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ASYMMETRIC ALL-PAY AUCTIONS, MONOTONE AND NON-MONOTONE EQUILIBRIUM.

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ABSTRACT. We re-visit the two-bidder, all-pay auction of Amann and Leininger (1996) allowing for interdependent values and correlation à la Lizzeri and Persico (2000) and Siegel (2014). We study both monotone and non-monotone pure strategy equilibria (MPSE and NPSE): First, we show the allocation and bidding strategies of MPSE can be obtained in the same manner as in the independent private values environment. For correlated private values, the allocation is the same regardless of correlation. For common-values, the allocation is determined by the signals' percentiles. Second, we present three conditions: local single-crossing, single-crossing and increasing, which are respectively: necessary; sufficient; necessary and sufficient for the existence of MPSE. Third, we exhibit common-value families of examples that violate the local single-crossing and thus lack MPSE. We also construct a correlated private values example, where the slightest amount of correlation breaks down MPSE that exists under independence. Lastly, we explicitly obtain NPSE for quadratic valuations in cases where no MPSE exists.

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1. INTRODUCTION

In rent-seeking contests, distinct individuals may entertain different estimates of the prize. Such estimates maybe of varying precision or accuracy; possibly they maybe interdependent and/or correlated.

In this paper we model rent-seeking contests as all-pay auctions. Our aim here is limited to provide a tractable characterization of pure strategy (monotone or *not monotone*) equilibria (henceforth MPSE and NPSE) of the (first-price) all-pay auction with two (possibly asymmetric) players with interdependent valuations and correlated, continuous signals.

Our model can be viewed either as an extension of Amann and Leininger (1996) as we add correlation and interdependent values, or alternatively, as specialization of Lizzeri and Persico (2000) to the all-pay auction. It is also closely related to Siegel (2014) who studies a discrete signals model. However, we study non-monotone equilibria which starkly departs from previous studies of the all-pay auctio. Araujo et al. (2008) study non-monotonic equilibria in other auction formats but their assumptions rule out non-monotonic equilibrium in the all-pay auction. A notable exception is Rentschler and Turocy (2014), who characterize symmetric non-monotone equilibria .

As Siegel (2014), we do not restrict attention to affiliated signals, as in Lizzeri and Persico (2000) or to independent signals as in, Araujo et al. (2008). We allow for positive or negative correlated signals. Speaking plainly, affiliation is (mostly) useless – in the context of all-pay auctions with interdependent valuations. In Lemma 1, we prove that any equilibrium with correlated signals must also be an equilibrium of some all-pay auction with independent signals (but different valuations).

After Lemma 1 is established, the characterization of MPSE is a straightforward application of the recursive algorithm of Amann and Leininger (1996) for the independent private values case.[1] The algorithm first solves for the allocation rule or tying function and next computes bid functions. It can be implemented with any available differential equation solver, which are available in all computer algebra systems[2]. Siegel (2014)'s algorithm may be viewed as the analogous version for discrete type spaces.

In any MPSE, the allocation rule (*i.e.* the assignment of the object given the signal of the players) only depends on the players' expected values for the object conditional on their signals. In particular: For the *correlated private values* environment, the allocation rule is the same regardless of the nature of the correlation in the sense it coincides with the allocation of the independent private values case with the same marginal distributions. For the *common-value* environment, the allocation is dictated by the percentiles of the distribution of the agents' signals[3]: when agent 1 gets a signal in the p-percentile, he bids the same as when agent 2 gets a signal in the p-percentile.[4]

[1] See also Parreiras (2006) for an application to the first-price auction, common-value, affiliated case.

[2] See Mathematica, Maple, or for an open-source alternative, see Sage.

[3] In the context of discrete signals, Siegel (2014) obtain this result.

[4] Corollary: if signals are conditionally (on the value) independent, winning probabilities of both agents are identical. Einy et al. (2013) and Warneryd (2013) independently obtain this. They study models where one agent's signals is Blackwell's sufficient for the other's, which implies condition independence.

In Section 4, we show economically interesting all-pay auctions have MPSE when signals are independent but even "small doses" of correlation "break-down" the MPSE. That motivates us to study non-monotonic equilibria. For the class of quadratic valuations, we provide an explicit equilibrium strategies of NPSE when no MPSE exists. Discrete pooling (*i.e.* two or more types placing the same bid) allows the bidders incentives to be properly aligned without requiring different tie-break rules to assure existence.

2. THE MODEL

There are two agents, $i = 1, 2$. Let V_i be the random variable describing the value of the object for player i. Let X_1 and X_2 be the agents' signals. The conditional expected value is $v_i(x, y) \stackrel{\text{def}}{=} \mathbb{E}[V_i | X_1 = x, X_2 = y]$. The cumulative distribution of X_i is F_i and, $F_{i|j}$ is the conditional cumulative distribution of X_i given X_j. The lower-case f denotes the respective probability density function. Finally, we also define

$$\lambda_1(x, y) \stackrel{\text{def}}{=} v_1(x, y) \cdot f_{X_2|X_1}(x|y) \text{ and}$$
$$\lambda_2(x, y) \stackrel{\text{def}}{=} v_2(x, y) \cdot f_{X_1|X_2}(y|x).$$

We assume:

CONTINUITY: F_{X_i} is absolutely continuous and λ_i is piecewise continuous.
UNIFORM MARGINALS: Without any loss of generality, $X_i \sim \mathrm{U}[0, 1]$.
FULL SUPPORT & POSITIVE VALUATIONS: For all $(x, y) \in [0, 1]^2$, $\lambda_i(x, y) > 0$.

Remark 1. To see uniform marginals does not entail loss of generality: let S_i be the original signal, we always can re-parametrize signals by taking as the new signal, $X_i = F_{S_i}(S_i)$.

Remark 2. Since marginal distributions are uniform $[0, 1]$, by Baye's rule, we have $f_{X_1|X_2}(x|y) = f_{X_1,X_2}(x, y) = f_{X_2|X_1}(y|x)$ and consequently, $\frac{\lambda_2(x,y)}{\lambda_1(x,y)} = \frac{v_2(x,y)}{v_1(x,y)}$.

As we are interested in non-monotone equilibria, unlike the previous literature, we do not necessarily assume $\lambda_i(x, y)$ increasing in x.[5]

In this paper, we assume a tie breaking rule that favors bidder 1. This is done of pure notational convenience. With two bidders, $\lambda_i > 0$, under any random tie-breaking rule, ties do not occur in equilibrium.

With interdependent valuations, there is no loss of generality in assuming independent signals in the context of the all-pay auction as the reasoning below shows.

THE FICTITIOUS AUCTION: Given an all-pay auction, the corresponding fictitious (or auxiliary) auction is the all-pay auction where signals are independently and uniformly distributed on the unit interval, and expected conditional valuations are $\lambda_i(x, y)$.

Lemma 1. *The fictitious auction and the original auction are payoff equivalent.*

[5]See Amann and Leininger (1996), Krishna and Morgan (1997), Lizzeri and Persico (2000), Araujo et al. (2008), and Siegel (2014).

Proof of lemma 1. Pick any strategy profile $\mathbf{b} = (b_1(x), b_2(y))$ then the bidders' payoffs from the fictitious auction are:[6]

$$\tilde{U}_1(b|x) = \int_{\{y:b_2(y)\le b\}} \lambda_1(x,y)dy - b = \int_{\{y:b_2(y)\le b\}} v_1(x,y)f_{2|1}(y|x)dy - b = U_1(b|x),$$
$$\tilde{U}_2(b|y) = \int_{\{x:b_1(x)<b\}} \lambda_2(x,y)dx - b = \int_{\{x:b_1(x)<b\}} v_2(x,y)f_{1|2}(x|y)dx - b = U_2(b|y). \qquad \blacksquare$$

As a result, best reply correspondences in the fictitious and original auctions coincide, so do their equilibria sets.

3. Monotone Equilibrium

In this section, we study monotone equilibrium. Without loss of generality, we focus on the equilibrium bidding strategies that are weakly increasing and have no mass point at any positive bid.[7] The equilibrium bids might have a mass point at zero for one bidder. Note that that a uniform upper bound $\bar{b}$ applies for the equilibrium bids of the two bidders. Without loss of generality, we assume bidder 1's equilibrium bids have no mass point. Let $\phi_1(b)$ and $\phi_2(b)$, $b \in [0, \bar{b}]$ denote the inverse bidding functions. Note that bidder 2's equilibrium bids might have a mass point at zero. In this case, we define $\phi_2(0) = \sup_{b_2(y)=0}\{y\}$. The *tying function*[8] Q (or allocation rule) maps the type of player 1 to the type of player 2 that bids the same in equilibrium, that is $Q(\phi_1(b)) \overset{\text{def}}{=} \phi_2(b)$. Note $Q(0) = \phi_2(0)$.

