v^0 - b)z(\epsilon) \leq \epsilon z(\epsilon)$$

which implies that $v^0 - b \leq \epsilon$ for all $\epsilon > 0$. Thus, $b \geq v^0$, and hence $\underline{b} \geq v^0$.

Together with the assumption of weakly undominated strategies, this implies that $H(b|0)$ is a mass point on v^0. In addition, it must be that $H(b|1)$ is also a mass point on v^0. The reason is that the weight on $H(b|1)$ on the left-hand side of (R4.3) is zero, so increasing $H(b|1)$ only increase the right-hand side of (R4.3), as well as increasing (R4.1).

Now we claim that if (R4.3) is not satisfied almost surely when $H(b|m) < 1/m$ for some m, then it is possible to strictly improve the objective and decrease revenue. The proof is as in the unknown values case. Let

$$G(b) = \frac{1}{\sum_{m=1}^{n} p^M(m)} \sum_{m=1}^{n} p^M(m) \left[\int_{x=-\infty}^{b} H(x|m)dx - (v^1 - b)(m-1)H(b|m) \right]$$

denote the slack in the constraint at b. Suppose there is a non-null set X for which $G(b) > 0$ and $H(b|m) < 1$ for some m. Then we can define the alternative solution

$$\tilde{H}(b|m) = H(b|m) + \begin{cases} 0 \text{ if } b \notin x; \\ \min\left\{1 - H(b|m), \frac{G(b)}{(m-1)(v^1-b)}\right\} \text{ otherwise} \end{cases}$$

Note that $\tilde{H}(b|m) > H(b|m)$ on a non-null set. Thus, the objective (R4.1) must be larger with $\tilde{H}$ than under H. In addition,

$$\begin{aligned}(v^1 - b)\sum_{m=1}^{n} p^M(m)(m-1)\tilde{H}(b|m) &\leq (v^1 - b)\sum_{m=1}^{n} p^M(m)(m-1)H(b|m) + \sum_{m=1}^{n} p^M(m)G(b) \\ &= \sum_{m=1}^{n} p^M(m) \int_{x=-\infty}^{b} H(x|m)dx \\ &\leq \sum_{m=1}^{n} p^M(m) \int_{x=-\infty}^{b} \tilde{H}(x|m)dx,\end{aligned}$$

so that the new solution is feasible as well.

Now let us show that (18) must be satisfied almost surely. Suppose that X is some non-null set on which $H(b|m) < 1/m$ but $H(b|m') > 0$ for some $m' > m$. Let

$$\phi(b) = \min\left\{p^M(m)\frac{1 - H(b|m)}{2}, p^M(m')H(b|m')\right\}.$$

As before, $\phi(b)$ must be strictly positive on X. Now define the perturbed solution

$$\tilde{H}(b|m'') = \begin{cases} H(b|m'') \text{ if } m'' \notin \{m, m'\}; \\ H(b|m) + \frac{1}{p^M(m)}\phi(b) \text{ if } m'' = m; \\ H(b|m') - \frac{1}{p^M(m')}\phi(b) \text{ if } m'' = m'. \end{cases}$$

We claim that $\tilde{H}$ is feasible for (R4) and induces weakly higher objective. For feasibility, observe that the right-hand side of (R4.3) is unchanged (as has the objective). On the other hand, the change in the left-hand side is

$$\begin{aligned}
&(v^1 - b) \sum_{m''=1}^{n} p^M(m'')(m'' - 1)(\tilde{H}(b|m'') - H(b|m'')) \\
&= (v^1 - b) \left[p^M(m)(m-1)\left(\tilde{H}(b|m) - H(b|m)\right) + p^M(m')(m'-1)\left(\tilde{H}(b|m') - H(b|m')\right)\right] \\
&= (v^1 - b)\phi(b)\,(m - m') < 0.
\end{aligned}$$

Since (R4.3) was weakly satisfied under H, it must be strictly satisfied for a positive measure of b such that $\phi(b) > 0$, and moreover it is clear that $H(b|m) < 1/m$ for all such b. From the definition of $\phi(b)$, it is also clear that $\tilde{H}(b|m) \in [0, 1]$. Thus, $\tilde{H}$ is feasible. On the other hand, the change in the objective is just

$$\left(\frac{1}{m} - \frac{1}{m'}\right) \int_{x=-\infty}^{v^1} \phi(x)dx > 0,$$

so expected revenue has decreased.