Proposition 1. *The tying function solves the differential equation,*

$$Q'(x) = \frac{v_2(x, Q(x))}{v_1(x, Q(x))} \quad \textit{and} \quad Q(1) = 1.$$

Also $b_1(x) = \int_0^x \lambda_2(z, Q(z))\, dz, \forall x \in [0,1]$, *and* $b_2(y) = \int_{Q(0)}^y \lambda_1(Q^{-1}(z), z)\, dz, \forall y \in [Q(0), 1]$.

Proof. First-order conditions for the two bidders' optimization problems are respectively

$$\lambda_1(x, \phi_2(b))\, \phi_2'(b) - 1 \;=\; 0, \forall x, \quad \text{and} \quad \lambda_2(\phi_1(b), y)\, \phi_1'(b) - 1 = 0, \forall y \ge \phi_2(0). \tag{3.1}$$

$\forall b \in [0, \bar{b}]$, we have $x = \phi_1(b)$ and $y = \phi_2(b)$ at equilibrium. (3.1) gives

$$\begin{aligned} \lambda_1(\phi_1(b), \phi_2(b))\, \phi_2'(b) - 1 &= 0 \\ \text{and} \quad \lambda_2(\phi_1(b), \phi_2(b))\, \phi_1'(b) - 1 &= 0. \end{aligned}$$

From $Q(\phi_1(b)) \overset{\text{def}}{=} \phi_2(b)$, we have $Q'(\phi_1(b)) \cdot \phi_1'(b) = \phi_2'(b)$. Thus $Q'(\phi_1(b)) = \frac{\phi_2'(b)}{\phi_1'(b)} = \frac{\lambda_2(\phi_1(b), \phi_2(b))}{\lambda_1(\phi_1(b), \phi_2(b))}$, which leads to $Q'(x) = \frac{\lambda_2(x, Q(x))}{\lambda_1(x, Q(x))}$. Finally, remark 2 implies $Q'(x) = \frac{v_2(x, Q(x))}{v_1(x, Q(x))}$.

Together with the boundary condition, $Q(\cdot)$ can be pinned down.

The first order conditions can be rewritten as

$$b_1'(x) = \lambda_2(x, Q(x)),$$

[6] Recall remark 2 and the tie-breaking rule that favors bidder 1.

[7] Analogous results hold for decreasing equilibrium.

[8] See Amann and Leininger (1996) or Parreiras (2006).

which leads to

$$b_1(x) = \int_0^x \lambda_2(z, Q(z))\, dz, \forall x \in [0,1].$$

Similarly, we have[9]

$$b_2(y) = \int_{Q(0)}^{y} \lambda_1(Q^{-1}(z), z)\, dz, \forall y \in [Q(0), 1].$$

■

Proposition 1 says that valuation interdependence, as opposed to the signals' correlation, is the only factor that matters for determining the tying function. Below, we illustrate this remark in a couple of interesting environments:

Corollary 1. CORRELATED PRIVATE VALUES. *If the signals' marginal distributions remains the same but the joint distribution (and possibly correlation) of signals varies. In any monotone equilibrium, the tying function remains the same as in the independent private values environment.*

Corollary 2. COMMON-VALUES. *In any monotone equilibrium, the tying function is the identity.*[10]

Note that with common values, we have $Q(0) = 0$. Without re-scaling and with common values, the tying function is $Q(x) = F_2^{-1}(F_1(x))$. In the statistical literature, this Q is also known as quantile-quantile plot, or simply $Q-Q$ plot.

To establish existence (or not) of a monotone equilibrium, we define for player 1:
(LOCAL SINGLE CROSSING) For all x, $\lambda_1(\cdot, Q(x))$ is non-decreasing in neighborhood of x.
(INCREASING)[11] For all $x, z \in [0,1]$, $\int_{Q(z)}^{Q(x)} (\lambda_1(x,y) - \lambda_1(Q^{-1}(y), y))\, dy \geq 0$.
(SINGLE CROSSING) For all $\hat{x} < x < \tilde{x}$: $\lambda_1(\hat{x}, Q(x)) < \lambda_1(x, Q(x)) < \lambda_1(\tilde{x}, Q(x))$. And similarly define the analogous conditions for player 2.[12] We say a condition holds iff it holds for both players.

Proposition 2. LOCAL SINGLE-CROSSING *is necessary,* SINGLE-CROSSING *is sufficient and* INCREASING *is necessary and sufficient for* $b_1(x) = \int_0^x \lambda_2(z, Q(z))dz$ *and*
$b_2(y) = b_1(Q^{-1}(y))$ *be an increasing equilibrium.*

Proof. Recall that marginals are uniformly distributed on $[0,1]$, i.e. $f_1(x) = f_2(y) = 1$. Thus, the function $\lambda_1(\cdot,\cdot)$ satisfies LOCAL SINGLE-CROSSING if and only if $v_1(x, \phi_2(b)) f_{1,2}(x, \phi_2(b))$ is non-decreasing in x at $x = \phi_1(b)$ where $f_{1,2}$ denotes the joint density function of (x, y). Differentiating the identity, $v_1(\phi_1(b), \phi_2(b)) f_{1,2}(\phi_1(b), \phi_2(b)) \phi_2'(b) - 1 = 0$, with respect to b, and assuming $\phi_1' > 0$, the local single-crossing at $x = \phi_1(b)$ is equivalent to the second-order condition for 1's optimal bid. Clearly, local single crossing is necessary.

The argument to establish SINGLE-CROSSING is sufficient is standard.[13]

[9] For $y \in [0, Q(0)]$, we set $Q^{-1}(y) \equiv 0$.

[10] Siegel (2014) obtains this result for the discrete signals case.

[11] Araujo et al. (2008) consider a version of INCREASING for the symmetric, independent signals case.

[12] For player 2, the respective conditions are: (LOCAL SINGLE CROSSING) For all y, $\lambda_2(Q^{-1}(y), \cdot)$ is non-decreasing in neighborhood of y. (INCREASING) For all $y, z \in [0,1]$, $\int_{Q^{-1}(z)}^{Q^{-1}(y)} (\lambda_2(x,y) - \lambda_2(x, Q(x)))\, dx \geq 0$. (SINGLE CROSSING) For all $\hat{y} < y < \tilde{y}$: $\lambda_2(Q^{-1}(y), \hat{y}) < \lambda_2(Q^{-1}(y), y) < \lambda_2(Q^{-1}(y), \tilde{y})$.

[13] See Krishna and Morgan (1997, p. 351), Lizzeri and Persico (2000, p. 104) or Athey (2001).

Let $U_1(b_1(\tilde{x})|x)$ denotes bidder 1's expected payoff when his type is x and he bids like type $\tilde{x}$. We have $\forall x, \tilde{x} \in [0,1]$,

$$\begin{aligned} U_1(b_1(\tilde{x})|x) &= \int_0^{Q(\tilde{x})} \lambda_1(x,y)dy - b_2(Q(\tilde{x})) = \int_0^{Q(\tilde{x})} \lambda_1(x,y)dy - \int_{Q(0)}^{Q(\tilde{x})} \lambda_1(Q^{-1}(y),y)dy \\ &= \int_0^{Q(0)} \lambda_1(x,y)dy + \int_{Q(0)}^{Q(\tilde{x})} \left[\lambda_1(x,y) - \lambda_1(Q^{-1}(y),y)\right] dy. \end{aligned}$$

By SINGLE-CROSSING, $y < Q(x)$ implies $\lambda_1(x,y) - \lambda_1(Q^{-1}(y),y) > 0$ and conversely, $y > Q(x)$ implies $\lambda_1(x,y) - \lambda_1(Q^{-1}(y),y) < 0$. Therefore, $\tilde{x} = x$ maximizes $U_1(b_1(\tilde{x})|x)$.