Having established that $H(b|0)$ and $H(b|1)$ are mass points on 0, it must be that there is an interval $[v^0, b^2]$ for which (R4.3) is solved with $H(b|1) = 1$ and with $H(b|m) = 0$ for all $m > 2$. In particular,

$$(v^1 - b)p^M(2)H(b|2) = \int_{x=v^0}^{b} \left[p^M(1) + p^M(2)H(x|2)\right] dx.$$

The solution of this differential equation is

$$H(b|2) = \frac{p^M(1)}{p^M(2)} \frac{v^1 - v^0}{(v^1 - b)^2},$$

which explodes as $b \uparrow v^1$. Thus, it must hit $1/2$ before b reaches v^1, and at that point (b^2) we change over to solving (R4.3) for equality with $H(b|3)$.

Inductively, there will be ranges $B_m = [b^{m-1}, b^m]$ over which (R4.3) is solved as an equality with $H(b|m') = 1/m'$ for $m' < m$ and $H(b|m') = 0$ for $m' > m$. The differential equation is:

$$\begin{aligned}(v^1 - b)&\left[\sum_{m'=1}^{m-1} p^M(m')\frac{m'-1}{m'} + p^M(m)(m-1)H(b|m)\right] \\ &= \sum_{m'=1}^{m-1} p^M(m')\left[\int_{x=-\infty}^{b^{m'}} H(x|m')dx + \frac{b - b^{m'}}{m'}\right] \\ &+ p^M(m)\int_{x=b^{m-1}}^{b} H(x|m)dx.\end{aligned}$$

All of the terms corresponding to $m' < m$ can be treated as constants, so that this differential equation can be rewritten as

$$(v^1 - b)\left[C_1^m + (m-1)H(b|m)\right] = C_2^m + \int_{x=b^{m-1}}^{b} H(x|m)dx,$$

which rearranges to

$$(v^1 - b)(m-1)H(b|m) - \int_{x=b^{m-1}}^{b} H(x|m)dx = C_2^m - C_1^m(v^1 - b).$$

The solution to the differential equation is of the form

$$\int_{x=b^{m-1}}^{b} H(x|m)dx = C_3^m(v^1 - b)^{-\frac{1}{m-1}} - C_2^m + C_1^m(v^1 - b).$$

Thus,

$$H(b|m) = C_3^m\frac{1}{m-1}(v^1 - b)^{-\frac{m-2}{m-1}} - C_1^m.$$

The coefficient C_3^m is chosen so that $H(b^{m-1}|m) = 0$, so

$$C_3^m = \frac{m-1}{m}C_1^m(v^1 - b^{m-1})^{\frac{m-2}{m-1}}.$$

Clearly, $H(b|m)$ is blowing up as $b \uparrow v^1$, so given that $b^{m-1} < v^1$, we must have $H(b|m)$ hit $1/m$ at $b^m < v^1$ as well. This completes the construction of the solution to the relaxed problem. □

Theorem 5. *The solution to the binary known value relaxed problem, together with the losing bid distributions defined by* (19)*, constitute a BCE. This BCE attains a tight lower bound on revenue and a tight upper bound on total bidder surplus over all binary known value BCE.*

Proof of Theorem 5. Recall that the losing bid distributions are defined by

$$p^M(m')(m'-1)L(b|b',m')H(db'|m') = \frac{1}{(v^1-b')^2}\sum_{l=1}^{m} p^M(l)\int_{x=-\infty}^{b}(v^1-x)H(dx|l).$$

Clearly, these functions are monotonically increasing in b, and if $b=b'$ and $m=m'$, then this formula reduces to

$$\begin{aligned} p^M(m)(m-1)L(b|b,m)H(db|m) &= \frac{1}{(v^1-b)^2}\sum_{l=1}^{m} p^M(l)\int_{x=-\infty}^{b}(v^1-x)H(dx|l) \\ &= \frac{1}{(v^1-b)^2}\sum_{l=1}^{m} p^M(l)\left[\int_{x=-\infty}^{b} H(x|l)dx + (v^1-b)H(b|l)\right]. \end{aligned}$$

Substituting in (R4.3), this further reduces to

$$p^M(m)(m-1)L(b|b,m)H(db|m) = \frac{1}{v^1-b}\sum_{l=1}^{m} p^M(l)lH(b|l).$$

Since (R4.3) must hold as an equality, we can differentiate both sides to obtain

$$(v^1-b)p^M(m)(m-1)H(db|m) = \sum_{l=1}^{m} p^M(l)lH(b|l),$$

where m is the almost-surely unique l such that $b\in[b^{l-1},b^l]$. Thus, we conclude that $L(b|b,m)$ must be 1.