Let $U_2(b_2(\tilde{y})|y)$ denotes bidder 2's expected payoff when his type is y and he bids like type $\tilde{y}$. Note when $\tilde{y} \in [0, Q(0)]$, for all y, we have $U_2(b_2(\tilde{y})|y) = U_2(0|y) = 0$. Now, for all $y \in [0,1]$ and $\tilde{y} \in (Q(0),1]$, we have:

$$\begin{aligned} U_2(b_2(\tilde{y})|y) &= \int_0^{Q^{-1}(\tilde{y})} \lambda_2(x,y)dx - b_1(Q^{-1}(\tilde{y})) = \int_0^{Q^{-1}(\tilde{y})} \lambda_2(x,y)dx - \int_0^{Q^{-1}(\tilde{y})} \lambda_2(x,Q(x))dx \\ &= \int_0^{Q^{-1}(\tilde{y})} \left[\lambda_2(x,y) - \lambda_2(x,Q(x))\right] dx. \end{aligned}$$

By SINGLE-CROSSING, when $x < Q^{-1}(y)$, we have $\lambda_2(x,y) - \lambda_2(x,Q(x)) > 0$. In the same manner, when $x > Q^{-1}(y)$, we have $\lambda_2(x,y) - \lambda_2(x,Q(x)) < 0$. Therefore, $\tilde{y} = y$ maximizes $U_2(b_2(\tilde{y})|y)$.

Finally to establish INCREASING is necessary and sufficient, we write the difference in bidder 1's payoffs $U_1(b_1(\tilde{x})|x) - U_1(b_1(x)|x) = \int_{Q(x)}^{Q(\tilde{x})} \left[\lambda_1(x,y) - \lambda_1(Q^{-1}(y),y)\right] dy$. The incentive compatible condition, for all x and $\tilde{x}$, $U_1(b_1(\tilde{x})|x) - U_1(b_1(x)|x) \leq 0$, is equivalent to INCREASING. Moreover, as $U_1(0|x) \geq 0$, the individual rational constrained is satisfied. In the same manner, for bidder 2, $U_2(b_2(\tilde{y})|y) - U_2(b_2(y)|y) = \int_{Q^{-1}(y)}^{Q^{-1}(\tilde{y})} \left[\lambda_2(x,y) - \lambda_2(x,Q(x))\right] dx$. As before, the incentive compatible condition, for all y and $\tilde{y}$, $U_2(b_2(\tilde{y})|y) - U_2(b_2(y)|y) \leq 0$, is equivalent to INCREASING. Lastly, $U_2(0|y) = 0$ implies individual rationality is also satisfied. ∎

Often[14] the following monotonicity assumption, or one of its variants, is used:

(M) $\lambda_i(x,y)$ is increasing in x for all y. We have M $\Rightarrow$ SINGLE CROSSING $\Rightarrow$ INCREASING $\Rightarrow$ LOCAL SINGLE CROSSING.

In section C of the Appendix, we characterize explicitly (in the context of quadratic valuations) the parameter space regions where each of the conditions in proposition 2 holds.

4. Non-Existence of Monotone Equilibrium

When signals are correlated, the all-pay auction may lack monotone equilibria. Even for "small doses" of correlation, provided the support of signals is sufficiently large.

[14] See Krishna and Morgan (1997), Lizzeri and Persico (2000), Araujo et al. (2008) or Siegel (2014)

Example 1. (CORRELATED PRIVATE VALUES) The signals (X_1, X_2) follow a truncated, symmetric, bivariate normal distribution specified by (μ, σ^2, ρ) and truncation points $\mu - M$ and $\mu + M$. Valuations are $v_i(x) = \exp(h(x_i))$ for $i = 1, 2$ where h is a given increasing function.

Proposition 3. *If $h'(x) < \frac{\rho(\mu - x)}{\sigma^2(1+\rho)}$ for some x, there is no increasing equilibrium for example 1.*

Proof. As players are symmetric, by Proposition 1, if a monotone, pure strategy equilibrium exists then it must be symmetric. However, using the fact that $X_j|X_i \sim \mathcal{N}\left((1-\rho)\mu + \rho X_i, (1-\rho^2)\sigma^2\right)$ we obtain $\left.\frac{\partial}{\partial x} v_i(x, y) \cdot f_{X_j|X_i}(y|x)\right|_{y=x} \geq 0 \Leftrightarrow h'(x) \geq \frac{\rho}{\sigma^2(1+\rho)}(\mu - x)$. ∎

As a result, the symmetric monotone equilibrium is not robust to the introduction of a small degree of correlation for a family of examples:

Corollary 3. *Assume $\|h'\|_\infty < K$ then for any $\rho > 0$ there is $M_0 > 0$ such that the private values model of example 1 has no monotone equilibrium for $M > M_0$.*

For common-values we have similar non-existence problems. Consider a setting as in Matthews (1984): signals and the value are affiliated; signals are unbiased estimates of the value; conditional on the value, signals are independent; the parameter θ measures the precision of players' information.

Example 2. (COMMON-VALUE) The all-pay auction has no monotone equilibrium if:

(1) The value is lognormal , $V \sim \ln \mathcal{N}(\mu, \tau^{-1})$, and conditional on the value, signals are the value plus some additive (normal) noise, $S_i|V \overset{iid}{\sim} \mathcal{N}(V, \theta^{-1})$; or
(2) The value follows the Pareto distribution , $V \sim \texttt{Pareto}(\omega, \alpha)$, and conditional signals are the value with a multiplicative (beta distributed) noise, $S_i|V \overset{iid}{\sim} V \cdot \mathrm{B}(\theta, 1) \cdot \frac{\theta+1}{\theta}$; or
(3) The value follows an inverse gamma distribution, $V \sim \mathrm{Inv}{-}\Gamma(\alpha, \beta)$; and conditional signals follow an exponential law, $S_i|V \overset{iid}{\sim} \mathrm{Exp}(\theta\, V^{-1}) \cdot \theta^{-1}$.

See Appendix A for a proof of Example 2.

5. NON-MONOTONE EQUILIBRIA

Consider a pure strategy equilibrium profile in which every bid strategy, $b_i(\cdot)$ with $i = 1, 2$, is piecewise monotone, that is, $b_i'(\cdot)$ (which exists almost everywhere in this case) changes sign a finite number of times[15]. We partition player 1's type space into finite intervals $[0, 1] = \bigcup_{k=1}^{n_1} T_k^1$ such these intervals are maximal with respect the property each restriction $b_1|_{T_k^1}(\cdot)$ is monotone. For exposition purposes, let's focus on the case where $b_1(\cdot)$ is increasing in odd intervals and decreasing otherwise. The cases where one or both of the $b_1(\cdot)$ is increasing in even intervals are analogous.

Now define the k_{th} local inverse bid function of player 1: $\phi_k^1 : b_1^{-1}(T_k^1) \to T_k^1$. For player 2, we also define everything in an analogous manner so we can express the payoff of a type x of player i

[15]This is related to the 'limited complexity strategies' of Athey's 1997 working-paper version of Athey (2001)

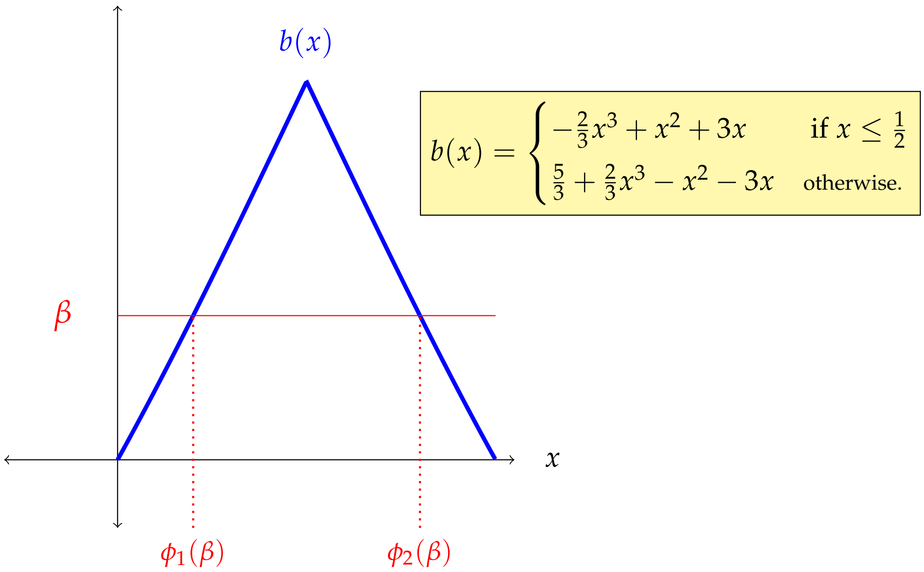

FIG. 1. A piecewise monotone strategy and its local inverse bids for the symmetric model with $v(x,y) = 3 - x - y(3 - 4x + 2x^2)$

who bids b as:[16]

$$U_1(b|x) = \left(\sum_{k=0}^{[\frac{n_2}{2}]} \int_{\phi^2_{2k}(b)}^{\phi^2_{2k+1}(b)} \lambda_1(x,y)dy \right) - b,$$

where $[\frac{n_2}{2}]$ denotes the largest integer that is smaller or equal to $\frac{n_2}{2}$. By convention, $\phi^2_0(b) = 0$, and for even n_2, $\phi^2_{n_2+1}(b) = 1$.