Now let us verify that incentive constraints are satisfied. We will use the same approach as before, showing that incentive compatibility is met almost surely with respect to the conditional distributions of values and bids given a bidder's own bid. In particular, a downward deviation from the recommended bid $b\in[b^{m-1},b^m]$ to some $x<b$ is suboptimal if

$$(v^1-x)L(x|b,m)^{m-1} \tag{29}$$

is increasing in x. $L(x|b,m)$ is continuous and almost everywhere differentiable, so a sufficient condition for (29) to be increasing is that

$$(v^1 - x)(m-1)L(dx|b,m) \geq L(x|b,m)$$

wherever $L(x|b,m)$ is differentiable. Up to a constant,

$$L(x|b,m) \propto \sum_{l=1}^{m} p^M(l) \int_{x=-\infty}^{b} (v^1 - x)H(dx|l),$$

so that the increasing condition is equivalent to

$$(v^1 - b)^2(m-1)p^M(m)H(db|m) \geq \sum_{l=1}^{m} p^M(l) \int_{x=-\infty}^{b} (v^1 - x)H(dx|l).$$

Again, from the differential form of (R4.3), the left-hand side can be rewritten as

$$\begin{aligned}(v^1 - b)\sum_{l=1}^{m} p^M(l)lH(b|l) &\geq (v^1 - b)\sum_{l=1}^{m} p^M(l)(l-1)H(b|l) \\ &= \sum_{l=1}^{m} p^M(l) \int_{x=-\infty}^{b} (v^1 - x)H(dx|l),\end{aligned}$$

where the second line follows from (R4.3). Thus, downward deviations are not attractive.

To see that upward deviations are not attractive, again we reverse engineer the incentive constraints that are used to derive (19). In particular, we can integrate (19) with respect to b', and then rearrange and differentiate with respect to b to conclude that (20) is satisfied, so that upward deviations are not attractive either.

Finally, it is obvious that the winning bid distribution induced by this equilibrium is equal to the solution to the relaxed problem (R4), and the latter generates a lower bound on revenue over all binary known value BCE. Thus, the BCE constructed above attains the lower bound on revenue. The allocation is also efficient, and so the BCE attains an upper bound on total bidder surplus as well. □

Proposition 3. *If $\frac{p_{11}}{p_{10}}$ is sufficiently large relative to $\frac{p_{22}}{p_{20}}$, the solution to the efficiency-constrained trinary known value relaxed problem is the unique $H(b|k,k')$ that satisfies* (R5.3) *with equality whenever* (R5.2) *is slack and also satisfies* (22).

Proof of Proposition 3. Arguments analogous to those for the unknown and binary values cases can be used to demonstrate the following facts:

(i) In any solution to the relaxed problem, no buyer wins with a bid less than v^0, and all buyers win with bids of v^0 when the second-highest value is v^0.

(ii) (R5.3.k) should bind whenever $H(b|k',k) < \frac{1+\mathbb{I}_{k\neq k'}}{2}$ for some k'.

The remaining piece to characterize the equilibrium is (22). Define T_1 to be the infimum of b such that

$$p_{11}H(x|1,1) + p_{21}H(x|2,1) = \frac{p_{11}}{2} + p_{21}$$

for all $x \geq b$. Analogously define T_2 to be the infimum b such that $H(x|2,2) = p_{22}/2$ for all $x \geq b$.

Now, suppose that there exists a non-null set of b for which (22) is violated and $b > T_2$. In other words, (R5.3.2) is slack, (R5.3.1) binds, and $H(b|2,1) > 0$ while $H(b|1,1) < 1/2$. In that case, we claim there is a simple perturbation that improves the objective: Let $\tilde{H}(b|k,k')$ be defined by

$$H(b|k,k') = H(b|k,k') + \begin{cases} 0 \text{ if } b < T_2 \text{ or } (k,k') = (2,2); \\ \frac{1}{p_{kk'}}\phi(b) \text{ otherwise,} \end{cases}$$

where

$$\phi(b) = \min\left\{p_{21}H(b|2,1), p_{11}\frac{1}{2}\left(\frac{1}{2} - H(b|1,1)\right)\right\}.$$