The main idea to solve for non-monotonic equilibria is to construct (and solve for) the *pooling functions*, which are similar to the tying function we saw on section 3. For expositional purposes[17], let's assume all local inverse bids are well-defined at b. For $x \in \phi^i_1(\mathbb{R}_+)$, we define $P^i_k(x) \stackrel{\text{def}}{=} \phi_k\Big(b_i(x)\Big)$, that is P^i_k maps the smallest type of player i to the k_{th} type (in an increasing order) that bids the same as type x of player i. Unlike the tying function, a pooling function maps types of the player to other types of the same player.

To make it more concrete, let's restrict attention to the symmetric case ($\lambda_1(x,y) = \lambda_2(y,x)$) and also assume that (at most) only two types are pooling in a symmetric equilibrium (so we write $P = P^1 = P^2$). Under certain regularity conditions, the pooling function solves the differential equation[18]:

$$P'(x) = \frac{\lambda_2(x,x) - \lambda_2(x,P(x))}{\lambda_2(P(x),x) - \lambda_2(P(x),P(x))}.$$

Once we solve for P we can reduce the first-order conditions to a single ODE. In general, it is hard to obtain a closed form solution for P. Moreover, the boundary conditions are not obvious, unlike in the monotone case, because now, one needs to characterize the region where pooling occurs. Despite the hurdles, in the next subsection we show how to compute the non-monotone equilibrium for the case of quadratic valuations.

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[16]For notational convenience, we assume b falls in the right range so the inverse biddign functions are well defined.

[17]See Appendix B for a more detailed and formal exposition.

[18]See Appendix B, proof of corollary 4.

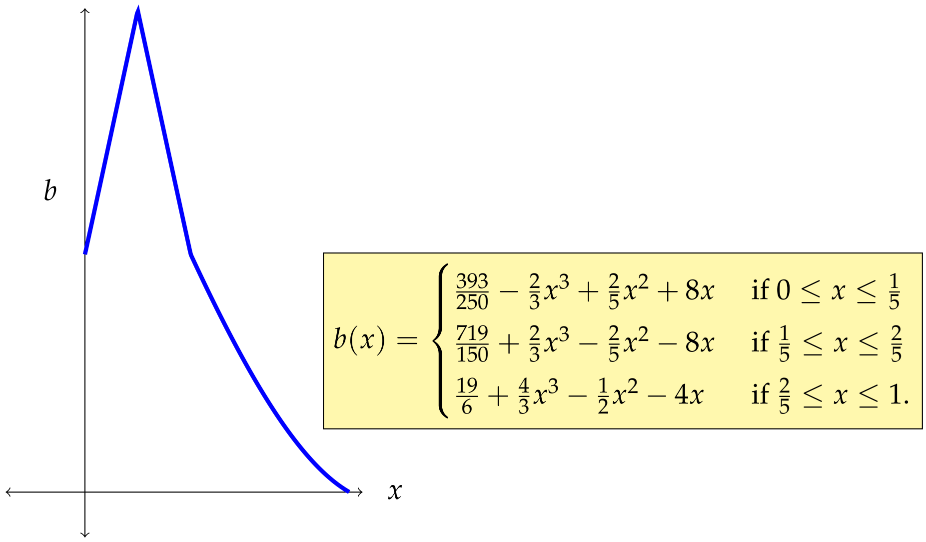


FIG. 2. Case $A = -1, C = -3, D = 1, F = 4$.

5.1. **Quadratic Preferences.** In this section we construct non-monotone equilibrium displaying mirror-symmetry. That is, the pooling function is a translation composed with a reflection: $P(x) = c - x$ for some suitable constant c. We assume:

QUADRATIC MODEL Players are symmetric, wlog. signals are independent and uniformly distributed in $[0, 1]$. Player i's value $v(x, y) = Ax^2 + By^2 + Cxy + Dx + Ey + F$, where x is the signal of player i and y is the signal of $-i$. We assume F is sufficient large so that[19], $v(x, y) > 0, \forall (x, y) \in [0, 1]^2$.

Notice we departed from our previous convention: hereafter if signals are (x, y), player 1's value is $v(x, y)$ and player 2's value is $v(y, x)$. Although the model allows for common-values, it does not assume it.

Some preliminaries and notation: Let $c \stackrel{\text{def}}{=} \frac{-2D}{2A+C}$ and notice it is the unique solution of:

$$v(x, x) + v(x, c - x) = v(c - x, x) + v(c - x, c - x).$$

We also need the auxiliary function, $\hat{\lambda}(x, y) \stackrel{\text{def}}{=} v(x, y) + v(x, c - y)$.

Remark 3. The function $\hat{\lambda}(x, y)$ satisfies $\hat{\lambda}(x, y) = \hat{\lambda}(c - x, y)$ for all x and y.

Remark 4. The derivative $\hat{\lambda}_x(x, y)$ is linear in x with $\hat{\lambda}_x(c/2, y) = 0$ for all y, and $\hat{\lambda}_{xx} = 4A$.

In the following propositions, *the equilibrium* refers to the symmetric equilibrium[20] bidding function where types $c/2 - t$ and $c/2 + t$ pool that is, $b(c/2 - t) = b(c/2 + t)$ for $t \in (0, \bar{t}]$, for some $\bar{t} > 0$.

[19] Notice this is assumption FULL SUPPORT & POSITIVE VALUATIONS.

[20] We conjecture the equilibrium is unique but we dot not claim uniqueness.

Proposition 4. *If $A, C < 0 < 2D < -2A - C$, the equilibrium is bell-shaped for types in $[0, c]$ and decreasing for types in $[c, 1]$. Moreover, it satisfies $b(0) = \int_c^1 v(x, x)dx$ and,*

$$b(x) = \begin{cases} b(0) + \int_0^x \hat{\lambda}(y, y)dy & \textit{if } 0 \le x \le c/2, \\ b(0) + \int_0^{c-x} \hat{\lambda}(y, y)dy & \textit{if } c/2 \le x \le c, \textit{ and} \\ \int_x^1 v(y, y)dy & \textit{if } c \le x \le 1. \end{cases}.$$

Proof. Since $\hat{\lambda}(x, y) > 0$, $b(\cdot)$ given above is increasing in $[0, c/2]$ and decreasing in $[0, 1]$. Let's verify that $b(\cdot)$ is indeed a NPSE by a direct mechanism approach. Let $U(z|x)$ be the payoff of a player with type x who bids as if his type were z. Using the expression of $b(\cdot)$, we have

$$U(z|x) = \begin{cases} U(c|x) + \int_0^z \left(\hat{\lambda}(x, y) - \hat{\lambda}(y, y)\right) dy & \text{if } 0 \le z \le c/2, \\ U(c - z|x) & \text{if } c/2 < z < c, \text{ and} \\ \int_z^1 \left(v(x, y) - v(y, y)\right) dy & \text{if } c \le z \le 1. \end{cases}$$

We want to show that $x = \underset{z}{\text{argmax}}\, U(z|x)$ for all x. Of course, the first-order condition is always satisfied since $U_z(x|x) = 0$. Moreover, as the direct mechanism is monotone and given the first-order condition holds, the single crossing condition, $U_{xz}(z|x) > 0$ for all x, is sufficient to guarantee incentive compatibility (*i.e.* no type wants to deviate to z). Unfortunately $U_{xz}(z|x) > 0$ does not always hold. Nonetheless, we still can use single-crossing arguments in many cases, as we show below.

(1) Claim: Types x in $[0, c]$ do not want to deviate to z in $[0, c]$. To prove the claim notice that,

$$U_{xz}(z|x) = \begin{cases} \hat{\lambda}_x(x, z) & \text{if } 0 \le z < c/2, \\ -\hat{\lambda}_x(x, c - z) & \text{if } c/2 < z < c, \text{ and} \\ -v_x(x, z) & \text{if } c < z \le 1. \end{cases}$$

and due to remark 4 and the fact $A < 0$, we also have:

$$\hat{x} < \frac{c}{2} < \tilde{x} \Rightarrow \hat{\lambda}_x(\hat{x}, z) > \hat{\lambda}_x(c/2, z) = 0 > \hat{\lambda}_x(\tilde{x}, z) \text{ for all } z,$$

which proves $U_{xz}(z|x) > 0$ for $x, z \in [0, c/2)$. As a result, no type in $[0, c/2)$ has an incentive to deviate to $z \in [0, c/2]$: given the first-order condition, $U_z(z|x) = 0$ for $z = x$, if some type x where to deviate to z, the deviation would be not optimal since either $U_z(z|x) > 0$ if $z < x$ or, $U_z(z|x) < 0$ if $z > x$.

Moreover, since $U(z|x) = U(c - z|x)$ for $z \in [c/2, c]$, it also follows that no type in $[0, c]$ has an incentive to deviate to $z \in [0, c]$.

(2) Claim: No type $x \in [0, c)$ wants to deviate to z in $[c, 1]$. Note that $U(x|x) \ge U(z|x), \forall x \in [0, c), \forall z \in [0, c]$. To establish the claim, we only need to show that $U_z(z|x) < 0$ for all $z \ge c > x$. And for $z \ge c > x$, we have $U_z(z|x) = v(z, z) - v(x, z) = (z - x)[A(z + x) + Cz + D]$
$< (z - x)[(A + C)c + D] = (z - x)[(A + C)\frac{2D}{|2A+C|} + D] < (z - x)[(A + C)\frac{D}{|A+C|} + D] = 0.$

(3) Claim: No type $x \in [c, 1]$ wants to deviate to z in $[c, 1]$. To prove this claim, we shall show the single crossing holds in the monotonic region, $U_{xz}(z|x) > 0$ for all $z \in [c, 1]$, thus no type has an

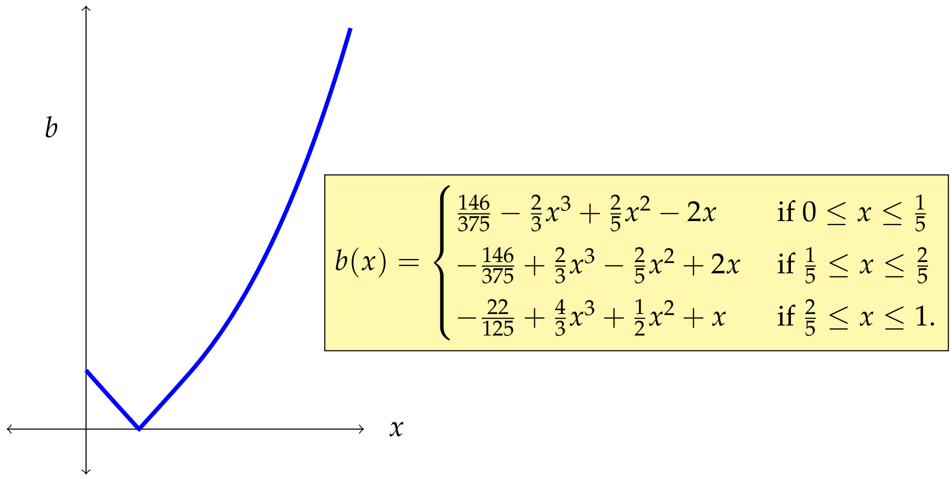


FIG. 3. Case $A = 1, C = 3, D = -1, F = 1$.

incentive to deviate to $z \in [c, 1]$:

$$-v_x(x,z) = -2Ax - D - Cz \ge -(2A + C)c - D = D > 0.$$

As a result, no type wants to deviate to $z \in [c, 1]$.

(4) Claim: Types x in $[c, 1]$ do not want to deviate to $z \in [0, c/2]$. To prove the claim, it suffices to show that $U_z(z|x) < 0$, for all z in $[0, c/2)$ since $U(x|x) \ge U(c|x) = U(0|x)$. Note that $U_z(z|x) = \hat{\lambda}(x,z) - \hat{\lambda}(z,z) = (x-z)[2A(x+z) + C \cdot c + 2D] \le (x-z)[2A \cdot c + C \cdot c + 2D] = 0$. It follows types x in $[c, 1]$ also do not want to deviate to $z \in [c/2, c]$ as $U(z|x) = U(c-z|x)$ for $z \in [c/2, c]$.

So far we proved the direct mechanism induced by the bid functions of the proposition is incentive compatible. It is also individual rational since $U(1|x) = 0$ for all x. ■

Proposition 5. *If $-2A - C < 2D < 0 < A, C$ the equilibrium is U-shaped for types in $[0, c]$ and increasing for types in $[c, 1]$. Moreover, it satisfies:*

$$b(x) = \begin{cases} \int_x^{c/2} \hat{\lambda}(z,z)dz & \textit{if } 0 \le x \le c/2, \\ \int_{c/2}^x \hat{\lambda}(z,z)dz & \textit{if } c/2 \le x \le c, \textit{ and} \\ b(0) + \int_c^x v(z,z)dz & \textit{if } c \le x \le 1. \end{cases}.$$

Proof. The proof is analogous to the previous one. Again as $\hat{\lambda}(\cdot) > 0$ and $v(\cdot) > 0$, the proposed bid function is decreasing in $[0, c/2]$ and increasing in $[c/2, 1]$. Let $U(z|x)$ be the payoff of a player with type x who bids as if his type were z. Using the expression of $b(\cdot)$, we have

$$U(z|x) = \begin{cases} \int_z^{c-z} v(x,y)dy - \int_z^{c/2} \hat{\lambda}(y,y)dy & \text{if } 0 \le z < c/2, \\ \int_{c-z}^{z} v(x,y)dy - \int_{c-z}^{c/2} \hat{\lambda}(y,y)dy & \text{if } c/2 < z < c, \text{ and} \\ \int_0^z v(x,y)dy - [b(0) + \int_c^z v(y,y)dy] & \text{if } c < z \le 1. \end{cases}$$

We want to show that $x = \operatorname*{argmax}_z U(z|x)$ for all x. Note

$$U_z(z|x) = \begin{cases} \hat{\lambda}(z,z) - \hat{\lambda}(x,z) & \text{if } 0 \leq z < c/2, \\ \hat{\lambda}(x,z) - \hat{\lambda}(c-z,c-z) & \text{if } c/2 < z < c, \text{ and} \\ v(x,z) - v(z,z) & \text{if } c < z \leq 1. \end{cases}$$

Once more, the first-order condition is always satisfied since $U_z(x|x) = 0$.

Now let's consider,

$$U_{xz}(z|x) = \begin{cases} -\lambda_x(x,z) & \text{if } 0 \leq z < c/2, \\ \lambda_x(x,z) & \text{if } c/2 < z < c, \text{ and} \\ v_x(x,z) & \text{if } c < z \leq 1. \end{cases}$$

By remark 4, and the fact $A < 0$, we have:

$$\hat{x} < \frac{c}{2} < \tilde{x} \Rightarrow \hat{\lambda}_x(\hat{x},z) < \hat{\lambda}_x(c/2,z) = 0 < \hat{\lambda}_x(\tilde{x},z) \text{ for all } z,$$

which proves $U_{xz}(z|x) > 0$ for $x, z \in [0, c/2)$.

(1) Claim: Types x in $[0,c]$ do not want to deviate to z in $[0,c]$. Consider a type in $[0,c/2)$. Based on similar arguments as in the proof of Proposition 4, these types do not want to deviate in this interval. Again, as $U(z|x) = U(c-z|x)$ for any $z \in [0,c/2]$ and all x, we conclude types in $[0,c]$ do not want to deviate to $[0,c]$

(2) Claim: No type $x \in [0,c)$ wants to deviate to z in $[c,1]$. To establish the claim, we only need to show that $U_z(z|x) < 0$ if $z \geq c > x$. For $z \geq c$, we have $U_z(z|x) = v(x,z) - v(z,z) = -(z-x)[A(z+x) + Cz + D] < -(z-x)[(A+C)c + D] = -(z-x)[(A+C)\frac{2|D|}{2A+C} + D]$
$< (z-x)[(A+C)\frac{|D|}{A+C} + D] = 0$.