Then clearly the left-hand side of (R5.3.1) has stayed the same while the right-hand side has increased, so that (R5.3.1) is slack. The right-hand side of (R5.3.2) has decreased, but

$$\begin{aligned}
&\int_{x=v^0}^{b} \left(p_{20} + p_{21}H(x|2,1) + p_{22}H(x|2,2)\right) dx \\
&\geq \int_{x=v^0}^{T_2} \left(p_{20} + p_{21}H(x|2,1) + p_{22}H(x|2,2)\right) dx \\
&= (v^2 - T_2)p_{22} = (v^2 - T_2)p_{22}H(T_2|2,2) \\
&\geq (v^2 - b)p_{22}H(b|2,2)
\end{aligned}$$

for $b > T_2$. Thus, this solution is clearly feasible. But (R5.3.1) is slack for a non-null set of b for which $H(b|1,1) < 1/2$, so that there is another perturbation that strictly increases revenue. Thus, for $b > T_2$, any optimal solution must satisfy (22).

Now we will use the assumption that $\frac{p_{11}}{p_{10}}$ is much larger than $\frac{p_{22}}{p_{20}}$ to argue that $T_1 > T_2$, for *any* feasible solution to the relaxed problem. Let

$$\alpha = \frac{p_{11}}{p_{10}}\frac{p_{20}}{p_{22}}.$$

A lower bound on T_1 is obviously achieved by the solution that satisfies (22), and in that solution, T_1 is at least as the infimum of b for which $H(x|1,1) = 1/2$ for all $x \geq b$, which we will denote by T_{11}. On the other hand, an

upper bound on T_2 is achieved by a solution where $H(b|2,1) = 0$ for all b. Thus, T_1 is at least the first time that $H(b|1,1)$ hits 1/2 when $H(b|1,1)$ solves

$$\begin{aligned}(v^1 - b)H(b|1,1) &= \int_{x=v^0}^{b} \left(\frac{p_{10}}{p_{11}} + H(x|1,1)\right) dx \\ &\leq \left(\frac{p_{10}}{p_{11}} + \frac{1}{2}\right)(b - v^0),\end{aligned}$$

and T_2 is no more than the first time that $H(b|2,2)$ hits 1/2 where $H(b|2,2)$ solves

$$\begin{aligned}(v^2 - b)H(b|2,2) &= \int_{x=v^0}^{b} \left(\frac{p_{20}}{p_{22}} + H(x|2,2)\right) dx \\ &= \int_{x=v^0}^{b} \left(\alpha\frac{p_{10}}{p_{11}} + H(x|2,2)\right) dx \\ &\geq \alpha\frac{p_{10}}{p_{11}}(b - v^0).\end{aligned}$$

Thus,

$$\begin{aligned}H(b|1,1) &\leq \left(\frac{p_{10}}{p_{11}} + \frac{1}{2}\right)\frac{b - v^0}{v^1 - b} \\ H(b|2,2) &\geq \alpha\frac{p_{10}}{p_{11}}\frac{b - v^0}{v^2 - b}.\end{aligned}$$

Clearly, if α is sufficiently large, then $H(b|2,2)$ will have to hit 1/2 before $H(b|1,1)$.

Now, let us argue that (22) must be satisfied for $b < T_2$, when $T_1 > T_2$. Suppose that there is a non-null set $X \subseteq [v^0, T_2]$ on which (22) is violated. We will construct a new, perturbed solution

$$\tilde{H}(b|k,k') = H(b|k,k') + \psi(b|k,k').$$

This perturbed solution will have $T_2 < \tilde{T}_2 < T_1$, and will satisfy $\psi(b|k,k') = 0$ for $b > \tilde{T}_2$. Moreover, we will perturb the solution in such a way that $\psi(b|2,1) \leq 0$, $\psi(b|2,2) \leq 0$, and $\psi(b|2,1) \geq 0$, but over the supports of these perturbations, (R5.3) will be satisfied as an equality. In particular, write

$$\Psi(b|k,k') = \int_{x=v^0}^{b} \psi(x|k,k')dx.$$

Note that at the solution H, (R5.3) holds with equality over the range $[v^0, T_2]$. For some $\tilde{T}_2$ just slightly larger than T_2, let $\widehat{H}(b|2,2)$ be the path that continues to solve (R5.3.2) as an equality on the range $[T_2, \tilde{T}_2]$, and is equal to 1 for $b > \tilde{T}_2$. Note that $\widehat{H}(\tilde{T}_2|2,2) > 1/2$ is infeasible for the original problem, which imposes $\widehat{H}(b|2,2) \in [0, 1/2]$, though it would a higher objective (since $\widehat{H} \geq H$).