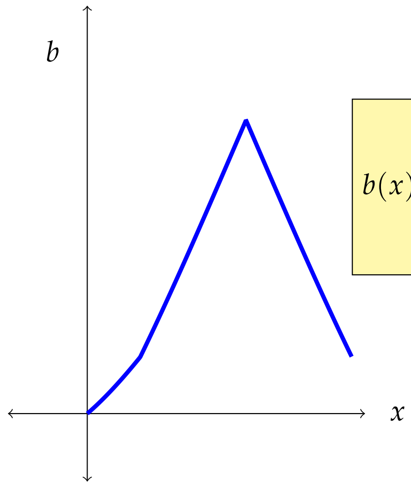
(3) Claim: No type $x \in [c,1]$ wants to deviate to z in $[c,1]$. To prove this claim, we shall show the single crossing holds in the monotonic region, $U_{xz}(z|x) > 0$ for all $z \in [c,1]$, thus no type has an incentive to deviate to $z \in [c,1]$:

$$v_x(x,z) = 2Ax + D + Cz \geq (2A+C)c + D = |D| > 0.$$

As a result, no type wants to deviate to $z \in [c,1]$.

(4) Claim: Types x in $[c,1]$ do not want to deviate to $z \in [0,c/2]$. To prove the claim, it suffices to show that $U_z(z|x) < 0$, for all $z \in [0,c/2)$ and $x \in [c,1]$, since $U(x|x) \geq U(c|x) = U(0|x)$. Note that $U_z(z|x) = \hat{\lambda}(z,z) - \hat{\lambda}(x,z) = -(x-z)[2A(x+z) + C \cdot c + 2D] > -(x-z)[2A \cdot c + C \cdot c + 2D] = 0$. It follows that types x in $[c,1]$ do not want to deviate to $z \in [c/2,c]$ since $U(z|x) = U(c-z|x)$ for $z \in [c/2,c]$.

Individual rationality always holds since $U(x|x) \geq U(c/2|x) = 0$. ■



$$b(x) = \begin{cases} \frac{146}{375} - \frac{2}{3}x^3 + \frac{2}{5}x^2 - 2x & \text{if } 0 \le x \le \frac{1}{5} \\ -\frac{146}{375} + \frac{2}{3}x^3 - \frac{2}{5}x^2 + 2x & \text{if } \frac{1}{5} \le x \le \frac{3}{5} \\ -\frac{22}{125} + \frac{4}{3}x^3 + \frac{1}{2}x^2 + x & \text{if } \frac{3}{5} \le x \le 1. \end{cases}$$

FIG. 4. Case $A = -1, C = -3, D = 3, F = 1$.

Proposition 6. *If $A, C < 0 < -2A - C < 2D < -4A - 2C$, the equilibrium is increasing for types in $[0, c-1]$ and bell shaped for types in $[c-1, 1]$. Moreover,*

$$b(x) = \begin{cases} \int_0^x v(y,y)dy & \textit{if } 0 \le x \le c-1, \\ b(c-1) + \int_{c-1}^x \hat{\lambda}(y,y)dy & \textit{if } c-1 < x \le c/2, \textit{ and} \\ b(c-1) + \int_x^1 \hat{\lambda}(y,y)dy & \textit{if } c/2 \le x \le 1. \end{cases}.$$

Proof. The proof is similar to the previous cases. In the direct mechanism induced by $b(\cdot)$, we have:

z range	$U(z\|x)$	$U_z(z\|x)$	$U_{xz}(z\|x)$
$[0, c-1)$	$\int_0^z (v(x,y) - v(y,y))\, dy$	$v(x,z) - v(z,z)$	$v_x(x,z)$
$(c-1, c/2)$	$U(c-1\|x) + \int_{c-1}^z (\hat{\lambda}(x,y) - \hat{\lambda}(y,y))\, dy$	$\hat{\lambda}(x,z) - \hat{\lambda}(z,z)$	$\hat{\lambda}_x(x,z)$
$(c/2, 1]$	$U(c-z\|x)$	$\hat{\lambda}(c-z, c-z) - \hat{\lambda}(x, c-z)$	$-\hat{\lambda}_x(x,z)$.

(1)

Claim: Types x in $[0, c-1]$ do not want to deviate to z in $[0, c-1]$. We only need to show that $U_{xz} = v_x(x,z) > 0, \forall x, z \in [0, c-1]$. Note that $v_x(x,z) = 2Ax + D + Cz \ge (2A + C)(c-1) + D = -(2A + C) - D > 0$.

(2) Claim: Types x in $[0, c-1]$ do not want to deviate to z in $[c/2, 1]$. Note that we have $U(x|x) \ge U(c-1|x) = U(1|x)$. To establish the claim, it suffices to show that $U_z(z|x) \ge 0$ for x in $[0, c-1]$ and z in $[c/2, 1]$. $U_z(z|x) = \hat{\lambda}(x,z) - \hat{\lambda}(z,z) = (x-z)[2A(x+z) + C \cdot c + 2D] > (x-z)[2A \cdot c + C \cdot c + 2D] = 0$. It further follows that types x in $[0, c-1]$ do not want to deviate to z in $[c-1, c/2]$ as $U(z|x) = U(c-z|x)$ for z in $[c-1, c/2]$.

(3) Claim: Types x in $[c-1, c/2]$ do not want to deviate to z in $[c-1, 1]$. It suffices to show types x in $[c-1, c/2]$ do not want to deviate to z in $[c-1, c/2]$. For this purpose, we only need to show that $U_{xz} \ge 0$ for $x, z \in [c-1, c/2]$.

Since $A < 0$, by remark 4 $\hat{\lambda}_x(x, \cdot) > 0$ if $x \in [c-1, c/2)$. Thus no types x in $[0, c/2]$ have a profitable deviation in $[c-1, c/2]$. Again, as $U(z|x) = U(c-z|x)$ for $z \in [c-1, c/2]$, we can conclude that, types in $[c-1, 1]$ do not have profitable deviations in $[c-1, 1]$.

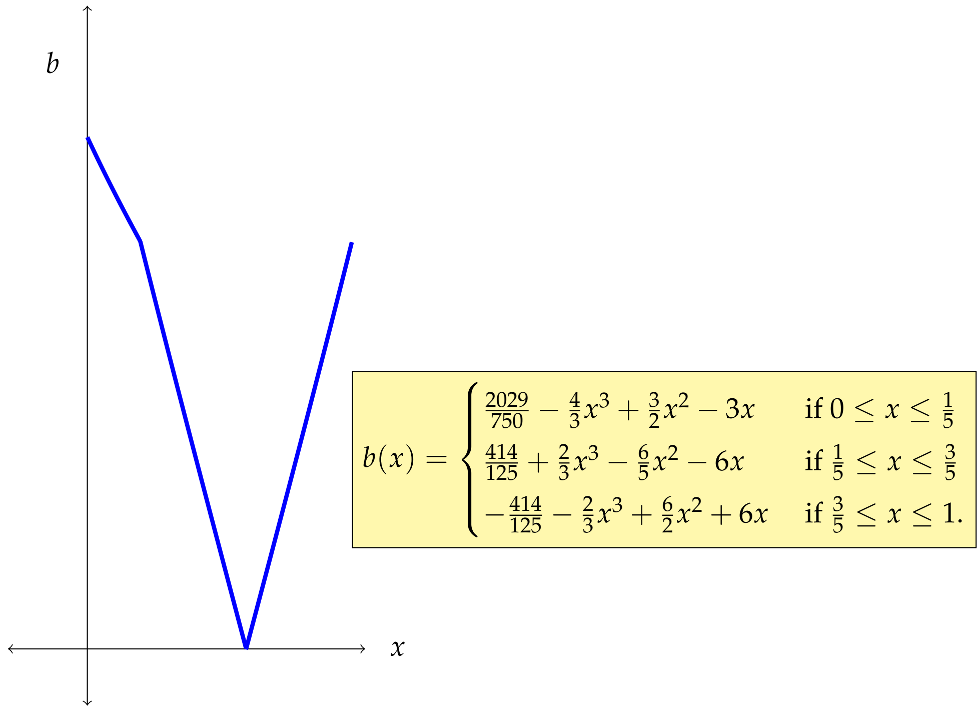


FIG. 5. Case $A = 1, C = 3, D = -3, F = 1$.

(4) Claim: Types x in $[c-1, c/2]$ do not want to deviate to z in $[0, c-1]$. Note $U(x|x) \geq U(c-1|x)$. It suffices to show $U_z(z|x) > 0$ for x in $[c-1, c/2]$ and z in $[0, c-1)$. $U_z(z|x) = v(x,z) - v(z,z) = (x-z)[A(z+x) + Cz + D] \geq (x-z)[A(c-1+c/2) + C(c-1) + D]$
$> (x-z)[A(c/2+c/2) + C(c/2) + D] = 0$. For the same reason, types x in $[c/2, 1]$ do not want to deviate to z in $[0, c-1]$. ∎

Proof. We need,

$$U_{xz}(z|x) = \begin{cases} -v_x(x,z) & \text{if } 0 \leq z \leq c-1, \\ -\hat{\lambda}_x(x,z) & \text{if } c-1 \leq z \leq c/2, \\ \hat{\lambda}_x(x,z) & \text{if } c/2 \leq z \leq 1, \end{cases}$$

Once more, we require the single-crossing, $U_{xz} > 0$, to hold for $z \in [0, c-1]$ and all x. Thus, $-v_x(x,z) = -2Ax - D - Cz \geq -2A - D + Cz \geq -2A - D - \max(0, C(c-1)) \geq -2A - D - \max(0, -C\frac{2D+2A+C}{2A+C}) > 0$. As before, the inequalities follow directly from the proposition's assumptions.

Since $A > 0$, by remark 4, $\hat{\lambda}_x(x, \cdot) < \hat{\lambda}_x(c/2, \cdot) = 0$ if $x \in [c-1, c/2)$. Thus, types x in $[0, c/2]$ do not have profitable deviations in $[c-1, c/2]$. Again, as $U(z|x) = U(c-z|x)$ for $z \in [c-1, c/2]$, we can conclude that, types in $[c-1, 1]$ do not have profitable deviations in $[c-1, 1]$. ∎

Proposition 6's dual is:

Proposition 7. *If $-4A - 2C < 2D < -2A - C < 0 < A, C$ the equilibrium is decreasing for types in $[0, c-1]$ and U-shaped for types in $[c-1, 1]$. Moreover,*

$$b(x) = \begin{cases} \int_x^{c-1} v(y,y)dy + \int_{c-1}^{c/2} \hat{\lambda}(y,y)dy & \text{if } 0 \le x \le c-1, \\ \int_x^{c/2} \hat{\lambda}(y,y)dy & \text{if } c-1 \le x \le c/2, \text{ and } \\ \int_{\frac{c}{2}}^{x} \hat{\lambda}(y,y)dy & \text{if } c/2 \le x \le 1. \end{cases}.$$

Proof. The proof is similar to the previous cases,

$$U(z|x) = \begin{cases} \int_z^1 v(x,y)dy - [\int_z^{c-1} v(y,y)dy + \int_{c-1}^{c/2} \hat{\lambda}(y,y)dy] & \text{if } 0 \le z \le c-1, \\ \int_z^{c-z} v(x,y)dy - \int_z^{c/2} \hat{\lambda}(y,y)dy & \text{if } c-1 \le z \le c/2, \\ U(c-z|x) & \text{if } c/2 \le z \le 1, \end{cases}$$

and so,

z range	$U_z(z\|x)$	$U_{xz}(z\|x)$
$[0, c-1)$	$-v(x,z) + v(z,z)$	$-v_x(x,z)$
$(c-1, c/2)$	$-\hat{\lambda}(x,z) + \hat{\lambda}(z,z)$	$-\hat{\lambda}_x(x,z)$
$(c/2, 1]$	$\hat{\lambda}(x, c-z) - \hat{\lambda}(c-z, c-z)$	$\hat{\lambda}_x(x,z)$

(1) Claim: Types x in $[0, c-1]$ do not want to deviate to z in $[0, c-1]$. We only need to show that $U_{xz} = -v_x(x,z) > 0, \forall x, z \in [0, c-1]$. Note that $-v_x(x,z) = -[2Ax + D + Cz] \geq -[(2A+C)(c-1)] - D = -(2A-C)c + (2A+C) - D = (2A+C) + D > 0$.

(2) Claim: Types x in $[0, c-1]$ do not want to deviate to z in $[c-1, c/2]$. Note that we have $U(z|x) \geq U(c-1|x)$. To establish the claim, it suffices to show that $U_z(z|x) \leq 0$ for x in $[0, c-1]$ and z in $[c-1, c/2]$. $U_z(z|x) = -\hat{\lambda}(x,z) + \hat{\lambda}(z,z) = (z-x)[2A(x+z) + C \cdot c + 2D] < (z-x)[2A \cdot c + C \cdot c + 2D] = 0$. It further follows that types x in $[0, c-1]$ do not want to deviate to z in $[c/2, 1]$ as $U(z|x) = U(c-z|x)$ for z in $[c-1, c/2]$.

(3) Claim: Types x in $[c/2, 1]$ do not want to deviate to z in $[c/2, 1]$. Since $A > 0$, by remark 4, $\hat{\lambda}_x(x, \cdot) > \hat{\lambda}_x(c/2, \cdot) = 0$ if $x \in (c/2, 1]$, which proves $U_{xz}(z|x) > 0$ for $x, z \in (c/2, 1]$. Thus, types x in $[c/2, 1]$ do not have profitable deviations in $[c/2, 1]$. Again, as $U(z|x) = U(c-z|x)$ for $z \in [c-1, c/2]$, we conclude types in $[c-1, 1]$ do not have profitable deviations in $[c-1, 1]$.

(4) Claim: Types x in $[c-1, c/2]$ do not want to deviate to z in $[0, c-1]$. Note $U(x|x) \geq U(c-1|x)$ for $x \in [c-1, c/2]$. To establish the claim, we show that $U_z(z|x) > 0$ for x in $[c-1, c/2]$ and z in $[0, c-1)$: $U_z(z|x) = -v(x,z) + v(z,z) = -(x-z)[A(z+x) + Cz + D] \geq -(x-z)[A(c-1+c/2) + C(c-1) + D] >$
$> -(x-z)[A(c/2 + c/2) + C(c/2) + D] = 0$. It further follows that types x in $[c/2, 1]$ do not want to deviate to z in $[0, c-1]$. ■

6. Concluding Remarks

We characterized monotone equilibrium of all-pay auctions in the continuous signals case, allowing for correlation and interdependent values.

We showed the monotone equilibrium may fail to be robust to arbitrarily small degrees of correlation. Motivated by that finding, we explicitly obtain non-monotone equilibrium for parametric families (*i.e.* quadratic valuations). For simplicity we considered the symmetric case, but it is possible to extend the results for the asymmetric case. We can also extend our results to allow for polynomial valuations, however, unlikely the quadratic case, the parameter space where mirror-symmetry non-monotone equilibria exists is not open.

The existence of pure strategy equilibria, however, remains an open question. Govindan and Wilson (2010) prove existence when players use behavioral strategies (*i.e.* conditional on the signal they choose a distribution over bids).

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Appendix A. Common Values Examples

Here we prove the common values models of example 2 lack monotone equilibria. As before, it suffices to show that for some x the local-single crossing condition is violated. So we first compute the conditional expected values and the corresponding conditional densities:

V	$S_1 \mid V$	$\mathrm{E}[V \mid S_1 = x, S_2 = y]$	$f_{S_2 \mid S_1}(y \mid x)$
$\ln \mathcal{N}(\mu, \tau^{-1})$	$\mathcal{N}(V, \theta^{-1})$	$\exp\left(\frac{\tau\mu+\theta x+\theta y+\frac{1}{2}}{\tau+2\theta}\right)$	$\mathcal{N}\left(\frac{\tau\mu+\theta x}{\tau+\theta}, \frac{\tau+2\theta}{\theta(\tau+\theta)}\right)$
$\mathtt{Pareto}(\omega, \alpha)$	$V \cdot \mathrm{B}(\theta, 1) \cdot \frac{\theta+1}{\theta}$	$\frac{\theta+1}{\theta} \frac{\alpha+2\theta}{\alpha+2\theta-1} \max(\omega, x, y)$	$\frac{(\alpha+\theta)\theta}{\alpha+2\theta} \frac{\omega^{(\alpha+\theta)\mathbf{1}_{[x<\omega]}}\, x^{(\alpha+\theta-2)\mathbf{1}_{[x>\omega]}}\, y^{\theta-1}}{\max(\omega, x, y)^{\alpha+2\theta}}$
$\mathrm{Inv}{-}\Gamma(\alpha, \beta)$	$\mathrm{Exp}(\theta V^{-1}) \cdot \theta^{-1}$	$\frac{\Gamma(\alpha+2\theta-1)}{\Gamma(\alpha+2\theta)}(x+y+\beta)$	$\frac{\Gamma(\alpha+2\theta)}{\Gamma(\alpha+\theta)\Gamma(\theta)} \frac{y^{\theta-1}\,(x+\beta)^{\alpha+\theta}}{(x+y+\beta)^{\alpha+2\theta}}$

For the Lognormal/Normal model, $v_1(x,y) \cdot f_{X_2|X_1}(y|x) = \exp\left(\frac{\tau\mu+\theta x+\theta y+\frac{1}{2}}{\tau+2\theta} - \frac{\left(y-\frac{\tau\mu+\theta x}{\tau+\theta}\right)^2}{2\frac{\tau+2\theta}{\theta(\tau+\theta)}}\right)$ and so $\operatorname{sign} \frac{\partial}{\partial x} v_1(x,y) \cdot f_{X_2|X_1}(y|x)\Big|_{y=x} = \operatorname{sign}\left(-\mu\tau\theta + \tau\theta x + \tau + \theta\right)$, which is negative for $x < M_0$ for some M_0.