The perturbed solution $\tilde{H}$ will satisfy (R5.3) as an equality as well, also over the range $[v^0, \tilde{T}_2]$. Thus, differencing (R5.3) across the perturbed solution and the alternative solution, we conclude that ψ has to satisfy

$$(v^1 - b)(p_{11}\psi(b|1,1) + p_{21}\psi(b|2,1)) = p_{11}\Psi(b|1,1)$$
$$(v^2 - b)p_{22}\psi(b|2,2) = p_{21}\Psi(b|2,1) + p_{22}\Psi(b|2,2).$$

Integrating these equations by parts, we conclude that

$$(v^1 - b)(p_{11}\Psi(b|1,1) + p_{21}\Psi(b|2,1)) = -p_{21}\int_{x=v^0}^{b} \Psi(x|2,1)dx$$
$$(v^2 - b)p_{22}\Psi(b|2,2) = p_{21}\int_{x=v^0}^{b} \Psi(x|2,1)dx$$

and so finally

$$p_{11}\Psi(b|1,1) = -p_{21}\left(\Psi(b|2,1) + \frac{1}{v^1 - b}\int_{x=v^0}^{b} \Psi(x|2,1)dx\right) \tag{30.1}$$
$$p_{22}\Psi(b|1,1) = p_{21}\frac{1}{v^2 - b}\int_{x=v^0}^{b} \Psi(x|2,1)dx. \tag{30.2}$$

Thus, once we have fixed the perturbation $\psi(b|2,1)$, it is possible to back out the perturbations $\Psi(b|1,1)$ and $\Psi(b|2,2)$ that maintain (R5.3) with equality. In addition, the above formulae imply that

$$p_{11}\Psi(b|1,1) + p_{21}\Psi(b|2,1) + p_{22}\Psi(b|2,2) = p_{21}\left(\frac{1}{v_2 - b} - \frac{1}{v_1 - b}\right)\int_{x=v^0}^{b} \Psi(x|2,1)dx.$$

The left-hand side of this equality, evaluated at $b = \tilde{T}_2$, is the change in the objective function at the perturbed solution relative to the alternative solution (since $\psi(b|k,k') = 0$ for $b > \tilde{T}_2$). Thus, if we can construct $\psi(b|2,1)$ so that the perturbed solution is feasible for the relaxed problem and such that $\int_{x=v^0}^{\tilde{T}_2} \Psi(x|2,1)dx < 0$, we will have demonstrated a perturbation that improves the objective.

We need to derive some bounds on how large perturbations might be that are induced in this manner. If $|\psi(b|2,1)| < \kappa$, then $|\Psi(b|2,1)| \leq b\kappa$, and $|\int_{x=v^0}^{b} \Psi(b|2,1)| \leq \frac{b^2}{2}\kappa$. Differentiating (30), we conclude that

$$p_{11}\psi(b|1,1) = -p_{21}\left(\psi(b|2,1) + \frac{1}{v^1 - b}\left(\frac{1}{v^1 - b}\Psi(b|2,1) + \int_{x=v^0}^{b} \Psi(x|2,1)dx\right)\right)$$
$$p_{22}\psi(b|2,2) = p_{21}\frac{1}{v^2 - b}\left(\Psi(b|2,1) + \frac{1}{v^2 - b}\int_{x=v^0}^{b} \Psi(x|2,1)dx\right)$$

Thus, if $|\psi(b|2,1)| < \kappa$, we conclude that

$$p_{11}|\psi(b|1,1)| \leq p_{21}\left(1 + \frac{1}{v^1 - b}\frac{b^2}{2} + \frac{b}{(v^1 - b)^2}\right)\kappa \tag{31.1}$$
$$p_{22}|\psi(b|2,2)| \leq p_{21}\frac{1}{v^2 - b}\left(b + \frac{b^2}{2(v^2 - b)}\right)\kappa. \tag{31.2}$$

Our perturbation will only involve $b \in [v^0, T_1]$, with $T_1 < v^1$, so that by choosing a perturbation $\psi(b|2,1)$ that is uniformly small, we can be guaranteed that the induced $\psi(b|2,2)$ and $\psi(b|1,1)$ will be uniformly small as well. This is necessary in order to ensure that $\tilde{H}$ will be feasible.

Now, let κ small, and let

$$X_\kappa = \left\{ b \in X \,\middle|\, H(b|2,1) \geq \kappa, H(b|1,1) \leq \frac{1}{2} - \kappa, H(b|2,2) \geq \kappa \right\}.$$

For κ sufficiently small, X_κ must be non-null as well. We will define ψ to be the perturbation generated by

$$\psi(b|2,1) = -\epsilon \mathbb{I}_{b \in X_\kappa}$$

for some κ and ϵ. Note that this perturbation induces $\psi_\kappa(b|2,2) \leq 0$, $\psi_\kappa(b|1,1) \geq 0$. Since $T_1 < v^1$, there is an ϵ small enough such that $\tilde{H}(b|1,1) \leq 1/2$, $\tilde{H}(b|2,2) \geq 0$, and $\tilde{H}(b|2,1) \geq 0$. Moreover, since $\psi(b|2,2)$ is continuous in ϵ, there will be an ϵ for which $\tilde{H}(b|2,2)$ hits $1/2$ at $\tilde{T}_2$, as long as $\tilde{T}_2$ is sufficiently close to T_2. In addition, by construction of the perturbations, $\tilde{H}(b|k,k')$ will satisfy (R5.3) for $b \in [0, \tilde{T}_2]$. In addition, the right-hand side of (R5.3.1) for $b > \tilde{T}_2$ is

$$\int_{x=v^0}^{b} (p_{10} + p_{11} H(x|1,1))\, dx + p_{11}\Psi(\tilde{T}_2|1,1) + p_{21}\Psi(\tilde{T}_2|2,1).$$

But since $\int_{x=v^0}^{\tilde{T}_2} \Psi(x|2,1)dx < 0$, as we argued above, it must be that $p_{11}\Psi(\tilde{T}_2|1,1) + p_{21}\Psi(\tilde{T}_2|2,1) > 0$. Thus, the right-hand side is larger than under the original solution H, and the paths that $H(b|1,1)$ and $H(b|2,1)$ followed for $b > \tilde{T}_2$ will still satisfy (R5.3). □

Corollary 1. *There exist distributions of values such that the welfare bounds from the the trinary known value relaxed problem are not tight.*

Proof of Corollary 1. Take as given an example $H(b|k,k')$ as constructed in the proof of Proposition 3, for the case in which α is large. Thus, $T_1 > T_2$, so that $X = \operatorname{supp} H(b|2,1) \setminus [v^0, T_2]$. Note that H is the almost surely unique solution to the relaxed problem, so that any other feasible solution has strictly lower revenue. As a result, if there were BCE that attained revenue close to that in the solution of the relaxed problem, the cumulative distributions of winning bids would have to converge to the solution to the relaxed problem as well.

Consider such a sequence of BCE with marginal winning bid distributions $H^l(\cdot|k,k')$ that converge to $H(\cdot|k,k')$ in the weak-* topology. We will argue that for such sequence, there exists an l large enough such that H^l cannot be induced by a BCE. Let L^l denote the probability that the v^1 type loses with a bid in X, let W_1^l denote the probability that the v^1 type wins with a bid in X, and let W_2^l denote the probability that the v^2 type wins with a bid in X.

Claim: L^l goes to 0 as l goes to ∞. Since the H distributions are absolutely continuous (above v^0), the probability that the v^1 type wins with a bid in X under the lth BCE must go to zero (since this is true under H), so W_1^l goes to 0. Thus, if L^l goes to $L > 0$, then the deviation which calls for always bidding T_1 when told to bid in X results in an asymptotic gain in surplus of at least

$$(v^1 - T_1)L^l - (T_1 - \min X)W_1^l \to (v^1 - T_2)L > 0,$$

since the winning bid recommendation in x must be at least $\min X$. For large enough l, this deviation would be profitable.

On the other hand, there is positive probability in the limit that type v^2 wins against v^1 with a bid in X, i.e., W_2^l goes to $W_2 > 0$. Consider the deviation in which the v^2 type bids $\min X$ whenever told to win with a bid above X. Let $\widehat{b}^l$ denote the average winning bid made by the v^2 type, conditional on winning with a bid in X. By weak-* convergence, $\widehat{b}^l$ converges to some $\widehat{b} > \min X$. The change in surplus from this deviation is

$$(\widehat{b}^l - \min X)(W_2^l - L^l) - (v^2 - \min X)L^l$$

which will be strictly positive for sufficiently large l.

Thus, it is impossible to have a sequence of BCE whose winning bid distributions converge weakly to the solution to the relaxed problem, and therefore, infimum revenue over all BCE must be bounded away from the solution to the relaxed problem. □