For the Pareto/Beta model, let's take $x > y > \omega$ so $v_1(x,y) \cdot f_{X_2|X_1}(y|x) = \frac{(\theta+1)(\alpha+2\theta)}{\alpha+2\theta-1} \cdot \frac{\alpha+\theta}{\alpha+2\theta} \cdot \frac{y^{\theta-1}}{x^{\theta+1}}$, which is clearly decreasing in x for all y.

For the Inverse Gamma/Exponential model, $v_1(x,y) \cdot f_{X_2|X_1}(y|x) = \frac{\Gamma(\alpha+2\theta-1)y^{\theta-1}}{\theta\Gamma(\alpha+\theta)\Gamma(\theta)} \cdot \frac{(x+\beta)^{\alpha+\theta}}{(x+y+\beta)^{\alpha+2\theta-1}}$, which is also decreasing in x when $\theta > 1$ and $x > M_0$ for some M_0.

Appendix B. Pooling Functions

The set of local inverse bids of player 1 that are well defined at b are $I(b) = \left\{k : b \in b_1(I_k^1)\right\}$. We similarly define $J(b)$ for player 2; and write $n_i(b) \stackrel{\text{def}}{=} \#I(b)$ and $n_j(b) \stackrel{\text{def}}{=} \#J(b)$ for the number of local inverse bids that are necessary to completely describe the system of first-order conditions at b. We also need to re-label indexes: $I(b) = \{i_1, i_2, \ldots, i_{n_1}\}$ and $J(b) = \{j_1, j_2, \ldots, j_{n_2}\}$ in order to describe the matrices: $\Lambda^1(b)$ and $\Lambda^2(b)$ with entries given by $\Lambda^1_{r,c}(b) = \lambda_1\left(\phi^1_{i_r}(b), \phi^2_{j_c}(b)\right)$ and $\Lambda^2_{r,c}(b) = \lambda_2\left(\phi^1_{i_c}(b), \phi^2_{j_r}(b)\right)$.

Definition 1. A piecewise-monotone equilibrium $\mathbf{b}$ is said *regular at b* if $n_1(b) = n_2(b) < +\infty$ is constant in a neighborhood[21] of b, and $\Lambda^1(b)$ and $\Lambda^2(b)$ are full-rank. In this case, we call b a *regular point* and say the equilibrium is regular if almost all b are regular.

Any monotone equilibrium is regular. Without regularity, there is little hope to pin-down the equilibrium using the first-order approach. Without regularity, the differential system corresponding to the FOCs is undetermined.

When only two types are pooling, the system is considerably simpler:

Lemma 2. *Consider a symmetric equilibrium that is regular at b, with $n_1(b) = n_2(b) = 2$. Its corresponding pooling functions satisfy:*

[21] In a regular equilibrium, $n_i(b)$ may vary with b but it can take at most a countable number of values.

$$\frac{\partial}{\partial y} P_k^2(x) = (-1)^{j_k - j_1} \frac{|L_k^1(x)|}{|L_1^1(x)|},$$

where the $(r,c)-$ *entry of the matrix* L_k^1 *is* $\lambda_1(P_{i_r}^1(x), P_{j_c}^2(y))$ *if* $c \neq k$ *and* 1 *for all* r *when* $c = k$.

Proof. The first-order condition for type x of player 1 who bids b is:

$$\sum_{l=1}^{n} (-1)^{j_l+1} \lambda_1 \left(x, \phi_{j_l}^2(b)\right) \frac{\partial \phi_{j_l}^2}{\partial b}(b) = 1 \tag{FOC}$$

As the FOC must be satisfied for $x = \phi_{i_k}^1(b)$ for all k, we obtain an ODE system, which in matrix form reads as, $\Lambda^1(b) \cdot \Phi^2 = \mathbf{1}$, where Φ^2 is the n–column vector $\left((-1)^{j_l+1} \frac{\partial \phi_{j_l}^2}{\partial b}(b)\right)$ and $\Lambda^1(b)$ is a square matrix with entries given by $\Lambda_{r,c}^1(b) = \lambda_1\left(\phi_{i_r}^1(b), \phi_{j_c}^2(b)\right) = \lambda_1\left(P_r^1(x), P_c^2(x)\right)$ for $y = Q(x)$ and $b = b_1(x)$. This system is invertible because of regularity and $n_1 = n_2$.

To apply Crammer's rule, we define L_k^1 – the matrix constructed by taking Λ^1 and replacing its' k_{th} column by a vector of ones, that is $L_k^1(b) = \left(\Lambda_{1,\ldots,k-1}^1(b), \mathbf{1}, \Lambda_{k+1,\ldots,n_i}^1(b)\right)$. By Cramer's rule,

$$\frac{\partial \phi_{j_l}^2}{\partial b}(b) = (-1) \frac{|L_k^1(x)|}{|\Lambda^1(x)|},$$

which together with $\frac{\partial}{\partial x} P_l^2(x) = \frac{\partial \phi_{j_l}^2}{\partial b} \Big/ \frac{\partial \phi_{j_1}^2}{\partial b}$ establishes the part of the lemma for the pooling functions of player 2. As the equilibrium is symmetric, $P_k^1 = P_k^2$ for all k. ∎

The case when only two types are pooling is much simpler since we have to characterize only one pooling function:

Corollary 4. *If are players are symmetric, in a symmetric equilibrium with* $n_1 = n_2 = 2$, *the pooling function satisfies,* $P'(x) = \dfrac{\lambda_1(x,x) - \lambda_1(P(x),x)}{\lambda_1(x,P(x)) - \lambda_1(P(x),P(x))} = \dfrac{\lambda_2(x,x) - \lambda_2(x,P(x))}{\lambda_2(P(x),x) - \lambda_2(P(x),P(x))}$.

Proof. By lemma 2 and since $y = x$ and $P = P^1 = P^2$ hold in a symmetric equilibrium,

$$\frac{\partial}{\partial x} P(x) = -\frac{\begin{vmatrix} \lambda_1(x,x) & 1 \\ \lambda_1(P(x),x) & 1 \end{vmatrix}}{\begin{vmatrix} 1 & \lambda_1(x,P(x)) \\ 1 & \lambda_1(P(x),P(x)) \end{vmatrix}} = -\frac{\lambda_1(x,x) - \lambda_1(P(x),x)}{\lambda_1(P(x),P(x)) - \lambda(x,P(x))}$$

∎

Appendix C. Parameter Space

In propositions 4,5,6, and 7, subsection 5.1, the equilibrium qualitative features depended only the A, C and D parameters of the valuation $v(x,y) = Ax^2 + By^2 + Cxy + Dx + Ey + F$. For this reason, our parameter space of interest is the set, $\mathcal{P} = \{(A,C,D) : (A,C,D) \in [-a,a] \times [-c,c] \times [-d,d]\}$ for $a,c,d > 0$.

We completely characterize the corresponding regions in $\mathcal{P}$ for which, each of the conditions in prop. 2 holds generically[22]. Our exhaustive characterization comes at a cost: our proofs are computational as we must rely on the cylindrical algebraic decomposition algorithm (see Basu et al., 2006), which is implemented in major computer algebra systems. In the online appendix, we provide the Mathematica© code.

Notation: $x^- \stackrel{\text{def}}{=} \min(x, 0)$.

M: $0 < 2A^- + C^- + D$.

SC: $0 < A^- + (A + C)^- + D$.

INCREASING: $0 < A^- + (A + C)^- + D$ or $-2(A + C)^- - (4A + C)^- < 3D < -3A - 3C^-$.

LOCAL SINGLE CROSSING: $0 < (2A + C)^- + D$.

[22] This is solely for the reader convenience: adding non-generic (lower dimensional, zero Lebesgue measure) components produces an extremely long characterization without adding any relevant information